

King Saud University
College of Sciences
Department of Mathematics

MATH 201
MULTIVARIABLE CALCULUS

CLASS NOTES
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Contents

1 Partial Derivatives **7**

1.1 Functions of several variables 7
1.1.1 Functions of two variables 7
1.1.2 Graphs 8
1.1.3 Level curves 9
1.1.4 Functions of three variables 11
1.1.5 EXERCISES 12
1.2 Limits and Continuity 13
1.2.1 Limits of Functions of Two Variables 13
1.2.2 Showing That a Limit Does Not Exist 13
1.2.3 Properties of Limits 14
1.2.4 Continuity 16
1.2.5 Limit and Continuity of a function of three variables . . . 17
1.2.6 EXERCISES 18
1.3 Partial Derivatives 20
1.3.1 Partial Derivatives of Functions of Two Variables 20
1.3.2 Interpretations of Partial Derivatives 20
1.3.3 Functions of Three Variables 22
1.3.4 Higher Derivatives 22
1.3.5 Partial Differential Equations 23
1.3.6 Differentiability 23
1.3.7 EXERCISES 26
1.4 Chain Rule 27
1.4.1 The Chain Rule (Case 1) 27
1.4.2 The Chain Rule (Case 2) 27
1.4.3 The Chain Rule (The General Case) 28
1.4.4 Implicit Differentiation 29
1.4.5 EXERCISES 30
1.5 Maximum and Minimum Values 31
1.5.1 Local Maximum and Minimum Values 31
1.5.2 Absolute Maximum and Minimum Values 32
1.5.3 EXERCISES 36
1.6 Lagrange Multipliers 37
1.6.1 Lagrange Multipliers (One Constraint) 37
1.6.2 EXERCISES 40

2 Multiple Integrals **41**
2.1 Double Integrals over Rectangles 41
2.1.1 Iterated Integrals 41
2.1.2 Volume 42
2.1.3 Average Value 43
2.1.4 EXERCISES 44
2.2 Double Integrals over General Regions 45
2.2.1 General Regions 45
2.2.2 Changing the Order of Integration 47
2.2.3 Properties of Double Integrals 48
2.2.4 EXERCISES 50
2.3 Double Integrals in Polar Coordinates 51
2.3.1 Double Integrals in Polar Coordinates 51
2.3.2 EXERCISES 54
2.4 Triple Integrals 55
2.4.1 Triple Integrals over Rectangular Boxes 55
2.4.2 Triple Integrals over General Regions 56
2.4.3 EXERCISES 61
2.5 Triple Integrals in Cylindrical Coordinates 62
2.5.1 Cylindrical Coordinates 62
2.5.2 Triple Integrals in Cylindrical Coordinates 64
2.5.3 EXERCISES 68
2.6 Triple Integrals in Spherical Coordinates 69
2.6.1 Spherical Coordinates 69
2.6.2 Triple Integrals in Spherical Coordinates 71
2.6.3 EXERCISES 75

3 Sequences, Series, and Power Series **77**
3.1 Sequences 77
3.1.1 Infinite Sequences 77
3.1.2 The Limit of a Sequence 77
3.1.3 Properties of Convergent Sequences 78
3.1.4 Monotonic and bounded sequences 79
3.2 Series 80
3.2.1 Infinite Series 80
3.2.2 Sum of a Geometric Series 80
3.2.3 Test for Divergence 81
3.2.4 Properties of Convergent Series 82
3.2.5 EXERCISES 83
3.3 The Integral Test 84
3.3.1 The Integral Test 84
3.3.2 EXERCISES 87
3.4 The Comparison Tests 88
3.4.1 The Direct Comparison Test 88
3.4.2 Limit Comparison Test 89
3.4.3 EXERCISES 91
3.5 Alternating Series and Absolute Convergence 92
3.5.1 Alternating Series 92
3.5.2 Absolute Convergence and Conditional Convergence . . . 93
3.5.3 EXERCISES 96

3.6 The Ratio and Root Tests 97
3.6.1 The Ratio Test 97
3.6.2 The Root Test 98
3.6.3 EXERCISES 100
3.7 Strategy for Testing Series 101
3.8 Power Series 103
3.8.1 Power Series 103
3.8.2 Interval of Convergence 104
3.8.3 EXERCISES 106
3.9 Representations of Functions as Power Series 107
3.9.1 Representations of Functions using Geometric Series . . . 107
3.9.2 Differentiation and Integration of Power Series 108
3.9.3 EXERCISES 110
3.10 Taylor and Maclaurin Series 111
3.10.1 Definitions of Taylor Series and Maclaurin Series 111
3.10.2 Remainder of a Taylor series 112
3.10.3 Taylor Series of Important Functions 113
3.10.4 New Taylor Series from Old 115
3.10.5 EXERCISES 117

Chapter 1

Partial Derivatives

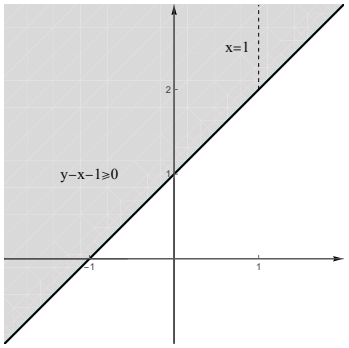
1.1 Functions of several variables

1.1.1 Functions of two variables

Definition: A function f of two variables is a map that assigns to each ordered pair of real numbers $(x, y) \in D \subseteq \mathbb{R}^2$ a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is $\{f(x, y)|(x, y) \in D\}$.

Example (1): If $f(x, y) = \dfrac{\sqrt{y - x - 1}}{x - 1}$, evaluate $f(2, 7)$, find the domain of f and sketch it.
Solution : $f(2, 7) = \dfrac{\sqrt{7 - 2 - 1}}{2 - 1} = \dfrac{\sqrt{4}}{1} = 2.$
The domain of f is the set $D = \{(x, y) \in \mathbb{R}^2 \mid y - x - 1 \geq 0 \ , \ x \neq 1\}$.
So, $D = \{(x, y) \in \mathbb{R}^2 \mid y \geq x + 1 \ , \ x \neq 1\}$.

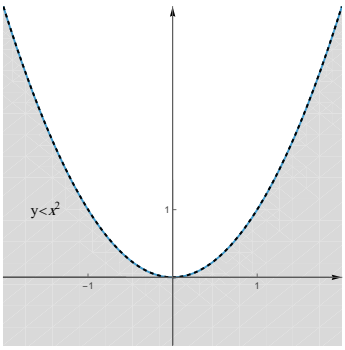
$y - x - 1 \geq 0 \implies y \geq x + 1$
represents the points on and above the line $y = x + 1$.
$x \neq 1$ means the point on the line $x = 1$ must be excluded from the domain .



Example (2): If $f(x, y) = \ln(x^2 - y)$, find the domain of f and sketch it.
Solution :

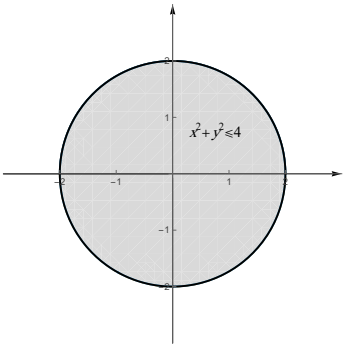
The domain of f is the set
$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y > 0\}$.
So, $D = \{(x, y) \in \mathbb{R}^2 \mid y < x^2\}$.

$x^2 - y > 0 \implies y < x^2$
represents the points below the parabola $y = x^2$.



Example (3): If $f(x, y) = \sqrt{4 - x^2 - y^2}$, find the domain and range of f.
Solution :
The domain of f is the set
$D = \{(x, y) \in \mathbb{R}^2 \mid 4 - x^2 - y^2 \geq 0\}$.
So, $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$.

$4 - x^2 - y^2 \geq 0 \implies x^2 + y^2 \leq 4$
represents the points on and inside the disk of center $(0, 0)$ and radius 2.

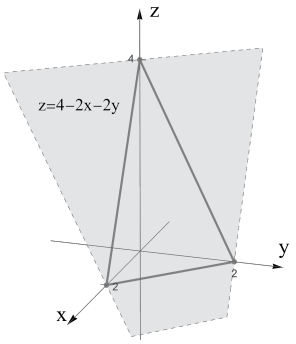


Note that $0 \leq \sqrt{4 - x^2 - y^2}$
and $\sqrt{4 - (x^2 + y^2)} \leq \sqrt{4} = 2$
So, the range of f is
$\{z \in \mathbb{R} \mid 0 \leq z \leq 2\} = [0, 2]$.

1.1.2 Graphs

Definition: If f is a function of two variables with domain $D \subseteq \mathbb{R}^2$, then the graph of f is the set of all points $(x, y, z) \in \mathbb{R}^3$, such that $z = f(x, y)$ and $(x, y) \in D$.

Example (4): Sketch the graph of the function $f(x, y) = 4 - 2x - 2y$.
Solution:
$z = 4 - 2x - 2y \implies 2x + 2y + z = 4$
It represents a plain.



To find the x-intercept, put $y = z = 0$
$2x = 4 \implies x = 2$.
So the x-intercept is $(2, 0, 0)$.

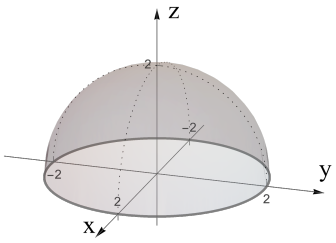
To find the y-intercept, put $x = z = 0$
$2y = 4 \implies y = 2$.
So the y-intercept is $(0, 2, 0)$.

To find the z-intercept, put $x = y = 0$
$z = 4$, So the z-intercept is $(0, 0, 4)$.

Example (5): Sketch the graph of the function $f(x,y) = \sqrt{4-x^2-y^2}$.
Solution:

$z = \sqrt{4-x^2-y^2}$
$\Longrightarrow \; z^2 = 4-x^2-y^2$
$\Longrightarrow \; x^2+y^2+z^2 = 2^2$
Note that $z = \sqrt{4-x^2-y^2} \geq 0$.

It represents the upper half of the sphere centered at the origin and its radius is 2.

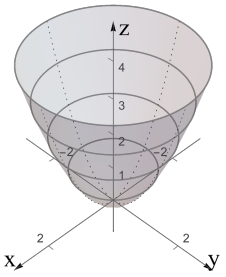


Example (6): Sketch the graph of the function $f(x,y) = x^2+y^2$.
Solution:

The domain of f is $\mathbb{R}^2$.

$z = x^2+y^2 \geq 0$
For each value of $z > 0$,
$x^2+y^2 = z$ represents a circle centered at the origin and its radius is $\sqrt{z}$.

$f(x,y) = x^2+y^2$ represents a paraboloid.



1.1.3 Level curves

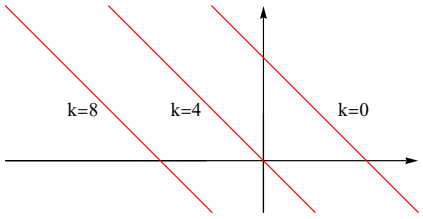
Definition: A level curve of a function $f(x,y)$ is the curve $f(x,y) = k$, where k is a constant (in the range of f).

Example (7): Sketch the level curves of the function $f(x,y) = 4-2x-2y$ for the values $k = 0, 4, 8$.
Solution:
$4-2x-2y = k$.
$\Longrightarrow \; 2x+2y = 4-k$

For $k=0: \; 2y+2x = 4$
$\Longrightarrow \; y = -x+2$

For $k=4: \; 2y+2x = 0$
$\Longrightarrow \; y = -x$

For $k=8: \; 2y+2x = -4$
$\Longrightarrow \; y = -x-2$

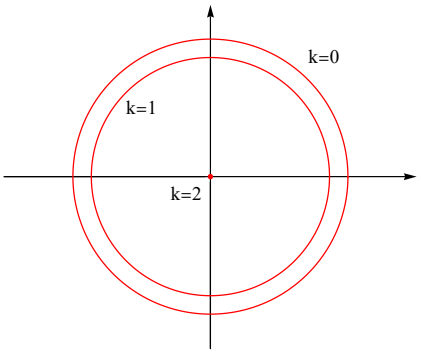


Example (8): Sketch the level curves of the function $f(x,y) = \sqrt{4-x^2-y^2}$ for the values $k = 0, 1, 2$.
Solution:
$\sqrt{4-x^2-y^2} = k.$
$\implies 4 - x^2 - y^2 = k^2$
$\implies x^2 + y^2 = 4 - k^2$

For $k = 0 : x^2 + y^2 = 4$
Circle: center is $(0,0)$, radius $= 2$.

For $k = 1 : x^2 + y^2 = 3$
Circle: center is $(0,0)$, radius $= \sqrt{3}$.

For $k = 2 : x^2 + y^2 = 0$
Just one point, the origin.



Example (9): Sketch the level curves of the function $f(x,y) = 4x^2 + y^2$ for the values $k = 0, 2, 4$.
Solution:

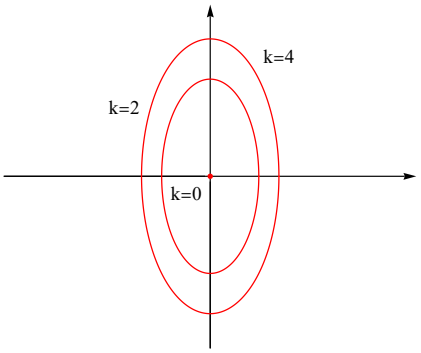
Note that the domain is $\mathbb{R}^2$.

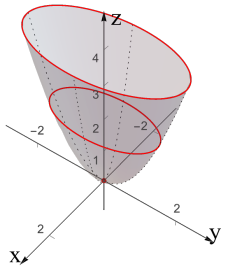
$4x^2 + y^2 = k.$

For $k = 0 : 4x^2 + y^2 = 0$
$\implies x = 0\ ,\ y = 0$
The level curve is the origin.

For $k > 0 : 4x^2 + y^2 = k$
$\implies \dfrac{x^2}{\left(\frac{k}{4}\right)} + \dfrac{y^2}{k} = 1$
The level curve is an ellipse centered at the origin, the major axis lies on the y-axis and the minor axis lies on the x-axis.

Note that the graph of f is an elliptic paraboloid.





1.1.4 Functions of three variables

Definition: A function f of two variables is a map that assigns to each ordered pair of real numbers $(x, y, z) \in D \subseteq \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. The set D is the domain of f and its range is $\{f(x, y, z)|(x, y, z) \in D\}$.

Example (10): Find the domain of $f(x, y, z) = \ln(z - x) + yz \sin x$.
Solution : $f(x, y, z)$ is defined when $z - x > 0$.
Therefore, $D = \left\{(x, y, z) \in \mathbb{R}^3 \mid z > x\right\}$.

Example (11): Find the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$.
Solution : $x^2 + y^2 + z^2 = k$, where $k \geq 0$.
If $k = 0$, then the level surface is just the origin $(0, 0, 0) \in \mathbb{R}^3$.
If $k > 0$, then the level surface is $x^2 + y^2 + z^2 = \left(\sqrt{k}\right)^2$, which is a sphere centered at the origin, and its radius is $\sqrt{k}$.

1.1.5 EXERCISES

1. Let $f(x,y) = x^2 \ln(x+y)$

   (a). Evaluate $f(3,1)$.

   (b). Find and sketch the domain of f.

   (c). Find the range of f.

2. Let $f(x,y,z) = \ln\left(z - \sqrt{x^2+y^2}\right)$.

   (a). Evaluate $f(4,23,6)$.

   (b). Find and describe the domain of f.

3. Find and sketch the domain of the following:

   (a). $f(x,y) = \sqrt{x-2} + \sqrt{y-1}$.

   (b). $f(x,y) = \ln(9 - x^2 - y^2)$.

   (c). $f(x,y) = \dfrac{\ln(2-x)}{1-x^2-y^2}$.

(*) Find and sketch the domain of the following:

   (a). $f(x,y) = \sqrt{25 - x^2 - y^2} + \ln(2+x)$.

   (b). $f(x,y) = \cos(x+y) + \dfrac{x^2 - y^2 - 3}{\sqrt{x+y-4}}$.

   (c). $f(x,y) = \sqrt{x+y^2} + \sqrt{y-x^2}$.

1.2 Limits and Continuity

1.2.1 Limits of Functions of Two Variables

Definition: Let f be a function of two variables whose domain $D \subseteq \mathbb{R}^2$ includes points arbitrarily close to (a, b). Then we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$,
if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$.

1.2.2 Showing That a Limit Does Not Exist

NOTE: If $f(x, y) \to L_1$ as $(x, y) \longrightarrow (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \longrightarrow (a, b)$ along a different path C_2, where $L_1 \neq L_2$, then $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Example (1): Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Solution: Note that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2} \qquad \left(\dfrac{0}{0}\right)$.

Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$.

$\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2} = \lim\limits_{x\to 0} \dfrac{x^2 - m^2x^2}{x^2 + m^2x^2} = \lim\limits_{x\to 0} \dfrac{x^2(1 - m^2)}{x^2(1 + m^2)} = \dfrac{1 - m^2}{1 + m^2}$.

Note that the limit depends only on m.

Let C_1 be the path $y = 0$ (where m=0) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2} = \dfrac{1 - 0}{1 + 0} = 1$.

Let C_2 be the path $y = x$ (where m=1) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2} = \dfrac{1 - 1}{1 + 1} = 0$.

Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Example (2): Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$ does not exist.

Solution: Note that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2} \qquad \left(\dfrac{0}{0}\right)$.

Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$.

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2} = \lim\limits_{x\to 0} \dfrac{mx^2}{x^2 + m^2x^2} = \lim\limits_{x\to 0} \dfrac{mx^2}{x^2(1 + m^2)} = \dfrac{m}{1 + m^2}$.

Note that the limit depends only on m.

Let C_1 be the path $y = 0$ (where m=0) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2} = \dfrac{0}{1 + 0} = 0$.

Let C_2 be the path $y = x$ (where m=1) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2} = \dfrac{1}{1 + 1} = \dfrac{1}{2}$.

Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$ does not exist.

Example (3): Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^4}$ does not exist.

Solution: Note that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4} \qquad \left(\dfrac{0}{0}\right)$.
Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$.
$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4} = \lim\limits_{x\to 0} \dfrac{x(m^2x^2)}{x^2+m^4x^4} = \lim\limits_{x\to 0} \dfrac{mx^3}{x^2(1+m^4\ x^2)} = \lim\limits_{x\to 0} \dfrac{mx}{1+m^4\ x^2}$=0
.
Note that the limit here depends on x and m .
Let C_1 be the path $y = 0$ (where m=0) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4} = 0$.
Let C_2 be the path $x = y^2$ (the parabola with vertex (0,0) and opens to the right) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4} = \lim\limits_{y\to 0} \dfrac{y^2\ y^2}{(y^2)^2+y^4} = \lim\limits_{y\to 0} \dfrac{y^4}{2y^4} = \dfrac{1}{2}$.
Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ does not exist.

Example (4): Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2}$ does not exist.
Solution: Note that $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2} \qquad \left(\dfrac{0}{0}\right)$.
Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$.
$\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{\sin(mx^2)}{x^2+m^2x^2} = \lim\limits_{x\to 0} \dfrac{\sin(mx^2)}{(1+m^2)x^2} = \dfrac{m}{1+m^2}$.
Note that the limit depends only on m.
Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2}$ does not exist.

1.2.3 Properties of Limits

If $f(x,y)$, $g(x,y)$ are two functions defined on $D \setminus \{(a,b)\}$, where $D \subseteq \mathbb{R}^2$, and $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L_1$, $\lim\limits_{(x,y)\to(a,b)} g(x,y) = L_2$, where $L_1, L_2 \in \mathbb{R}$ then
(1). $\lim\limits_{(x,y)\to(a,b)} [f(x,y) + g(x,y)] = L_1 + L_2$.
(2). $\lim\limits_{(x,y)\to(a,b)} [f(x,y) - g(x,y)] = L_1 - L_2$.
(3). $\lim\limits_{(x,y)\to(a,b)} k\ f(x,y) = k\ L_1$, where $k \in \mathbb{R}$.
(4). $\lim\limits_{(x,y)\to(a,b)} [f(x,y).g(x,y)] = L_1L_2$.
(5). $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L_1}{L_2}$, where $L_2 \neq 0$.
(6). If $P(x,y)$ is a polynomial in x, y then $\lim\limits_{(x,y)\to(a,b)} P(x,y) = P(a,b)$.

Example (5): Evaluate $\lim\limits_{(x,y)\to(2,1)} \left(x^2y^2 - 2xy + x + y - 1\right)$.
Solution: Note that $P(x,y) = x^2y^2 - 2xy + x + y - 1$ is a polynomial.
So, $\lim\limits_{(x,y)\to(2,1)} \left(x^2y^2 - 2xy + x + y - 1\right) = P(2,1)$
$= (2)^2(1)^2 - 2(2)(1) + 2 + 1 - 1 = 2$.

Example (6): Evaluate $\lim\limits_{(x,y)\to(-1,3)} \dfrac{2x^2y+1}{xy^3-2x}$.

Solution: Note that $P(x,y) = 2x^2y+1$ and $Q(x,y) = xy^3-2x$ are both polynomials.

$\lim\limits_{(x,y)\to(-1,3)} \dfrac{2x^2y+1}{xy^3-2x} = \dfrac{P(-1,3)}{Q(-1,3)} = \dfrac{2(-1)^2(3)+1}{(-1)(3)^3-2(-1)} = \dfrac{7}{-25}$.

Note that $Q(-1,3) \neq 0$.

Example (7): Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+y^2} = 0$.

First Solution: $\forall\ x, y \in \mathbb{R}^*$,

$y^2 \leq x^2+y^2 \implies \dfrac{y^2}{x^2+y^2} \leq 1$.

$0 \leq \left|\dfrac{2xy^2}{x^2+y^2}\right| = \dfrac{|2x|\ |y^2|}{|x^2+y^2|} = 2|x|\ \dfrac{y^2}{x^2+y^2} \leq 2|x|$.

Note that $(x,y) \to (0,0) \implies x \to 0$.

Since $\lim\limits_{(x,y)\to(0,0)} 0 = 0$ and $\lim\limits_{(x,y)\to(0,0)} 2|x| = 2(0) = 0$,

By Squeeze Theorem $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+y^2} = 0$.

Second Solution: Using polar coordinates,

$\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+y^2} = \lim\limits_{r\to0} \dfrac{2r^3\cos\theta\ \sin^2\theta}{r^2} = \lim\limits_{r\to0} 2r\cos\theta\sin^2\theta = 0.$

Note that $\lim\limits_{r\to0} 2r = 0$ and $\cos\theta\sin^2\theta$ is bounded .

Example (8): Discuss the existence of $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2}$.

First Solution: $\forall\ x, y \in \mathbb{R}^*$,

$0 \leq \left|\dfrac{x^3+y^3}{x^2+y^2}\right| \leq \left|\dfrac{x^3}{x^2+y^2}\right| + \left|\dfrac{y^3}{x^2+y^2}\right| = |x|\left|\dfrac{x^2}{x^2+y^2}\right| + |y|\left|\dfrac{y^2}{x^2+y^2}\right| \leq |x|+|y|$

Since $\lim\limits_{(x,y)\to(0,0)} |x| = 0$ and $\lim\limits_{(x,y)\to(0,0)} |y| = 0$,

By Squeeze Theorem $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2} = 0$.

Second Solution: Using polar coordinates,

$\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2} = \lim\limits_{r\to0} \dfrac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2} = \lim\limits_{r\to0} \dfrac{r^3\left(\cos^3\theta+\sin^3\theta\right)}{r^2}.$

$= \lim\limits_{r\to0} r\left(\cos^3\theta+\sin^3\theta\right) = 0$.

Note that $\lim\limits_{r\to0} r = 0$ and $\left(\cos^3\theta+\sin^3\theta\right)$ is bounded .

Example (9): Discuss the existence of $\lim\limits_{(x,y)\to(-1,0)} \dfrac{y(x+1)^2+y^2\sin(\pi x)}{(x+1)^2+y^2}$.

Solution: $\forall\ (x,y) \neq (-1,0)$,

$0 \leq \left|\dfrac{y(x+1)^2+y^2\sin(\pi x)}{(x+1)^2+y^2}\right| \leq |y|\left|\dfrac{(x+1)^2}{(x+1)^2+y^2}\right| + |\sin(\pi x)|\left|\dfrac{y^2}{(x+1)^2+y^2}\right|$

Therefore, $0 \leq \left|\dfrac{y(x+1)^2+y^2\sin(\pi x)}{(x+1)^2+y^2}\right| \leq |y| + |\sin(\pi x)|$

Since $\lim\limits_{(x,y)\to(-1,0)} |y| = 0$ and $\lim\limits_{(x,y)\to(-1,0)} |\sin(\pi x)| = \sin(-\pi) = 0$,
By Squeeze Theorem, $\lim\limits_{(x,y)\to(-1,0)} \dfrac{y(x+1)^2 + y^2\sin(\pi x)}{(x+1)^2+y^2} = 0$.

1.2.4 Continuity

Definition: Let $f(x,y)$ be a function of two variables defined on a set $D \subseteq \mathbb{R}^2$ and $(a,b) \in D$, If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, then f is continuous at (a,b).
If f is continuous at every point in D, then f is continuous on D .

Example (10): Discuss the continuity of $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$.
Solution:
f is not defined at $(0,0)$, so it is not continuous at $(0,0)$.
$\forall\ (a,b) \neq (0,0)$, $\lim\limits_{(x,y)\to(a,b)} f(x,y) = \lim\limits_{(x,y)\to(a,b)} \dfrac{x^2-y^2}{x^2+y^2} = \dfrac{a^2-b^2}{a^2+b^2} = f(a,b)$.
So, f is continuous on $D = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\} = \mathbb{R}^2 \setminus \{(0,0)\}$.

Example (11): Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$.
Solution:
$\forall\ (a,b) \neq (0,0)$, $\lim\limits_{(x,y)\to(a,b)} f(x,y) = \lim\limits_{(x,y)\to(a,b)} \dfrac{x^2-y^2}{x^2+y^2} = \dfrac{a^2-b^2}{a^2+b^2} = f(a,b)$.
f is defined at $(0,0)$ and $f(0,0) = 0$, but $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.
So, f is continuous on $D = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\} = \mathbb{R}^2 \setminus \{(0,0)\}$.

Example (12): Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{2xy^2}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$.
Solution:
$\forall\ (a,b) \neq (0,0)$, $\lim\limits_{(x,y)\to(a,b)} f(x,y) = \lim\limits_{(x,y)\to(a,b)} \dfrac{2xy^2}{x^2+y^2} = \dfrac{2ab^2}{a^2+b^2} = f(a,b)$.
f is defined at $(0,0)$ and $f(0,0) = 0$, also $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.
So, f is continuous on $\mathbb{R}^2$.

Example (13): Discuss the continuity of $f(x,y) = e^{-x^2-y^2}$.
Solution: $f(x,y) = e^{-x^2-y^2} = e^{-(x^2+y^2)}$.
f is defined on $\mathbb{R}^2$ and $\lim\limits_{(x,y)\to(a,b)} f(x,y) = e^{-(a^2+b^2)} = f(a,b)$.
Therefore, f is continuous on $\mathbb{R}^2$.

Example (14): Discuss the continuity of $f(x,y) = \tan^{-1}\left(\dfrac{y}{x}\right)$.
Solution:
f is not defined where $x = 0$.

$\forall\ (a,b)$ where $a \neq 0$, $\lim\limits_{(x,y)\to(a,b)} f(x,y) = \tan^{-1}\left(\dfrac{b}{a}\right) = f(a,b)$.
So, f is continuous on $D = \left\{(x,y) \in \mathbb{R}^2 \mid x \neq 0\right\}$ which is $\mathbb{R}^2$ except the y-axis.

1.2.5 Limit and Continuity of a function of three variables

Definition: Let f be a function of three variables whose domain $D \subseteq \mathbb{R}^3$ includes points arbitrarily close to (a,b,c). Then we say that the limit of $f(x,y,z)$ as (x,y,z) approaches (a,b,c) is L and we write $\lim\limits_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$,
if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that
if $(x,y,z) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta$
then $|f(x,y,z) - L| < \epsilon$.

