MATH 382 - Real Analysis (1)
Second Semester - 1446 H
Solution of the First Exam
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Question (1): [8 marks]

1. Give an example of the following:

   (i) A non-empty set $A \subset \mathbb{R}$ such that $\inf A = \min A$ and $\sup A \notin A$. [1]

   Solution :

   $A = [a, b)$ where $a, b \in \mathbb{R}$ and $a < b$.

   $A = [a, \infty)$ where $a \in \mathbb{R}$.

   (ii) Two infinite subsets $A \subset B$ and $A \sim B$. [1]

   Solution :

   $A = \mathbb{N}_1$ or $\mathbb{N}_2$, and $B = \mathbb{N}$.

   $A = \mathbb{N}$ and $B = \mathbb{Z}$.

   $A = (0, 1)$ and $B = (0, b)$, where $b \in \mathbb{R}$ and $b > 1$.

2. If A and B are two any non-empty upper bounded subsets of $\mathbb{R}$,

   Prove that : $\sup (A \cup B) = \max \{\sup A, \sup B\}$. [3]

   Solution :

   $A \subset A \cup B \implies \sup A \leq \sup (A \cup B)$.

   $B \subset A \cup B \implies \sup B \leq \sup (A \cup B)$.

   Hence, $\max \{\sup A, \sup B\} \leq \sup (A \cup B) \quad \longrightarrow \quad (1)$.

   If $x \in A \cup B \implies x \in A$ or $x \in B$

   $\implies x \leq \sup A$ or $x \leq \sup B \implies x \leq \max \{\sup A, \sup B\}$

   which means that $\max \{\sup A, \sup B\}$ is an upper bound of the set $A \cup B$

   Hence, $\sup (A \cup B) \leq \max \{\sup A, \sup B\} \quad \longrightarrow \quad (2)$

   From (1) and (2) : $\sup (A \cup B) = \max \{\sup A, \sup B\}$.

3. If A and B are denumerable subsets of $\mathbb{R}$, Prove that $A \times B$ is a denumerable set. [3]

   Solution :

Since A is denumerable then there exists a bijection $f : A \longrightarrow \mathbb{N}$.

also, since B is denumerable then there exists a bijection $g : B \longrightarrow \mathbb{N}$.

Define $h : A \times B \longrightarrow \mathbb{N}$ as : $h(a, b) = 2^{f(a)}3^{g(b)}$, $\forall\ a \in A, b \in B$.

Suppose $(a_1, b_1), (a_2, b_2) \in A \times B$:

$h(a_1, b_1) = h(a_2, b_2) \implies 2^{f(a_1)}3^{g(b_1)} = 2^{f(a_2)}3^{g(b_2)}$

$\implies 2^{f(a_1)} = 2^{f(a_2)}$ and $3^{g(b_1)} = 3^{g(b_2)}$

$\implies f(a_1) = f(a_2)$ and $g(b_1) = g(b_2)$

$\implies a_1 = a_2$ and $b_1 = b_2$ (since f and g are both injective).

$\implies (a_1, b_1) = (a_2, b_2)$.

Therefore, h is an injection.

Hence, $A \times B \sim R_h \subset \mathbb{N}$.

Since R_h is countable, then $A \times B$ is countable, and being infinite it is denumerable.

Question (2): [17 marks]

1. Give an example of the following:

   (i) A convergent sequence which is not monotonic. [1]

   Solution :

   The sequence $\left(\dfrac{(-1)^n}{n}\right)$.

   (ii) A divergent sequence which has a Cauchy subsequence. [1]

   Solution :

   The sequence $(x_n) = ((-1)^n)$ is divergent,

   the subsequence $(x_{2n}) = \left((-1)^{2n}\right)$ is convergent, so it is a Cauchy subsequence .

   (iii) An infinite set A such that $\hat{A} = \phi$. [1]

   Solution :

   $A = \mathbb{N}$ or $A = \mathbb{Z}$.

2. Prove that any convergent sequence is bounded. [2]

   Solution :

   Suppose the sequence (x_n) converges to x, Let $\epsilon = 1$, then there exists $N \in \mathbb{N}$ such that :

For $n \geq N \;:\; |x_n - x| < 1$

$\implies \; ||x_n| - |x|| < |x_n - x| < 1$

$\implies \; -1 < |x_n| - |x| < 1$

$\implies \; |x_n| < 1 + |x|$

Take $K = \{|x_1|, |x_2| . \cdots, |x_{N-1}|, 1 + |x|\}$

Then $K > 0$ and $|x_n| < K$, $\forall n \in \mathbb{N}$.

