MATH 382 - Real Analysis (1)
First Semester - 1447 H
Solution of the First Exam
Dr Tariq A. Alfadhel

Question (1): [2+2+3 = 7 marks]

1. (a). State the Completeness Axiom.

   Solution :

   Every non-empty upper bounded subset of $\mathbb{R}$ has a least upper bound in $\mathbb{R}$.

   Every non-empty lower bounded subset of $\mathbb{R}$ has a greatest lower bound in $\mathbb{R}$.

   (b). If $A = \left\{ q \in \mathbb{Q} : \sqrt{2} \le q < \sqrt{3} \right\}$, Find $\sup A$ and $\inf A$.

   Solution :

   $\sup A = \sqrt{3}$ and $\inf A = \sqrt{2}$.

2. If $A \subseteq \mathbb{R}$ is a non-empty set which is bounded above, and $k > 0$:

   Show that $\sup(kA) = k \ \sup(A)$.

   Solution :

   $\forall a \in A \ : \ a \le \sup A \implies ka \le k \sup A$.

   $k \sup A$ is an upper bound of the set kA, therefore, $\sup(kA) \le k \sup A$.

   $\forall n \in \mathbb{N} \ : \ \exists \ a_n \in A$ such that $\sup A - \dfrac{1}{kn} < a_n \le \sup A$

   $\implies k \sup A - \dfrac{1}{n} < k \ a_n \le \sup(kA) \implies k \sup A - \sup(kA) < \dfrac{1}{n}, \ \forall n \in \mathbb{N}$

   $\implies k \sup A - \sup(kA) \le 0 \implies k \sup A \le \sup(kA)$.

   Therefore, $\sup(kA) = k \ \sup(A)$.

3. If $x, y \in \mathbb{R}$ and $y > x > 0$. Prove that there exists $r \in \mathbb{Q}$ such that $x < r < y$.

   Solution :

   $y > x > 0 \implies y - x > 0 \implies \exists \ n \in \mathbb{N}$ such that $\dfrac{1}{n} < y - x$

   $\implies 1 < ny - nx \implies 1 + nx < ny$.

   Also, $x > 0 \implies nx > 0 \implies \exists \ m \in \mathbb{N}$ such that $m - 1 \le nx < m$

$\implies m \leq 1 + nx < m + 1.$

Now, $nx < m \leq 1 + nx < ny \implies nx < m < ny \implies x < \dfrac{m}{n} < y.$

Let $r = \dfrac{m}{n}$ then $r \in \mathbb{Q}$ and $x < r < y$.

Question (2): [17 marks]

1. Give an example of the following:

(a). A bounded sequence which is not convergent.

Solution :

The sequence $(x_n) = ((-1)^n)$

(b). A convergent sequence which is not monotonic.

Solution :

The sequence $\left(\dfrac{(-1)^n}{n}\right)$ or $\left(\dfrac{(-1)^n}{2^n}\right)$.

2. (a). If (x_n) converges to 0 and (y_n) is bounded. Prove that $(x_n y_n)$ is convergent.

Solution :

First Proof : Since (y_n) is bounded, then $\exists\ K > 0$ such that $|y_n| \leq K$, $\forall\ n \in \mathbb{N}$.

Let $\epsilon > 0$, since (x_n) converges to 0 then $\exists\ N \in \mathbb{N}$ such that

$\forall n \geq N\ :\ |x_n - 0| < \dfrac{\epsilon}{K}.$

Now, $\forall n \geq N\ :\ |x_n y_n - 0| = |x_n|\ \ |y_n| < \dfrac{\epsilon}{K}\ K = \epsilon.$

Therefore, $\lim\limits_{n\to\infty} x_n y_n = 0.$

Second Proof : Since (y_n) is bounded, then $\exists\ K > 0$ such that $|y_n| \leq K$, $\forall\ n \in \mathbb{N}$.

$0 \leq |x_n y_n| \leq K\ \ |x_n|$.

Since $x_n \longrightarrow 0$, then $|x_n| \longrightarrow 0$, therefore, $K\ \ |x_n| \longrightarrow 0$.

Hence $|x_n y_n| \longrightarrow 0$, therefore, $x_n y_n \longrightarrow 0$.

(b). Use (a) to calculate $\lim\limits_{n\to\infty} \dfrac{1 + \sin n}{n^3 + 2}$ (Justify your answer).

Solution :

Let $x_n = \dfrac{1}{n^3 + 2}$ and $y_n = 1 + \sin n$, $\forall\ n \in \mathbb{N}$.

$x_n = \dfrac{1}{n^3+2} \longrightarrow 0$ and $|y_n| = |1+\sin n| \leq 2$, so (y_n) is bounded.