Definition: Let $f(x,y,z)$ be a function of three variables defined on a set $D \subseteq \mathbb{R}^3$ and $(a,b,c) \in D$, If $\lim\limits_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c)$, then f is continuous at (a,b,c).
If f is continuous at every point in D, then f is continuous on D .

Example (15): Discuss the continuity of $f(x,y,z) = \dfrac{1}{1-x^2-y^2-z^2}$.
Solution:
f is not defined where $1 - x^2 - y^2 - z^2 = 0 \implies x^2 + y^2 + z^2 = 1$,
f is not continuous on the unit sphere.
$\forall\ (a,b,c) \in \mathbb{R}^3$ where $a^2 + b^2 + c^2 \neq 1$,
$\lim\limits_{(x,y,z)\to(a,b,c)} f(x,y,z) = \dfrac{1}{1-a^2+b^2+c^2} = f(a,b,c)$.
So, f is continuous on $D = \left\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \neq 1\right\}$ which is $\mathbb{R}^3$ except the unit sphere.

1.2.6 EXERCISES

1. Find the limit of the following:

   $(a).\ \lim\limits_{(x,y)\to(3,2)} \left(x^2y^3 - 4y^2\right)$ $(b).\ \lim\limits_{(x,y)\to(2,-1)} \dfrac{x^2y + yx^2}{x^2 - y^2}$

2. Show that the following limits do not exist:

   $(a).\ \lim\limits_{(x,y)\to(0,0)} \dfrac{y^2}{x^2+y^2}$ $(b).\ \lim\limits_{(x,y)\to(0,0)} \dfrac{(x+y)^2}{x^2+y^2}$

   $(c).\ \lim\limits_{(x,y)\to(1,1)} \dfrac{y-x}{1-y+\ln x}$

3. Discuss the existence of the following limits (and find its value if exists):

   $(a).\ \lim\limits_{(x,y)\to(2,3)} \dfrac{3x-2y}{4x^2-y^2}$ $(b).\ \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y\cos y}{x^2+y^4}$

4. Use the Squeeze Theorem to find the limit of the following :

   $(a).\ \lim\limits_{(x,y)\to(0,0)} xy\ \sin\left(\dfrac{1}{x^2+y^2}\right)$ $(b).\ \lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^2y^2z^2}{x^2+y^2+z^2}$

5. Determine the set of points of continuity of the following:

   (a). $f(x,y) = \begin{cases} \dfrac{x^2y^3}{2x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 1 & if \quad (x,y) = (0,0) \end{cases}$

   (b). $f(x,y) = \begin{cases} \dfrac{xy}{x^2+xy+y^2} & if \quad (x,y) \neq (0,0) \\ 1 & if \quad (x,y) = (0,0) \end{cases}$

(*) Show that the following limits do not exist:

$(a).\ \lim_{(x,y)\to(0,0)} \dfrac{x^3+y^2}{x^2+y^2}$ $(b).\ \lim_{(x,y)\to(0,0)} \dfrac{xy}{x^4+y^4}$

(*) Use Squeeze theorem to find $\lim_{(x,y)\to(0,0)} \dfrac{y^2\sin x + yx^2}{x^2+y^2}$

(*) Show that the limit is zero in the following:

$(a).\ \lim_{(x,y)\to(0,0)} \dfrac{x^5+y^5}{x^2+y^2}$ $(b).\ \lim_{(x,y)\to(0,0)} \dfrac{x^6+y^6}{x^4+y^4}$

$(c).\ \lim_{(x,y)\to(0,0)} \dfrac{x^2+2xy}{\sqrt{x^2+y^2}}$ $(d).\ \lim_{(x,y)\to(0,0)} \dfrac{\sin(x^3+y^3)}{x^2+y^2}$

1.3 Partial Derivatives

1.3.1 Partial Derivatives of Functions of Two Variables

Definition: Let $f(x,y)$ be a function of two variables defined on a set $D \subseteq \mathbb{R}^2$, If $\lim_{h\to 0} \dfrac{f(x+h,y)-f(x,y)}{h}$ and $\lim_{h\to 0} \dfrac{f(x,y+h)-f(x,y)}{h}$ exist, then the partial derivatives of f are denoted by f_x and f_y and are defined as

(1). $f_x(x,y) = \lim_{h\to 0} \dfrac{f(x+h,y)-f(x,y)}{h}$.

(2). $f_y(x,y) = \lim_{h\to 0} \dfrac{f(x,y+h)-f(x,y)}{h}$.

Notations for Partial Derivatives: If $z = f(x,y)$ then

(1). $f_x(x,y) = f_x = \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x} f(x,y) = \dfrac{\partial z}{\partial x} = D_x f$.

(2). $f_y(x,y) = f_y = \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y} f(x,y) = \dfrac{\partial z}{\partial y} = D_y f$.

Rule for Finding Partial Derivatives

(1). To find f_x : differentiate $f(x,y)$ with respect to x regarding y as a constant.

(2). To find f_y : differentiate $f(x,y)$ with respect to y regarding x as a constant.

Example (1): If $f(x,y) = x^2 - xy^3 - 3y^2$, find $f_x(1,1)$ and $f_y(1,1)$.

Solution:

(1). $f_x(x,y) = 2x - (1)y^3 - 0 = 2x - y^3$,

$f_x(1,1) = 2(1) - (1)^3 = 1$.

(2). $f_y(x,y) = 0 - x(3y^2) - 3(2y) = -3xy^2 - 6y$,

$f_y(1,1) = -3(1)(1)^2 - 6(1) = -9$.

Example (2): If $f(x,y) = \sin\left(\dfrac{2x}{1+3y}\right)$, Evaluate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

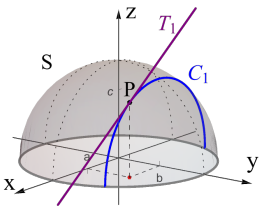
Solution:

(1). $\dfrac{\partial f}{\partial x} = \cos\left(\dfrac{2x}{1+3y}\right)\left(\dfrac{2}{1+3y}\right)$.

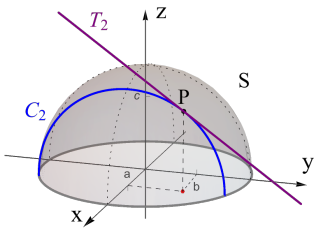
(2). $\dfrac{\partial f}{\partial y} = \cos\left(\dfrac{2x}{1+3y}\right)\left((2x)(-1)(1+3y)^{-2}(3)\right) = \cos\left(\dfrac{2x}{1+3y}\right)\left(\dfrac{-6x}{(1+3y)^2}\right)$.

1.3.2 Interpretations of Partial Derivatives

Let S be the surface represented by $z = f(x,y)$ and $f(a,b) = c$.
Then $P(a,b,c)$ lies on S.
Let C_1 be the curve where the plain $y = b$ intersects the surface S,
The slope of the tangent line T_1 to the curve C_1 at $P(a,b,c)$ is $f_x(a,b)$.

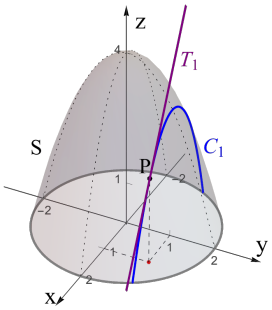


Let C_2 be the curve where the plain $x = a$ intersects the surface S,
The slope of the tangent line T_2 to the curve C_2 at $P(a, b, c)$ is $f_y(a, b)$.

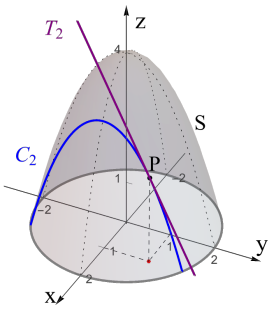


Example (3): If $f(x, y) = 4 - x^2 - y^2$, find $f_x(1, 1)$, $f_y(1, 1)$ and interpret them as slopes. Solution:

$z = 4 - x^2 - y^2$.
$f_x(x, y) = -2x$.
$f_x(1, 1) = -2(1) = -2$.
$f(1, 1) = 4 - 1 - 1 = 2$.
C_1 is the intersection of $z = 4 - x^2 - y^2$ and $y = 1$, and it is the parabola $z = 3 - x^2$.
The line T_1 is tangent to the curve C_1 at the point $P(1, 1, 2)$ and its slope is $f_x(1, 1) = -2$.



$z = 4 - x^2 - y^2$.
$f_y(x, y) = -2y$.
$f_y(1, 1) = -2(1) = -2$.
$f(1, 1) = 4 - 1 - 1 = 2$.
C_2 is the intersection of $z = 4 - x^2 - y^2$ and $x = 1$, and it is the parabola $z = 3 - y^2$.
The line T_2 is tangent to the curve C_2 at the point $P(1, 1, 2)$ and its slope is $f_y(1, 1) = -2$.



Example (4): If $x^3 + y^3 + z^3 + 6xyz + 4 = 0$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Solution:
Differentiating implicitly with respect to x .

$$3x^2 + 0 + 3z^2 \frac{\partial z}{\partial x} + 6y\left(z + x\frac{\partial z}{\partial x}\right) + 0 = 0$$

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}\left(3z^2 + 6xy\right) = -3x^2 - 6yz$$

$$\frac{\partial z}{\partial x} = \frac{-3x^2 - 6yz}{3z^2 + 6xy} = \frac{-x^2 - 2yz}{z^2 + 2xy} = -\frac{x^2 + 2yz}{z^2 + 2xy} \ .$$

Differentiating implicitly with respect to y .

$$0 + 3y^2 + 3z^2 \ \frac{\partial z}{\partial y} + 6x\left(z + y\frac{\partial z}{\partial y}\right) + 0 = 0$$

$$3y^2 + 3z^2 \ \frac{\partial z}{\partial y} + 6xz + 6xy\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y}\left(3z^2 + 6xy\right) = -3y^2 - 6xz$$

$$\frac{\partial z}{\partial y} = \frac{-3y^2 - 6xz}{3z^2 + 6xy} = \frac{-y^2 - 2xz}{z^2 + 2xy} = -\frac{y^2 + 2xz}{z^2 + 2xy} \ .$$

1.3.3 Functions of Three Variables

Definition: Let $f(x, y, z)$ be a function of three variables defined on a set $D \subseteq \mathbb{R}^3$, If $\lim\limits_{h\to 0} \dfrac{f(x+h, y, z) - f(x, y, z)}{h}$, $\lim\limits_{h\to 0} \dfrac{f(x, y+h, z) - f(x, y, z)}{h}$ and $\lim\limits_{h\to 0} \dfrac{f(x, y, z+h) - f(x, y, z)}{h}$ exist, then the partial derivatives of f are denoted by f_x f_y and f_z and are defined as

(1). $f_x(x, y, z) = \lim\limits_{h\to 0} \dfrac{f(x+h, y, z) - f(x, y, z)}{h}$.

(2). $f_y(x, y, z) = \lim\limits_{h\to 0} \dfrac{f(x, y+h, z) - f(x, y, z)}{h}$.

(3). $f_z(x, y, z) = \lim\limits_{h\to 0} \dfrac{f(x, y, z+h) - f(x, y, z)}{h}$.

Example (5): If $f(x, y, z) = e^{xy}\ln(z^2 + 1)$, find f_x, f_y and f_z.
Solution:

$f_x(x, y, z) = (e^{xy}y)\ln(z^2 + 1) = ye^{xy}\ln(z^2 + 1)$.

$f_y(x, y, z) = (e^{xy}x)\ln(z^2 + 1) = xe^{xy}\ln(z^2 + 1)$.

$$f_z(x, y, z) = e^{xy}\ \left(\frac{2z}{z^2 + 1}\right) = \frac{2ze^{xy}}{z^2 + 1} \ .$$

1.3.4 Higher Derivatives

Definition: If $z = f(x, y)$, then the second partial derivatives of f are

(1) $f_{xx} = (f_x)_x = \dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial x} = \dfrac{\partial f^2}{\partial x^2}$.

(2) $f_{xy} = (f_x)_y = \dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x} = \dfrac{\partial f^2}{\partial y \partial x}$.

(3) $f_{yx} = (f_y)_x = \dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial y} = \dfrac{\partial f^2}{\partial x \partial y}$.

(y) $f_{yy} = (f_y)_y = \dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial y} = \dfrac{\partial f^2}{\partial y^2}$.

Example (6): Find the second partial derivatives of $f(x, y) = x^3 + x^2y^2 - 2y^3$.
Solution:

$f_x(x,y) = 3x^2 + (2x)y^2 - 0 = 3x^2 + 2xy^2$.
$f_{xx}(x,y) = 3(2x) + 2y^2(1) = 6x + 2y^2$.
$f_{xy}(x,y) = 0 + 2x(2y) = 4xy$.
$f_y(x,y) = 0 + x^2(2y) - 2(3y^2) = 2x^2y - 6y^2$.
$f_{yx}(x,y) = 2y(2x) - 0 = 4xy$.
$f_{yy} = 2x^2(1) - 6(2y) = 2x^2 - 12y$.
NOTE : $f_{xy}(x,y) = f_{yx}(x,y)$.

Clairaut's Theorem: Suppose f is defined on a set $D \subseteq \mathbb{R}^2$ that contains the point (a,b), If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$.

Example (7): If $f(x,y,z) = \sin(2x - yz)$, find $f_{xyz}(x,y)$.
Solution:
$f_x(x,y,z) = \cos(2x - yz)(2) = 2\cos(2x - yz)$.
$f_{xy}(x,y,z) = 2\left[-\sin(2x - yz)\ (-z)\right] = 2z\sin(2x - yz)$.
$f_{xyz}(x,y,z) = (2)\sin(2x - yz) + 2z\left[\cos(2x - yz)\ (-y)\right]$
$= 2\sin(2x - yz) - 2yz\cos(2x - yz)$.

1.3.5 Partial Differential Equations

(1). **Laplace's Equation:** If $u(x,y)$ is a function of two variables and $u_{xx} + u_{yy} = 0$, then u satisfies the Laplace's equation, and u is called a harmonic function.

Example (8): Show that $u(x,y) = e^x \cos y$ is a solution of Laplace's equation.
Solution:
$u_x(x,y) = e^x \cos y$ and $u_{xx}(x,y) = e^x \cos y$.
$u_y(x,y) = -e^x \sin y$ and $u_{yy}(x,y) = -e^x \cos y$.
$u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0$.

(2). **Wave Equation:** If $u(x,t)$ is a function of two variables and $u_{tt} = a^2 u_{xx}$, then u satisfies the wave equation.

Example (9): Show that $u(x,t) = \sin(x - at)$ is a solution of the wave equation.
Solution:
$u_x(x,t) = \cos(x - at)$ and $u_{xx}(x,t) = -\sin(x - at)$.
$u_t(x,t) = -a\cos(x - at)$ and $u_{tt}(x,t) = -a^2\sin(x - at)$.
$u_{tt} = a^2(-\sin(x - at)) = a^2 u_{xx}$.

1.3.6 Differentiability

Definition: If $z = f(x,y)$ and $(a,b) \in D_f$, if x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$, then the increment of z is
$\Delta z = f\left(a + \Delta x, b + \Delta y\right) - f(a,b)$.

Definition: If $z = f(x,y)$, then f is differentiable at (a,b), if
$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \sqrt{(\Delta x)^2 + (\Delta y)^2}\,\epsilon\,(\Delta x, \Delta y)$,
where $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \epsilon\,(\Delta x, \Delta y) = 0$.

NOTE:
If $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{f\,(a+\Delta x, b+\Delta y) - f(a,b) - f_x(a,b)\Delta x - f_y(a,b)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$, then
f is differentiable at (a,b).

Theorem: If f is differentiable at (a,b), then f is continuous at (a,b).

NOTE: If f is not continuous at (a,b), then f is not differentiable at (a,b).

Example (10): Show that $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$
is not differentiable at $(0,0)$.
Solution:
$\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = \lim\limits_{r\to 0} \dfrac{r\cos\theta\ r\sin\theta}{r^2}$
$= \lim\limits_{r\to 0} \dfrac{r^2\ \cos\theta\ \sin\theta}{r^2} = \cos\theta\ \sin\theta.$
The limit depends on θ, so the limit does not exist.
f is not continuous at $(0,0)$, hence, f is not differentiable at $(0,0)$.

Example (11): Show that $f(x,y) = \begin{cases} \dfrac{xy^2}{\sqrt{x^2+y^2}} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$
is differentiable at $(0,0)$.
Solution:
$f(0,0) = 0$.
$f\,(0+\Delta x, 0+\Delta y) = f\,(\Delta x, \Delta y) = \dfrac{\Delta x\,(\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}.$
$f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0}{h} = 0.$
$f_y(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,0+h) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0}{h} = 0.$
$\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{f\,(\Delta x, \Delta y) - f(0,0) - f_x(0,0)\Delta x - f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$
$= \lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{\left(\frac{\Delta x(\Delta y)^2}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\right)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{\Delta x\,(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} = 0.$
Note that $0 \leq \left| \dfrac{\Delta x\,(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \right| = |\Delta x| \left| \dfrac{(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \right| \leq |\Delta x|$

By the squeeze theorem $\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\left|\dfrac{\Delta x\,(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2}\right|=0,$

So, $\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\dfrac{\Delta x\,(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2}=0.$
Therefore, f is differentiable at $(0,0)$.

Example (12): Show that $f(x,y)=\begin{cases}\dfrac{y^2\sin x+yx^2}{x^2+y^2} & \text{if} \quad (x,y)\neq(0,0)\\ 0 & \text{if} \quad (x,y)=(0,0)\end{cases}$
is not differentiable at $(0,0)$.
Solution:
$f(0,0)=0.$
$f\left(0+\Delta x,0+\Delta y\right)=f\left(\Delta x,\Delta y\right)=\dfrac{(\Delta y)^2\sin\left(\Delta x\right)+\Delta y\left(\Delta x\right)^2}{\left(\Delta x\right)^2+\left(\Delta y\right)^2}.$
$f_x(0,0)=\lim\limits_{h\to0}\dfrac{f(0+h,0)-f(0,0)}{h}=\lim\limits_{h\to0}\dfrac{0}{h}=0.$
$f_y(0,0)=\lim\limits_{h\to0}\dfrac{f(0,0+h)-f(0,0)}{h}=\lim\limits_{h\to0}\dfrac{0}{h}=0.$
$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{f\left(\Delta x,\Delta y\right)-f(0,0)-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{\left(\Delta x\right)^2+\left(\Delta y\right)^2}}$$
$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\left(\frac{(\Delta y)^2\sin(\Delta x)+\Delta y(\Delta x)^2}{(\Delta x)^2+(\Delta y)^2}\right)}{\sqrt{\left(\Delta x\right)^2+\left(\Delta y\right)^2}}=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{(\Delta y)^2\sin\left(\Delta x\right)+\Delta y\left(\Delta x\right)^2}{\left[\left(\Delta x\right)^2+\left(\Delta y\right)^2\right]^{\frac{3}{2}}}$$
Taking the path $\Delta y=\Delta x$:
$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{(\Delta y)^2\sin\left(\Delta x\right)+\Delta y\left(\Delta x\right)^2}{\left[\left(\Delta x\right)^2+\left(\Delta y\right)^2\right]^{\frac{3}{2}}}=\lim_{\Delta x\to0}\frac{(\Delta x)^2\sin\left(\Delta x\right)+\Delta x\left(\Delta x\right)^2}{\left[\left(\Delta x\right)^2+\left(\Delta x\right)^2\right]^{\frac{3}{2}}}$$
$$=\lim_{\Delta x\to0}\frac{\left(\Delta x\right)^2\left[\sin\left(\Delta x\right)+\Delta x\right]}{\left[2\left(\Delta x\right)^2\right]^{\frac{3}{2}}}=\lim_{\Delta x\to0}\frac{\left(\Delta x\right)^2\left[\sin\left(\Delta x\right)+\Delta x\right]}{\sqrt{8}\left(\Delta x\right)^3}=\lim_{\Delta x\to0}\frac{\sin\left(\Delta x\right)+\Delta x}{\sqrt{8}\Delta x}$$
$$=\lim_{\Delta x\to0}\frac{1}{\sqrt{8}}\left(\frac{\sin\left(\Delta x\right)}{\Delta x}+\frac{\Delta x}{\Delta x}\right)=\frac{1}{\sqrt{8}}(1+1)=\frac{2}{\sqrt{8}}=\frac{1}{\sqrt{2}}\neq0.$$
Therefore, f is not differentiable at $(0,0)$.

1.3.7 EXERCISES

1. Find the first partial derivatives of the function.

   $(a).\ f(x,y) = x^4 - 5xy^3$ $\quad (b).\ g(x,y) = x^3 \sin y$

   $(c).\ w(u,v) = \dfrac{u}{v^2}$ $\quad (d).\ u(r,\theta) = \sin(r\cos\theta)$

   $(e).\ w(x,y,z) = y\tan(x+2z)$

2. If $f(x,y) = y\sin^{-1}(xy)$, find $f_y\left(1, \dfrac{1}{2}\right)$.

3. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$:

   $(a).\ z = f(x) + g(y)$ $\quad (b).\ z = f(x+y)$

4. Find all the second partial derivatives of $f(x,y) = x^4y - 2x^3y^2$.

5. Verify that the conclusion of Clairaut's Theorem holds for
   $u(x,y) = x^4y^3 - y^4$.

6. If $f(x,y) = x^4y^2 - x^3y$, find f_{xxx} and f_{xyx}.

(*) Discuss the differentiability of the following functions at the given points:

(a). $f(x,y) = \begin{cases} \dfrac{x^2(y-2)}{x^2+(y-2)^2} & \text{if} \quad (x,y) \neq (0,2) \\ 0 & \text{if} \quad (x,y) = (0,2) \end{cases}$ at (0,2).

(b). $f(x,y) = \begin{cases} \dfrac{x^2y - xy^2}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$ at (0,0).

(c). $f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$ at (0,0).

1.4 Chain Rule

1.4.1 The Chain Rule (Case 1)

Theorem: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and $\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\,\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\,\dfrac{dy}{dt}$.

Example (1): If $f(x, y) = x^3y + 2xy^3$, where $x = \sin 2t$ and $y = \cos t$, find $\dfrac{dz}{dt}$ at $t = 0$.
Solution:
$\dfrac{\partial f}{\partial x} = 3x^2y + 2y^3$, $\dfrac{\partial f}{\partial y} = x^3 + 6xy^2$.
$\dfrac{dx}{dt} = 2\cos 2t$, $\dfrac{dy}{dt} = -\sin t$.
$\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\,\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\,\dfrac{dy}{dt}$
$= (3x^2y + 2y^3)(2\cos 2t) + (x^3 + 6xy^2)(-\sin t)$
$\left.\dfrac{dz}{dt}\right|_{t=0} = (0 + 2)(2) + (0 + 0)(0) = 4$.

1.4.2 The Chain Rule (Case 2)

Theorem: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t. Then z is a differentiable function of t and $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x}\,\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\,\dfrac{\partial y}{\partial s}$,
$\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\,\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\,\dfrac{\partial y}{\partial t}$

Example (2): If $f(x, y) = e^x \sin y$, where $x = s^2 + t^2$ and $y = s^2t^2$, find $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$.
Solution:
$\dfrac{\partial f}{\partial x} = e^x \sin y$, $\dfrac{\partial f}{\partial y} = e^x \cos y$.
$\dfrac{\partial x}{\partial s} = 2s$, $\dfrac{\partial y}{\partial s} = 2st^2$.
(1). $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x}\,\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\,\dfrac{\partial y}{\partial s} = (e^x \sin y)\,(2s) + (e^x \cos y)\,(2st^2)$
$= 2se^{s^2+t^2} \sin(s^2t^2) + 2st^2e^{s^2+t^2} \cos(s^2t^2)$.
Also, $\dfrac{\partial x}{\partial t} = 2t$, $\dfrac{\partial y}{\partial s} = 2ts^2$.
(2). $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\,\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\,\dfrac{\partial y}{\partial t} = (e^x \sin y)\,(2t) + (e^x \cos y)\,(2ts^2)$
$= 2te^{s^2+t^2} \sin(s^2t^2) + 2ts^2e^{s^2+t^2} \cos(s^2t^2)$.

1.4.3 The Chain Rule (The General Case)

Theorem: Suppose that u is a differentiable function of the n variables $x_1, x_2, \cdots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \cdots, t_m$. Then u is a function of $t_1, t_2, \cdots, t_m$ and
$\dfrac{\partial u}{\partial t_i} = \dfrac{\partial u}{\partial x_1}\dfrac{\partial x_1}{\partial t_i} + \dfrac{\partial u}{\partial x_2}\dfrac{\partial x_2}{\partial t_i} + \cdots + \dfrac{\partial u}{\partial x_n}\dfrac{\partial x_n}{\partial t_i}$, for each $i = 1, 2, \cdots, m$.

Example (3): If $u(x, y, z) = x^2y^2 + yz^3$, where $x = r^2se^t$, $y = rse^{-t}$ and $z = rs^2 \sin t$, find $\dfrac{\partial u}{\partial s}$ when $r = 1$, $s = 1$ and $t = 0$.
Solution:
$\dfrac{\partial u}{\partial x} = 2xy^2$, $\dfrac{\partial u}{\partial y} = 2x^2y + z^3$ and $\dfrac{\partial u}{\partial z} = 3yz^2$.
$\dfrac{\partial x}{\partial s} = r^2e^t$, $\dfrac{\partial y}{\partial s} = re^{-t}$ and $\dfrac{\partial z}{\partial s} = 2rs\sin t$.
So, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial s}$
$\dfrac{\partial u}{\partial s} = (2xy^2)(r^2e^t) + (2x^2y + z^3)(re^{-t}) + (3yz^2)(2rs\sin t)$
When $r = 1, s = 1, t = 0$, $\dfrac{\partial u}{\partial s} = (2)(1) + (2)(1) + (0)(0) = 4$.

Example (4): If $w = f(x, y)$ is differentiable at (x, y) and $x = s + t$, $y = s - t$.
Show that $\dfrac{\partial w}{\partial s}\dfrac{\partial w}{\partial t} = \left(\dfrac{\partial w}{\partial x}\right)^2 - \left(\dfrac{\partial w}{\partial y}\right)^2$.
$\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial s} = f_x(1) + f_y(1) = f_x + f_y.$
$\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t} = f_x(1) + f_y(-1) = f_x - f_y.$
$\dfrac{\partial w}{\partial s}\dfrac{\partial w}{\partial t} = (f_x + f_y)(f_x - f_y) = (f_x)^2 - (f_y)^2 = \left(\dfrac{\partial w}{\partial x}\right)^2 - \left(\dfrac{\partial w}{\partial y}\right)^2$.

Example (5): If $z = f(s^2 - t^2, t^2 - s^2)$ is differentiable at (s, t),
Show that $t\dfrac{\partial z}{\partial s} + s\dfrac{\partial z}{\partial t} = 0$.
Solution:
Let $z = f(x, y)$, where $x = s^2 - t^2$ and $y = t^2 - s^2$.
$\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} = f_x(2s) + f_y(-2s) = 2sf_x - 2sf_y$.
$\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} = f_x(-2t) + f_y(2t) = -2tf_x + 2tf_y$.
$t\dfrac{\partial z}{\partial s} + s\dfrac{\partial z}{\partial t} = t(2sf_x - 2sf_y) + s(-2tf_x + 2tf_y)$
$= 2stf_x - 2stf_y - 2stf_x + 2stf_y = 0$.

Example (6): If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s$, $y = 3rs$. Find $\dfrac{\partial^2 z}{\partial r^2}$.
Solution:
$\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial r} = f_x(2r) + f_y(3s) = 2rf_x + 3sf_y.$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(2rf_x + 3sf_y\right) = 2f_x + 2r\left(\frac{\partial f_x}{\partial r}\right) + 3s\left(\frac{\partial f_y}{\partial r}\right)$$
$$= 2f_x + 2r\left(\frac{\partial f_x}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial r}\right) + 3s\left(\frac{\partial f_y}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial r}\right)$$
$$= 2f_x + 2r\left(f_{xx}(2r) + f_{xy}(3s)\right) + 3s\left(f_{yx}(2r) + f_{yy}(3s)\right)$$
$$= 2f_x + 4r^2 f_{xx} + 6rsf_{xy} + 6rsf_{yx} + 9s^2 f_{yy} \ .$$

1.4.4 Implicit Differentiation

Let $F(x,y) = 0$ where $y = f(x)$, using chain rule $\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$

$\implies F_x(1) + F_y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{F_x}{F_y}$.