Therefore, the sequence (x_n) is bounded .

3. Discuss the convergence of the sequence $(\cos(n\pi))$. [2]

   Solution : Let $(x_n) = (\cos(n\pi))$, consider the subsequences

   $x_{2n} = \cos(2n\pi) = 1 \;\longrightarrow\; 1$.

   $x_{2n+1} = \cos((2n+1)\pi) = -1 \;\longrightarrow\; -1$.

   Therefore, the sequence $(\cos(n\pi))$ is divergent.

4. Find $\lim\limits_{n\to\infty} \dfrac{2 + \sin n}{n^3 + 1}$. (Justify your answer) [2]

   Solution : Let $x_n = \dfrac{2 + \sin n}{n^3 + 1} = (2 + \sin n)\;\dfrac{1}{n^3 + 1} = a_n\; b_n$, $\forall n \in \mathbb{N}$.

   $|a_n| = |2 + \sin n| \leq 2 + |\sin n| \leq 2 + 1 = 3$, $\forall n \in \mathbb{N}$, so (a_n) is bounded.

   Also, $b_n \;\longrightarrow\; 0$, Therefore $x_n = a_n . b_n \;\longrightarrow\; 0$.

5. If (x_n) and (y_n) are Cauchy sequences, prove that $(x_n y_n)$ is a Cauchy sequence. [3]

   Solution :

   Since (x_n) is Cauchy then it is bounded, so $|x_n| < K_1$, where $K_1 > 0$.

   Since (y_n) is Cauchy then it is bounded, so $|y_n| < K_2$, where $K_2 > 0$.

   Let $\epsilon > 0$ be given :

   Since (x_n) is Cauchy then there exists $N_1 \in \mathbb{N}$ such that :

   $\forall n, m \geq N_1 \;:\; |x_n - x_m| < \epsilon$.

   Since (y_n) is Cauchy then there exists $N_2 \in \mathbb{N}$ such that :

   $\forall n, m \geq N_2 \;:\; |y_n - y_m| < \epsilon$.

Take $N = \max\{N_1, N_2\}$, then $\forall n, m \geq N$:

$|x_n y_n - x_m y_m| = |x_n y_n - x_m y_n + x_m y_n - x_m y_m|$

$= |y_n (x_n - x_m) + x_m (y_n - y_m)|$

$\leq |y_n| \; |x_n y - x_m| + |x_m| \; |y_n - y_m|$

$\leq K_2 \; \epsilon + K_1 \; \epsilon = (K_1 + K_2) \; \epsilon = c \; \epsilon$, where $c = K_1 + K_2 > 0$.

Therefore, $(x_n y_n)$ is a Cauchy sequence .

6. If $0 < a < b$, find $\lim\limits_{n\to\infty} \sqrt[n]{a+b}$. [2]

   Solution :

   $0 < a < b \implies a + b > 0$, Therefore, $\lim\limits_{n\to\infty} \sqrt[n]{a+b} = \lim\limits_{n\to\infty} (a+b)^{\frac{1}{n}} = 1$.

   Note that, if $c > 0$, then $\lim\limits_{n\to\infty} c^{\frac{1}{n}} = 1$. (see Example 3.8, page 78).

7. If $x_1 = 1$ and $x_{n+1} = \sqrt{4x_n + 5}$,$\forall n \in \mathbb{N}$, show that (x_n) is monotonic and bounded, then find its limit. [3]

   Solution :

   First - Showing that (x_n) is an increasing sequence :

   (i). $x_1 = 1 \leq 3 = x_2$.

   (ii). Suppose $x_{n-1} \leq x_n$.

   (iii) Proving that $x_n \leq x_{n+1}$:

   $x_{n-1} \leq x_n \implies 4x_{n-1} \leq 4x_n \implies 4x_{n-1} + 5 \leq 4x_n + 5$

   $\implies \sqrt{4x_{n-1} + 5} \leq \sqrt{4x_n + 5} \implies x_n \leq x_{n+1}$.

