MATH 382 - Real Analysis (1)
Second Semester - 1446 H
Solution of the First Exam
Dr Tariq A. Alfadhel

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Question (1): [8 marks]

1. Give an example of the following:

   (i) A non-empty set $A \subset \mathbb{R}$ such that $\inf A = \min A$ and $\sup A \notin A$. [1]

   Solution :

   $A = [a, b)$ where $a, b \in \mathbb{R}$ and $a < b$.

   $A = [a, \infty)$ where $a \in \mathbb{R}$.

   (ii) Two infinite subsets $A \subset B$ and $A \sim B$. [1]

   Solution :

   $A = \mathbb{N}_1$ or $\mathbb{N}_2$, and $B = \mathbb{N}$.

   $A = \mathbb{N}$ and $B = \mathbb{Z}$.

   $A = (0, 1)$ and $B = (0, b)$, where $b \in \mathbb{R}$ and $b > 1$.

2. If A and B are two any non-empty upper bounded subsets of $\mathbb{R}$,

   Prove that : $\sup(A \cup B) = \max\{\sup A, \sup B\}$. [3]

   Solution :

   $A \subset A \cup B \implies \sup A \leq \sup(A \cup B)$.

   $B \subset A \cup B \implies \sup B \leq \sup(A \cup B)$.

   Hence, $\max\{\sup A, \sup B\} \leq \sup(A \cup B) \quad \longrightarrow \quad (1).$

   If $x \in A \cup B \implies x \in A$ or $x \in B$

   $\implies x \leq \sup A$ or $x \leq \sup B \implies x \leq \max\{\sup A, \sup B\}$

   which means that $\max\{\sup A, \sup B\}$ is an upper bound of the set $A \cup B$

   Hence, $\sup(A \cup B) \leq \max\{\sup A, \sup B\} \quad \longrightarrow \quad (2)$

   From (1) and (2) : $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

3. If A and B are denumerable subsets of $\mathbb{R}$, Prove that $A \times B$ is a denumerable set. [3]

   Solution :

Since A is denumerable then there exists a bijection $f : A \longrightarrow \mathbb{N}$.

also, since B is denumerable then there exists a bijection $g : B \longrightarrow \mathbb{N}$.

Define $h : A \times B \longrightarrow \mathbb{N}$ as : $h(a,b) = 2^{f(a)}3^{g(b)}$, $\forall\, a \in A, b \in B$.

Suppose $(a_1, b_1), (a_2, b_2) \in A \times B$:

$h(a_1, b_1) = h(a_2, b_2) \implies 2^{f(a_1)}3^{g(b_1)} = 2^{f(a_2)}3^{g(b_2)}$

$\implies 2^{f(a_1)} = 2^{f(a_2)}$ and $3^{g(b_1)} = 3^{g(b_2)}$

$\implies f(a_1) = f(a_2)$ and $g(b_1) = g(b_2)$

$\implies a_1 = a_2$ and $b_1 = b_2$ (since f and g are both injective).

$\implies (a_1, b_1) = (a_2, b_2)$.

Therefore, h is an injection.

Hence, $A \times B \sim R_h \subset \mathbb{N}$.

Since R_h is countable, then $A \times B$ is countable, and being infinite it is denumerable.

Question (2): [17 marks]

1. Give an example of the following:

(i) A convergent sequence which is not monotonic. [1]

Solution :

The sequence $\left(\dfrac{(-1)^n}{n}\right)$.

(ii) A divergent sequence which has a Cauchy subsequence. [1]

Solution :

The sequence $(x_n) = ((-1)^n)$ is divergent,

the subsequence $(x_{2n}) = \left((-1)^{2n}\right)$ is convergent, so it is a Cauchy subsequence .

(iii) An infinite set A such that $\hat{A} = \phi$. [1]

Solution :

$A = \mathbb{N}$ or $A = \mathbb{Z}$.

2. Prove that any convergent sequence is bounded. [2]

Solution :

Suppose the sequence (x_n) converges to x, Let $\epsilon = 1$, then there exists $N \in \mathbb{N}$ such that :

For $n \geq N$: $|x_n - x| < 1$

$\implies$ $||x_n| - |x|| < |x_n - x| < 1$

$\implies$ $-1 < |x_n| - |x| < 1$

$\implies$ $|x_n| < 1 + |x|$

Take $K = \{|x_1|, |x_2|. \cdots, |x_{N-1}|, 1 + |x|\}$

Then $K > 0$ and $|x_n| < K$, $\forall n \in \mathbb{N}$.

Therefore, the sequence (x_n) is bounded .

3. Discuss the convergence of the sequence $(\cos(n\pi))$. [2]

   Solution : Let $(x_n) = (\cos(n\pi))$, consider the subsequences

   $x_{2n} = \cos(2n\pi) = 1 \longrightarrow 1$.

   $x_{2n+1} = \cos((2n+1)\pi) = -1 \longrightarrow -1$.

   Therefore, the sequence $(\cos(n\pi))$ is divergent.