Example (7): If $x^2 + y^2 = 5xy$, Find $\frac{dy}{dx}$.
Solution:
$x^2 + y^2 = 5xy \implies x^2 + y^2 - 5xy = 0$, Let $F(x,y) = x^2 + y^2 - 5xy$ then $F(x,y) = 0$.
$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x - 5y}{2y - 5x}$.

Let $F(x,y,z) = 0$ where $z = f(x,y)$, then $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$.

Example (8): If $x^3 + y^3 + z^2 + 3xyz - 1 = 0$, Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
Solution:
Let $F(x,y,z) = x^3 + y^3 + z^2 + 3xyz - 1$ then $F(x,y,z) = 0$.
$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 3yz}{2z + 3xy}$.
$\frac{\partial z}{\partial x} = -\frac{F_y}{F_z} = -\frac{3y^2 + 3xz}{2z + 3xy}$.

1.4.5 EXERCISES

1. Find $\dfrac{dw}{dt}$ of the following:

   (a). $w = xy^3 - x^2y$, where $x = t^2 + 1$ and $y = t^2 - 1$.

   (b). $w = \sin x \cos y$, where $x = \sqrt{t}$ and $y = \dfrac{1}{t}$.

   (c). $w = xe^{\frac{y}{z}}$, where $x = t^2$, $y = 1 - t$ and $z = 1 + 2t$.

   (d). $w = \ln \sqrt{x^2 + y^2 + z^2}$, where $x = \sin t$, $y = \cos t$ and $z = \tan t$.

2. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ of the following:

   (a). $z = (x - y)^5$, where $x = s^2t$ and $y = st^2$.

   (b). $z = \tan^{-1}\left(x^2 + y^2\right)$, where $x = s\ \ln t$ and $y = t\ e^s$.

   (c). $z = \dfrac{\sin\theta}{r}$, where $r = st$ and $\theta = s^2 + t^2$.

3. If $z = x^4 + x^2y$, where $x = s + 2t - u$ and $y = stu^2$,

   Find $\dfrac{\partial z}{\partial s}, \dfrac{\partial z}{\partial t}$ and $\dfrac{\partial z}{\partial u}$ at $s = 4$, $t = 2$, $u = 1$.

4. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, Show that

   $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

5. If $z = f(x + at) + g(x - at)$, Show that $\dfrac{\partial^2 z}{\partial t^2} = a^2\ \dfrac{\partial^2 z}{\partial x^2}$.

(*) If $xe^{yz} - 2ye^{xz} + 3ze^{xy} = 1$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

1.5 Maximum and Minimum Values

1.5.1 Local Maximum and Minimum Values

Definition: If f is a differentiable function of two variables x and y, then the gradient of f is the vector function ∇f and it is defined by
$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j}$.

Definition: If f is a differentiable function of three variables x,y and z, then the gradient of f is the vector function ∇f and it is defined by
$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j} + \dfrac{\partial f}{\partial z}\vec{k}$.

Definition (Critical point): If $f(x,y)$ is a function of two variables then $(a,b) \in D_f$ is a critical point if both $f_x(a,b) = 0$ and $f_y(a,b) = 0$, or f is not differentiable at (a,b).

Definition (Local maximum and Local minimum):
(i) A function f of two variables has a local maximum at $(a,b) \in D_f$ if $f(x,y) \le f(a,b)$, for all points (x,y) in some disk with center (a,b).
(ii) A function f of two variables has a local minimum at $(a,b) \in D_f$ if $f(x,y) \ge f(a,b)$, for all points (x,y) in some disk with center (a,b).

Theorem: If $f(x,y)$ has a local maximum or minimum at $(a,b) \in D_f$ and the first-order partial derivatives of f exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Second Derivatives Test: Suppose the second partial derivatives of $f(x,y)$ are continuous on a disk with center $(a,b) \in D_f$, and suppose that
$f_x(a,b) = 0$ and $f_y(a,b) = 0$ [so (a,b) is a critical point of f].
Let $D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$.
(1). If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
(2). If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
(3). If $D < 0$, then (a,b) is a saddle point.

NOTES:
(1). If $D = 0$, then the test gives no information.
(2). $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}f_{yx} = f_{xx}f_{yy} - [f_{xy}]^2$.

Example (1): Find the local maximum and minimum values and saddle points of $f(x,y) = 6xy - 2x^3 + y^2$.
Solution:
$f_x(x,y) = 6y - 6x^2$ and $f_y(x,y) = 6x + 2y$.
$f_x(x,y) = 0 \implies 6y - 6x^2 = 0 \implies y = x^2$.
$f_y(x,y) = 0 \implies 6x + 2y = 0 \implies y = -3x$.
$f_x(x,y) = f_y(x,y) \implies x^2 = -3x \implies x^2 + 3x = 0$
$\implies x(x+3) = 0 \implies x = 0\ ,\ x = -3 \implies y = 0\ ,\ y = 9$.
So, the critical points are $(0,0)$ and $(-3,9)$.

$f_{xx}(x,y) = -12x$, $f_{xy}(x,y) = 6$ and $f_{yy}(x,y) = 2$.

First- At the point $(0,0)$:
$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - [f_{x,y}(0,0)]^2 = (0)(2) - (6)^2 = -36 < 0.$
Therefore, $(0,0)$ is a saddle point.

Second- At the point $(-3,9)$:
$D(-3,9) = f_{xx}(-3,9)f_{yy}(-3,9) - [f_{x,y}(-3,9)]^2 = (36)(2) - (6)^2 = 36 > 0.$
Since $f_{xx}(-3,9) = 36 > 0$, then f attains a local minimum at $(-3,9)$,
and its value is $f(-3,9) = 6(-3)(9) - 2(-3)^3 + (-9)^2 = -162 + 54 + 81 = -27.$

Example (2): Find the local maximum and minimum values and saddle points of $f(x,y) = x^3 - y^3 - 3x + 3y + 5$.
Solution:
$f_x(x,y) = 3x^2 - 3$ and $f_y(x,y) = -3y^2 + 3$.
$f_x(x,y) = 0 \implies 3x^2 - 3 = 0 \implies x^2 - 1 = 0 \implies x = \pm 1$.
$f_y(x,y) = 0 \implies -3y^2 + 3 = 0 \implies y^2 - 1 = 0 \implies y = \pm 1$.
So, the critical points are $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$.

$f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = 0$ and $f_{yy}(x,y) = -6y$.

First- At the point $(1,1)$:
$D(1,1) = f_{xx}(1,1)f_{yy}(1,1) - [f_{x,y}(1,1)]^2 = (6)(-6) - (0)^2 = -36 < 0.$
Therefore, $(1,1)$ is a saddle point.

Second- At the point $(1,-1)$:
$D(1,-1) = f_{xx}(1,-1)f_{yy}(1,-1) - [f_{x,y}(1,-1)]^2 = (6)(6) - (0)^2 = 36 > 0.$
Since $f_{xx}(1,-1) = 6 > 0$, then f attains a local minimum at $(1,-1)$,
and its value is $f(1,-1) = 1 + 1 - 3 - 3 + 5 = 1.$

Third- At the point $(-1,1)$:
$D(-1,1) = f_{xx}(-1,1)f_{yy}(-1,1) - [f_{x,y}(-1,1)]^2 = (-6)(-6) - (0)^2 = 36 > 0.$
Since $f_{xx}(-1,1) = -6 < 0$, then f attains a local maximum at $(-1,1)$,
and its value is $f(-1,1) = -1 - 1 + 3 + 3 + 5 = 9.$

Fourth- At the point $(-1,-1)$:
$D(-1,-1) = f_{xx}(-1,-1)f_{yy}(-1,-1) - [f_{x,y}(-1,-1)]^2$
$= (-6)(6) - (0)^2 = -36 < 0.$
Therefore, $(-1,-1)$ is a saddle point.

1.5.2 Absolute Maximum and Minimum Values

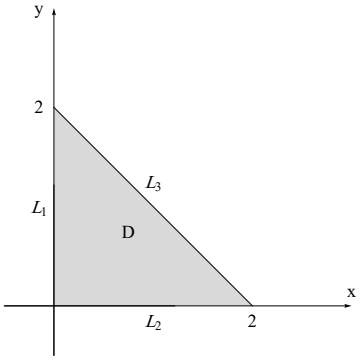
Definition: Let (a,b) be a point in the domain D of a function f of two variables. Then $f(a,b)$ is the
(1). absolute maximum value of f on D if $f(a,b) \geq f(x,y)$ for all (x,y) in D.
(2). absolute minimum value of f on D if $f(a,b) \leq f(x,y)$ for all (x,y) in D.

Theorem: If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

NOTES: To find the absolute maximum and minimum values of a continuous function $f(x, y)$ on a closed, bounded set $D \subseteq \mathbb{R}^2$:
1. Find the values of f at the critical points of f in the interior of D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps (1) and (2) is the absolute maximum value of f on D, and the smallest of these values is the absolute minimum value of f on D.

Example (3): Find the absolute maximum and minimum values of the function $f(x, y) = xy + 7$ on the plain region bounded by the graphs of the lines $x = 0$, $y = 0$ and $y + x = 2$.
Solution:

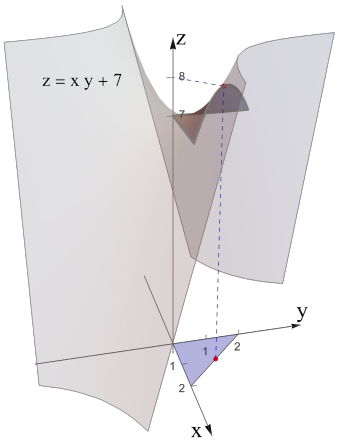
$f_x(x, y) = y$ and $f_y(x, y) = x$.
$f_x(x, y) = 0 \implies y = 0$.
$f_y(x, y) = 0 \implies x = 0$.
The critical point is $(0, 0)$.
Note that the critical point
is not inside the given region.



Let L_1 be the line $x = 0$:
then $f(x, y) = f(0, y) = 7$.
$f(x, y) = 7$ for all $(x, y) \in L_1$.

Let L_2 be the line $y = 0$:
then $f(x, y) = f(x, 0) = 7$.
$f(x, y) = 7$ for all $(x, y) \in L_2$.

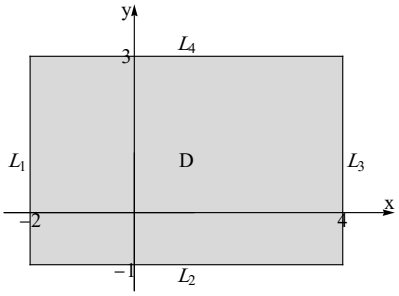
Let L_3 be the line $y + x = 2$:
then $y = -x + 2$ where $0 \le x \le 2$.
$f(x, y) = f(x, -x + 2)$
$= x(-x + 2) + 7 = -x^2 + 2x + 7$.
$f'(x) = -2x + 2$,
$f'(x) = 0 \implies -2x + 2 = 0$
$\implies x = 1$.
So, $y = -2 + 1 = 1$.
$f(1, 1) = (1)(1) + 7 = 8$.
Also, $f(0, 2) = 7$ and $f(2, 0) = 7$.



The absolute maximum is 8, and f takes it at $(1, 1)$.
The absolute minimum is 7, and f takes it at any point on $L_1 \cup L_2$.

Example (4): Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + 2xy + 3y^2$ on the closed and bounded region
$D = \{(x, y) \in \mathbb{R}^2 : -2 \le x \le 4 \ , \ -1 \le y \le 3\}$.

Solution:



$f_x(x,y) = 2x + 2y.$
$f_y(x,y) = 2x + 6y.$
$f_x(x,y) = 0 \implies x = -y.$
$f_y(x,y) = 0 \implies x = -3y.$
$-y = -3y \implies y = 0 \implies x = 0.$
The critical point is $(0,0)$.
$f(0,0) = (0)^2 + 2(0)(0) + 3(0)^3 = 0.$

Let L_1 be the line between $(-2,-1)$ and $(-2,3)$.
On L_1 : $x = -2$ and $-1 \le y \le 3$.
$f(x,y) = f(-2,y) = 4 - 4y + 3y^2 \implies f'(y) = -4 + 6y$.
$f'(y) = 0 \implies 6y = 4 \implies y = \dfrac{2}{3}.$
Note that $\left(-2, \dfrac{2}{3}\right) \in L_1.$
$f\left(-2, \dfrac{2}{3}\right) = (-2)^2 + 2(-2)\left(\dfrac{2}{3}\right) + 3\left(\dfrac{2}{3}\right)^2 = 4 - \dfrac{8}{3} + \dfrac{4}{3} = \dfrac{8}{3}.$

Let L_2 be the line between $(-2,-1)$ and $(4,-1)$.
On L_2 : $y = -1$ and $-2 \le x \le 4$.
$f(x,y) = f(x,-1) = x^2 - 2x + 3 \implies f'(x) = 2x - 2$.
$f'(x) = 0 \implies 2x = 2 \implies x = 1.$
Note that $(1,-1) \in L_2$.
$f(1,-1) = (1)^2 + 2(1)(-1) + 3(-1)^2 = 1 - 2 + 3 = 2.$

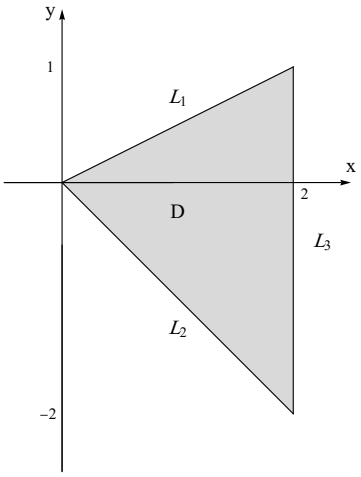
Let L_3 be the line between $(4,-1)$ and $(4,3)$.
On L_3 : $x = 4$ and $-1 \le y \le 3$.
$f(x,y) = f(4,y) = 16 + 8y + 3y^2 \implies f'(y) = 8 + 6y$.
$f'(y) = 0 \implies 6y = -8 \implies y = -\dfrac{4}{3}.$
Note that $\left(4, -\dfrac{4}{3}\right) \notin L_3.$

Let L_4 be the line between $(-2,3)$ and $(4,3)$.
On L_4 : $y = 3$ and $-2 \le x \le 4$.
$f(x,y) = f(x,3) = x^2 + 6x + 27 \implies f'(x) = 2x + 6$.
$f'(x) = 0 \implies 2x = 2 \implies x = -3.$
Note that $(-3,) \notin L_4$.

Evaluating $f(x,y)$ at the four corners of D :
$f(-2,-1) = (-2)^2 + 2(-2)(-1) + 3(-1)^2 = 4 + 4 + 3 = 11$.
$f(-2,3) = (-2)^2 + 2(-2)(3) + 3(3)^2 = 4 - 12 + 27 = 19$.
$f(4,-1) = (4)^2 + 2(4)(-1) + 3(-1)^2 = 16 - 8 + 3 = 11$.
$f(4,3) = (4)^2 + 2(4)(3) + 3(3)^2 = 16 + 24 + 27 = 67$.

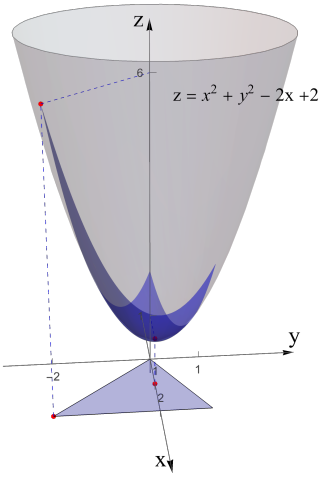
The absolute maximum is 67, and f takes it at $(4,3)$.
The absolute minimum is 0, and f takes it at $(0,0)$.

Example (5): Find the absolute maximum and minimum values of the function $f(x,y) = x^2 + y^2 - 2x + 2$ on the closed region with vertices $(0,0)$, $(2,1)$ and $(2,-2)$.
Solution:



$f_x(x,y) = 2x - 2$ and $f_y(x,y) = 2y$.
$f_x(x,y) = 0 \implies x = 1$.
$f_y(x,y) = 0 \implies y = 0$.
The critical point is $(1,0)$.
$f(1,0) = 1 + 0 - 2 + 2 = 1$.

Let L_1 be the line passing through $(0,0)$ and $(2,1)$ then $y = \frac{x}{2}$.
$f\left(x, \frac{x}{2}\right) = x^2 + \frac{x^2}{4} - 2x + 2$.
$f(x) = \frac{5}{4}x^2 - 2x + 2$.
$f'(x) = \frac{5}{2}x - 2$.
$f'(x) = 0 \implies x = \frac{4}{5}$ and $y = \frac{2}{5}$.
$f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{16}{25} + \frac{4}{25} - \frac{8}{5} + 2$
$= \frac{16}{25} + \frac{4}{25} - \frac{40}{25} + \frac{50}{25} = \frac{30}{25} = \frac{6}{5}$.



Let L_2 be the line passing through $(0,0)$ and $(2,-2)$ then $y = -x$.
$f(x,-x) = x^2 + x^2 - 2x + 2$
$f(x) = 2x^2 - 2x + 2$.
$f'(x) = 4x - 2$.
$f'(x) = 0 \implies x = \frac{1}{2}$ and $y = -\frac{1}{2}$.
$f\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} - 1 + 2 = \frac{3}{2}$.

Let L_3 be the line passing through $(2,-2)$ and $(2,1)$ then $x = 2$.
$f(2,y) = 4 + y^2 - 4 + 2 = y^2 + 2$, so $f(y) = y^2 + 2 \implies f'(y) = 2y$.
$f'(y) = 0 \implies y = 0$ and $x = 2$.
$f(2,0) = 4 + 0 - 4 + 2 = 2$.

Evaluating $f(x,y)$ at the three corners of D :
$f(0,0) = 0 + 0 - 0 + 2 = 2$.
$f(2,1) = 4 + 1 - 4 + 2 = 3$.
$f(2,-2) = 4 + 4 - 4 + 2 = 6$.

The absolute maximum is 6, and f takes it at $(2,-2)$.
The absolute minimum is 1, and f takes it at $(1,0)$.

1.5.3 EXERCISES

1. Find the local maximum and minimum values and saddle point(s) of the functions:

   (a). $f(x,y) = x^2 + xy + y^2 + y$.

   (b). $f(x,y) = x^3 + y^3 + 3xy$.

   (c). $f(x,y) = 2 - x^4 + 2x^2 - y^2$.

   (d). $f(x,y) = x^4 - 2x^2 + y^3 - 3y$.

2. Find the absolute maximum and minimum values of $f(x,y) = x^2 + y^2 - 2x$ on the closed region with vertices $(2,0)$, $(0,2)$ and $(0,-2)$.

(*) Find the local maximum and minimum values and saddle point(s) of the functions:

   (a). $f(x,y) = 4x^3 - 2x^2y + y^2$.

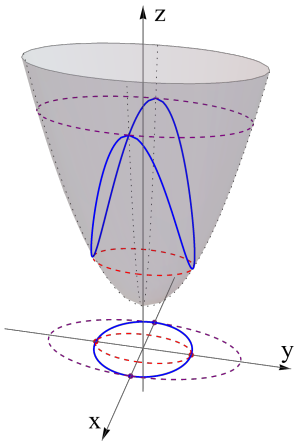
   (b). $f(x,y) = 2x^3 - 3x^2 + 3y^2 - 6xy + 2$.

   (c). $f(x,y) = x^4 + y^3 + 32x - 3y$.

1.6 Lagrange Multipliers

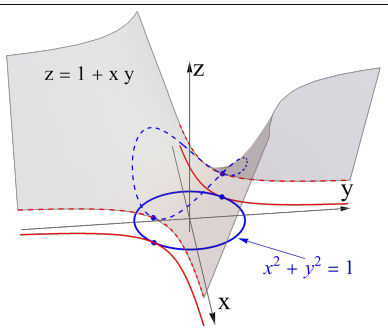
1.6.1 Lagrange Multipliers (One Constraint)

To find the extreme values of $f(x,y)$ subject to a constraint $g(x,y)=k$.
Let $z=f(x,y)$ be the gray surface, and $g(x,y)$ is the blue curve representing the constraint in the xy-plain.
Note that the level curves $f(x,y)$ touches $g(x,y)$ at the points where $f(x,y)$ have minimum and maximum values.
This means that $\nabla f(x,y)$ is parallel to $\nabla g(x,y)$ at these point.
Find these points by solving the equation: $\nabla f(x,y)=\lambda\nabla g(x,y)$,
where $\lambda\in\mathbb{R}$ and $\nabla g(x,y)\neq 0$.
Evaluate $f(x,y)$ at these points, The largest is the maximum value of f and the smallest is the minimum value of f.

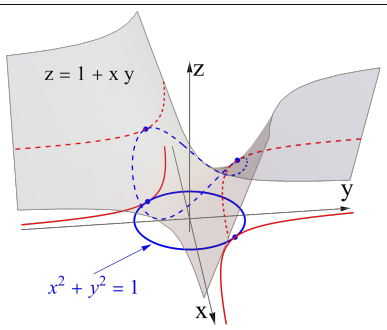


Example (1): Find the extreme values of the function $f(x,y)=1+xy$ on the circle $x^2+y^2=1$.
Solution:
$f(x,y)=1+xy$ and $g(x,y)=x^2+y^2-1$.
$\nabla f(x,y)=\lambda\nabla g(x,y) \implies \langle y,x\rangle=\lambda\langle 2x,2y\rangle=\langle 2\lambda x,2\lambda y\rangle$
$\implies \begin{cases} y=2\lambda x \\ x=2\lambda y \end{cases} \implies x=2\lambda(2\lambda x)=4\lambda^2 x \implies x\left(1-4\lambda^2\right)=0$
$\implies x=0\ ,\ \lambda=\pm\dfrac{1}{2}$.
If $x=0$ then $y=2\lambda(0)=0$, but $(0,0)$ does not lie on the unit circle, so $(0,0)$ is excluded.
If $\lambda=\pm\dfrac{1}{2} \implies y=2\left(\pm\dfrac{1}{2}\right)x \implies y=\pm x$.
From the constraint : $x^2+y^2=1 \implies 2x^2=1 \implies x=\pm\dfrac{1}{\sqrt{2}}$.
This means $\dfrac{1}{2}+y^2=1 \implies y=\pm\dfrac{1}{\sqrt{2}}$.
So, There are 4 points $\left(\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right),\left(\dfrac{1}{\sqrt{2}},-\dfrac{1}{\sqrt{2}}\right),\left(-\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)$ and $\left(-\dfrac{1}{\sqrt{2}},-\dfrac{1}{\sqrt{2}}\right)$.
$f\left(\dfrac{1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)=1+\dfrac{1}{\sqrt{2}}\dfrac{1}{\sqrt{2}}=1+\dfrac{1}{2}=\dfrac{3}{2}$.
$f\left(\dfrac{1}{\sqrt{2}},\dfrac{-1}{\sqrt{2}}\right)=1+\dfrac{1}{\sqrt{2}}\dfrac{-1}{\sqrt{2}}=1-\dfrac{1}{2}=\dfrac{1}{2}$.
$f\left(\dfrac{-1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)=1+\dfrac{-1}{\sqrt{2}}\dfrac{1}{\sqrt{2}}=1-\dfrac{1}{2}=\dfrac{1}{2}$.
$f\left(\dfrac{-1}{\sqrt{2}},\dfrac{-1}{\sqrt{2}}\right)=1+\dfrac{-1}{\sqrt{2}}\dfrac{-1}{\sqrt{2}}=1+\dfrac{1}{2}=\dfrac{3}{2}$.

The maximum value is $\frac{3}{2}$, and f takes it at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.
The minimum value is $\frac{1}{2}$, and f takes it at $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.



Minimum value of $z = 1 + xy$ on $x^2 + y^2 = 1$



Maximum value of $z = 1 + xy$ on $x^2 + y^2 = 1$

Note: to find the extreme values of $f(x, y, z)$ subject to a constraint $g(x, y, z) = k$, where $k \in \mathbb{R}$.
(1). Find the points that satisfy the equation $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$
$f_x(x, y, z) = \lambda g_x(x, y, z)$, $f_y(x, y, z) = \lambda g_y(x, y, z)$ and $f_z(x, y, z) = \lambda g_z(x, y, z)$.
(2). Evaluate $f(x, y, z)$ at these points, The largest is the maximum value of f and the smallest is the minimum value of f.

Example (2): Find the points on the sphere $x^2 + y^2 + z^2 = 1$ that are closest to and farthest from the point $(2, 2, 2)$.
Solution:
Let $f(x, y, z)$ be the function of the square of the distance between any point in the sphere and the point $(2, 2, 2)$, then $f(x, y, z) = (x-2)^2 + (y-2)^2 + (z-2)^2$.
Let $g(x, y, z) = x^2 + z^2 + y^2 - 1 = 0$ be the constraint.

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \implies \langle 2(x-2), 2(y-2), 2(z-2)\rangle = \lambda \langle 2x, 2y, 2z\rangle$$

$$\begin{cases} 2(x-2) = 2\lambda x \\ 2(y-2) = 2\lambda y \\ 2(z-2) = 2\lambda z \end{cases} \implies \begin{cases} x - 2 = \lambda x \\ y - 2 = \lambda y \\ z - 2 = \lambda z \end{cases} \implies \begin{cases} x(1-\lambda) = 2 \\ y(1-\lambda) = 2 \\ z(1-\lambda) = 2 \end{cases}$$

If $\lambda = 1$ then $x - 2 = x \implies -2 = 0$, so $\lambda \neq 1$.
Therefore, $x(1-\lambda) = y(1-\lambda) = z(1-\lambda) \implies x = y = z$.
So, $x^2 + y^2 + z^2 = 1 \implies 3x^2 = 1 \implies x = \pm\frac{1}{\sqrt{3}}$

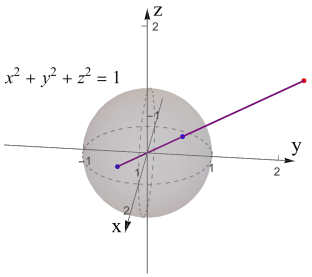
Hence, $x = y = z = \pm\frac{1}{\sqrt{3}}$.
The required points are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$.

$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}} - 2\right)^2 + \left(\frac{1}{\sqrt{3}} - 2\right)^2 + \left(\frac{1}{\sqrt{3}} - 2\right)^2 = 3\left(2 - \frac{1}{\sqrt{3}}\right)^2.$$

$$f\left(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)=\left(\frac{-1}{\sqrt{3}}-2\right)^2+\left(\frac{-1}{\sqrt{3}}-2\right)^2+\left(\frac{-1}{\sqrt{3}}-2\right)^2=3\left(2+\frac{1}{\sqrt{3}}\right)^2.$$

The point $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ is the closest to $(2,2,2)$.

The point $\left(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)$ is the farthest to $(2,2,2)$.



Example (3): Find the maximum volume of a rectangular box without a lid, where its surface area equals 12 cm^2.
Solution:
Suppose the sides of the rectangular box are x, y and z.
The volume of the rectangular box is $V(x,y,z)=xyz$, subject to the constraint $2xz+2yz+xy=12$.
The constraint is $g(x,y,z)=2xz+2yz+xy-12=0$.
$\nabla V(x,y,z)=\lambda\nabla g(x,y,z) \implies \langle yz,xz,xy\rangle=\lambda\langle 2z+y,2z+x+2x+2z\rangle$

$$\begin{cases} yz=\lambda(2z+y) \\ xz=\lambda(2z+x) \\ xy=\lambda(2x+2y) \end{cases} \implies \begin{cases} xyz=\lambda(2xz+xy) & \longrightarrow (1) \\ xyz=\lambda(2yz+xy) & \longrightarrow (2) \\ xyz=\lambda(2xz+2yz) & \longrightarrow (3) \end{cases}$$

If $\lambda=0$ then $V(x,y,z)=0$.
If $x=0$, $y=0$ or $z=0$, then $V(x,y,z)=0$.