   Second - Showing that (x_n) is bounded above by 5 :

   (i). $x_1 = 1 \leq 5$.

   (ii). Suppose $x_n \leq 5$.

   (iii) Proving that $x_{n+1} \leq 5$:

   $x_{n+1} = \sqrt{4x_n + 5} \leq \sqrt{4(5) + 5} = \sqrt{25} = 5$.

   Since (x_n) is an increasing and bounded above, then it converges to l .

   Third - Finding the value of l :

   $x_{n+1} = \sqrt{4x_n + 5} \implies l = \sqrt{4l + 5} \implies l^2 = 4l + 5$

   $\implies l^2 - 4l - 5 = 0 \implies (l-5)(l+1) = 0 \implies l = 5 \; , \; l = -1.$

Note that $x_n \geq 1$, $\forall n \in \mathbb{N}$, so $l = -1$ is excluded.

Therefore, $x_n \longrightarrow 5$.

Bonus Question: If (x_n) is an increasing sequence of positive terms which has a convergent subsequence, Prove that (x_n) is convergent.

Solution :

Since (x_n) is an increasing sequence, then it is enough to show that it is bounded above.

Suppose that (x_{n_k}) is the convergent subequence, then (x_{n_k}) is bounded,
$|x_{n_k}| = x_{n_k} \leq M$, $\forall\ n_k \in \mathbb{N}$, where $M > 0$.
$\forall\ k \in \mathbb{N} : k \leq n_k \implies x_k < x_{n_k}$ (since (x_n) is increasing) .
$\implies x_k < x_{n_k} \leq M$, $\forall\ k \in \mathbb{N}$.
Therefore, the sequence (x_n) is bounded above, Hence, it is convergent.

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Solution of the Second Exam
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Question (1): [(1+1)+3+2+3 = 10 marks]

1. Give an example of the following:

   (i) A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that the limit does not exist at any point.

   Solution:

   $f(x) = \begin{cases} a & , \quad x \in \mathbb{Q} \\ -a & , \quad x \in \mathbb{Q}^c \end{cases}$, where $a \in \mathbb{R}^*$.

   (ii) Two different increasing functions such that their product is not increasing.

   Solution:

   $f(x) = x$ and $g(x) = x^3$ on $[-1, 0]$.

   Both f, g are increasing, but $(fg)(x) = x^4$ is decreasing on $[-1, 0]$.

2. Let $f : D \longrightarrow \mathbb{R}$ and $c \in \hat{D}$, if $\lim_{x \to c} f(x) = l$, Prove that for every sequence (x_n) in D such that $x_n \neq c$ for any $n \in \mathbb{N}$ and $x_n \longrightarrow c$, the sequence $(f(x_n))$ converges to l.

   Solution:

   Let $\epsilon > 0$ be given, since $\lim_{x \to c} f(x) = l$, then there exists $\delta > 0$ such that :

   $x \in D \ , \ 0 < |x - c| < \delta \implies |f(x) - l| < \epsilon$.

   Let (x_n) be any sequence in D such that $x_n \neq c$ for any $n \in \mathbb{N}$ and $x_n \longrightarrow c$, then for this $\delta > 0$ there exists $N \in \mathbb{N}$ such that

   $\forall n \geq N \ : \ |x_n - c| < \delta$.

   Since $x_n \neq c$ for any $n \in \mathbb{N}$, then $0 < |x_n - c|$.

   Therefore, $\forall n \geq N \ : \ 0 < |x_n - c| < \delta \implies |f(x_n) - l| < \epsilon$.

   Hence, the sequence $(f(x_n))$ converges to l.

3. Discuss the existence of $\lim_{x \to 0} \cos\left(\frac{4}{x}\right)$.

   Solution:

Let $x_n = \dfrac{4}{2n\pi}$, then $x_n \neq 0$, $\forall\ n \in \mathbb{N}$, and $x_n \longrightarrow 0$.

Let $y_n = \dfrac{4}{\pi + 2n\pi}$, then $y_n \neq 0$, $\forall\ n \in \mathbb{N}$, and $y_n \longrightarrow 0$.

$\cos\left(\dfrac{4}{x_n}\right) = \cos(2n\pi) = 1 \longrightarrow 1$.

$\cos\left(\dfrac{4}{y_n}\right) = \cos(\pi + 2n\pi) = -1 \longrightarrow -1$.

