MATH 343 · Group Theory

Chapter 5 Assigned Problems

Exercises 2, 5, 8, 27, 46, and 47

Solutions based on Gallian, Contemporary Abstract Algebra, 10th edition

Exercise 2: Question

Question

Let α and β be the following permutations in S_8.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{pmatrix}$$

Write α, β, and αβ as products of disjoint cycles and as products of 2-cycles.

Exercise 2: Solution (1 of 2)

α and β in disjoint cycle form, and then as products of 2-cycles.

Disjoint cycles: follow each symbol until it returns

Trace α

$$1 \to 2 \to 3 \to 4 \to 5 \to 1, \qquad 6 \to 7 \to 8 \to 6$$

Trace β

$$1 \to 1, \qquad 2 \to 3 \to 8 \to 4 \to 7 \to 2, \qquad 5 \to 6 \to 5$$

$$\alpha = (12345)(678), \qquad \beta = (23847)(56)$$

2-cycles: $(a_1 a_2 \cdots a_k) = (a_1 a_k)(a_1 a_{k-1}) \cdots (a_1 a_2)$

$$\alpha = (15)(14)(13)(12)(68)(67)$$

$$\beta = (27)(24)(28)(23)(56)$$

The fixed point 1 of β is left out, as usual.

Exercise 2: Solution (2 of 2)

The product αβ, read right to left: β acts first, then α.

x	1	2	3	4	5	6	7	8
β(x)	1	3	8	7	6	5	2	4
α(β(x))	2	4	6	8	7	1	3	5

Follow the bottom row from 1: 1 → 2 → 4 → 8 → 5 → 7 → 3 → 6 → 1, a single 8-cycle.

$$\alpha\beta = (12485736) = (16)(13)(17)(15)(18)(14)(12)$$

A parity check

α uses 4 + 2 = 6 transpositions (even) and β uses 4 + 1 = 5 (odd). So αβ must be odd, and it is: 7 transpositions.

Exercise 5: Question

Question

What is the order of each of the following permutations?

(a) $(124)(357)$ (b) $(124)(3567)$ (c) $(124)(35)$

(d) $(124)(357869)$ (e) $(1235)(24567)$ (f) $(345)(245)$

Exercise 5: Solution (1 of 2)

Parts (a)–(d) are already in disjoint cycle form, so Theorem 5.3 applies directly.

$$|\sigma| = \operatorname{lcm}(\text{lengths of the disjoint cycles of } \sigma) \qquad \text{(Theorem 5.3)}$$

Part	Permutation	Cycle lengths	Order
(a)	(124)(357)	3, 3	lcm(3, 3) = 3
(b)	(124)(3567)	3, 4	lcm(3, 4) = 12
(c)	(124)(35)	3, 2	lcm(3, 2) = 6
(d)	(124)(357869)	3, 6	lcm(3, 6) = 6

Parts (e) and (f) are different

There the two cycles share symbols, so Theorem 5.3 does not apply yet. Reading off lcm(4, 5) = 20 and lcm(3, 3) = 3 would give two wrong answers. Multiply first.

Exercise 5: Solution (2 of 2)

Parts (e) and (f): multiply right to left, then use Theorem 5.3.

(e) (1235)(24567)

x	1	2	3	4	5	6	7
after (24567)	1	4	3	5	6	7	2
after (1235)	2	4	5	1	6	7	3

$$(1235)(24567) = (124)(3567), \qquad \text{order } \operatorname{lcm}(3,4) = 12$$

(f) (345)(245)

x	1	2	3	4	5
after (245)	1	4	3	5	2
after (345)	1	5	4	3	2

$$(345)(245) = (25)(34), \qquad \text{order } \operatorname{lcm}(2,2) = 2$$

Exercise 8: Question

Question

Determine whether the following permutations are even or odd.

$$\text{(a)}\ (135) \qquad \text{(b)}\ (1356) \qquad \text{(c)}\ (13567)$$

$$\text{(d)}\ (12)(134)(152) \qquad \text{(e)}\ (1243)(3521)$$

Exercise 8: Solution

Even or odd? Count 2-cycles: a k-cycle is a product of k − 1 of them.

$$(a_1 a_2 \cdots a_k) = (a_1 a_k)(a_1 a_{k-1}) \cdots (a_1 a_2)$$

Part	Permutation	As 2-cycles	Count	Parity
(a)	(135)	(15)(13)	2	even
(b)	(1356)	(16)(15)(13)	3	odd
(c)	(13567)	(17)(16)(15)(13)	4	even
(d)	(12)(134)(152)	(12)(14)(13)(12)(15)	1 + 2 + 2 = 5	odd
(e)	(1243)(3521)	(13)(14)(12)(31)(32)(35)	3 + 3 = 6	even

A check on (d) and (e)

In disjoint form, (12)(134)(152) = (15)(234), which uses 1 + 2 = 3 two-cycles, and (1243)(3521) = (354), which uses 2. Same parities, as Theorem 5.5 guarantees.

Exercise 27: Question

Question

Show that if H is a subgroup of S_n, then either every member of H is an even permutation or exactly half of the members are even.

Gallian uses this result again in Chapter 24.

Exercise 27: Solution (1 of 2)

$H \leq S_n$: either every member of H is even, or exactly half of them are.

Set-up

Split H into E = {even members of H} and O = {odd members of H}. If O is empty, every member of H is even and there is nothing more to prove.