If $\lambda\neq 0$, From equations (1) and (2) :
$\lambda(2xz+xy)=\lambda(2yz+xy) \implies 2xz+xy=2yz+xy \implies 2xz=2yz \implies x=y.$

From equations (2) and (3), and $x=y$:
$\lambda(2yz+xy)=\lambda(2xz+2yz) \implies 2xz+x^2=2xz+2xz$
$\implies x^2=2xz \implies x^2-2xz=0 \implies x(x-2z)=0 \implies x=2z.$

From the equation of the constraint and $x=y=2z$:
$4z^2+4z^2+4z^2=12 \implies z^2=1 \implies z=1$ and $x=y=2z=2.$

The sides of the rectangular box are 2, 2 and 1, and its maximum volume is 4 cm^3.

1.6.2 EXERCISES

1. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint:

   (a). $f(x,y) = x^2 - y^2$, $x^2 + y^2 = 1$.

   (b). $f(x,y) = xye^{-x^2-y^2}$, $2x - y = 0$.

   (c). $f(x,y,z) = xy^2z$, $x^2 + y^2 + z^2 = 4$.

   (d). $f(x,y,z) = x^4 + y^4 + z^4$, $x^2 + y^2 + z^2 = 1$.

2. Find the extreme values of f on the region described by the inequality:

   (a). $f(x,y) = x^2 + y^2 + 4x - 4y$, $x^2 + y^2 \leq 9$.

   (b). $f(x,y) = e^{-xy}$, $x^2 + 4y^2 \leq 1$.

3. Show that the problem of finding the minimum value of f subject to the given constraint can be solved using Lagrange multipliers, but f does not have a maximum value with that constraint:

   (a). $f(x,y) = x^2 + y^2$, $xy = 1$.

   (b). $f(x,y,z) = x^2 + 2y^2 + 3z^2$, $x + 2y + 3z = 10$.

Chapter 2

Multiple Integrals

2.1 Double Integrals over Rectangles

2.1.1 Iterated Integrals

Suppose that $f(x,y)$ is integrable on the rectangle $R=[a,b]\times[c,d]$, then
$\iint_R f(x,y)\ dA=\int_a^b\int_c^d f(x,y)\ dydx=\int_a^b\left(\int_c^d f(x,y)\ dy\right)\ dx.$

Example (1): Evaluate the integral:
(a). $\int_0^2\int_1^2 x^2y\ dydx$, (b). $\int_1^2\int_0^2 x^2y\ dxdy.$
Solution:
(a). $\int_0^2\int_1^2 x^2y\ dydx=\int_0^2\left(\int_1^2 x^2y\ dy\right)\ dx=\int_0^2\left(x^2\int_1^2 y\ dy\right)\ dx$
$=\int_0^2 x^2\ \left[\frac{y^2}{2}\right]_1^2\ dx=\int_0^2 x^2\ \left[\frac{4}{2}-\frac{1}{2}\right]\ dx$
$=\frac{3}{2}\int_0^2 x^2\ dx=\frac{3}{2}\ \left[\frac{x^3}{3}\right]_0^2=\frac{3}{2}\left[\frac{8}{3}-0\right]=4.$
(b). $\int_1^2\int_0^2 x^2y\ dxdy=\int_1^2\left(\int_0^2 x^2y\ dx\right)dy=\int_1^2\left(y\int_0^2 x^2\ dx\right)dy.$
$=\int_1^2 y\ \left[\frac{x^3}{3}\right]_0^2 dy=\int_1^2 y\ \left[\frac{8}{3}-0\right]_0^2 dy=\frac{8}{3}\int_1^2 y\ dy$
$=\frac{8}{3}\left[\frac{y^2}{2}\right]_1 2=\frac{8}{3}\left[\frac{4}{2}-\frac{1}{2}\right]=\frac{8}{3}\ \frac{3}{2}=4.$

Fubini's Theorem: If f is continuous on the rectangle $R=\{(x,y)\mid a\le x\le b\ ,\ c\le y\le d\}$,
Then $\iint_R f(x,y)\ dA=\int_a^b\int_c^d f(x,y)\ dydx=\int_c^d\int_a^b f(x,y)\ dxdy.$

Example (2): Evaluate the integral $\iint_R y\cos(xy)\ dA$,
where $R=[0,1]\times\left[0,\frac{\pi}{2}\right]$.

Solution: Using Fubini's Theorem.

$$\iint_R y\cos(xy)\ dA = \int_0^{\frac{\pi}{2}}\int_0^1 y\cos(xy)\ dxdy = \int_0^{\frac{\pi}{2}}\left(\int_0^1 y\cos(xy)\ dx\right)\ dy$$
$$= \int_0^{\frac{\pi}{2}} [\sin(xy)]_0^1\ dy = \int_0^{\frac{\pi}{2}} [\sin y - \sin(0)]\ dy = \int_0^{\frac{\pi}{2}} \sin y\ dy$$
$$= [-\cos y]_0^{\frac{\pi}{2}} = -\cos\left(\frac{\pi}{2}\right) - (-\cos(0)) = 0 + 1 = 1\ .$$

Note : Solving $\iint_R y\cos(x,y)\ dA = \int_0^1\int_0^{\frac{\pi}{2}} y\cos(xy)\ dydx$ is hard, it needs integration by parts.

2.1.2 Volume

If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle $R = [a,b]\times[c,d]$ and below the surface $z = f(x,y)$ is

$$\mathrm{V} = \iint_R f(x,y)\ dA = \int_a^b\int_c^d f(x,y)\ dydx = \int_c^d\int_a^b f(x,y)\ dxdy\ .$$

Example (3): Find the volume of the solid S that is bounded by $z = x^2+y^2+1$, the plains $x = 1$ and $y = 3$, and the three coordinate plains.
Solution:
Note that S is the solid that lies under the surface $z = x^2 + y^2 + 1$ and above the square $R = [0,1]\times[0,3]$.

$$\mathrm{V} = \iint_R (x^2+y^2+1)\ dA = \int_0^1\int_0^3 (x^2+y^2+1)\ dydx$$
$$= \int_0^1 \left[x^2y + \frac{y^3}{3} + y\right]_0^3\ dx = \int_0^1 \left[(3x^2+9+3) - (0+0+0)\right]\ dx$$
$$= \int_0^1 (3x^2+12)\ dx = \left[x^3+12x\right]_0^1 = (1+12) - (0+0) = 13\ .$$

Corollary: If $f(x,y) = g(x)h(y)$ is continuous on the rectangle $R = [a,b]\times[c,d]$, then $\iint_R f(x,y)\ dA = \iint_R g(x)h(y)\ dA = \int_a^b g(x)\ dx\ \int_c^d h(y)\ dy$.

Example (4): Evaluate $\iint_R x\cos y\ dA$, where $R = [0,2]\times\left[0,\frac{\pi}{2}\right]$.
Solution:

$$\iint_R x\cos y\ dA = \int_0^2\int_0^{\frac{\pi}{2}} x\cos y\ dydx = \left(\int_0^2 x\ dx\right)\left(\int_0^{\frac{\pi}{2}}\cos y\ dy\right)$$
$$= \left[\frac{x^2}{2}\right]_0^2 [\sin y]_0^{\frac{\pi}{2}} = [2-0][1-0] = 2\ .$$

Example (5): Evaluate $\int_0^1\int_1^2 \left(e^y e^{-x} + y^2\right)\ dx\ dy$.
Solution:

$$\int_0^1\int_1^2 \left(e^y e^{-x} + y^2\right)\ dx\ dy = \int_0^1\int_1^2 e^y e^{-x}\ dx\ dy + \int_0^1\int_1^2 y^2\ dx\ dy$$

$$= \left(\int_1^2 e^{-x}\ dx\right)\left(\int_0^1 e^{y}\ dy\right) + \left(\int_1^2 dx\right)\left(\int_0^1 y^2\right)$$
$$= \left[-e^{-x}\right]_1^2 \left[e^{y}\right]_0^1 + [x]_1^2 \left[\frac{y^3}{3}\right]_0^1 = \left(-e^{-2} - \left(-e^{-1}\right)\right)(e-1) + (2-1)\left(\frac{1}{3} - 0\right)$$
$$= (e-1)\left(\frac{1}{e} - \frac{1}{e^2}\right) + \frac{1}{3}\ .$$

2.1.3 Average Value

If $f(x,y)$ is defined on a rectangle R then its average value is $f_{avg} = \dfrac{1}{A(R)} \iint_R f(x,y)\ dA$, where $A(R)$ is the area of the rectangle R.

Example (6): Evaluate f_{avg} of $f(x,y) = x\cos y$ on $R = [0,2] \times \left[0, \dfrac{\pi}{2}\right]$.
Solution:
$f_{avg} = \dfrac{1}{A(R)} \iint_R x\cos y\ dA = \dfrac{2}{(2-0)\left(\frac{\pi}{2} - 0\right)} = \dfrac{2}{\pi}\ .$

2.1.4 EXERCISES

1. Calculate the iterated integrals :

   (a). $\int_1^4 \int_0^2 (6x^2y - 2x)\ dy\ dx$.

   (b). $\int_{-3}^1 \int_1^2 (x^2 + y^{-2})\ dy\ dx$.

   (c). $\int_{-3}^3 \int_0^{\frac{\pi}{2}} (y + y^2 \cos x)\ dx\ dy$.

   (d). $\int_1^3 \int_1^5 \frac{\ln y}{xy}\ dy\ dx$.

   (e). $\int_0^1 \int_0^1 xy\sqrt{x^2 + y^2}\ dy\ dx$.

   (f). $\int_0^1 \int_0^1 v(u + v^2)^4\ du\ dv$.

2. Calculate the double integrals :

   (a). $\iint_R x \sec^2 y\ dA$, where $R = \left\{(x, y)|\ 0 \le x \le 2\ ,\ 0 \le y \le \frac{\pi}{4}\right\}$.

   (b). $\iint_R \frac{xy^2}{x^2 + 1}\ dA$, where $R = \{(x, y)|\ 0 \le x \le 1\ ,\ -3 \le y \le 3\}$.

   (c). $\iint_R \frac{1}{1 + x + y}\ dA$, where $R = [1, 3] \times [1, 2]$.

3. Find the volume of the solid that lies under the plain $4x+6y-2z+15 = 0$ and above the rectangle $R = \{(x, y)|\ -1 \le x \le 2\ ,\ -1 \le y \le 1\}$.

4. Find the volume of the solid that lies under $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

2.2 Double Integrals over General Regions

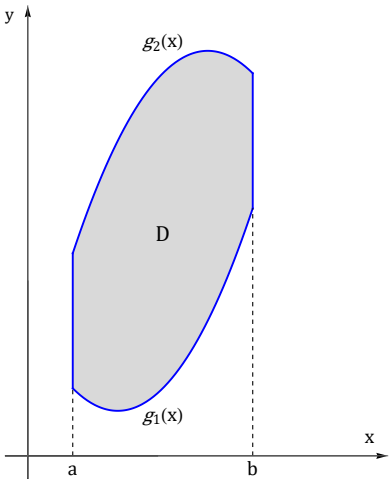
2.2.1 General Regions

First - Regions of Type I:

Let D be the region
$\{(x,y) \mid a \le x \le b\ ,\ g_1(x) \le y \le g_2(x)\}$

If f is continuous on D, then

$$\iint_D f(x,y)\ dA$$
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dydx$$
$$= \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\ dy \right)\ dx$$

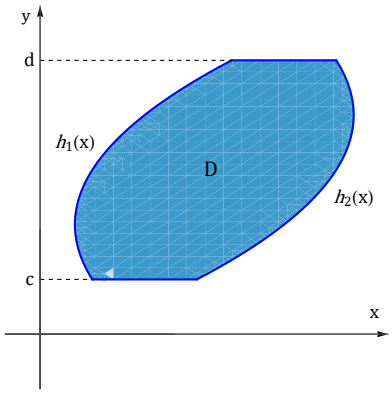


Second - Regions of Type II:

Let D be the region
$\{(x,y) \mid c \le y \le d\ ,\ h_1(y) \le x \le h_2(y)\}$

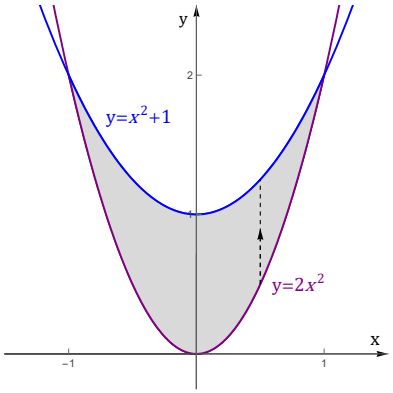
If f is continuous on D, then

$$\iint_D f(x,y)\ dA$$
$$= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dxdy$$
$$= \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\ dx \right)\ dy$$



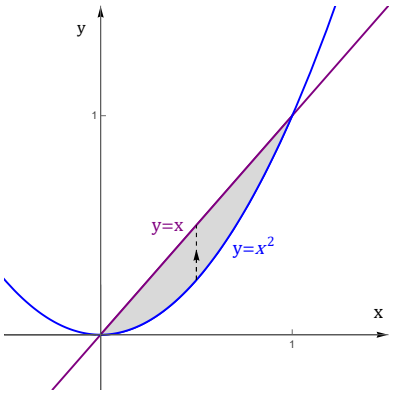
Example (1): Evaluate $\iint_D 2xy\ dA$, where D is the region bounded by the graphs of $y = 2x^2$ and $y = x^2 + 1$.
Solution:

Points of intersection:
$2x^2 = x^2 + 1 \implies x^2 = 1$
$\implies x = \pm 1$
So, D is the region where $-1 \le x \le 1$ and $2x^2 \le y \le x^2 + 1$



$$\iint_D 2xy \, dA = \int_{-1}^{1} \int_{2x^2}^{x^2+1} 2xy \, dy \, dx$$
$$= \int_{-1}^{1} x \left[y^2\right]_{2x^2}^{x^2+1} \, dx$$
$$= \int_{-1}^{1} x \left[(x^2+1)^2 - (2x^2)^2\right] \, dx$$
$$= \int_{-1}^{1} x \left(x^4 + 2x^2 + 1 - 4x^4\right) \, dx$$
$$= \int_{-1}^{1} x \left(-3x^4 + 2x^2 + 1\right) \, dx = \int_{-1}^{1} \left(-3x^5 + 2x^3 + x\right) dx$$
$$= \left[-\frac{x^6}{2} + \frac{x^4}{2} + \frac{x^2}{2}\right]_{-1}^{1}$$
$$= \left(-\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) - \left(-\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) = 0 \, .$$

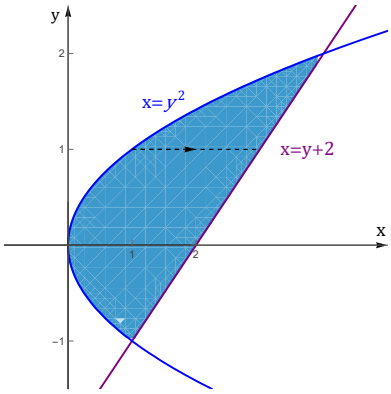
Example (2): Evaluate $\iint_D (x^2 + y^2) \, dA$, where D is the region bounded by the graphs of $y = x^2$ and $y = x$.
Solution:
Points of intersection:
$x^2 = x \implies x^2 - x = 0$
$\implies x(x-1) = 0 \implies x = 0 \, , \; x = 1$
So, D is the region where $0 \le x \le 1$ and $x^2 \le y \le x$



$$\iint_D (x^2 + y^2) \, dA$$
$$= \int_0^1 \int_{x^2}^{x} (x^2 + y^2) \, dy \, dx$$
$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_{x^2}^{x} \, dx$$
$$= \int_0^1 \left[\left(x^3 + \frac{x^3}{3}\right) - \left(x^4 - \frac{x^6}{3}\right)\right] \, dx$$
$$= \int_0^1 \left(-\frac{x^6}{3} - x^4 + \frac{4x^3}{3}\right) \, dx = \left[-\frac{x^7}{21} - \frac{x^5}{5} + \frac{x^4}{3}\right]_0^1$$
$$= \left(-\frac{1}{21} - \frac{1}{5} + \frac{1}{3}\right) - (0 + 0 + 0) = \frac{-5 - 21 + 35}{105} = \frac{9}{105} = \frac{3}{35} \, .$$

Example (3): Evaluate $\iint_D xy \, dA$, where D is the region bounded by the graphs of $x = y^2$ and $x = y + 2$.

Solution:
Points of intersection:
$y^2 = y + 2 \implies y^2 - y - 2 = 0$
$\implies (y-2)(y+1) = 0$
$\implies y = -1\ ,\ y = 2$
So, D is the region where $-1 \le y \le 2$ and $y^2 \le x \le y + 2$

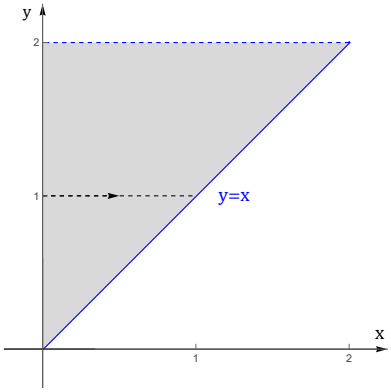


$$\iint_D xy\ dA = \int_{-1}^{2}\int_{y^2}^{y+2} xy\ dx\ dy$$
$$= \int_{-1}^{2} y\left[\frac{x^2}{2}\right]_{y^2}^{y+2} dy$$
$$= \frac{1}{2}\int_{-1}^{2} y\left[(y+2)^2 - (y^2)^2\right]\ dy$$
$$= \frac{1}{2}\int_{-1}^{2} y\left(y^2 + 4y + 4 - y^4\right)\ dy$$
$$= \frac{1}{2}\int_{-1}^{2} \left(-y^5 + y^3 + 4y^2 + 4y\right)\ dy = \frac{1}{2}\left[-\frac{y^6}{6} + \frac{y^4}{4} + \frac{4y^3}{3} + 2y^2\right]_{-1}^{2}$$
$$= \frac{1}{2}\left[\left(-\frac{64}{6} + \frac{16}{4} + \frac{32}{3} + 8\right) - \left(-\frac{1}{6} + \frac{1}{4} - \frac{4}{3} + 2\right)\right] = \frac{45}{8}\ .$$

2.2.2 Changing the Order of Integration

Example (4): Change the order of integration to evaluate $\int_0^2\int_x^2 e^{5+y^2}\ dy\ dx$.

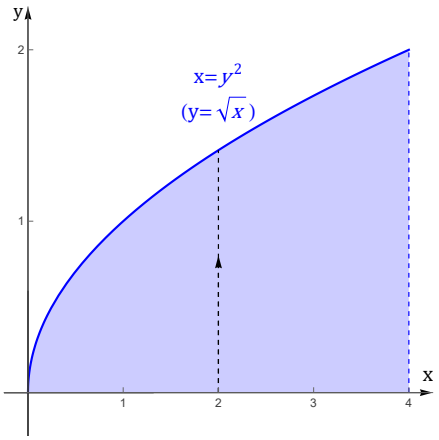
Solution:
D is the region where $0 \le x \le 2$ and $x \le y \le 2$.
Or, $0 \le y \le 2$ and $0 \le x \le y$.



$$\int_0^2\int_x^2 e^{5+y^2}\ dy\ dx$$
$$= \int_0^2\int_0^y e^{5+y^2}\ dx\ dy$$
$$= \int_0^2 e^{5+y^2}[x]_0^y\ dy = \int_0^2 ye^{5+y^2}\ dy$$
$$= \frac{1}{2}\int_0^2 e^{5+y^2}(2y)\ dy$$
$$= \frac{1}{2}\left[e^{5+y^2}\right]_0^2 = \frac{1}{2}\left(e^9 - e^5\right).$$

Example (5): Change the order of integration to evaluate $\int_0^2 \int_{y^2}^4 \cos\left(x^{\frac{3}{2}}\right)\ dx\ dy$.

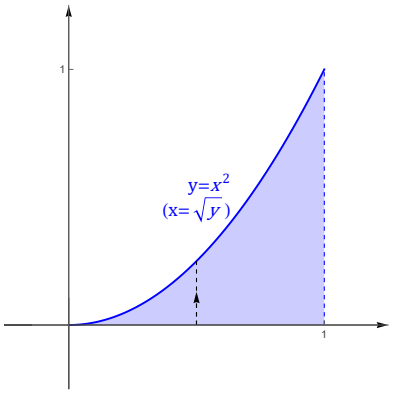
Solution:
D is the region where $0 \le y \le 2$ and $y^2 \le x \le 4$.
Or, $0 \le x \le 4$ and $0 \le y \le \sqrt{x}$.



$$\int_0^2 \int_{y^2}^4 \cos\left(x^{\frac{3}{2}}\right)\ dx\ dy$$
$$= \int_0^4 \int_0^{\sqrt{x}} \cos\left(x^{\frac{3}{2}}\right)\ dy\ dx$$
$$= \int_0^4 \cos\left(x^{\frac{3}{2}}\right)\ [y]_0^{\sqrt{x}}\ dx$$
$$= \frac{2}{3}\int_0^4 \cos\left(x^{\frac{3}{2}}\right)\ \left(\frac{3}{2}x^{\frac{1}{2}}\right)\ dx$$
$$= \frac{2}{3}\left[\sin\left(x^{\frac{3}{2}}\right)\right]_0^4 = \frac{2}{3}\left[\sin(8) - \sin(0)\right] = \frac{2}{3}\sin(8)\ .$$

Example (6): Change the order of integration to evaluate $\int_0^1 \int_{\sqrt{y}}^1 \sin\left(x^3\right)\ dx\ dy$.

Solution:
D is the region where $0 \le y \le 1$ and $\sqrt{y} \le x \le 1$.
Or, $0 \le x \le 1$ and $0 \le y \le x^2$.



$$\int_0^1 \int_{\sqrt{y}}^1 \sin\left(x^3\right)\ dx\ dy$$
$$= \int_0^1 \int_0^{x^2} \sin\left(x^3\right)\ dy\ dx$$
$$= \int_0^1 \sin\left(x^3\right)\ [y]_0^{x^2}\ dx$$
$$= \int_0^1 \sin\left(x^3\right)\ x^2\ dx$$
$$= \frac{1}{3}\int_0^1 \sin\left(x^3\right)\ (3x^2)\ dx = \frac{1}{3}\left[-\cos(x^3)\right]_0^1\ .$$
$$= \frac{1}{3}\left[-\cos(1) - (-\cos(0))\right] = \frac{1-\cos(1)}{3}\ .$$

2.2.3 Properties of Double Integrals

If $f(x,y)$ and $g(x,y)$ are both integrable on $D \subseteq \mathbb{R}^2$, then

(1). $\iint_D [f(x,y) + g(x,y)]\ dA = \iint_D f(x,y)\ dA + \iint_D g(x,y)\ dA$.

(2). $\iint_D k\ f(x,y)\ dA = k \iint_D f(x,y)\ dA$, where $k \in \mathbb{R}$.

(3). If $f(x,y) \ge g(x,y)$ on D, then $\iint_D f(x,y)\ dA \ge \iint_D g(x,y)\ dA$.

(4). If $D = D_1 \cup D_2$, where $D_1 \cap D_2 = \phi$, then
$\iint_D f(x,y)\ dA = \iint_{D_1} f(x,y)\ dA + \iint_{D_2} f(x,y)\ dA.$

(5). $\iint_D 1\ dA = A(D)$, where $A(D)$ is the area of the region D.

(6). If $m \leq f(x,y) \leq M$ on D, then $m\ A(D) \leq \iint_D f(x,y)\ dA \leq M\ A(D)$.

2.2.4 EXERCISES

1. Evaluate the iterated integrals :

   (a). $\int_1^5 \int_0^x (8x - 2y)\ dy\ dx$ (b). $\int_0^2 \int_0^{y^2} x^2 y\ dx\ dy$

   (c). $\int_0^1 \int_0^{e^x} \sqrt{1 + e^x}\ dy\ dx$

2. Evaluate $\iint_D 2y\ dA$, where D is the region bounded by the graphs of $y = 3x - x^2$ and $y = x$.

3. Evaluate the double integrals:

   (a). $\iint_D \frac{y}{x^2 + 1}\ dA$, $D = \left\{(x, y) \mid 0 \le x \le 4\ ,\ 0 \le y \le \sqrt{x}\right\}$.

   (b). $\iint_D (2x + y)\ dA$, $D = \{(x, y) \mid 1 \le y \le 2\ ,\ y - 1 \le x \le 1\}$.

   (c). $\iint_D x\ dA$, D is enclosed by the lines $y = x$, $y = 0$ and $x = 1$.

   (d). $\iint_D xy\ dA$, D is enclosed by the curves $y = x^2$ and $y = 3x$.

   (e). $\iint_D x \cos y\ dA$, D is bounded by the $y = 0$, $y = x^2$ and $x = 1$.

   (f). $\iint_D y^2\ dA$, D is the triangular region with vertices $(0, 1)$, $(1, 2)$ and $(4, 1)$.

4. Evaluate the integral by reversing the order of integration:

   (a). $\int_0^1 \int_{3y}^3 e^{x^2}\ dx\ dy$ (b). $\int_0^1 \int_{x^2}^1 \sqrt{y}\ \sin y\ dy\ dx$

   (c). $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1}\ dy\ dx$ (d). $\int_0^2 \int_{\frac{y}{2}}^1 y\ \cos(x^3 - 1)\ dx\ dy$

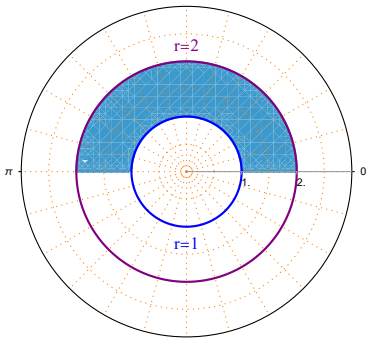
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2.3 Double Integrals in Polar Coordinates

2.3.1 Double Integrals in Polar Coordinates

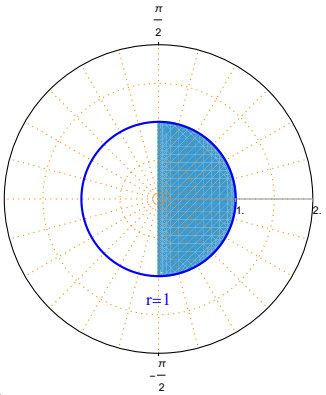
If f is continuous on the polar region R given by $a \le r \le b$ and $\theta_1 \le \theta \le \theta_2$, where $0 \le \theta_2 - \theta_1 \le 2\pi$, then $\iint_R f(x,y)\ dA = \int_{\theta_1}^{\theta_2} \int_a^b f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta$.