Therefore, $\lim\limits_{x\to 0} \cos\left(\dfrac{4}{x}\right)$ does not exist.

4. If f is increasing on (a, b) and unbounded above,

   Prove that $\lim\limits_{x\to b^-} f(x) = \infty$.

   Solution:

   To show that : $\forall M > 0$ there exists $\delta > 0$ such that :

   $\forall x \in (a, b)$, $0 < b - x < \delta \implies f(x) \geq M$.

   Since f is unbounded above on (a, b), then there exist $x_0 \in (a, b)$ such that $f(x_0) \geq M$.

   Let $\delta = b - x_0$ then $\delta > 0$.

   $\forall x \in (a, b)$, $0 < b - x < \delta \implies 0 < b - x < b - x_0 \implies x > x_0$

   $\implies f(x) > f(x_0) \geq M$. (since f is increasing).

   Therefore, $\lim\limits_{x\to b^-} f(x) = \infty$.

Question (2): [(1+1)+3+2+(2+2)+(2+2) = 15 marks]

1. Give an example of the following:

   (i) A function f not continuous at one point, but $|f|$ is continuous at this point.

   Solution:

   $$f(x) = \begin{cases} a & , \quad x \geq 0 \\ -a & , \quad x < 0 \end{cases} \text{ , where } a \in \mathbb{R}^* .$$

   f is not continuous at $x = 0$, but $|f(x)| = |a|$ is continuous at $x = 0$.

   (ii) Two different functions f, g , where f is continuous at c and g is not continuous at c, but fg is continuous at c.

Solution:

$f(x) = x^2$ is continuous at $c = 0$ and $g(x) = \dfrac{1}{x}$ is not continuous at $c = 0$.

$(fg)(x) = x$ is continuous at $c = 0$.

2. Let $f : D \longrightarrow \mathbb{R}$, $g : E \longrightarrow \mathbb{R}$, and $f(D) \subseteq E$. If f is continuous at $c \in D$ and g is continuous at $f(c)$. Prove that $g \circ f$ is continuous at c.

   Solution:

   Let (x_n) be any sequence in D such that $x_n \longrightarrow c$,

   Since f is continuous at c then $f(x_n) \longrightarrow f(c)$.

   The sequence $(f(x_n))$ is in E and $f(x_n) \longrightarrow f(c)$,

   Since g is continuous at $f(c)$ the $g(f(x_n)) \longrightarrow g(f(c))$.

   Therefore, $(g \circ f)(x_n) \longrightarrow (g \circ f)(c)$, and $g \circ f$ is continuous at c.

3. If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and $f(x) > 0$, $\forall x \in [a, b]$.

   Prove that there exists $\alpha > 0$ such that $f(x) > \alpha$, $\forall x \in [a, b]$.

   Solution:

   Since f is continuous on a closed and bounded interval, then f attains its minimum at a point $x_0 \in [a, b]$, That is $f(x) \geq f(x_0)$, $\forall x \in [a, b]$.

   Since $f(x) > 0$, $\forall x \in [a, b]$ then $f(x_0) > 0$.

   Take $\alpha = \dfrac{f(x_0)}{2}$, then $\alpha > 0$ and $f(x) \geq f(x_0) > \alpha$, $\forall x \in [a, b]$.

4. (i) State the Intermediate Value Property of continuous functions.

   Solution:

   If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and $\lambda \in \mathbb{R}$ lies between $f(a)$ and $f(b)$ then there exists $c \in (a, b)$ such that $f(c) = \lambda$.

   (ii) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, $f(a) = b$, $f(b) = a$, where $a, b \in \mathbb{R}$ and $b > a$. Prove that f has a fixed point.

   Solution:

   Define $g : [a, b] \longrightarrow \mathbb{R}$ as $g(x) = f(x) - x$.

   Since f is continuous onn $\mathbb{R}$ then g is continuous on $[a, b]$.

$g(a) = f(a) - a = b - a > 0$ and $g(b) = f(b) - b = a - b < 0$.

That is $g(b) < 0 < g(a)$. By I.V.P there exist $c \in (a, b)$ such that $g(c) = 0$.

Therefore, $f(c) - c = 0 \implies f(c) = c$.