Otherwise fix one odd permutation σ in H, and define

$$\lambda: E \to O, \qquad \lambda(\alpha) = \sigma\alpha$$

λ lands in O

σα is in H, by closure. It is odd: an odd number of 2-cycles followed by an even number is an odd number of 2-cycles.

λ is one-to-one

$$\sigma\alpha = \sigma\alpha' \Rightarrow \alpha = \alpha' \qquad \text{(left cancellation)}$$

Exercise 27: Solution (2 of 2)

The map $\lambda(\alpha) = \sigma\alpha$ from the even members E to the odd members O.

λ is onto

Take γ in O. The inverse σ^{-1} is in H and is odd (write the 2-cycles of σ in reverse order). So $\sigma^{-1}\gamma$ is in H and is even, that is, $\sigma^{-1}\gamma \in E$, and

$$\lambda\left(\sigma^{-1}\gamma\right) = \sigma\sigma^{-1}\gamma = \gamma$$

Conclusion

λ is a bijection, so |E| = |O| and |H| = |E| + |O| = 2|E|.

Exactly half of the members of H are even.

Theorem 5.7 is the case $H = S_n$ with σ = (12). For other H, (12) need not lie in H, so σ must come from H itself. Example: ⟨(1234)⟩ = {ε, (1234), (13)(24), (1432)} has the two even members ε and (13)(24).

Exercise 46: Question

Question

Let α be a 12-cycle. Describe the cycle structure of α^2, α^3, α^4 and α^6.

More generally, if α is an n-cycle and k divides n, describe the cycle structure of α^k.

Exercise 46: Solution (1 of 2)

The powers $\alpha^2, \alpha^3, \alpha^4, \alpha^6$ of a 12-cycle α.

Name the symbols so that $\alpha = (1, 2, 3, \ldots, 12)$. Then α^k sends i to i + k, counting modulo 12 with 12 in place of 0.

Power	Disjoint cycle form	Structure
α^2	(1, 3, 5, 7, 9, 11)(2, 4, 6, 8, 10, 12)	two 6-cycles
α^3	(1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12)	three 4-cycles
α^4	(1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12)	four 3-cycles
α^6	(1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12)	six 2-cycles

In each case α^k splits into k disjoint cycles of length 12/k.

Exercise 46: Solution (2 of 2)

The general case: α an n-cycle and k a positive divisor of n.

Write $\alpha = (a_0, a_1, \ldots, a_{n-1})$, with indices read modulo n. Then $\alpha(a_i) = a_{i+1}$, so

$$\alpha^k(a_i) = a_{i+k}$$

Starting at a_r, repeated application of α^k visits the indices r, r + k, r + 2k, …, and first returns to r after n/k steps, because $r + jk \equiv r \pmod{n}$ exactly when n divides jk, that is, when n/k divides j. So α^k has the cycle

$$(a_r,\ a_{r+k},\ a_{r+2k},\ \ldots,\ a_{r+n-k}), \qquad r = 0,1,\ldots,k-1$$

Since k divides n, every index in this cycle is $\equiv r \pmod{k}$, so the k cycles are disjoint and together use all n symbols.

α^k is a product of k disjoint cycles, each of length n/k.

Check: Theorem 5.3 gives $|\alpha^k| = n/k$, which agrees with $|\alpha^k| = n/\gcd(n, k)$ from Theorem 4.2.

Exercise 47: Question

Question

Let α be an n-cycle and k a positive integer.

Prove that α^k is an n-cycle if and only if gcd(n, k) = 1.

Exercise 47: Solution (1 of 2)

α an n-cycle ($n \geq 2$): α^k is an n-cycle exactly when $\gcd(n, k) = 1$.

As in Exercise 46, write $\alpha = (a_0, a_1, \ldots, a_{n-1})$ with indices modulo n, so

$$\alpha^k(a_i) = a_{i+k}$$

Reduce to one cycle

α^k fixes every symbol other than $a_0, \ldots, a_{n-1}$. So α^k is an n-cycle exactly when its cycle through a_0 contains all n of these symbols.

That cycle is $\langle k \rangle$ in $\mathbb{Z}_n$

$$a_0 \rightarrow a_k \rightarrow a_{2k} \rightarrow a_{3k} \rightarrow \cdots$$

The indices reached are the multiples of k modulo n, which form the cyclic subgroup $\langle k \rangle$ of $\mathbb{Z}_n$. The cycle contains all n symbols exactly when $\langle k \rangle = \mathbb{Z}_n$.

Exercise 47: Solution (2 of 2)

α an n-cycle (n ≥ 2): α^k is an n-cycle exactly when gcd(n, k) = 1.

$$\alpha^k \text{ is an } n\text{−cycle} \Leftrightarrow \langle k \rangle = \mathbb{Z}_n \Leftrightarrow \gcd(n, k) = 1$$

The last step is Corollary 4 of Theorem 4.2: k generates $\mathbb{Z}_n$ exactly when gcd(n, k) = 1.

So α^k is an n-cycle if and only if gcd(n, k) = 1.

Example: α = (1, 2, …, 12)

k	gcd(12, k)	α^k
5	1	(1, 6, 11, 4, 9, 2, 7, 12, 5, 10, 3, 8), a 12-cycle
8	4	(1, 9, 5)(2, 10, 6)(3, 11, 7)(4, 12, 8), four 3-cycles

In general α^k is gcd(n, k) disjoint cycles of length n/gcd(n, k). Exercise 46 is the case k | n.