Example (1): Evaluate $\iint_R (4x^2 + 3y)\ dA$, where R is the region in the upper half-plain bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
Solution:
R is the region where $1 \le r \le 2$ and $0 \le \theta \le \pi$.



$$\iint_R (4x^2 + 3y)\ dA$$
$$= \int_0^{\pi} \int_1^2 (4r^2\cos^2\theta + 3r\sin\theta)\ r\ dr\ d\theta$$
$$= \int_0^{\pi} \int_1^2 (4r^3\cos^2\theta + 3r^2\sin\theta)\ dr\ d\theta$$
$$= \int_0^{\pi} \left[r^4\cos^2\theta + r^3\sin\theta\right]_1^2\ d\theta$$
$$= \int_0^{\pi} \left[(16\cos^2\theta + 8\sin\theta) - (\cos^2\theta + \sin\theta)\right]\ d\theta = \int_0^{\pi} (15\cos^2\theta + 7\sin\theta)\ d\theta$$
$$= \int_0^{\pi} \left[15\left(\frac{1+\cos 2\theta}{2}\right) + 7\sin\theta\right]\ d\theta = \frac{15}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi} + 7\left[-\cos\theta\right]_0^{\pi}$$
$$= \frac{15}{2}\left[(\pi + 0) - (0 + 0)\right] + 7[-(-1) - (-1)] = \frac{15\pi}{2} + 14.$$

Example (2): Evaluate $\int_{-1}^{1} \int_0^{\sqrt{1-y^2}} \frac{8(x^2+y^2)}{9 + (x^2+y^2)^2}\ dx\ dy$
Solution:
R is the region where $-1 \le y \le 1$ and $0 \le x \le \sqrt{1-y^2}$
In polar coordinates. $0 \le r \le 1$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

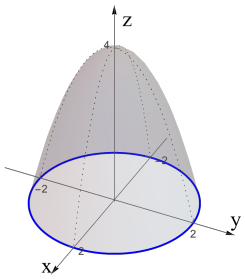


$$\int_{-1}^{1} \int_0^{\sqrt{1-y^2}} \frac{8(x^2+y^2)}{9 + (x^2+y^2)^2}\ dx\ dy$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \frac{8r^2}{9 + r^4} r\ dr\ d\theta$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 2\left(\frac{4r^3}{9 + r^4}\right)\ dr\ d\theta$$
$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\ln(9 + r^4)\right]_0^1\ d\theta = 2\left(\ln 10 - \ln 9\right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta$$

$$= 2\ln\left(\frac{10}{9}\right)[\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\ln\left(\frac{10}{9}\right)\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 2\pi\ln\left(\frac{10}{9}\right).$$

Example (3): Find the volume of the solid bounded by $z = 4 - x^2 - y^2$ and $z = 0$
Solution:

The surface of intersection is :
$4 - x^2 - y^2 = 0 \implies x^2 + y^2 = 4$.
R is the region inside the circle centered at the origin with radius 2.
In polar coordinates, $0 \le r \le 2$ and $0 \le \theta \le 2\pi$.



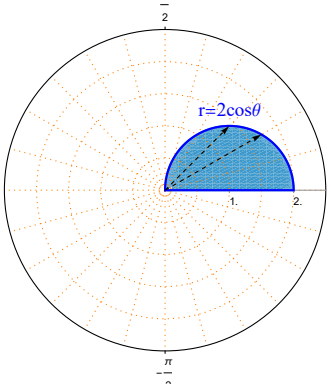
$$\mathrm{V} = \int_0^{2\pi}\int_0^2 (4 - r^2)\ r\ dr\ d\theta$$
$$= \int_0^{2\pi}\int_0^2 (4r - r^3)\ dr\ d\theta$$
$$= \int_0^{2\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi}[(8-4)-(0-0)]\ d\theta$$
$$= 4\int_0^{2\pi} d\theta = 4\,[\theta]_0^{2\pi} = 4\,(2\pi - 0) = 8\pi\ .$$

NOTE: If f is continuous on the polar region $R = \{(r,\theta)|\theta_1 \le \theta \le \theta_2\ ,\ g_1(\theta) \le r \le g_2(\theta)\}$, then $\iint_R f(x,y)\ dA = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta$.

Example (4): Evaluate $\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)^{\frac{1}{2}}\ dy\ dx$.
Solution:

R is the region where $0 \le x \le 2$ and $0 \le y \le \sqrt{2x - x^2}$.
$y = \sqrt{2x - x^2} \implies y^2 = 2x - x^2$
$\implies x^2 - 2x + y^2 = 0$
$\implies (x^2 - 2x + 1) + y^2 = 1$
$\implies (x-1)^2 + y^2 = 1$.

$y = \sqrt{2x - x^2}$ is the upper-half of the circle centered at $(1,0)$ with radius 1.



Note that : $y = \sqrt{2x - x^2} \implies y^2 = 2x - x^2 \implies x^2 + y^2 = 2x$
$\implies r^2 = 2r\cos\theta \implies r = 2\cos\theta$.

In polar coordinates, $0 \le r \le 2\cos\theta$ and $0 \le \theta \le \dfrac{\pi}{2}$.

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)^{\frac{1}{2}}\ dy\ dx = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r\ r\ dr\ d\theta = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\ dr\ d\theta$$
$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \cos^3\theta\ d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \cos^2\theta\ \cos\theta\ d\theta$$
$$= \frac{8}{3}\int_0^{\frac{\pi}{2}} \left(1-\sin^2\theta\right)\cos\theta\ d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \left(\cos\theta - \sin^2\theta\ \cos\theta\right)\ d\theta$$
$$= \frac{8}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{\frac{\pi}{2}} = \frac{8}{3}\left[\left(1-\frac{1}{3}\right) - (0-0)\right] = \frac{16}{9}.$$

2.3.2 EXERCISES

1. Evaluate the given integral by changing to polar coordinates.

   (a). $\iint_R (2x - y)\ dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.

   (b). $\iint_R e^{-x^2-y^2}\ dA$, where R is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis.

   (c). $\iint_R \cos\sqrt{x^2 + y^2}\ dA$, where R is the disk with center the origin and radius 2.

2. Use polar coordinates to find the volume of the given solid.

   (a). Under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leq 25$.

   (b). Below the cone $z = \sqrt{x^2 + y^2}$ and above the ring $1 \leq x^2 + y^2 \leq 4$.

   (c). Below the plain $2x + y + z = 4$ and above the disk $x^2 + y^2 \leq 1$.

   (d). Inside the sphere $x^2+y^2+z^2 = 16$ and outside the cylinder $x^2+y^2 = 4$.

   (e). Above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

   (f). Bounded by the paraboloids $z = 6 - x^2 - y^2$ and $z = 2x^2 + 2y^2$.

3. Evaluate the iterated integral by converting to polar coordinates.

   (a). $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\ dy\ dx$.

   (b). $\int_0^{\frac{1}{2}} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\ dx\ dy$.

2.4 Triple Integrals

2.4.1 Triple Integrals over Rectangular Boxes

Fubini's Theorem:
If f is continuous on the rectangular box $E = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_E f(x,y,z)\ dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\ dx\ dy\ dz = \int_r^s \int_a^b \int_c^d f(x,y,z)\ dy\ dx\ dz$$
$$= \int_c^d \int_r^s \int_a^b f(x,y,z)\ dx\ dz\ dy = \int_c^d \int_a^b \int_r^s f(x,y,z)\ dz\ dx\ dy$$
$$= \int_a^b \int_r^s \int_c^d f(x,y,z)\ dy\ dz\ dx = \int_a^b \int_c^d \int_r^s f(x,y,z)\ dz\ dy\ dx\ .$$

Example (1): Evaluate $\iiint_E xyz^2\ dV$, where $E = [0,1] \times [0,2] \times [0,3]$.
Solution:

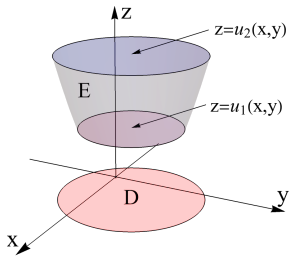
$$\iiint_E xyz^2\ dV = \int_0^1 \int_0^2 \int_0^3 xyz^2\ dz\ dy\ dx = \int_0^1 \int_0^2 xy \left[\frac{z^3}{3}\right]_0^3 dy\ dx$$
$$= \left[\frac{27}{3} - \frac{0}{3}\right] \int_0^1 \int_0^2 xy\ dy\ dx = 9 \int_0^1 x \left[\frac{y^2}{2}\right]_0^2 dx = 9 \int_0^1 x \left[\frac{2^2}{2} - \frac{0}{2}\right] dx$$
$$= 9 \int_0^1 2x\ dx = 9 \left[x^2\right]_0^1 = 9[1-0] = 9\ .$$

Example (2): Evaluate $\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^1 \left(2xz\cos(y)e^{x^2} + 3x^2\sin(y)e^z\right)\ dx\ dy\ dz$.
Solution:

$$\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^1 \left(2xz\cos(y)e^{x^2} + 3x^2\sin(y)e^z\right)\ dx\ dy\ dz$$
$$= \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^1 \left(2xz\cos(y)e^{x^2}\right)\ dx\ dy\ dz + \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^1 \left(3x^2\sin(y)e^z\right)\ dx\ dy\ dz$$
$$= \left(\int_0^1 2xe^{x^2}\ dx\right)\left(\int_0^{\frac{\pi}{2}} \cos(y)\ dy\right)\left(\int_0^1 z\ dz\right)$$
$$+ \left(\int_0^1 3x^2\ dx\right)\left(\int_0^{\frac{\pi}{2}} \sin(y)\ dy\right)\left(\int_0^1 e^z\ dz\right)$$
$$= \left[e^{x^2}\right]_0^1 \left[\sin(y)\right]_0^{\frac{\pi}{2}} \left[\frac{z^2}{2}\right]_0^1 + \left[x^3\right]_0^1 \left[-\cos(y)\right]_0^{\frac{\pi}{2}} \left[e^z\right]_0^1$$
$$= \left(e - e^0\right)\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right)\left(\frac{1}{2} - 0\right) + (1-0)\left(-\cos\left(\frac{\pi}{2}\right) - (-\cos(0))\right)\left(e - e^0\right)$$
$$= (e-1)(1-0)\left(\frac{1}{2}\right) + (1)\,(0+1)\,(e-1) = \frac{e-1}{2} + (e-1) = \frac{3(e-1)}{2}\ .$$

2.4.2 Triple Integrals over General Regions

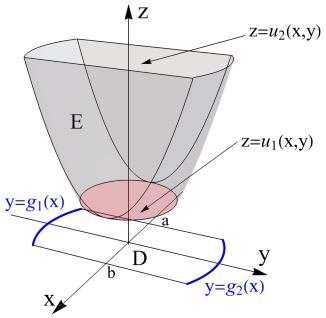
First - Regions of Type I:
Case (1) :
Let E be the region where
$(x, y) \in D$ and
$u_1(x, y) \leq z \leq u_2(x, y)$.



If f is continuous on E, then

$$\iiint_E f(x, y, z)\, dV$$
$$= \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right) dA.$$

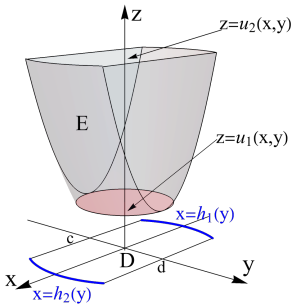
Case (2) :
Let E be the region where
$a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$
and $u_1(x, y) \leq z \leq u_2(x, y)$.



If f is continuous on E, then

$$\iiint_E f(x, y, z)\, dV$$
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

Case (3) :
Let E be the region where
$c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$
and $u_1(x, y) \leq z \leq u_2(x, y)$.

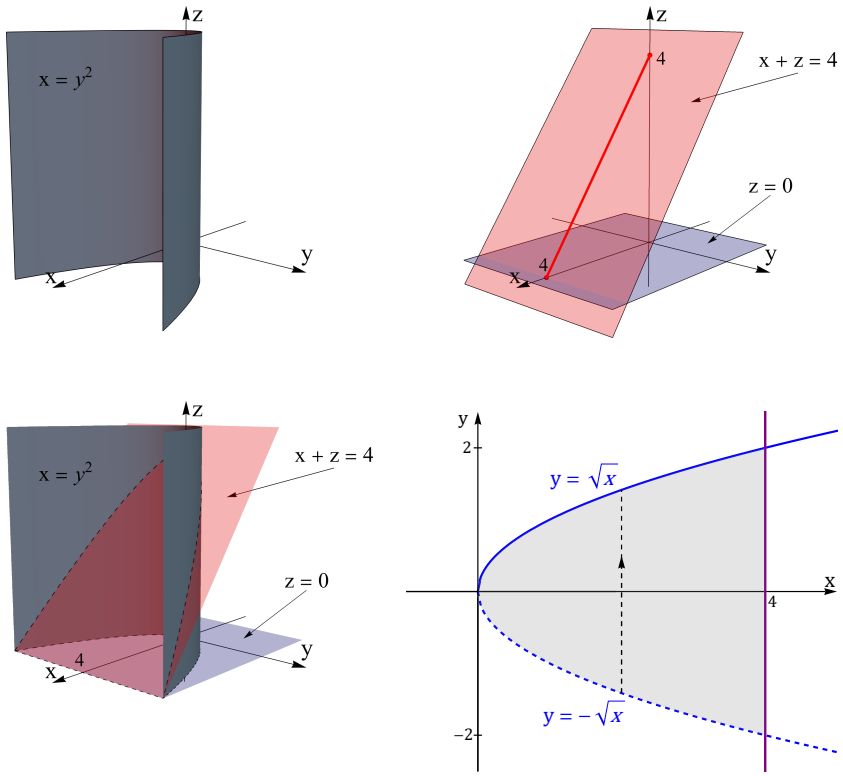


If f is continuous on E, then

$$\iiint_E f(x, y, z)\, dV$$
$$= \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dx\, dy.$$

NOTE: $\iiint_E 1\, dV = \mathrm{V(E)}$, where V(E) is the volume of the region E.

Example (3): Find the volume of the solid bounded by the cylinder $x = y^2$ and the plains $z = 0$ and $x + z = 4$.
Solution:



Let E be the region bounded by the cylinder $x = y^2$ and the plains $z = 0$ and $x + z = 4$.
Note that $z = 0$ intersects $x + z = 4$ at the line where $x = 4$ and $z = 4$.
On $E : 0 \le x \le 4\ ,\ -\sqrt{x} \le y \le \sqrt{x}\ ,\ 0 \le z \le 4 - x\ .$

$$\text{V(E)} = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{4-x} 1\ dz\ dy\ dx = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} [z]_0^{4-x}\ dy\ dx = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} (4-x)\ dy\ dx$$

$$= \int_0^4 (4-x)\ [y]_{-\sqrt{x}}^{\sqrt{x}}\ dx = \int_0^4 2\sqrt{x}\ (4-x)\ dx = 2\int_0^4 \left(4x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)\ dx$$

$$= 2\left[\frac{8}{3}\ x^{\frac{3}{2}} - \frac{2}{5}\ x^{\frac{5}{2}}\right]_0^4 = 2\left[\frac{8}{3}\ (4)^{\frac{3}{2}} - \frac{2}{5}\ (4)^{\frac{5}{2}}\right] = 2\left[\frac{8}{3}\ (2^3) - \frac{2}{5}(2^5)\right]$$

$$2\left[\frac{2^6}{3} - \frac{2^6}{5}\right] = 2^7\ \left(\frac{1}{3} - \frac{1}{5}\right) = 2^7\ \left(\frac{2}{15}\right) = \frac{2^8}{15} = \frac{256}{15}.$$

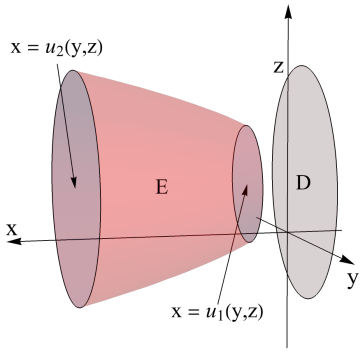
Another Solution :
On $E : -2 \le y \le 2\ ,\ y^2 \le x \le 4\ ,\ 0 \le z \le 4 - x.$

$$\text{V(E)} = \int_{-2}^{2} \int_{y^2}^{4} \int_0^{4-x} 1\ dz\ dx\ dy\ .$$

Second - Regions of Type II:

Let E be the region where
$(y,z) \in D$ and
$u_1(y,z) \le x \le u_2(y,z)$.

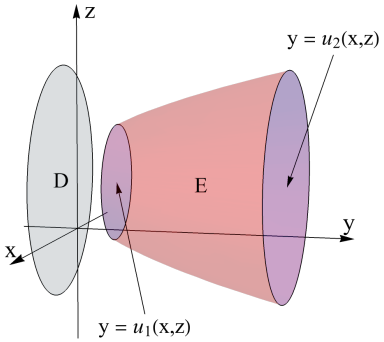
If f is continuous on E, then
$\iiint_E f(x,y,z)\ dV$
$= \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\ dx \right)\ dA.$



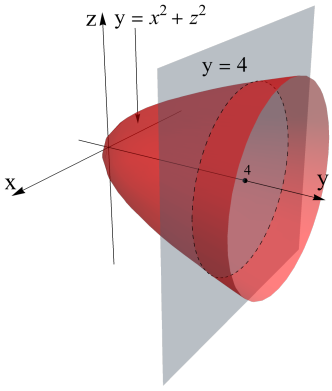
Third - Regions of Type III:

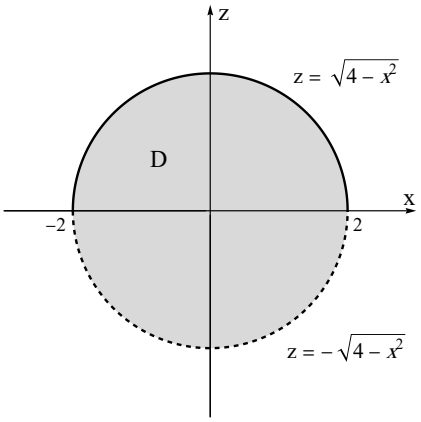
Let E be the region where
$(x,z) \in D$ and
$u_1(x,z) \le y \le u_2(x,z)$.

If f is continuous on E, then
$\iiint_E f(x,y,z)\ dV$
$= \iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\ dy \right)\ dA.$



Example (4): Evaluate $\iiint_E \sqrt{x^2+z^2}\ dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plain $y = 4$.
Solution:





Note that $y = x^2 + z^2$ intersects $y = 4$ at $x^z + z^2 = 4$.
So, E is the region where $x^2 + z^2 \le y \le 4$, $-\sqrt{4-x^2} \le z \le \sqrt{4-x^2}$ and

$-2 \le x \le 2$.

$$\iiint_E \sqrt{x^2+z^2}\, dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\, dz\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [y]_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dz\, dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4-(x^2+z^2)\right] \sqrt{x^2+z^2}\, dz\, dx$$

Using the polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$,

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4-(x^2+z^2)\right] \sqrt{x^2+z^2}\, dz\, dx = \int_0^{2\pi}\int_0^2 (4-r^2)\, r\, r\, dr\, d\theta$$

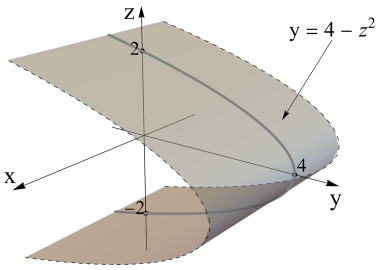
$$= \int_0^{2\pi}\int_0^2 (4-r^2)\, r^2\, dr\, d\theta = \int_0^{2\pi}\int_0^2 (4r^2-r^4)\, dr\, d\theta = \int_0^{2\pi} \left[4\frac{r^3}{3} - \frac{r^5}{5}\right]_0^2 d\theta$$

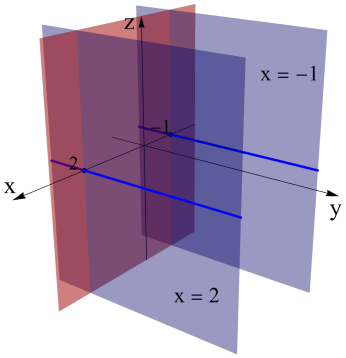
$$= \left(4\,\frac{2^3}{3} - \frac{2^5}{5}\right)(2\pi - 0) = 2\pi\left(\frac{2^5}{3} - \frac{2^5}{5}\right) = 2^6\left(\frac{1}{3} - \frac{1}{5}\right)\pi = \frac{128\pi}{15}.$$

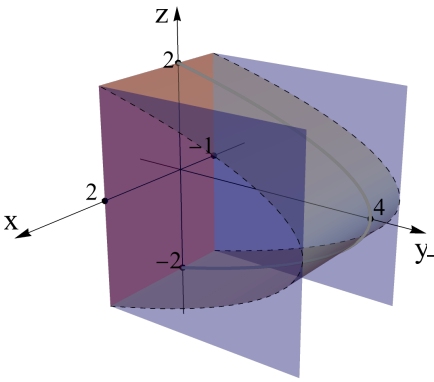
Example (5): Find the volume of the solid bounded by the surfaces $y = 4-z^2$, $x = -1$, $x = 2$ and $y = 0$.

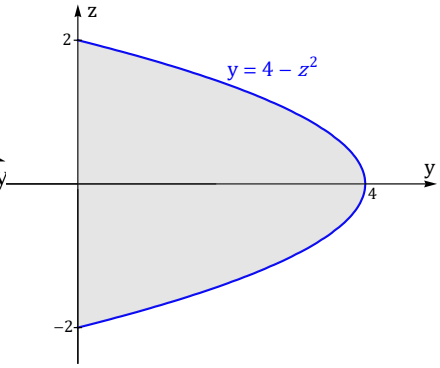
Solution:

Let E be the region bounded by the surfaces $y = 4 - z^2$, $x = -1$, $x = 2$ and $y = 0$.









The surface $y = 4-z^2$ intersects the plain $y = 0$ at the two lines passing through $z = \pm 2$.
On $E : -1 \le x \le 2$, $0 \le y \le 4 - z^2$ and $-2 \le z \le 2$.

$$\mathrm{V(E)} = \iiint_E dV = \int_{-1}^{2}\int_{-2}^{2}\int_{0}^{4-z^2} dy\ dz\ dx$$

$$= \int_{-1}^{2}\int_{-2}^{2} [y]_0^{4-z^2}\, dz\ dx = \int_{-1}^{2}\int_{-2}^{2} (4-z^2) dz\ dx$$

$$= \int_{-1}^{2} \left[4z - \frac{z^3}{3}\right]_{-2}^{2} dx = \left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right] \int_{-1}^{2} dx$$

$$= \left(16 - \frac{16}{3}\right)\ (2-(-1)) = 3\left(16 - \frac{16}{3}\right) = 48 - 16 = 32\ .$$

2.4.3 EXERCISES

1. Evaluate the iterated integral.

   (a). $\int_0^1 \int_y^{2y} \int_0^{x+y} 6xyz \; dz \; dx \; dy$ (b). $\int_1^2 \int_0^{2z} \int_0^{\ln x} x \; e^{-y} \; dy \; dx \; dz$

2. Evaluate the triple integral.

   (a). $\iiint_E y \; dV$, where

   $E = \{(x, y, z) \mid 0 \leq x \leq 3 \; , \; 0 \leq y \leq x \; , \; x - y \leq z \leq x + y\}$.

   (b). $\iiint_E \frac{1}{x^3} \; dV$, where

   $E = \left\{(x, y, z) \mid 0 \leq y \leq 1 \; , \; 0 \leq z \leq y^2 \; , \; 1 \leq x \leq z + 1\right\}$.

   (c). $\iiint_E 6xy \; dV$, where where E lies under the plain $z = 1 + x + y$ and above the region in the xy-plain bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$.

3. Use a triple integral to find the volume of the given solid.

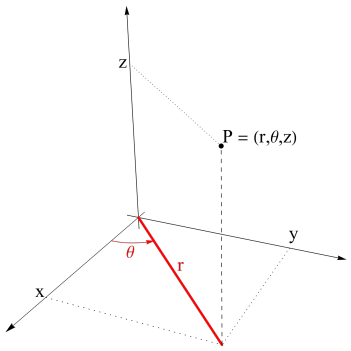
   (a). The tetrahedron enclosed by the coordinate plains and the plain $2x + y + z = 4$.

   (b). The solid enclosed by the paraboloids $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$.

2.5 Triple Integrals in Cylindrical Coordinates

2.5.1 Cylindrical Coordinates

If $P(x, y, z)$ is a point in $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$, then its cylindrical coordinates $P(r, \theta, z)$ are :
$r = \sqrt{x^2 + y^2}$,
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$, where $x \neq 0$,
and $z = z$.
Note: $r \in \mathbb{R}$ and $\theta \in [0, 2\pi]$.



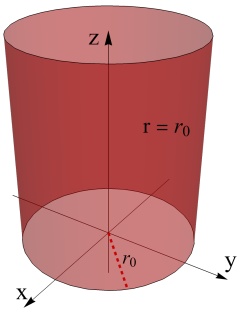
If $P(r, \theta, z)$ is given, the the Cartesian coordinates are :
$x = r\cos\theta$, $y = r\sin\theta$ and $z = z$.

Important equations in Cylindrical coordinates

(1). $r = r_0$, where $r_0 \neq 0$.

$r = r_0 \implies r^2 = r_0^2$
$\implies x^2 + y^2 = r_0^2$.

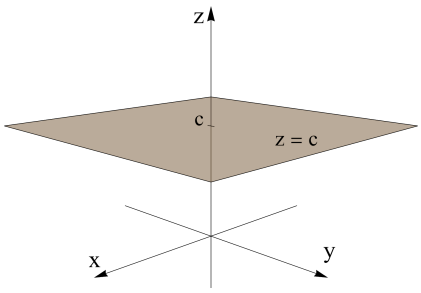
A cylinder,
centered at the origin,
and its radius is r_0.



(2). $z = c$, where $c \in \mathbb{R}$.

All the points (x, y, c) in $\mathbb{R}^3$,
where $x, y \in \mathbb{R}$.

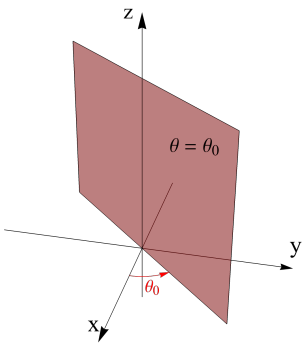
A horizontal plain,
parallel to the xy-plain,
and passes through $(0, 0, c)$.



(3). $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi]$.

$\theta = \theta_0 \implies \tan(\theta) = \tan(\theta_0)$,
$\implies \dfrac{y}{x} = \tan(\theta_0)$
$\implies y = \tan(\theta_0)\, x$.

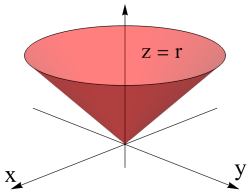
A vertical plain,
passes through the origin.



(4). $z = r$, where $r \neq 0$.

$z = r \implies z = \sqrt{x^2 + y^2}$.

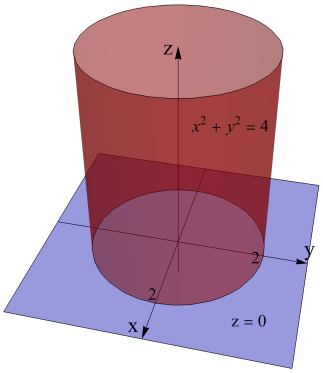
A Cone.