5. (i) Show that $f(x) = \sqrt{x}$ satisfies Lipschitz condition on $[1, \infty)$.

Solution:

$\forall\ x, y \in [1, \infty)$, $|f(x) - f(y)| = \left|\sqrt{x} - \sqrt{y}\right| = \left|\left(\sqrt{x} - \sqrt{y}\right)\ \dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right|$

$|f(x) - f(y)| = \left|\dfrac{x - y}{\sqrt{x} + \sqrt{y}}\right| = \dfrac{1}{\sqrt{x} + \sqrt{y}}\ |x - y|.$

$x, y \in [1.\infty)\ :\ x \geq 1$ and $y \geq 1 \implies \sqrt{x} \geq 1$ and $\sqrt{y} \geq 1$

$\implies \sqrt{x} + \sqrt{y} \geq 2 \implies \dfrac{1}{\sqrt{x} + \sqrt{y}} \leq 2$.

Therefore, $|f(x) - f(y)| = \left|\sqrt{x} - \sqrt{y}\right| \leq \dfrac{1}{2}\ |x - y|.$

$f(x)$ satisfies Lipschitz condition on $[1, \infty)$.

(ii) Show that $f(x) = \dfrac{1}{x^3}$ is not uniformly continuous on $(0, \infty)$.

Solution:

Let $x_n = \dfrac{1}{2n}$ and $t_n = \dfrac{1}{n}$ for every $n \in \mathbb{N}$, then

$|x_n - t_n| = \left|\dfrac{1}{2n} - \dfrac{1}{n}\right| = \left|\dfrac{-1}{2n}\right| = \dfrac{1}{2n} \longrightarrow 0.$

But $|f(x_n) - f(t_n)| = \left|\dfrac{1}{\left(\frac{1}{2n}\right)^3} - \dfrac{1}{\left(\frac{1}{n}\right)^3}\right| = \left|8n^3 - n^3\right| = 7n^3 \longrightarrow \infty.$

Therefore, $f(x) = \dfrac{1}{x^3}$ is not uniformly continuous on $(0, \infty)$.

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Solution of the Final Exam
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Question (1): [2+2+2+2 = 8 marks]

1. If $A \subseteq \mathbb{R}$ is a non-empty set which is bounded above, and $k > 0$:

   Show that $\sup(kA) = k\ \sup(A)$.

   Solution :

   $\forall a \in A\ :\ a \leq \sup A \implies ka \leq k\ \sup A$

   So, $k\ \sup A$ is an upper bound of the set kA.

   Therefore, $\sup(kA) \leq k\ \sup A \qquad \longrightarrow\ (1)$.

   $\forall n \in \mathbb{N}$, there exists $a_n \in A$ such that $\sup A - \dfrac{1}{n} \leq a_n$.

   So, $k \sup A - \dfrac{k}{n} \leq k\ a_n \leq \sup(kA)$

   $\implies\ k\ \sup A - \dfrac{k}{n} \leq \sup(kA)\ ,\ \forall n \in \mathbb{N}$

   Therefore, $k\ \sup A \leq \sup(kA) \qquad \longrightarrow\ (2)$.

   From (1) and (2) : $\sup(kA) = k\ \sup(A)$.

2. Let $S \subseteq \mathbb{R}$ be a non-empty set which is bounded below. Show that there exists a sequence (x_n) in S which converges to $u = \inf S$.

   Solution :

   $\forall n \in \mathbb{N}$ there exists $x_n \in S$ such that $u = \inf S \leq x_n \leq u + \dfrac{1}{n}$.

   The sequence (x_n) is in S and $0 \leq |x_n - u| \leq \dfrac{1}{n}\ ,\ \forall n \in \mathbb{N}$.

   Therefore, $x_n\ \longrightarrow\ u = \inf S$.

3. If (x_n) is convergent, show that it is a Cauchy sequence.

   Solution :

   Suppose that $x_n \longrightarrow x_0$.

   Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that : $\forall n \geq N\ :\ |x_n - x_0| \leq \dfrac{\epsilon}{2}$.

   So, $\forall n, m \geq N$:

$$|x_n - x_m| = |x_n - x_0 + x_0 - x_m| \leq |x_n - x_0| + |x_m - x_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ .$$

Therefore, (x_n) is a Cauchy sequence.

4. (i) Give an example of an unbounded sequence that has a convergent subsequence.