4. Find $\lim_{n\to\infty} \dfrac{2+\sin n}{n^3+1}$. (Justify your answer) [2]

   Solution : Let $x_n = \dfrac{2+\sin n}{n^3+1} = (2+\sin n)\ \dfrac{1}{n^3+1} = a_n\ b_n$, $\forall n \in \mathbb{N}$.

   $|a_n| = |2+\sin n| \leq 2 + |\sin n| \leq 2+1 = 3$, $\forall n \in \mathbb{N}$, so (a_n) is bounded.

   Also, $b_n \longrightarrow 0$, Therefore $x_n = a_n.b_n \longrightarrow 0$.

5. If (x_n) and (y_n) are Cauchy sequences, prove that $(x_n y_n)$ is a Cauchy sequence. [3]

   Solution :

   Since (x_n) is Cauchy then it is bounded, so $|x_n| < K_1$, where $K_1 > 0$.

   Since (y_n) is Cauchy then it is bounded, so $|y_n| < K_2$, where $K_2 > 0$.

   Let $\epsilon > 0$ be given :

   Since (x_n) is Cauchy then there exists $N_1 \in \mathbb{N}$ such that :

   $\forall n, m \geq N_1$: $|x_n - x_m| < \epsilon$.

   Since (y_n) is Cauchy then there exists $N_2 \in \mathbb{N}$ such that :

   $\forall n, m \geq N_2$: $|y_n - y_m| < \epsilon$.

Take $N = \max\{N_1, N_2\}$, then $\forall n, m \geq N$:

$|x_n y_n - x_m y_m| = |x_n y_n - x_m y_n + x_m y_n - x_m y_m|$

$= |y_n (x_n - x_m) + x_m (y_n - y_m)|$

$\leq |y_n| \ \ |x_n y - x_m| + |x_m| \ \ |y_n - y_m|$

$\leq K_2 \ \epsilon + K_1 \ \epsilon = (K_1 + K_2) \ \epsilon = c \ \epsilon$, where $c = K_1 + K_2 > 0$.

Therefore, $(x_n y_n)$ is a Cauchy sequence .

6. If $0 < a < b$, find $\lim\limits_{n\to\infty} \sqrt[n]{a+b}$. [2]

   Solution :

   $0 < a < b \implies a + b > 0$, Therefore, $\lim\limits_{n\to\infty} \sqrt[n]{a+b} = \lim\limits_{n\to\infty} (a+b)^{\frac{1}{n}} = 1$.

   Note that, if $c > 0$, then $\lim\limits_{n\to\infty} c^{\frac{1}{n}} = 1$. (see Example 3.8, page 78).

7. If $x_1 = 1$ and $x_{n+1} = \sqrt{4x_n + 5}$,$\forall n \in \mathbb{N}$, show that (x_n) is monotonic and bounded, then find its limit. [3]

   Solution :

   First - Showing that (x_n) is an increasing sequence :

   (i). $x_1 = 1 \leq 3 = x_2$.

   (ii). Suppose $x_{n-1} \leq x_n$.

   (iii) Proving that $x_n \leq x_{n+1}$:

   $x_{n-1} \leq x_n \implies 4x_{n-1} \leq 4x_n \implies 4x_{n-1} + 5 \leq 4x_n + 5$

   $\implies \sqrt{4x_{n-1} + 5} \leq \sqrt{4x_n + 5} \implies x_n \leq x_{n+1}$.

   Second - Showing that (x_n) is bounded above by 5 :

   (i). $x_1 = 1 \leq 5$.

   (ii). Suppose $x_n \leq 5$.

   (iii) Proving that $x_{n+1} \leq 5$:

   $x_{n+1} = \sqrt{4x_n + 5} \leq \sqrt{4(5) + 5} = \sqrt{25} = 5$.

   Since (x_n) is an increasing and bounded above, then it converges to l .

   Third - Finding the value of l :

   $x_{n+1} = \sqrt{4x_n + 5} \implies l = \sqrt{4l + 5} \implies l^2 = 4l + 5$

   $\implies l^2 - 4l - 5 = 0 \implies (l-5)(l+1) = 0 \implies l = 5 \ , \ l = -1.$

Note that $x_n \geq 1$, $\forall n \in \mathbb{N}$, so $l = -1$ is excluded.

Therefore, $x_n \longrightarrow 5$.

Bonus Question: If (x_n) is an increasing sequence of positive terms which has a convergent subsequence, Prove that (x_n) is convergent.
Solution :
Since (x_n) is an increasing sequence, then it is enough to show that it is bounded above.
Suppose that (x_{n_k}) is the convergent subequence, then (x_{n_k}) is bounded,
$|x_{n_k}| = x_{n_k} \leq M$, $\forall\ n_k \in \mathbb{N}$, where $M > 0$.
$\forall\ k \in \mathbb{N} :\ k \leq n_k \implies x_k < x_{n_k}$ (since (x_n) is increasing) .
$\implies x_k < x_{n_k} \leq M\ ,\ \forall\ k \in \mathbb{N}$.
Therefore, the sequence (x_n) is bounded above, Hence, it is convergent.