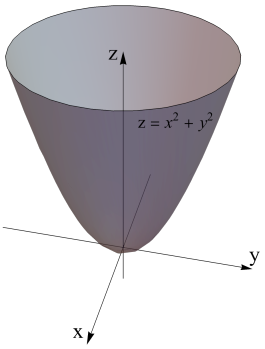


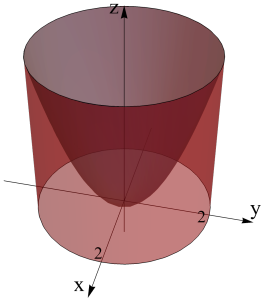
2.5.2 Triple Integrals in Cylindrical Coordinates

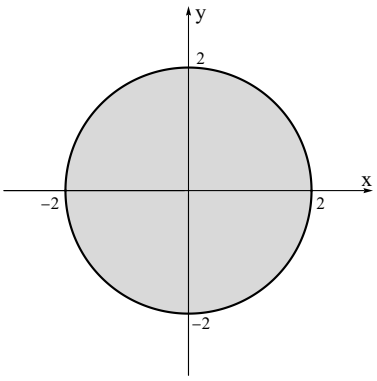
Suppose f is continuous on $E = \{(x, y, z) \mid (x, y) \in D \ , \ u_1(x, y) \leq z \leq u_2(x, y)\}$, where D is given in polar coordinates by $D = \{(r, \theta) \mid \theta_1 \leq \theta \leq \theta_2 \ , \ r_1(\theta) \leq r \leq r_2(\theta)\}$, then $\iiint_E f(x, y, z) \ dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) \ r \ dz \ dr \ d\theta.$

Example (1): Find the volume of the solid within $x^2 + y^2 = 4$, bounded above by $z = x^2 + y^2$ and below by $z = 0$.
Solution:









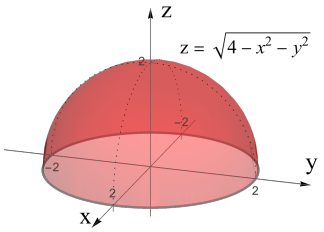
$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi \ , \ 0 \leq r \leq 2 \ , \ 0 \leq z \leq x^2 + y^2 = r^2\}.$

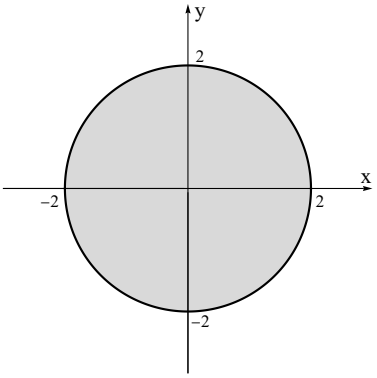
Volume $= \iiint_E dV = \int_0^{2\pi} \int_0^2 \int_0^{x^2+y^2} r \ dz \ dr \ d\theta = \int_0^{2\pi} \int_0^2 \int_0^{r^2} r \ dz \ dr \ d\theta$

$= \int_0^{2\pi} \int_0^2 [z]_0^{r^2} \ r \ dr \ d\theta = \int_0^{2\pi} \int_0^2 (r^2 - 0) r \ dr \ d\theta = \int_0^{2\pi} \int_0^2 r^3 \ dr \ d\theta$

$= \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^2 d\theta = \left[\frac{2^4}{4} - 0\right] \int_0^{2\pi} d\theta = 4(2\pi - 0) = 8\pi.$

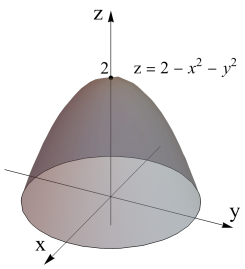
Example (2): Evaluate $\iiint_E z\ dx\ dy\ dz$, where $E = \left\{(x,y,z) \mid 0 \le z \le \sqrt{4-x^2-y^2}\right\}$.
Solution:
$z = \sqrt{4-x^2-y^2}$ is the upper-half of the sphere centered at the origin with radius 2.
$\sqrt{4-x^2-y^2} = 0 \implies x^2+y^2 = 4$, the upper-half of the sphere intersects the plain $z = 0$ at the circle centered at the origin with center 2.
$E = \left\{(r,\theta,z) \mid 0 \le \theta \le 2\pi\ ,\ 0 \le r \le 2\ ,\ 0 \le z \le \sqrt{4-x^2-y^2} = \sqrt{4-r^2}\right\}$.

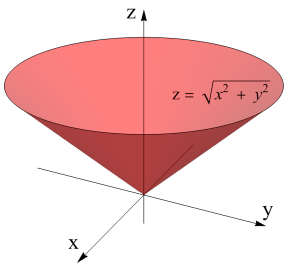


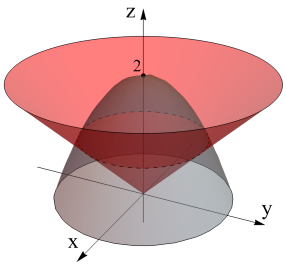


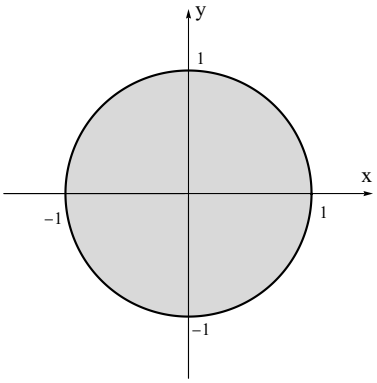
$$\iiint_E z\ dx\ dy\ dz = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{4-r^2}} z\ dz\ r\ drd\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 \left[z^2\right]_0^{\sqrt{4-r^2}} r\ dr\ d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\int_0^2 (4-r^2)r\ dr\ d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 (4r-r^3)\ dr\ d\theta = \frac{1}{2}\int_0^{2\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta$$
$$= \frac{1}{2}(8-4)\int_0^{2\pi} d\theta = 2(2\pi - 0) = 4\pi\ .$$

Example (3): Find the volume of the solid bounded above by $z = 2-x^2-y^2$ and below by $z = \sqrt{x^2+y^2}$.
Solution:
$z = 2-x^2-y^2 = 2-r^2$ intersects $z = \sqrt{x^2+y^2} = r$ at :
$2-r^2 = r \implies r^2+r-2 = 0 \implies (r+2)(r-1) = 0 \implies r = 1$.
(Note that $r = -2$ is excluded because $r \ge 0$).









$E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi \ , \ 0 \le r \le 1 \ , \ r \le z \le 2 - r^2\}$.

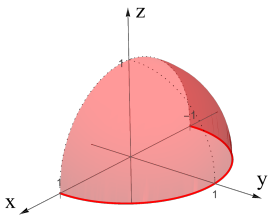
$$\text{Volume} = \iiint_E dV = \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} dz \ r \ dr \ d\theta = \int_0^{2\pi} \int_0^1 [z]_r^{2-r^2} \ r \ dr \ d\theta$$

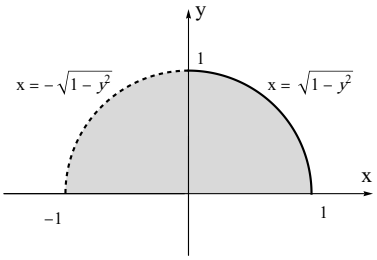
$$= \int_0^{2\pi} \int_0^1 (2 - r^2 - r) r \ dr \ d\theta = \int_0^{2\pi} \int_0^1 (-r^3 - r^2 + 2r) \ dr \ d\theta$$

$$= \int_0^{2\pi} \left[-\frac{r^4}{4} - \frac{r^3}{3} + r^2 \right]_0^1 d\theta = \left(-\frac{1}{4} - \frac{1}{3} + 1 \right) \int_0^{2\pi} d\theta = \frac{5}{12}(2\pi - 0) = \frac{5\pi}{6}.$$

Example (4): Evaluate the integral $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz \ dx \ dy$

Solution:
Note that $0 \le y \le 1$, $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$ and $0 \le z \le \sqrt{1-x^2-y^2}$.
$z = \sqrt{1-x^2-y^2} \implies x^2 + y^2 + z^2 = 1$ represents the upper half of the unit sphere.
$x = \sqrt{1-y^2} \implies x^2 + y^2 = 1$ represents the right half of the unit circle.
$x = -\sqrt{1-y^2} \implies x^2 + y^2 = 1$ represents the left half of the unit circle.





In cylindrical coordinates: $0 \le \theta \le \pi$, $0 \le r \le 1$ and $0 \le z \le \sqrt{1-r^2}$.

$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz \ dx \ dy = \int_0^{\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} dz \ r \ dr \ d\theta$$

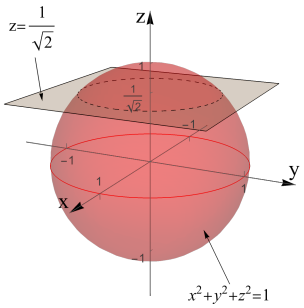
$$= \int_0^{\pi} \int_0^1 \sqrt{1-r^2} \ r \ dr \ d\theta = -\frac{1}{2} \int_0^{\pi} \int_0^1 \left(1 - r^2\right)^{\frac{1}{2}} \ (-2r) \ dr \ d\theta$$

$$= -\frac{1}{2} \left[\frac{2}{3} \left(1 - r^2\right)^{\frac{3}{2}} \right]_0^1 \int_0^{\pi} d\theta = -\frac{1}{2} \left(0 - \frac{2}{3} \right) \ (\pi - 0) = \frac{\pi}{3}.$$

Example (5): Find the volume of the solid inside the sphere $x^2+y^2+z^2=1$ and above the plain $z=\dfrac{1}{\sqrt{2}}$.

Solution:

Note that $x^2+y^2+z^2=1$ intersects $z=\dfrac{1}{\sqrt{2}}$ at $x^2+y^2=\dfrac{1}{2}$ which is a circle centered at the origin and its radius is $\dfrac{1}{\sqrt{2}}$.



The solid is bounded above by the upper-half of the sphere, and below by the plain.

$x^2+y^2+z^2=1$

$\Longrightarrow\ z=\sqrt{1-x^2-y^2}=\sqrt{1-r^2}.$

The cylindrical coordinates of the solid : $0\le r\le\dfrac{1}{\sqrt{2}}$, $0\le\theta\le 2\pi$

and $\dfrac{1}{\sqrt{2}}\le z\le\sqrt{1-r^2}$.

$$\mathrm{V}=\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\int_{\frac{1}{\sqrt{2}}}^{\sqrt{1-r^2}} dz\ r\ dr\ d\theta=\left(\int_0^{2\pi}d\theta\right)\left(\int_0^{\frac{1}{\sqrt{2}}}\int_{\frac{1}{\sqrt{2}}}^{\sqrt{1-r^2}} dz\ r\ dr\right)$$

$$=[\theta]_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}[z]_{\frac{1}{\sqrt{2}}}^{\sqrt{1-r^2}}\ r\ dr=(2\pi-0)\int_0^{\frac{1}{\sqrt{2}}}\left(\sqrt{1-r^2}-\frac{1}{\sqrt{2}}\right)r\ dr$$

$$=2\pi\int_0^{\frac{1}{\sqrt{2}}}\left(r\sqrt{1-r^2}-\frac{r}{\sqrt{2}}\right)\ dr=2\pi\left[-\frac{1}{2}\ \frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{r^2}{2\sqrt{2}}\right]_0^{\frac{1}{\sqrt{2}}}$$

$$=2\pi\left[\left(-\frac{1}{2}\frac{2}{3}\left(1-\frac{1}{2}\right)^{\frac{3}{2}}-\frac{1}{4\sqrt{2}}\right)-\left(-\frac{1}{2}\frac{2}{3}-0\right)\right]$$

$$=2\pi\left(-\frac{1}{3}\frac{1}{2\sqrt{2}}-\frac{1}{4\sqrt{2}}+\frac{1}{3}\right)=\frac{2\pi}{3}\left(-\frac{1}{2\sqrt{2}}-\frac{3}{4\sqrt{2}}+1\right)$$

$$=\frac{2\pi}{3}\left(1-\frac{5}{4\sqrt{2}}\right).$$

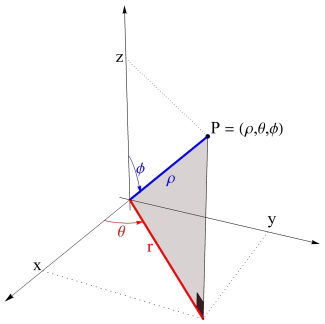
2.5.3 EXERCISES

1. Evaluate $\iiint_E \sqrt{x^2+y^2}\ dV$, where E is the region that lies inside the cylinder $x^2+y^2=16$ and between the plains $z=-5$ and $z=4$.
2. Evaluate $\iiint_E z\ dV$, where E is enclosed by the paraboloid $z=x^2+y^2$ and the plain $z=4$.
3. Evaluate $\iiint_E (x+y+z)\ dV$, where E is the solid in the first octant that lies under the paraboloid $z=4-x^2-y^2$.
4. Find the volume of the solid that lies within both the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$.
5. Find the volume of the solid that is enclosed by the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=2$.
6. Find the volume of the solid that lies between the paraboloid $z=x^2+y^2$ and the sphere $x^2+y^2+z^2=2$.
7. Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\ dz\ dx\ dy$.
8. Evaluate $\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}}\int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\ dz\ dx\ dy$.

2.6 Triple Integrals in Spherical Coordinates

2.6.1 Spherical Coordinates

If $P(x, y, z)$ is a point in $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$,
then its spherical coordinates
$P(\rho, \theta, \phi)$ are :
$\rho = \sqrt{x^2 + y^2 + z^2}$,
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$, where $x \neq 0$,
and $\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$, where $\rho \neq 0$.
Note: $\rho \geq 0$, $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$.

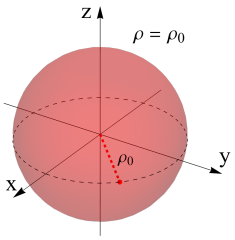


Note that if $P(\rho, \theta, \phi)$ is given then :
$\sin\phi = \frac{r}{\rho} \implies r = \rho\sin\phi$ and $\cos\phi = \frac{z}{\rho} \implies z = \rho\cos\phi$.
So, $x = r\cos\theta = \rho\cos\theta\sin\phi$ and $y = r\sin\theta = \rho\sin\theta\sin\phi$.
Important equations in Spherical coordinates

(1). $\rho = \rho_0$, where $\rho_0 > 0$.

$\rho = \rho_0 \implies \rho^2 = \rho_0^2$
$\implies x^2 + y^2 + z^2 = r_0^2$.

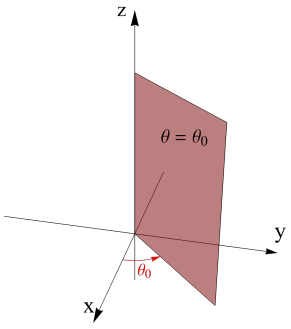
A sphere,
centered at the origin,
and its radius is ρ_0.



(2). $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi]$.

Since $\phi \in [0, \pi]$ then
$\sin\phi \geq 0$.
If $\theta \in [0, \pi]$ then :
$y = \rho\sin\theta\sin\phi \geq 0$
$\theta = \theta_0$ represents a half-plain.

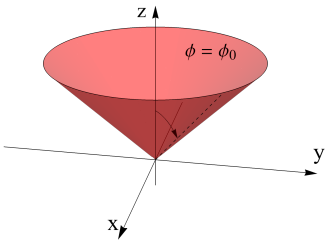
If $\theta \in [\pi, 2\pi]$ then :
$y = \rho\sin\theta\sin\phi \leq 0$
$\theta = \theta_0$ represents the other half-plain.



(3). $\phi = \phi_0$, where $0 < \phi_0 < \dfrac{\pi}{2}$.

$\cot\phi = \dfrac{z}{r} \implies z = r\ \cot\phi,$
$\implies z = \cot\phi\ \sqrt{x^2+y^2}$

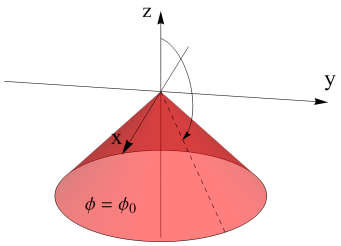
Since $\phi \in \left(0, \dfrac{\pi}{2}\right)$ then
$\cot\phi > 0.$
$\phi = \phi_0$ represents an upper cone.



(4). $\phi = \phi_0$, where $\dfrac{\pi}{2} < \phi_0 < \pi$.

$\cot\phi = \dfrac{z}{r} \implies z = r\ \cot\phi,$
$\implies z = \cot\phi\ \sqrt{x^2+y^2}$

Since $\phi \in \left(\dfrac{\pi}{2}, \pi\right)$ then
$\cot\phi < 0.$
$\phi = \phi_0$ represents a lower cone.



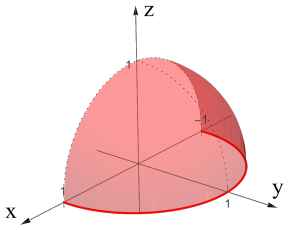
2.6.2 Triple Integrals in Spherical Coordinates

$$\iiint_E f(x,y,z)\ dV = \int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2}\int_a^b f(\rho,\theta,\phi)\ \rho^2\sin\phi\ d\rho\ d\theta\ d\phi\ ,$$
where $E = \{(\rho,\theta,\phi)\mid a\le\rho\le b\ ,\ \theta_1\le\theta\le\theta_2\ ,\ \phi_1\le\phi\le\phi_2\}$.

Example (1): Evaluate the integral $\displaystyle\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\ dx\ dy$

Solution: Referring to Example(4) in section (2.5).

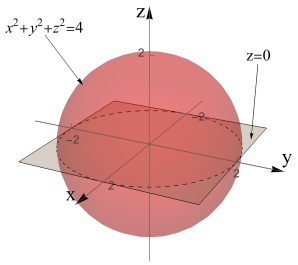
In Spherical coordinates :
$0\le\rho\le 1$, $0\le\theta\le\pi$, and $0\le\phi\le\dfrac{\pi}{2}$.



$$\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\ dx\ dy$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^1 \rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$
$$= \left(\int_0^1 \rho^2\ d\rho\right)\left(\int_0^{\frac{\pi}{2}}\sin\phi\ d\phi\right)\left(\int_0^{\pi} d\theta\right)$$
$$= \left[\frac{\rho^3}{3}\right]_0^1\ [-\cos\phi]_0^{\frac{\pi}{2}}\ \ [\theta]_0^{\pi} = \left(\frac{1}{3}-0\right)\ \left(-\cos\left(\frac{\pi}{2}\right)+\cos(0)\right)\ (\pi-0) = \frac{\pi}{3}\ .$$

Example (2): Evaluate the integral $\displaystyle\iiint_E 4z\ dV$, where E is the region bounded above by $x^2+y^2+z^2=4$ and below by $z=0$.
Solution:

Note that $x^2+y^2+z^2=4$ intersects $z=0$ at $x^2+y^2=4$ which is a circle centered at the origin and its radius is 2.

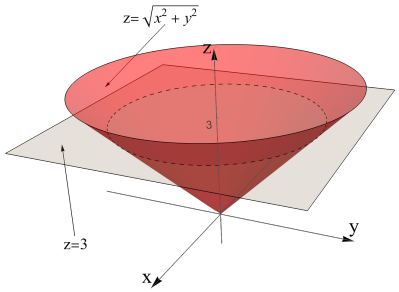


E in spherical coordinates:
$0\le\rho\le 2$, $0\le\theta\le 2\pi$ and $0\le\phi\le\dfrac{\pi}{2}$.
$z=\rho\cos\phi \implies 4z = 4\rho\cos\phi$.

$$\iiint_E 4z\ dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^2 (4\rho\cos\phi)\ \rho^2\sin\phi\ d\rho\ d\phi\ d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^2 (4\rho^3)\ (\cos\phi\ \sin\phi)\ d\rho\ d\phi\ d\theta$$
$$= \left(\int_0^2 4\rho^3\ d\rho\right)\left(\int_0^{\frac{\pi}{2}}\cos\phi\ \sin\phi\ d\phi\right)\left(\int_0^{2\pi} d\theta\right) = \left[\rho^4\right]_0^2\ \left[\frac{\sin^2\phi}{2}\right]_0^{\frac{\pi}{2}}\ \ [\theta]_0^{2\pi}$$
$$= (16-0)\ \left(\frac{1}{2}-0\right)\ (2\pi-0) = 16\pi\ .$$

Example (3): Evaluate $\iiint_E \sqrt{x^2+y^2+z^2}\, dV$, where E is the region bounded above by the plain $z=3$ and below by the cone $z=\sqrt{x^2+y^2}$.
Solution:



Note that $z=\sqrt{x^2+y^2}$ intersects $z=3$ at $x^2+y^2=9$ which is a circle centered at the origin and its radius is 3.

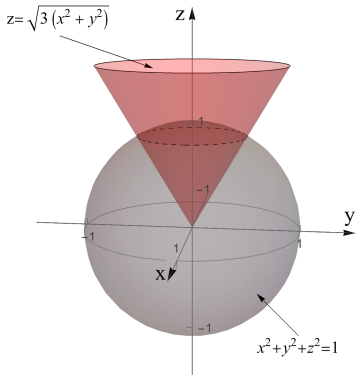
$$z=\sqrt{x^2+y^2}=r$$
$$\implies \rho\cos\phi=\rho\sin\phi$$
$$\implies \cos\phi=\sin\phi$$
$$\implies \phi=\frac{\pi}{4}.$$

$$z=3 \implies \rho\cos\phi=3 \implies \rho=\frac{3}{\cos\phi}.$$

The spherical coordinates of E : $0\le\rho\le\dfrac{3}{\cos\phi}$, $0\le\theta\le 2\pi$ and $0\le\phi\le\dfrac{\pi}{4}$.

$$\iiint_E \sqrt{x^2+y^2+z^2}\, dV=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\frac{3}{\cos\phi}} \rho\,\rho^2\,\sin\phi\, d\rho\, d\phi\, d\theta$$
$$=\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{4}}\int_0^{\frac{3}{\cos\phi}} \rho^3\sin\phi\, d\rho\, d\phi\right)=[\theta]_0^{2\pi}\int_0^{\frac{\pi}{4}}\left[\frac{\rho^4}{4}\right]_0^{\frac{3}{\cos\phi}}\sin\phi\, d\phi$$
$$=(2\pi-0)\,\frac{3^4}{4}\int_0^{\frac{\pi}{4}}(\cos\phi)^{-4}\sin\phi\, d\phi=\frac{81\pi}{2}\left[\frac{-(\cos\phi)^{-3}}{-3}\right]_0^{\frac{\pi}{4}}=\frac{27\pi}{2}\left(2\sqrt{2}-1\right)$$
.

Example (4): Find the volume of the solid inside the sphere $x^2+y^2+z^2=1$ and inside the cone $z=\sqrt{3(x^2+y^2)}$.
Solution:



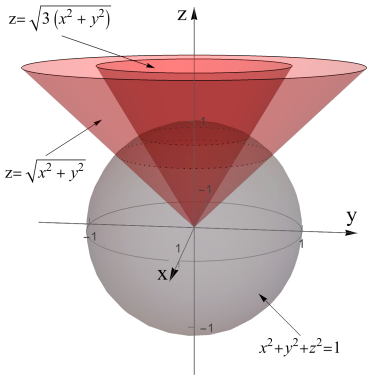
Note that $x^2+y^2+z^2=1$ intersects $z=\sqrt{3(x^2+y^2)}$ at
$3x^2+3y^2=1-x^2-y^2$
$\implies 4x^2+4y^2=1$
$\implies x^2+y^2=\dfrac{1}{4}$
which is a circle centered at the origin and its radius is $\dfrac{1}{2}$.

$$z=\sqrt{3(x^2+y^2)}=\sqrt{3}r \implies$$
$$\rho\cos\phi=\sqrt{3}\rho\sin\phi \implies \tan\phi=\frac{1}{\sqrt{3}}$$
$$\implies \phi=\frac{\pi}{6}.$$

The spherical coordinates of the solid : $0\le\rho\le 1$, $0\le\theta\le 2\pi$ and $0\le\phi\le\dfrac{\pi}{6}$.

$$\mathrm{V} = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^1 \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$
$$= \left(\int_0^1 \rho^2 \; d\rho\right)\left(\int_0^{\frac{\pi}{6}} \sin\phi \; d\phi\right)\left(\int_0^{2\pi} d\theta\right)$$
$$= \left[\frac{\rho^3}{3}\right]_0^1 [-\cos\phi]_0^{\frac{\pi}{6}} \, [\theta]_0^{2\pi} = \left(\frac{1}{3}-0\right)\left(-\frac{\sqrt{3}}{2}-(-1)\right)(2\pi-0)$$
$$= \frac{2\pi}{3}\left(1-\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}\left(2-\sqrt{3}\right).$$

Example (5) Find the volume of the solid bounded by the surfaces $x^2+y^2+z^2=1$, $z=\sqrt{3(x^2+y^2)}$ and $z=\sqrt{x^2+y^2}$.
Solution:



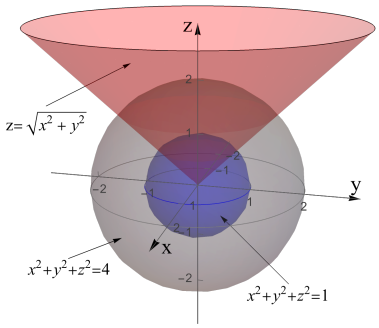
Note that $x^2+y^2+z^2=1$ is the unit sphere.

$$z=\sqrt{3(x^2+y^2)}=\sqrt{3}r \implies$$
$$\rho\cos\phi=\sqrt{3}\rho\sin\phi \implies \tan\phi=\frac{1}{\sqrt{3}}$$
$$\implies \phi=\frac{\pi}{6}.$$
$$z=\sqrt{x^2+y^2}=r \implies$$
$$\rho\cos\phi=\rho\sin\phi \implies \tan\phi=1$$
$$\implies \phi=\frac{\pi}{4}.$$

The spherical coordinates of the solid :
$0\le\rho\le 1$, $0\le\theta\le 2\pi$ and $\frac{\pi}{6}\le\phi\le\frac{\pi}{4}$.

$$\mathrm{V} = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\int_0^1 \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$
$$= \left(\int_0^1 \rho^2 \; d\rho\right)\left(\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin\phi \; d\phi\right)\left(\int_0^{2\pi} d\theta\right)$$
$$= \left[\frac{\rho^3}{3}\right]_0^1 [-\cos\phi]_{\frac{\pi}{6}}^{\frac{\pi}{4}} \, [\theta]_0^{2\pi} = \left(\frac{1}{3}-0\right)\left(-\frac{\sqrt{2}}{2}-(-\frac{\sqrt{3}}{2})\right)(2\pi-0)$$
$$= \frac{2\pi}{3}\left(\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3}\left(\sqrt{3}-\sqrt{2}\right).$$

Example (6): Evaluate $\iiint_E z \; dV$, where E is the region bounded by the surfaces $x^2+y^2+z^2=1$, $x^2+y^2+z^2=4$ and $z=\sqrt{x^2+y^2}$.
Solution:



Note that $x^2 + y^2 + z^2 = 1$ is the unit sphere.

$x^2 + y^2 + z^2 = 4$ is a sphere centered at the origin and its radius equals 2.

$z = \sqrt{x^2 + y^2} = r \implies$
$\rho\cos\phi = \rho\sin\phi \implies \tan\phi = 1$
$\implies \phi = \dfrac{\pi}{4}.$

The spherical coordinates of E :
$1 \le \rho \le 1$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \dfrac{\pi}{4}$.

$$\iiint_E z\, dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^2 (\rho\cos\phi)\rho^2\sin\phi\; d\rho\; d\phi\; d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^2 \rho^3\;\sin\phi\cos\phi\; d\rho\; d\phi\; d\theta = \left(\int_1^2 \rho^3\; d\rho\right)\left(\int_0^{\frac{\pi}{4}} \sin\phi\cos\phi\; d\phi\right)\left(\int_0^{2\pi} d\theta\right)$$

$$= \left[\frac{\rho^4}{4}\right]_1^2\left[\frac{(\sin\phi)^2}{2}\right]_0^{\frac{\pi}{4}}[\theta]_0^{2\pi} = \left(\frac{16}{4} - \frac{1}{4}\right)\left[\frac{\left(\frac{1}{\sqrt{2}}\right)^2}{2} - \frac{0}{2}\right](2\pi - 0)$$

$$= \frac{15}{4}\;\frac{1}{4}\;2\pi = \frac{15\pi}{8}\;.$$

2.6.3 EXERCISES

1. Change from rectangular to spherical coordinates:

   (a). $(3, 3, 0)$. (b). $\left(1, -\sqrt{3}, 2\sqrt{3}\right)$.

2. Identify the surface whose equation is $\rho = \cos\phi$.

3. Sketch the solid described by the given inequalities:

   (a). $\rho \leq 1$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq \dfrac{\pi}{6}$. (b). $\rho \leq 2$, $\rho \leq \csc\phi$.

4. Evaluate $\iiint_E \left(x^2 + y^2 + z^2\right)^2 \, dV$, where E is the ball with center the origin and radius 5.

5. Evaluate $\iiint_E y^2 z^2 \, dV$, where E lies above the cone $\phi = \dfrac{\pi}{3}$ and below the sphere $\rho = 1$.

6. Evaluate $\iiint_E x e^{x^2+y^2+z^2} \, dV$, where E is the portion of the unit ball $x^2 + y^2 + z^2 \leq 1$ that lies in the first octant.

7. Find the volume of the part of the ball $\rho \leq a$ that lies between the cones $\phi = \dfrac{\pi}{6}$ and $\phi = \dfrac{\pi}{3}$.