   Solution :

   Let (x_n) be the sequence where $x_{2n-1} = 2n - 1$ and $x_{2n} = \dfrac{1}{2n}$, $\forall n \in \mathbb{N}$.

   (x_n) is not bounded since (x_{2n-1}) is not bounded, and the subsequence (x_{2n}) converges to zero.

   (ii) Give an example of a countable set A such that $\hat{A}$ is not countable.

   Solution :

   $A = \mathbb{Q}$ is a countable set and $\hat{A} = \mathbb{R}$ is not countable.

Question (2): [3+3+2 = 8 marks]

1. If $f : D \longrightarrow \mathbb{R}$, $c \in \hat{D}$ and $\lim\limits_{x \to c} f(x) = l$.

   Show that f is bounded in a neighborhood of c .

   Solution :

   Let $\epsilon = 1$, since $\lim\limits_{x \to c} f(x) = l$, then there exists δ such that :

   $\forall x \in D \ : \ 0 < |x - c| < \delta \implies |f(x) - l| < 1$

   $\implies ||f(x)| - |l|| \leq |f(x) - l| < 1 \implies -1 < |f(x)| - |l| < 1$

   $\implies -1 + |l| < |f(x)| < 1 + |l| \implies |f(x)| < 1 + |l|$.

   Let $U = (x - \delta, x + \delta)$, then U is a neighborhood of c and

   $|f(x)| < 1 + |l|$ for all $x \in U \setminus \{c\}$.

   If $c \in U$ take $M = \max\{f(c), 1 + |l|\}$, otherwise take $M = 1 + |l|$.

   Therefore, $|f(x)| \leq M$ for all $x \in U$.

2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

   If f has a limit at some point in $\mathbb{R}$. Prove that

   (i) f has a limit at every point in $\mathbb{R}$.

   Solution :

Note that $f(0) = f(0+0) = f(0) + f(0) = 2f(0) \implies f(0) = 0$.

Also, $0 = f(0) = f(x-x) = f(x) + f(-x) \implies f(-x) = -f(x)$.

Suppose that $\lim\limits_{x \to c} f(x) = l$, where $c \in \mathbb{R}$.

Let $t \in \mathbb{R}$, where $t \neq c$, and let (x_n) be any sequence in $\mathbb{R}$ such that $x_n \neq t$ for all $n \in \mathbb{N}$ and $x_n \longrightarrow t$, then $x_n - t + c \longrightarrow c$.

Since $\lim\limits_{x \to c} f(x) = l$, then $f(x_n - t + c) \longrightarrow l$

$\implies f(x_n) - f(t) + f(c) \longrightarrow l \implies f(x_n) \longrightarrow l + f(t) - f(c)$.

Therefore, $\lim\limits_{x \to t} f(x) = l + f(t) - f(c)$, and f has a limit at any point in $\mathbb{R}$.

(ii) $\lim\limits_{x \to 0} f(x) = 0$.

Solution :

Suppose that $\lim\limits_{x \to 0} f(x) = l_0$, let (x_n) be any sequence in $\mathbb{R}$ such that $x_n \neq 0$ for all $n \in \mathbb{N}$ and $x_n \longrightarrow 0$, then $f(x_n) \longrightarrow l_0$.

Note that the sequence $(2x_n)$ also in $\mathbb{R}$, $2x_n \neq 0$ for all $n \in \mathbb{N}$ and $2x_n \longrightarrow 0$, then $f(2x_n) = 2f(x_n) \longrightarrow l_0 \implies f(x_n) \longrightarrow \frac{l_0}{2}$.

By the uniqueness of the limit, $l_0 = \frac{l_0}{2} \implies 2l_0 = l_0 \implies l_0 = 0$.

Therefore, $\lim\limits_{x \to 0} f(x) = 0$.

3. If $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = l > 0$ and $\lim\limits_{x \to \infty} f(x) = \infty$, show that $\lim\limits_{x \to \infty} g(x) = \infty$.