Using (a) : $\lim\limits_{n\to\infty} \dfrac{1+\sin n}{n^3+2} = \lim\limits_{n\to\infty} \left(\dfrac{1}{n^3+2} \, (1+\sin n) \right) = 0.$

3. State the squeeze (sandwich) theorem for sequences and prove it.

   Solution :

   Squeeze (sandwich) Theorem : If $x_n \leq y_n \leq z_n$, $\forall \, n \geq N_0$, where $N_0 \in \mathbb{N}$, and $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = l$, then $\lim\limits_{n\to\infty} y_n = l$.

   Proof : Let $\epsilon > 0$, since $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = l$, then $\exists \, N_1, N_2 \in \mathbb{N}$ such that $\forall \, n \geq N_1 \;:\; |x_n - l| < \epsilon \implies l - \epsilon < x_n < l + \epsilon$

   and $\forall \, n \geq N_2 \;:\; |z_n - l| < \epsilon \implies l - \epsilon < z_n < l + \epsilon$.

   Let $N = \max\{N_0, N_1, N_2\}$, then $\forall \, n \geq N$:

   $l - \epsilon < x_n \leq y_n \leq z_n < l + \epsilon \implies l - \epsilon < y_n < l + \epsilon \implies |y_n - l| < \epsilon.$

   Therefore, $\lim\limits_{n\to\infty} y_n = l$.

4. Calculate $\lim\limits_{n\to\infty} \sqrt[n]{2^n + 3^n}$.

   Solution : $\forall \, n \in \mathbb{N}$,

   $3^n < 2^n + 3^n < 3^n + 3^n = 2 \; (3)^n \implies \sqrt[n]{3^n} < \sqrt[n]{2^n + 3^n} < \sqrt[n]{2 \; (3)^n}$

   $\implies 3 < \sqrt[n]{2^n + 3^n} < 3 \; \sqrt[n]{2}.$

   Since $\lim\limits_{n\to\infty} \sqrt[n]{2} = 1$, using squeeze theorem, $\lim\limits_{n\to\infty} \sqrt[n]{2^n + 3^n} = 3$.

5. If (x_n) is strictly increasing and bounded above, Prove that it is convergent.

   Solution :

   Let $A = \{x_n : n \in \mathbb{N}\}$.

   Since (x_n) is strictly increasing, then A is an infinite set, so $A \neq \phi$.

   Since (x_n) is bounded above, then the set A is bounded above. By Completeness Axiom $\sup A$ exists, let $x_0 = \sup A$.

   To show that $x_n \longrightarrow x_0$:

   Let $\epsilon > 0$, then $x_0 - \epsilon$ is not an upper bound of A, then $\exists \; x_N \in A$ such that $x_0 - \epsilon < x_N \leq x_0 = \sup A$.

   $\forall \, n \geq N \;:\; x_N < x_n$ since (x_n) is strictly increasing, and $x_n \leq x_0$ since $x_0 = \sup A$.

Therefore, $\forall\ n \geq N\ :\ x_N < x_n\ :\ x_0 - \epsilon < x_N < x_n < x_0$

$\implies \forall\ n \geq N\ :\ |x_n - x_0| < \epsilon$. Hence $\lim\limits_{n\to\infty} x_n = x_0$.

6. If $x_1 = 1$ and $x_{n+1} = \dfrac{1}{3}(x_n + 5)$, $\forall n \in \mathbb{N}$,

Show that (x_n) is monotonic and bounded, then compute its limit.

Solution :

First - Showing that (x_n) is an increasing sequence :

(i). $x_1 = 1 \leq \dfrac{1}{3}(1+5) = 2 = x_2$.

(ii). Suppose $x_{n-1} \leq x_n$.

(iii) Proving that $x_n \leq x_{n+1}$:

$x_{n-1} \leq x_n \implies x_{n-1} + 5 \leq x_n + 5 \implies \dfrac{1}{3}(x_{n-1}+5) \leq \dfrac{1}{3}(x_n + 5)$

$\implies x_n \leq x_{n+1}$.

Second - Showing that (x_n) is bounded above by $\dfrac{5}{2}$:

(i). $x_1 = 1 \leq \dfrac{5}{2}$.

(ii). Suppose $x_n \leq \dfrac{5}{2}$.

(iii) Proving that $x_{n+1} \leq \dfrac{5}{2}$:

$x_{n+1} = \dfrac{1}{3}(x_n + 5) \leq \dfrac{1}{3}\left(\dfrac{5}{2} + 5\right) = \dfrac{1}{3}\left(\dfrac{15}{2}\right) = \dfrac{5}{2}$.

Since (x_n) is an increasing and bounded above, then it converges to l .

Third - Finding the value of l :

$x_{n+1} = \dfrac{1}{3}(x_n + 5) \implies l = \dfrac{1}{3}(l+5) \implies 3l = l + 5$

$\implies 2l = 5 \implies l = \dfrac{5}{2}$.

Therefore, $x_n \longrightarrow \dfrac{5}{2}$.