Chapter 3

Sequences, Series, and Power Series

3.1 Sequences

3.1.1 Infinite Sequences

Definition: An infinite sequence is a function $f : \mathbb{N} \longrightarrow \mathbb{R}$, where $f(n) = a_n$.

Notation: The sequence $\{a_1, a_2, \cdots\}$ is also denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

Example (1): Some sequences can be defined by giving a formula for the n^{th} term.
(a). $a_n = \dfrac{1}{2^n}$ is the n^{th} term of the sequence $\left\{\dfrac{1}{2^n}\right\} = \left\{\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \cdots\right\}$.
(b). $\left\{\dfrac{(-1)^n(n+2)}{5^n}\right\} = \left\{-\dfrac{3}{5}, \dfrac{4}{25}, -\dfrac{5}{125}, \cdots\right\}$.

Example (2): The Fibonacci sequences is defined recursively by
$f_1 = 1\ ,\ f_2 = 1\ ,\ f_n = f_{n-1} + f_{n-2}\ ,\ n \geq 3$.
$\{f_n\} = \{1, 1, 2, 3, 5, 8, 13, \cdots\}$.

3.1.2 The Limit of a Sequence

Definition: A sequence $\{a_n\}$ has the limit $L \in \mathbb{R}$, if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that : if $n \geq N$ then $|a_n - L| < \epsilon$, and we write $\lim\limits_{n\to\infty} a_n = L$.

Definition: A sequence $\{a_n\}$ goes to ∞, if for every positive real number M there exists an $N \in \mathbb{N}$ such that : if $n \geq N$ then $a_n \geq M$, and we write $\lim\limits_{n\to\infty} a_n = \infty$.

3.1.3 Properties of Convergent Sequences

Theorem: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ where $n \in \mathbb{N}$ then $\lim_{x\to\infty} a_n = L$.

Example (3): Calculate $\lim_{n\to\infty} \dfrac{\ln n}{n}$.
Solution : Let $f(x) = \dfrac{\ln x}{x}$.
$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{\ln x}{x} = \lim_{x\to\infty} \dfrac{1}{x} = 0$ (By L'Hôpital's Rule).
Therefore, $\lim_{n\to\infty} \dfrac{\ln n}{n} = 0$.

Theorem: Suppose $\lim_{n\to\infty} a_n = L_1$, $\lim_{n\to\infty} b_n = L_2$ and $c \in \mathbb{R}$, then:
(1). $\lim_{n\to\infty} c = c$.
(2). $\lim_{n\to\infty} a_n \pm b_n = L_1 \pm L_2$.
(3). $\lim_{n\to\infty} c\ a_n = cL_1$.
(4). $\lim_{n\to\infty} a_n.b_n = L_1.L_2$.
(5). $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L_1}{L_2}$, where $L_2 \neq 0$.

Power Law: $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$, if $p > 0$ and $a_n > 0$.

Squeeze Theorem for sequences: If $a_n \leq b_n \leq c_n$ for $n \geq N_0$
and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Example (4): Calculate $\lim_{n\to\infty} \dfrac{n!}{n^n}$.
Solution:
$0 \leq \dfrac{n!}{n^n} = \dfrac{1.2.3\cdots n}{n.n.n\cdots n} = \dfrac{1}{n}\left(\dfrac{2.3\cdots n}{n.n\cdots n}\right) \leq \dfrac{1}{n}(1) = \dfrac{1}{n}$
Since $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \dfrac{1}{n} = 0$, then $\lim_{n\to\infty} \dfrac{n!}{n^n} = 0$ (By Squeeze Theorem).

Theorem: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Example (5): Calculate $\lim_{n\to\infty} \dfrac{(-1)^n}{n}$.
Solution: Let $a_n = \dfrac{(-1)^n}{n}$.
$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|\dfrac{(-1)^n}{n}\right| = \lim_{n\to\infty} \dfrac{|(-1)^n|}{|n|} = \lim_{n\to\infty} \dfrac{1}{n} = 0$.
Therefore $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{(-1)^n}{n} = 0$.

Theorem: If $\lim_{n\to\infty} a_n = L$, and the function f is continuous at L ,
then $\lim_{n\to\infty} f(a_n) = f(L)$.

Example (6): Calculate $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)$.
Solution:
$\lim_{n\to\infty} \frac{\pi}{n} = \lim_{n\to\infty} \pi \frac{1}{n} = \pi(0) = 0$.
Since the cosine function is continuous at 0 , then $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$.

Important Note: The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$, and divergent for all other values of r.
$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad -1 < r < 1 \\ 1 & \text{if} \quad r = 1 \end{cases}$.

3.1.4 Monotonic and bounded sequences

Definition:
(1). A sequence $\{a_n\}$ is called increasing if $a_{n+1} \geq a_n$ for all $n \geq 1$.
(2). A sequence $\{a_n\}$ is called decreasing if $a_{n+1} \leq a_n$ for all $n \geq 1$.
(3). A sequence is called monotonic if it is either increasing or decreasing.

Definition:
(1). A sequence $\{a_n\}$ is bounded above if there exists a real number M such that $a_n \leq M$ for all $n \geq 1$.
(2). A sequence $\{a_n\}$ is bounded below if there exists a real number m such that $a_n \geq m$ for all $n \geq 1$.
(3). If a sequence is bounded above and below, then it is called a bounded sequence.

Monotonic Sequence Theorem
Every bounded, monotonic sequence is convergent. In particular :
(1). A sequence that is increasing and bounded above converges.
(2). A sequence that is decreasing and bounded below converges.

Important Notes :
(1). If $a_n \leq b_n$ for some $n \geq N_0$, $\{a_n\}$ is an increasing sequence and $\{b_n\}$ is convergent, then $\{a_n\}$ is convergent.
(2). If $a_n \leq b_n$ for some $n \geq N_0$, if $\lim_{n\to\infty} a_n = \infty$, then $\lim_{n\to\infty} b_n = \infty$.

3.2 Series

3.2.1 Infinite Series

Definition:
(1). The sum of the terms of the sequence $\{a_n\}$ is called an infinite series (or a series) and it is denoted by $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

(2). $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$ is called the n^{th} partial sum of the series.

Definition:
If the sequence $\{s_n\}$ of the partial sums of the series $\sum_{n=1}^{\infty} a_n$ is convergent, and $\lim_{n\to\infty} s_n = s \in \mathbb{R}$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} a_n = s$.
The number s is called the sum of the series.
If the sequence $\{s_n\}$ is divergent, then the series is divergent.

Example (1): Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Solution : Note that $\frac{1}{n} - \frac{1}{n+1} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n(n+1)}$.

$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$

$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

So, $s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1}$.

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$.

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

3.2.2 Sum of a Geometric Series

Definition : $\sum_{n=0}^{\infty} ar^n$ (where $a \neq 0$ and $r \in \mathbb{R}$) is called a geometric series.

Note that $s_n = a + ar + ar^2 + \cdots + ar^n$, $rs_n = ra + ar^2 + ar^3 + \cdots + ar^n + ar^{n+1}$.

$s_n - rs_n = a - ar^{n+1} \implies (1-r)s_n = a\left(1 - a^{n+1}\right) \implies s_n = \frac{a\left(1 - a^{n+1}\right)}{1-r}$

where $r \neq 1$.

Sum of a geometric series :

(1). If $|r| < 1$, then $\sum_{n=0}^{\infty} ar^n$ is convergent and its sum is $\frac{a}{1-r}$.

(2). If $|r| \geq 1$, then $\sum_{n=0}^{\infty} ar^n$ is divergent.

Example (2): Find the sum of the geometric series $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \cdots$.

Solution : $a = 3$.

$r = \frac{a_2}{a_1} = \frac{-\frac{3}{2}}{3} = -\frac{1}{2}$.

the sum of the geometric series is $\frac{a}{1-r} = \frac{3}{1-\left(-\frac{1}{2}\right)} = \frac{3}{\left(\frac{3}{2}\right)} = 2$.

Example (3): Does the series $\sum_{n=1}^{\infty} 3^{2n} 5^{1-n}$ converge or diverge?.

Solution :

$\sum_{n=1}^{\infty} 3^{2n} 5^{1-n} = \sum_{n=1}^{\infty} 9^n 5^{1-n} = \sum_{n=1}^{\infty} \frac{9^n}{5^{n-1}}$

$= \sum_{n=1}^{\infty} \frac{9\ \left(9^{n-1}\right)}{5^{n-1}} = \sum_{n=1}^{\infty} 9\ \left(\frac{9}{5}\right)^{n-1} = \sum_{n=0}^{\infty} 9\ \left(\frac{9}{5}\right)^{n}$.

Since $|r| = \left|\frac{9}{5}\right| = \frac{9}{5} > 1$, then the geometric series diverges.

3.2.3 Test for Divergence

Example (4): Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Solution :

$s_2 = s_{2^1} = 1 + \frac{1}{2}$.

$s_4 = s_{2^2} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$.

$s_8 = s_{2^3} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$

$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}$.

So, $s_{2^n} > 1 + \frac{n}{2}$ (By induction).

$\lim_{n\to\infty} s_{2^n} \geq \lim_{n\to\infty} \left(1 + \frac{n}{2}\right) = \infty$.

Therefore, The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem : If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Important Note : The converse of the last theorem is not true.
In other words, if $\lim_{n\to\infty} a_n = 0$ then this does not mean that $\sum_{n=1}^{\infty} a_n$ is convergent.
For example $\lim_{n\to\infty} \frac{1}{n} = 0$, while $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Test of divergence : If $\lim_{n\to\infty} a_n$ does not exist, or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example (5): Show that the series $\sum_{n=1}^{\infty} \frac{n^3}{2n^3+1}$ is divergent.
Solution :
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3}{2n^3+1} = \frac{1}{2} \neq 0$.
Therefore, the series $\sum_{n=1}^{\infty} \frac{n^3}{2n^3+1}$ is divergent.

3.2.4 Properties of Convergent Series

Theorem : If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent series and $c \in \mathbb{R}$ then :
(1). The series $\sum_{n=1}^{\infty} (ca_n)$ is convergent, and $\sum_{n=1}^{\infty} (ca_n) = c\sum_{n=1}^{\infty} a_n$.
(2). The series $\sum_{n=1}^{\infty} (a_n + b_n)$ is convergent, and $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
(3). The series $\sum_{n=1}^{\infty} (a_n - b_n)$ is convergent, and $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

Example (6): Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{4}{n(n+1)} + \frac{2}{3^n} \right)$.
Solution :
From Example (1), $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.
$\sum_{n=1}^{\infty} \frac{2}{3^n} = \sum_{n=1}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^{n-1} = \sum_{n=0}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^{n} = \frac{2}{3} \frac{1}{1-\frac{1}{3}} = \frac{2}{3} \frac{3}{2} = 1$.
$\sum_{n=1}^{\infty} \left(\frac{4}{n(n+1)} + \frac{2}{3^n} \right) = \sum_{n=1}^{\infty} \frac{4}{n(n+1)} + \sum_{n=1}^{\infty} \frac{2}{3^n}$
$= 4 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{2}{3^n} = 4(1) + 1 = 5$.

3.2.5 EXERCISES

1. Let $a_n = \dfrac{2n}{3n+1}$.

   (a). Determine whether $\{a_n\}$ is convergent.

   (b). Determine whether $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

2. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

   (1). $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\pi^n}$ (2). $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$ (3). $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n}$

   (4). $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$ (5). $\displaystyle\sum_{n=1}^{\infty} \frac{6\ 2^{2n-1}}{3^n}$

3. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

   (1). $\displaystyle\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$ (2). $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2-2n+5}$ (3). $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4+e^{-n}}$

   (4). $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+4^n}{e^n}$ (5). $\displaystyle\sum_{n=1}^{\infty} (\sin 100)^n$ (6). $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n}$

   (7). $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$

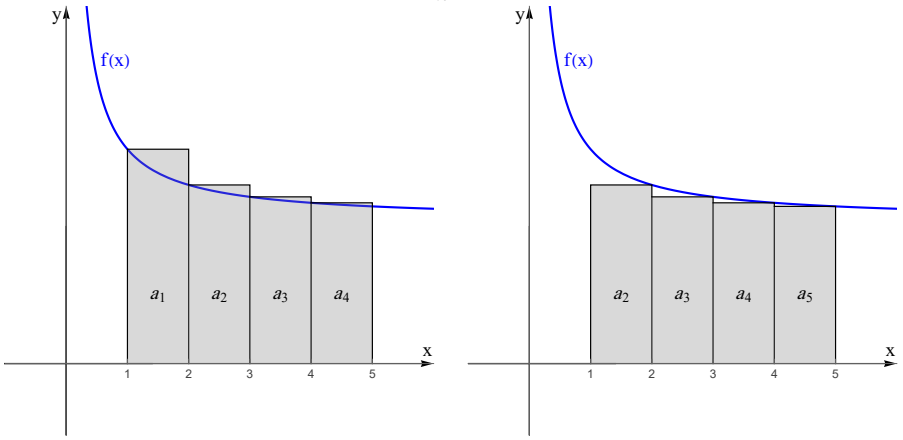
3.3 The Integral Test

3.3.1 The Integral Test

Theorem (The Integral Test) :
Suppose f is a positive, decreasing continuous function defined on $[1,\infty)$, and let $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(1). If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent .

(2). If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent .



Example (1): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Solution :
Let $f(x) = \frac{1}{x^2+1}$, the f is a positive continuous function on $[1,\infty)$.
$f'(x) = \frac{-2x}{(x^2+1)^2} < 0$ on the interval $[1,\infty)$, then f is decreasing on $[1,\infty)$.
$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \left[\tan^{-1} x\right]_1^t$
$= \lim_{t\to\infty} \left[\tan^{-1}(t) - \tan^{-1}(1)\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$
So, $\int_1^{\infty} \frac{1}{x^2+1}\,dx$ is convergent, then $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent.

Example (2): Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n\ \ln n}$.

Solution :
Let $f(x) = \frac{1}{x\ \ln x}$, the f is a positive continuous function on $[2,\infty)$.

$f'(x) = \dfrac{-(1+\ln x)}{(x\ \ln x)^2} < 0$ on $[2,\infty)$, then f is decreasing on $[2,\infty)$.

$\displaystyle\int_2^\infty \frac{1}{x\ \ln x}\,dx = \lim_{t\to\infty}\int_2^t \frac{\left(\frac{1}{x}\right)}{\ln x}\,dx = \lim_{t\to\infty}[\ln(\ln x)]_2^t$

$= \lim_{t\to\infty}[\ln(\ln t) - \ln(\ln 2)] = \infty$. (Note that $\lim_{t\to\infty}\ln(t) = \infty$).

So, $\displaystyle\int_2^\infty \frac{1}{x\ \ln x}\,dx$ is divergent, then $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ \ln n}$ is divergent.

Definition : (The p-series)

The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ where $p\in\mathbb{R}$ is called a p-series.

Theorem :

The p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent if $p>1$, and divergent if $p\le 1$.

Proof:

(1). If $p<0$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\dfrac{1}{n^p} = \lim_{n\to\infty} n^{-p} = \infty$. (Note: $-p>0$).

Since $\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ is divergent.

(2). If $p=0$, then $a_n = \dfrac{1}{n^0} = 1$, hence $\lim_{n\to\infty} a_n = 1 \neq 0$.

Therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ is divergent.

(3). If $0<p<1$, then using the integral test:

Note that $f(x) = \dfrac{1}{x^p}$ is a positive, decreasing continuous function on $[1,\infty)$.

$\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^t$

$= \lim_{t\to\infty}\left[\dfrac{t^{1-p}}{1-p} - \dfrac{1}{1-p}\right] = \infty$. (Note : $1-p>0$).

Since $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ is divergent.

(4). If $p=1$, then using the integral test:

Note that $f(x) = \dfrac{1}{x}$ is a positive, decreasing continuous function on $[1,\infty)$.

$\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}[\ln x]_1^t = \lim_{t\to\infty}[\ln(t) - \ln(1)] = \infty$.

Since $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

(5). If $p>1$, then using the integral test:

Note that $f(x) = \dfrac{1}{x^p}$ is a positive, decreasing continuous function on $[1,\infty)$.

$\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty}\left[\frac{1}{(1-p)x^{p-1}}\right]_1^t$

$= \lim_{t\to\infty}\left[\dfrac{1}{(1-p)\ t^{p-1}} - \dfrac{1}{1-p}\right] = 0 - \dfrac{1}{1-p} = \dfrac{1}{p-1}$.

Since $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

Example (3): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Solution :
$\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a p-series, where $p = 4 > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent.

Example (4): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

Solution :
$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a p-series, where $p = \frac{1}{2} < 1$, then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

Example (5): Discuss the convergence of the series $\sum_{n=1}^{\infty} n\ e^{-n^2}$.

Solution : Let $f(x) = x\ e^{-x^2}$ then f is a positive continuous function on $[1, \infty)$.
$f'(x) = e^{-x^2} - 2x^2\ e^{-x^2} = \left(1 - 2x^2\right)\ e^{-x^2} < 0$ on $[1, \infty)$.
Therefore, f is decreasing on $[1, \infty)$. Using the integral test :

$$\int_1^\infty x\ e^{-x^2}\,dx = \lim_{t\to\infty}\left(-\frac{1}{2}\int_1^t e^{-x^2}\ (-2x)\,dx\right) = \lim_{t\to\infty}\left(-\frac{1}{2}\left[e^{-x^2}\right]_1^t\right)$$

$$= \lim_{t\to\infty}\left(-\frac{1}{2}\left[e^{-t^2} - e^{-1}\right]\right) = \lim_{t\to\infty}\left(-\frac{1}{2}\left[\frac{1}{e^{t^2}} - \frac{1}{e}\right]\right) = -\frac{1}{2}\left(0 - \frac{1}{e}\right) = \frac{1}{2e}\ .$$

Since $\int_1^\infty x\ e^{-x^2}\,dx$ is convergent, then $\sum_{n=1}^{\infty} n\ e^{-n^2}$ is convergent.

3.3.2 EXERCISES

1. Use the Integral Test to determine whether the series is convergent or divergent.

   (1). $\sum_{n=1}^{\infty} n^{-3}$ (2). $\sum_{n=1}^{\infty} \frac{2}{5n-1}$ (3). $\sum_{n=2}^{\infty} \frac{n^2}{n^3+1}$

   (4). $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ (5). $\sum_{n=2}^{\infty} \frac{1}{n\left(\ln n\right)^3}$

2. Determine whether the series is convergent or divergent.

   (1). $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ (2). $\sum_{n=3}^{\infty} n^{-0.9999}$ (3). $\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\cdots$

   (4). $\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2}$ (5). $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ (6). $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\cdots$

   (7). $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ (8). $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ (9). $\sum_{n=1}^{\infty} \frac{1}{n^2+n^3}$

3.4 The Comparison Tests

3.4.1 The Direct Comparison Test

Theorem : (The Direct Comparison Test)
Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.
(1). If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent.
(2). If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

Example (1): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{|\sin n|}{2+3^n}$.
Solution :
For all $n \geq 1$: $\frac{|\sin n|}{2+3^n} \leq \frac{1}{2+3^n} < \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$.
Since $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is convergent (a geometric series with $r = \frac{1}{3} < 1$), then $\sum_{n=1}^{\infty} \frac{|\sin n|}{2+3^n}$ is also convergent (by The direct comparison test).

Example (2): Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{2}{\sqrt{n}-1}$.
Solution :
For all $n \geq 2$: $\sqrt{n}-1 < \sqrt{n} \implies \frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$
$\implies \frac{2}{\sqrt{n}-1} > \frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$.
Since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is divergent (a p-series, with $p = \frac{1}{2} \leq 1$), then $\sum_{n=2}^{\infty} \frac{2}{\sqrt{n}-1}$ is also divergent (by The direct comparison test) .

Example (3): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^4+1}$.
Solution :
For all $n \geq 1$: $e^{-n} = \frac{1}{e^n} < 1 \implies \frac{e^{-n}}{n^4+1} < \frac{1}{n^4+1} < \frac{1}{n^4}$.
Since $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent(a p-series, with $p = 4 > 1$), then $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^4+1}$ is also convergent.

Example (4): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Solution :
Since the function $f(x) = \ln(x)$ is increasing on $[1, \infty)$, then for all $n \geq 3$:
$\ln(n) \geq \ln(3) > \ln(e) = 1 \implies \dfrac{\ln n}{n} > \dfrac{1}{n}$.
Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (a p-series, with $p = 1 \leq 1$), then $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is also divergent.

3.4.2 Limit Comparison Test

Theorem: (Limit comparison test)
Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ are series with positive terms.
If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = c$, where $c \in \mathbb{R}$ and $c > 0$, then either both series converge or both series diverge.

Example (1): Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$.
Solution :
Note that $a_n = \dfrac{1}{2^n - 1}$. Let $b_n = \dfrac{1}{2^n}$.
$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\left(\frac{1}{2^n - 1}\right)}{\left(\frac{1}{2^n}\right)} = \lim\limits_{n\to\infty} \dfrac{2^n}{2^n - 1} = 1 > 0$.
Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent (a geometric series with $r = \dfrac{1}{2} < 1$), then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ is convergent (by limit comparison test).

Example (2): Discuss the convergence of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2 - 1}}$.
Solution :
Note that $a_n = \dfrac{1}{\sqrt[3]{n^2 - 1}}$. Let $b_n = \dfrac{1}{\sqrt[3]{n^2}} = \dfrac{1}{n^{\frac{2}{3}}}$.
$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\sqrt[3]{n^2}}{\sqrt[3]{n^2 - 1}} = \lim\limits_{n\to\infty} \sqrt[3]{\dfrac{n^2}{n^2 - 1}} = \sqrt[3]{1} = 1 > 0$.
Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{2}{3}}}$ is divergent (a p-series, with $p = \dfrac{2}{3} \leq 1$), then $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2 - 1}}$ is divergent (by limit comparison test).

Example (3): Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2 + 3n}{3^n(n^2 + 2)}$.
Solution :

Note that $a_n = \dfrac{5n^2+3n}{3^n(n^2+2)}$. Let $b_n = \dfrac{5n^2}{3^n\ n^2} = \dfrac{5}{3^n}$.

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \left(\dfrac{5n^2+3n}{3^n(n^2+2)} . \dfrac{3^n}{5} \right) = \lim\limits_{n\to\infty} \dfrac{5n^2+3n}{5n^2+10} = 1 > 0$.

Since $\sum\limits_{n=1}^{\infty} \dfrac{5}{3^n} = \sum\limits_{n=1}^{\infty} 5 \left(\dfrac{1}{3} \right)^n$ is convergent (a geometric series with $r = \dfrac{1}{3} < 1$), then $\sum\limits_{n=1}^{\infty} \dfrac{5n^2+3n}{3^n(n^2+2)}$ is convergent.

Example (4): Discuss the convergence of the series $\sum\limits_{n=1}^{\infty} \dfrac{n+3n^2}{\sqrt{1+n^5}}$.

Solution :

Note that $a_n = \dfrac{3n^2+n}{\sqrt{n^5+1}}$. Let $b_n = \dfrac{3n^2}{\sqrt{n^5}} = \dfrac{3n^2}{n^{\frac{5}{2}}} = \dfrac{3}{n^{\frac{1}{2}}}$.

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \left(\dfrac{3n^2+n}{\sqrt{n^5+1}} . \dfrac{n^{\frac{1}{2}}}{3} \right) = \lim\limits_{n\to\infty} \dfrac{3n^{\frac{5}{2}}+n^{\frac{3}{2}}}{3\sqrt{n^5+1}} = 1 > 0$.

Since $\sum\limits_{n=1}^{\infty} \dfrac{3}{n^{\frac{1}{2}}}$ is divergent (a p-series, with $p = \dfrac{1}{2} < 1$), then $\sum\limits_{n=1}^{\infty} \dfrac{n+3n^2}{\sqrt{1+n^5}}$ is divergent.

3.4.3 EXERCISES

1. (a). Use the Direct Comparison Test to show that the first series converges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n}{n^3+5} , \qquad \sum_{n=2}^{\infty} \frac{1}{n^2} .$$

(b). Use the Limit Comparison Test to show that that the first series converges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n}{n^3-5} , \qquad \sum_{n=2}^{\infty} \frac{1}{n^2} .$$

2. (a). Use the Direct Comparison Test to show that the first series diverges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n^2+n}{n^3-2} , \qquad \sum_{n=2}^{\infty} \frac{1}{n} .$$

(b). Use the Limit Comparison Test to show that that the first series diverges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n^2-n}{n^3+2} , \qquad \sum_{n=2}^{\infty} \frac{1}{n} .$$

3. Which of the following inequalities can be used to show that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ converges?

(a). $\dfrac{n}{n^3+1} \geq \dfrac{1}{n^3+1}$ (b). $\dfrac{n}{n^3+1} \leq \dfrac{1}{n}$ (c). $\dfrac{n}{n^3+1} \leq \dfrac{1}{n^2}$.

4. Determine whether the series converges or diverges.

(1). $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+8}$ (2). $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ (3). $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$

(4). $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ (5). $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ (6). $\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$

(7). $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ (8). $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos n}{e^n}$ (9). $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

(10). $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3+n}$ (11). $\displaystyle\sum_{n=1}^{\infty} \frac{2+\sin n}{n^2}$ (12). $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

3.5 Alternating Series and Absolute Convergence

3.5.1 Alternating Series

Definition : If $a_n \geq 0$, the the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is called an alternating series.

Theorem : (Alternating Series Test)
If $a_n \geq 0$ and the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ satisfies :
(1). The sequence $\{a_n\}$ is decreasing, i.e. $a_{n+1} \leq a_n$ for all $n \geq 1$.
(2). $\lim_{n\to\infty} a_n = 0$.
Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent.

Example (1): Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$.
Solution :
(1). For all $n \geq 1$: $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$.
The sequence $\{a_n\} = \left\{\frac{1}{n}\right\}$ is decreasing.
(2). $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$.
By the alternating series test, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is convergent .

Example (2): Discuss the convergence of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n}{4n^2-1}$.
Solution :
(1). Let $f(x) = \frac{2x}{4x^2-1}$, then for $x \geq 1$:
$f'(x) = \frac{2(4x^2-1)-2x(8x)}{(4x^2-1)^2} = \frac{8x^2-2-16x^2}{(4x^2-1)^2} = \frac{-2-8x^2}{(4x^2-1)^2} < 0$.
The sequence $\{a_n\} = \left\{\frac{2n}{4n^2-1}\right\}$ is decreasing.
(2). $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2n}{4n^2-1} = 0$.
By the alternating series test, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n}{4n^2-1}$ is convergent .

Example (3): Discuss the convergence of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n+1}$.
Solution :
$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2} \neq 0$.

So, the alternating series test does not apply.
Note that if $b_n = (-1)^{n+1}\dfrac{n}{2n+1}$, then $\lim\limits_{n\to\infty} b_{2n} = \dfrac{1}{2}$ and $\lim\limits_{n\to\infty} b_{2n+1} = -\dfrac{1}{2}$.
So, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} (-1)^{n+1}\dfrac{n}{2n+1}$ does not exist.
Therefore, $\sum\limits_{n=1}^{\infty} (-1)^{n+1}\dfrac{n}{2n+1}$ is divergent (by test of divergence).

3.5.2 Absolute Convergence and Conditional Convergence

Definition : (Absolute Convergence)
A series $\sum\limits_{n=1}^{\infty} a_n$ is called absolutely convergent if the series of absolute values $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent .

Example (4): The series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^p}$, where $p > 1$ is absolutely convergent.
Solution :
$\sum\limits_{n=1}^{\infty} \left|\dfrac{(-1)^{n+1}}{n^p}\right| = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ is convergent (a p-series, with $p > 1$).
Therefore, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^p}$ is absolutely convergent.

Definition : (Conditional Convergence)
A series $\sum\limits_{n=1}^{\infty} a_n$ is called conditionally convergent if it is convergent but not absolutely convergent. That is, if $\sum\limits_{n=1}^{\infty} a_n$ is convergent but $\sum\limits_{n=1}^{\infty} |a_n|$ is divergent.

Example (5): The series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}$ is conditionally convergent.
Solution :
Note that $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}$ is convergent (see Example (1)).
But $\sum\limits_{n=1}^{\infty} \left|\dfrac{(-1)^{n+1}}{n}\right| = \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ is divergent.
Therefore, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}$ is conditionally convergent.