Solution :

Let $\epsilon = \frac{l}{2} > 0$, since $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = l > 0$, then $\exists\ N_1 > 0$ such that

$$\forall x \geq N_1 \ : \ \left| \frac{f(x)}{g(x)} - l \right| < \frac{l}{2} \implies -\frac{l}{2} < \frac{f(x)}{g(x)} - l < \frac{l}{2} \implies \frac{f(x)}{g(x)} < \frac{3l}{2}$$

$$\implies \frac{3l}{2}\, g(x) > f(x) \implies g(x) > \frac{2}{3l}\, f(x) \ .$$

Let $M > 0$ be given, since $\lim\limits_{x \to \infty} f(x) = \infty$, then $\exists\ N_2 > 0$ such that

$$\forall x \geq N_2 \ : \ f(x) > \frac{3l}{2}\, M \ .$$

Take $N = \max\{N_1, N_2\}$, then

$$\forall\ x > N \ : \ g(x) > \frac{2}{3l}\, f(x) > \frac{2}{3l}\left(\frac{3l}{2}\, M\right) = M \implies g(x) > M \ .$$

Therefore, $\lim_{x \to \infty} g(x) = \infty$.

Question (3): [3+3+2+2 = 10 marks]

1. If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, Prove that f is bounded.

   Solution :

   Let $I = [a, b]$.

   Suppose f is not bounded on $f(I)$, then $\forall\ n \in \mathbb{N}$, $\exists\ x_n \in I$ such that $|f(x_n)| \geq n$. The sequence $(f(x_n))$ is not convergent.

   The sequence (x_n) is in I, and I is bounded, then it has a convergent subsequence (x_{n_k}), suppose $x_{n_k} \longrightarrow x_0$, since I is closed then $x_0 \in I$.

   Since f is continuous on I then $f(x_{n_k}) \longrightarrow f(x_0)$.

   So, the subsequence $(f(x_{n_k}))$ is convergent and hence it is bounded, which is a contradiction, since $f(x_{n_k}) \geq n_k \geq k$, $\forall\ k \in \mathbb{N}$.

   Therefore, f is bounded on I .

2. If $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are both continuous, $f(a) > g(a)$ and $f(b) < g(b)$, where $a, b \in \mathbb{R}$ and $a < b$. Show that there exists $c \in \mathbb{R}$ such that $f(c) = g(c)$.

   Solution :

   Define $h : \mathbb{R} \longrightarrow\ \mathbb{R}$ by $h(x) = f(x) - g(x)$, $\forall\ x \in \mathbb{R}$.

   Since f and g are both continuous on $\mathbb{R}$, then h is also continuous on $\mathbb{R}$.

   $h(a) = f(a) - g(a) > 0$ and $h(b) = f(b) - g(b) < 0$.

   So, $h(b) < 0 < h(a)$, By the intermediate value property, $\exists\ c \in (a, b)$ such that $h(c) = 0 \implies f(c) - g(c) = 0 \implies f(c) = g(c)$.

3. Show that $f(x) = e^x$ is not uniformly continuous on $[1, \infty)$.

   Solution :

   Take $x_n = \ln(n + 3)$ and $t_n = \ln(n + 2)$, $\forall\ n \in \mathbb{N}$.

   The sequences (x_n) and (t_n) are both in $[1, \infty)$.

   $|x_n - t_n| = |\ln(n + 3) - \ln(n + 2)| = \left|\ln\left(\frac{n+3}{n+2}\right)\right| \longrightarrow\ \ln(1) = 0$.

   $|f(x_n) - f(t_n)| = \left|e^{\ln(n+3)} - e^{\ln(n+2)}\right| = |(n + 3) - (n + 2)| = 1 \longrightarrow 1$.

   $|f(x_n) - f(t_n)| \nrightarrow\ 0$.

   Therefore, $f(x) = e^x$ is not uniformly continuous on $[1, \infty)$.

4. If $f : D \longrightarrow \mathbb{R}$ is continuous, prove that the set $\{x \in D \ : \ f(x) = 0\}$ is closed in D .

Solution :

Let $A = \{x \in D \ : \ f(x) = 0\}$, then $A = f^{-1}(\{0\})$.

Since the set $\{0\}$ is closed in $\mathbb{R}$, and f is continuous on D, then the set $f^{-1}(\{0\})$ is closed in D .

Therefore, $A = \{x \in D \ : \ f(x) = 0\}$ is closed in D .

Question (4): [2+3+3+3+3 = 14 marks]

1. Let f be differentiable at a. Find $\lim\limits_{x \to a} \dfrac{a^n \ f(x) - x^n \ f(a)}{x - a}$, where $n \in \mathbb{N}$.

   Solution :

   $$\lim_{x \to a} \frac{a^n \ f(x) - x^n \ f(a)}{x - a} = \lim_{x \to a} \frac{a^n \ f(x) - a^n \ f(a) + a^n \ f(a) - x^n \ f(a)}{x - a}$$

   $$= \lim_{x \to a} \frac{a^n \ [f(x) - f(a)] - (x^n - a^n) \ f(a)}{x - a}$$

   $$= \lim_{x \to a} \left[a^n \ \left(\frac{f(x) - f(a)}{x - a} \right) - \left(\frac{x^n - a^n}{x - a} \right) \ f(a) \right] = a^n \ f'(a) - na^{n-1} \ f(a)$$

   Note that $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = f'(a)$ (since f is differentiable at a) .

   Also, $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = n \ a^{n-1}$ (by L'Höpital's rule) .

2. If f is continuous on $[a, b]$, differentiable on (a, b) and $f'(x) = 0$ for all $x \in (a, b)$. Show that f is constant on $[a, b]$.

   Solution :

   To show that $f(x) = f(y), \forall \ x, y \in [a, b]$ and $x \neq y$.

   Apply Mean Value Theorem to f on the interval I between x and y, then $\exists \ c \in I$ such that $\dfrac{f(x) - f(y)}{x - y} = f'(c) = 0 \implies f(x) - f(y) = 0$ $\implies f(x) = f(y) \ , \ \forall \ x, y \in I$.

   Therefore, f is constant on $[a, b]$

3. Show that $\left|\tan^{-1} x - \tan^{-1} y\right| \leq |x - y|, \forall \ x, y \in \mathbb{R}$.

Solution :

$\forall\ x, y \in \mathbb{R}$ and $x \neq y$, apply Mean Value Theorem to $f(x) = \tan^{-1} x$ on the interval I between x and y, then $\exists\ c \in I$ such that

$$\frac{\tan^{-1} x - \tan^{-1} y}{x - y} = \frac{1}{1+c^2} \implies \left|\frac{\tan^{-1} x - \tan^{-1} y}{x - y}\right| = \left|\frac{1}{1+c^2}\right| \leq 1$$

$$\frac{\left|\tan^{-1} x - \tan^{-1} y\right|}{|x - y|} \leq 1 \implies \left|\tan^{-1} x - \tan^{-1} y\right| \leq |x - y| \ .$$

4. Define two functions f and g, in a neighborhood of zero such that

$\lim_{x \to 0} \frac{f(x)}{g(x)} = 0$ but $\lim_{x \to 0} \frac{f'(x)}{g'(x)}$ does not exist.

Solution :

Let $f(x) = x$ and $g(x) = x^2 + 1$.

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x}{x^2+1} = \frac{0}{0+1} = 0 \ .$$

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{1}{2x} \text{ does not exist.}$$

5. Show that $\left|\cos x - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)\right| \leq \frac{1}{120}$, for all $|x| \leq 1$.

Solution :

Using Taylor's Theorem on $f(x) = \cos x$ at $x_0 = 0$.

$$\begin{array}{lll} f(x) = \cos x & f(0) = \cos(0) = 1 & \\ f'(x) = -\sin x & f'(0) = -\sin(0) = 0 & \\ f''(x) = -\cos x & f''(x) = -\cos(0) = -1 & \\ f^{(3)}(x) = \sin x & f^{(3)}(0) = \sin(0) = 0 & \\ f^{(4)}(x) = \cos x & f^{(4)}(0) = \cos(0) = 1 & \\ f^{(5)}(x) = -\sin x & f^{(5)}(c) = -\sin(c) & \text{, where } c \in (0, x) \end{array}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{-\sin(c)\ x^5}{5!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \frac{-\sin(c)\ x^5}{120}$$

$$\cos x - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) = \frac{-\sin(c)\ x^5}{120}$$

$$\left|\cos x - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)\right| = \left|\frac{-\sin(c)\ x^5}{120}\right| = \frac{|-\sin(c)|\ |x|^5}{120}$$

$$\left|\cos x - \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)\right| \leq \frac{(1)(1)^5}{120} = \frac{1}{120} \text{ , for all } |x| \leq 1 \ .$$