Theorem :
If a series is absolutely convergent then it is convergent.

Example (6): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$.

Solution :

$\left|\frac{\cos n}{n^2}\right| = \frac{|\cos n|}{|n^2|} \leq \frac{1}{n^2}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent then by direct comparison test $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ is convergent.

Therefore, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent and hence convergent.

Example (7): Discuss the convergence of the following series :

(a). $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1}$ (b). $\sum_{n=1}^{\infty} (-1)^n \frac{1}{1+n\sqrt{n}}$ (c). $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{(2n-3)^2}$.

Solution :

(a). $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{n+1}{n^2+1}\right| = \sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$

Let $a_n = \frac{n+1}{n^2+1}$, put $b_n = \frac{1}{n}$, then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n+1}{n^2+1}\frac{n}{1}\right) = \lim_{n\to\infty} \frac{n^2+n}{n^2+1} = 1 > 0$.

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, then $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{n+1}{n^2+1}\right|$ is divergent.

(1). Let $f(x) = \frac{x+1}{x^2+1}$, then for $x \geq 1$:

$f'(x) = \frac{(1)(x^2+1) - (x+1)(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2-2x}{(x^2+1)^2}$

$= \frac{-x^2-2x+1}{(x^2+1)^2} = \frac{1-(x^2+2x)}{(x^2+1)^2} < 0.$

(Note that $1-(x^2+2x) < 0$ when $x \geq 1$).

Therefore, the sequence $\left\{\frac{n+1}{n^2+1}\right\}$ is decreasing.

(2). $\lim_{n\to\infty} \frac{n+1}{n^2+1} = 0.$

From (1) and (2), the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1}$ is convergent.

Hence, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1}$ is conditionally convergent.

(b). $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{1+n\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{1+n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{1+n^{\frac{3}{2}}}$.

Let $a_n = \frac{1}{1+n^{\frac{3}{2}}}$, put $b_n = \frac{1}{n^{\frac{3}{2}}}$, then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{\frac{3}{2}}}{n^{\frac{3}{2}}+1} = 1 > 0.$

Since $\sum_{n=1}^{\infty} \frac{1}{1+n^{\frac{3}{2}}}$ is convergent, then $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{1+n\sqrt{n}}\right|$ is convergent.

Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{1+n\sqrt{n}}$ is absolutely convergent.

(c). Let $a_n = (-1)^n \frac{n^2+1}{(2n-3)^2} = (-1)^n \frac{n^2+1}{4n^2-12n+9}$

$\lim_{n\to\infty} a_{2n} = \frac{1}{4}$ and $\lim_{n\to\infty} a_{2n+1} = -\frac{1}{4}$.

Hence $\lim_{n\to\infty} a_n$ does not exist.

Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{(2n-3)^2}$ is divergent.

3.5.3 EXERCISES

1. Test the series for convergence or divergence.

   (1). $\frac{2}{3}-\frac{2}{5}+\frac{2}{7}-\frac{2}{9}+\frac{2}{11}-\cdots$ (2). $\frac{1}{\ln 3}-\frac{1}{\ln 4}+\frac{1}{\ln 5}-\frac{1}{\ln 6}+\frac{1}{\ln 7}-\cdots$

   (3). $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3+5n}$ (4). $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n+1}}$ (5). $\sum_{n=1}^{\infty}(-1)^n\ \frac{3n-1}{2n+1}$

   (6). $\sum_{n=1}^{\infty}(-1)^n e^{-n}$ (7). $\sum_{n=1}^{\infty}(-1)^{n+1}\ \frac{n^2}{n^3+4}$ (8). $\sum_{n=1}^{\infty}(-1)^n\ \frac{n}{2^n}$

   (9). $\sum_{n=1}^{\infty}\frac{n\cos(n\pi)}{2^n}$ (10). $\sum_{n=1}^{\infty}(-1)^n\ \sin\left(\frac{\pi}{n}\right)$ (11). $\sum_{n=1}^{\infty}(-1)^n\ \cos\left(\frac{\pi}{n}\right)$

2. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

   (1). $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$ (2). $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt[3]{n^2}}$ (3). $\sum_{n=0}^{\infty}(-1)^{n+1}\ \frac{n^2}{n^2+1}$

   (4). $\sum_{n=1}^{\infty}\frac{(-1)^n}{5n+1}$ (5). $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$ (6). $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$

   (7). $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$

3.6 The Ratio and Root Tests

3.6.1 The Ratio Test

Theorem : (the Ratio Test)
(1). If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
(2). If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
(3). If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the ratio test is inconclusive.

Example (1): Discuss the convergence of the series $\sum_{n=1}^{\infty}(-1)^n\frac{n^4}{2^n}$.
Solution :
$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^4}{2^{n+1}}\ \frac{2^n}{n^4}\right| = \lim_{n\to\infty}\frac{1}{2}\left(\frac{n+1}{n}\right)^4 = \frac{1}{2}(1)^4 = \frac{1}{2} < 1$.
Therefore, $\sum_{n=1}^{\infty}(-1)^n\frac{n^4}{2^n}$ is absolutely convergent.

Example (2): Discuss the convergence of the series $\sum_{n=1}^{\infty}(-1)^n\frac{5^n}{n!}$.
Solution :
$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5^{n+1}}{(n+1)!}\ \frac{n!}{5^n}\right| = \lim_{n\to\infty}\left|\frac{5^{n+1}}{n!\ (n+1)}\ \frac{n!}{5^n}\right|$
$= \lim_{n\to\infty}\frac{5}{n+1} = 0 < 1$.
Therefore, $\sum_{n=1}^{\infty}(-1)^n\frac{5^n}{n!}$ is absolutely convergent.

Example (3): Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{n^n}{n!}$.
Solution :
$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{(n+1)!}\ \frac{n!}{n^n}\right] = \lim_{n\to\infty}\left[\frac{(n+1)^n\ (n+1)}{n!\ (n+1)}\ \frac{n!}{n^n}\right]$
$= \lim_{n\to\infty}\frac{(n+1)^n}{n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$.
Therefore, $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ is divergent.
Note that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, using L'Hôpital's rule.

Example (4): Discuss the convergence of the series $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{3^n}$.

Solution : For $n \geq 2$,

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\ln(n+1)}{3^{n+1}}\ \frac{3^n}{\ln(n)}\right| = \lim_{n\to\infty}\frac{1}{3}\ \frac{\ln(n+1)}{\ln(n)}$

$= \lim_{n\to\infty}\frac{1}{3}\ \left(\frac{\frac{1}{n+1}}{\frac{1}{n}}\right) = \lim_{n\to\infty}\frac{1}{3}\ \frac{n}{n+1} = \frac{1}{3}(1) = \frac{1}{3} < 1$.

Therefore, $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{3^n}$ is convergent.

Notes :

(1). The harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{\frac{1}{n+1}}{\frac{1}{n}}\right) = \lim_{n\to\infty}\frac{n}{n+1} = 1$.

(2). The series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent.

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right) = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = (1)^2 = 1$.

3.6.2 The Root Test

Theorem : (the Root Test)

(1). If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty}a_n$ is absolutely convergent (and therefore convergent).

(2). If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty}a_n$ is divergent.

(3). If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, the root test is inconclusive.

Example (5): Discuss the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{n+3}{3n+2}\right)^n$.

Solution :

$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n+3}{3n+2}\right)^n\right|} = \lim_{n\to\infty}\left[\left(\frac{n+3}{3n+2}\right)^n\right]^{\frac{1}{n}}$

$= \lim_{n\to\infty}\frac{n+3}{3n+2} = \frac{1}{3} < 1$.

Therefore, $\sum_{n=1}^{\infty}\left(\frac{n+3}{3n+2}\right)^n$ is absolutely convergent.

Example (6): Discuss the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{2n+1}{n+3}\right)^n$.

Solution :

$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{2n+1}{n+3}\right)^n\right|} = \lim_{n\to\infty}\left[\left(\frac{2n+1}{n+3}\right)^n\right]^{\frac{1}{n}}$

$= \lim_{n\to\infty}\frac{2n+1}{n+3} = 2 > 1$.

Therefore, $\sum_{n=1}^{\infty}\left(\frac{2n+1}{n+3}\right)^n$ is divergent.

Example (7): Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{2^{3n+1}}{n^n}$.

Solution :

$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{2^{3n+1}}{n^n}\right|} = \lim_{n\to\infty}\left(\frac{2^{3n+1}}{n^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{2^{3+\frac{1}{n}}}{n} = 0 < 1$.

Therefore, $\sum_{n=1}^{\infty}\frac{2^{3n+1}}{n^n}$ is absolutely convergent.

Another solution :

$\sum_{n=1}^{\infty}\frac{2^{3n+1}}{n^n} = \sum_{n=1}^{\infty}\frac{(2)\ 2^{3n}}{n^n} = \sum_{n=1}^{\infty}\frac{(2)\ \left(2^3\right)^n}{n^n} = \sum_{n=1}^{\infty}2\ \left(\frac{2^3}{n}\right)^n = \sum_{n=1}^{\infty}2\ \left(\frac{8}{n}\right)^n$

$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|2\ \left(\frac{8}{n}\right)^n\right|} = \lim_{n\to\infty}\sqrt[n]{2\ \left(\frac{8}{n}\right)^n}$

$= \lim_{n\to\infty}\left[(2)^{\frac{1}{n}}\ \left(\frac{8}{n}\right)\right] = (2)^0\ (0) = (1)(0) = 0 < 1$.

Therefore, $\sum_{n=1}^{\infty}\frac{2^{3n+1}}{n^n}$ is absolutely convergent.

Example (8): Discuss the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^n$.

Solution :

$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n+1}{n}\right)^n\right|} = \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^n\right]^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n+1}{n} = 1.$

So, The root test is inconclusive.

But $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \neq 0$.

Therefore, $\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^n$ is divergent (by test of divergence).

3.6.3 EXERCISES

1. What can you say about the series $\sum_{n=1}^{\infty} a_n$ an in each of the following cases?

   (a). $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 8$ (b). $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0.8$ (c).$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

2. Use the Ratio Test to determine whether the series is convergent or divergent.

   (1). $\sum_{n=1}^{\infty} \frac{n}{5^n}$ (2). $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ (3). $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{2^n n^3}$

   (4). $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$ (5). $\sum_{n=1}^{\infty} \frac{1}{n!}$ (6). $\sum_{n=1}^{\infty} \frac{n}{100^n}$

   (7). $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

3. Use the Root Test to determine whether the series is convergent or divergent.

   (1). $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$ (2). $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$ (3). $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

3.7 Strategy for Testing Series

(1). Divergence test :

If $\lim_{n\to\infty} a_n \neq 0$, or $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ is divergent.

$\sum_{n=1}^{\infty} \frac{n^2+2n}{n^2+4}$ is divergent, because $\lim_{n\to\infty} \frac{n^2+2n}{n^2+4} = 1 \neq 0$.

$\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{n+2}$ is divergent, because $\lim_{n\to\infty} (-1)^n \frac{n-1}{4n+2}$ does not exist.

(2). p-series : It has the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$:

If $p > 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent, like $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$.

If $p \leq 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent, like $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

(3). Geometric series : It has the form $\sum_{n=0}^{\infty} a\, r^n$ where $a \neq 0$ and $r \in \mathbb{R}$:

If $|r| < 1$ then $\sum_{n=0}^{\infty} a\, r^n$ is convergent, like $\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$.

If $|r| \geq 1$ then $\sum_{n=0}^{\infty} a\, r^n$ is divergent, like $\sum_{n=1}^{\infty} \frac{3^n}{2}$.

(4). Comparison test :

It is used when the series has a form that is similar to a p-series or a geometric series, it can be used also when a_n is a rational function or an algebraic function of n.

It is used on series of positive terms, it can be used on $\sum_{n=1}^{\infty} |a_n|$ to test for absolute convergence.

$\sum_{n=1}^{\infty} \frac{3}{n+2}$ is divergent, comparing it with $\sum_{n=1}^{\infty} \frac{1}{n}$.

$\sum_{n=1}^{\infty} \frac{2}{7^n+1}$ is convergent, comparing it with $\sum_{n=1}^{\infty} \frac{1}{7^n}$.

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+5}$ is convergent, comparing it with $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$.

(5). Alternating series test :

It is used on the series $\sum_{n=1}^{\infty} (-1)^n a_n$, where the sequence $\{a_n\}$ is decreasing and

$\lim_{n\to\infty} a_n = 0$. Note that if $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} (-1)^n a_n$ is absolutely convergent.
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+2}$ satisfies all the conditions of the alternating series test, hence it is absolutely convergent. Also, $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^3+2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^3+2}$ is convergent (using comparison test), hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+2}$ is absolutely convergent.

(6). The Ratio test :
it is used on series involving factorial or other products like a constant raised to a power n.
It is not used on p-series or rational functions or algebraic functions of n.
$\sum_{n=1}^{\infty} \frac{2^n}{n!}$ is convergent since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$.

(7). The Root test :
It is used on series of the form $\sum_{n=1}^{\infty} (a_n)^n$.
$\sum_{n=1}^{\infty} \left(\frac{3n^2+1}{4n^2+5}\right)^n$ is convergent, since $\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n^2+1}{4n^2+5}\right)^n} = \frac{3}{4} < 1$.

(8). The Integral test :
It is used when $a_n = f(n)$, where f is a decreasing function on $[1,\infty)$ and $\int_1^{\infty} f(x)\,dx$ is easy to calculate.
$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent, since $f(x) = \frac{1}{x(\ln x)^2}$ is decreasing on $[2,\infty)$ and
$\int_2^{\infty} \frac{1}{x(\ln x)^2} = \lim_{t\to\infty} \int_2^t (\ln x)^{-2} \frac{1}{x}\,dx = \lim_{t\to\infty} \left[\frac{(\ln x)^{-1}}{-1}\right]_2^t = \lim_{t\to\infty} \left[\frac{-1}{\ln x}\right]_2^t$
$= \lim_{t\to\infty} \left[\frac{-1}{\ln t} - \left(\frac{-1}{\ln 2}\right)\right] = 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}$.

3.8 Power Series

3.8.1 Power Series

Definition : A power series is a series of the form $\sum_{n=1}^{\infty} c_n x^n$, where x is a variable and c_n's are real constants called the coefficients.

Notes :

(1). If $c_n = 1$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} c_n x^n = \sum_{n=1}^{\infty} x^n$ is the geometric series and it converges when $|x| < 1$ and diverges when $|x| > 1$.

If $x = \frac{1}{2}$, then $\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges to 2.

If $x = 2$, then $\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} 2^n$ diverges.

(2). $\sum_{n=1}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-c)^2 + \cdots$ is called a power series in $(x-a)$ or a power series centered at a or a power series about a.

Example (1): For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$ converge?

Solution :

Let $a_n = \frac{(x-2)^n}{n}$, using the ratio test :

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-2)^{n+1}}{n+1} \frac{n}{(x-2)^n}\right| = \lim_{n\to\infty} \frac{n}{n+1}|x-2| = |x-2|$.

The series converges when $|x-2| < 1 \implies -1 < x-2 < 1 \implies 1 < x < 3$.

The series diverges when $|x-2| > 1 \implies x-2 < -1$ or $x-2 > 1$

$\implies x < 1$ or $x > 3$.

If $x = 1$ then $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is the convergent alternating harmonic series.

If $x = 3$ then $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Therefore, $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$ converges when $x \in [1, 3)$ or $1 \leq x < 3$.

Example (2): For what values of x does the series $\sum_{n=1}^{\infty} n!\, x^n$ converge?

Solution :

Let $a_n = n!\, x^n$, using the ratio test :

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right| = \lim_{n\to\infty} (n+1)\, |x|$

If $x \neq 0$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$. Therefore the series diverges when $x \neq 0$.

If $x = 0$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$. Therefore the series converges when $x = 0$.

Example (3): For what values of x does the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converge?

Solution :
Let $a_n = \frac{x^n}{n!}$, using the ratio test :
$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!}\ \frac{n!}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1.$
Therefore, the series converges for all $x \in \mathbb{R}$.

3.8.2 Interval of Convergence

Theorem : For a power series $\sum_{n=1}^{\infty} c_n(x-a)^n$, there are only three possibilities:
(1). The series converges only when $x = a$.
(2). The series converges for all $x \in \mathbb{R}$.
(3). There is a positive real number R such that the series converges when $|x - a| < R$ and diverges when $|x - a| > R$.

Notes :
(1). The positive real number R is called the radius of convergence. If the series converges only when $x = a$ then $R = 0$, and if the series converges for all $x \in \mathbb{R}$ then $R = \infty$.
(2). The interval of convergence of a power series is the interval that consists of all values of x for which the series converges.
If the series converges only when $x = a$ then the interval of convergence consists of only one point a.
If the series converges for all $x \in \mathbb{R}$ then the interval of convergence is $(-\infty, \infty)$.
If the radius of convergence is R, then interval of convergence is $(a - R, a + R)$.

Example (4): Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$.

Solution :
Let $a_n = \frac{(x-2)^n}{n^2+1}$, using the ratio test :
$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-2)^{n+1}}{(n+1)^2+1}\ \frac{n^2+1}{(x-2)^n}\right| = \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1}\ |x-2|$
$= \lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2}\ |x-2| = (1)\ |x-2| = |x-2|.$
The series converges when $|x - 2| < 1$, so the radius of convergence is $R = 1$.
Therefore, the series converges on $(1, 3)$.
If $x = 1$, then $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ which is convergent.

If $x = 3$, then $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1} = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ which is convergent.
Therefore, the interval of convergence is $[1, 3]$.

Example (5): Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n\ 5^n}$.
Solution :
Let $a_n = \frac{(2x-6)^n}{n\ 5^n}$, using the ratio test :
$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2x-6)^{n+1}}{(n+1)\ 5^{n+1}} \ \frac{n\ 5^n}{(2x-6)^n}\right| = \lim_{n\to\infty} \frac{n}{5(n+1)}\ |2x-6|$
$= \lim_{n\to\infty} \frac{n}{5n+5}\ |2x-6| = \frac{1}{5}\ |2x-6| = \frac{|2x-6|}{5}$.
The series converges when $\frac{|2x-6|}{5} < 1 \implies 2|x-3| < 5 \implies |x-3| < \frac{5}{2}$, so the radius of convergence is $R = \frac{5}{2}$.
Therefore, the series converges on $\left(3-\frac{5}{2}, 3+\frac{5}{2}\right) = \left(\frac{1}{2}, \frac{11}{2}\right)$.
If $x = \frac{1}{2}$, then $\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n\ 5^n} = \sum_{n=1}^{\infty} \frac{(-5)^n}{n\ 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is convergent.
If $x = \frac{11}{2}$, then $\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n\ 5^n} = \sum_{n=1}^{\infty} \frac{5^n}{n\ 5^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent.
Therefore, the interval of convergence is $\left[\frac{1}{2}, \frac{11}{2}\right)$.

Example (6): Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{n}{7^n}(x+3)^n$.
Solution :
Let $a_n = \frac{n}{7^n}(x+3)^n$, using the ratio test :
$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(x+3)^{n+1}}{7^{n+1}} \ \frac{7^n}{n(x+3)^n}\right|$
$= \lim_{n\to\infty} \frac{n+1}{7n}|x+3| = \frac{1}{7}|x+3| = \frac{|x+3|}{7}$.
The series converges when $\frac{|x+3|}{7} < 1 \implies |x+3| < 7$, so the radius of convergence is $R = 7$.
Therefore, the series converges on $(-3-7, -3+7) = (-10, 4)$.
If $x = -10$, then $\sum_{n=1}^{\infty} \frac{n}{7^n}(x+3)^n = \sum_{n=1}^{\infty} (-1)^n n$ which is divergent.
If $x = 4$, then $\sum_{n=1}^{\infty} \frac{n}{7^n}(x+3)^n = \sum_{n=1}^{\infty} n$ which is divergent.
Therefore, the interval of convergence is $(-10, 4)$.

3.8.3 EXERCISES

1. Find the radius of convergence and interval of convergence of the power series.

(1). $\sum_{n=1}^{\infty} \frac{x^n}{n}$ (2). $\sum_{n=1}^{\infty} (-1)^n n x^n$ (3). $\sum_{n=1}^{\infty} \sqrt{n} x^n$

(4). $\sum_{n=1}^{\infty} \frac{n}{5^n} x^n$ (5). $\sum_{n=1}^{\infty} \frac{5^n}{n} x^n$ (6). $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$

(7). $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$ (8). $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ (9). $\sum_{n=1}^{\infty} n^n x^n$

(10). $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n} x^n$ (11). $\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$ (12). $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

(13). $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$ (14). $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$

3.9 Representations of Functions as Power Series

3.9.1 Representations of Functions using Geometric Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \text{ when } |x| < 1.$$

Example (1): Express $\dfrac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.
Solution :

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \text{ , where } \left|-x^2\right| < 1.$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - \cdots \text{ , where } \left|-x^2\right| < 1 \text{ .}$$

$$\left|-x^2\right| < 1 \implies \left|x^2\right| < 1 \implies |x| < 1 \text{ .}$$

If $x = \pm 1$, the series $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n}$ diverge.
Therefore, The interval of convergence is $(-1, 1)$.

Example (2): Express $\dfrac{x^2}{1-x^6}$ as the sum of a power series and find the interval of convergence.
Solution :

$$\frac{x^2}{1-x^6} = x^2 \, \frac{1}{1-x^6} = x^2 \sum_{n=0}^{\infty} \left(x^6\right)^n = x^2 \sum_{n=0}^{\infty} x^{6n} = \sum_{n=0}^{\infty} x^{6n+2} \text{ , where } \left|x^6\right| < 1.$$

$$\left|x^6\right| < 1 \implies |x| < 1 \text{ .}$$

If $x = \pm 1$, the series $\displaystyle\sum_{n=0}^{\infty} x^{6n+2}$ diverge.
Therefore, The interval of convergence is $(-1, 1)$.

Example (3): Express $\dfrac{x}{5-x}$ as the sum of a power series and find the interval of convergence.
Solution :

$$\frac{x}{5-x} = \frac{x}{5\left(1-\frac{x}{5}\right)} = \frac{x}{5} \, \frac{1}{1-\frac{x}{5}} = \frac{x}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n = \frac{x}{5} \sum_{n=0}^{\infty} \frac{x^n}{5^n} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}} \text{ ,}$$

where $\left|\dfrac{x}{5}\right| < 1$.

$$\left|\frac{x}{5}\right| < 1 \implies \frac{|x|}{5} < 1 \implies |x| < 5 \text{ .}$$

If $x = \pm 5$, the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}}$ diverge.
Therefore, The interval of convergence is $(-5, 5)$.

Example (4): Express $\dfrac{1}{2-x}$ as power series in $x-1$, and find the interval of convergence.
Solution :
$\dfrac{1}{2-x} = \dfrac{1}{1-x+1} = \dfrac{1}{1-(x-1)} = \sum_{n=0}^{\infty} (x-1)^n$, where $|x-1|<1$.
$|x-1|<1 \implies -1<x-1<1 \implies 0<x<2$.
If $x=0$ or $x=2$, the series $\sum_{n=0}^{\infty} (x-1)^n$ diverge.
Therefore, The interval of convergence is $(0,2)$.

3.9.2 Differentiation and Integration of Power Series

Theorem : If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R>0$, then the function f defined by
$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$
is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and
(1). $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.
(2). $\int f(x)\,dx = C + c_0(x-a) + c_1\dfrac{(x-a)^2}{2} + c_2\dfrac{(x-a)^3}{3} + \cdots$
$= C + \sum_{n=0}^{\infty} c_n \dfrac{(x-a)^{n+1}}{n+1}$, where C is a constant.
The radii of convergence of the power series in (1) and (2) are both R.

Notes : Equations (1) and (2) in the last theorem can be rewritten as :
(1). $\dfrac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty}\left[\dfrac{d}{dx}c_n(x-a)^n\right]$.
(2). $\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]\,dx = \sum_{n=0}^{\infty}\left[\int c_n(x-a)^n\,dx\right]$.

Example (5): Express $\dfrac{1}{(1-x)^2}$ as a power series, and find its radius of convergence.
Solution :
$\dfrac{d}{dx}\left[\dfrac{1}{1-x}\right] = \dfrac{d}{dx}\left[(1-x)^{-1}\right] = (-1)(1-x)^{-2}(-1) = \dfrac{1}{(1-x)^2}$.
Note that $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$, where $|x|<1$.
$\dfrac{1}{(1-x)^2} = \dfrac{d}{dx}\left[\dfrac{1}{1-x}\right] = \dfrac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\dfrac{d}{dx}\left[x^n\right] = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}$.

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots .$$

Since the radius of convergence of $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ is $R = 1$ then the radius of convergence of $\dfrac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ is also $R = 1$.

Example (6): Express $\ln(1+x)$ as a power series, and find its radius of convergence.
Solution :
$\int \dfrac{1}{1+x}\, dx = \ln(1+x) + c$

Note that $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$, where $|x| < 1$.

$$\ln(1+x) = \int \frac{1}{1+x}\, dx = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n\right] dx = \sum_{n=0}^{\infty} \left[\int (-1)^n x^n \, dx\right]$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C \text{ , where } |x| < 1.$$

Put $x = 0$ in the last equation : $\ln(1+0) = 0 + C \implies C = 0$.

Therefore, $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots$.

The radius of convergence of $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n}$ is $R = 1$.

Example (7): Express $\tan^{-1} x$ as a power series, and find its radius of convergence.
Solution :
$\int \dfrac{1}{1+x^2}\, dx = \tan^{-1} x + c$.

From Example (1), $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, where $|x| < 1$.

$$\tan^{-1} x = \int \frac{1}{1+x^2}\, dx = \int \left[\sum_{n=0}^{\infty} (-1)^n x^{2n}\right] dx = \sum_{n=0}^{\infty} \int \left[(-1)^n x^{2n}\right] dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C \text{ , where } |x| < 1.$$

Put $x = 0$ in the last equation : $\tan^{-1}(0) = 0 + C \implies C = 0$.

Therefore, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots$.

The radius of convergence of $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ is $R = 1$.

3.9.3 EXERCISES

1. Find a power series representation for the function and determine the interval of convergence.

   (1). $f(x) = \dfrac{x}{x+1}$ (2). $f(x) = \dfrac{1}{1-x^2}$ (3). $f(x) = \dfrac{5}{1-4x^2}$

   (4). $f(x) = \dfrac{2}{3-x}$ (5). $f(x) = \dfrac{4}{2x+3}$ (6). $f(x) = \dfrac{x^2}{x^4+16}$

   (7). $f(x) = \dfrac{x}{2x^2+1}$ (8). $f(x) = \dfrac{x-1}{x+2}$

2. (a). Use differentiation to find a power series representation for

   $f(x) = \dfrac{1}{(x+1)^2}$, What is the radius of convergence?.

   (b). Use part (a) to find a power series for $f(x) = \dfrac{1}{(x+1)^3}$.

   (c). Use part (b) to find a power series for $f(x) = \dfrac{x^2}{(x+1)^3}$.

3. Find a power series representation for the function and determine the radius of convergence.

   (1). $f(x) = \ln(5-x)$ (2). $f(x) = x^2 \tan^{-1}(x^3)$.

3.10 Taylor and Maclaurin Series

3.10.1 Definitions of Taylor Series and Maclaurin Series

Theorem : If f has a power series representation at a, that is $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, where $|x-a| < R$, then the coefficient c_n is given by $c_n = \dfrac{f^{(n)}(a)}{n!}$.

Therefore, $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$,

or $f(x) = f(a) + \dfrac{f'(a)}{1!}(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \dfrac{f'''(a)}{3!}(x-a)^3 + \cdots$.

Definition : $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$ is called the Taylor series of the function f at a (or about a, or centered at a).

Definition : If $a = 0$ in the Taylor series of f, then $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!}x^n$ is called the Maclaurin series of f. In this case :

$f(x) = f(0) + \dfrac{f'(0)}{1!}x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 + \cdots$.

Example (1): Find the Maclaurin Series and its radius of convergence for the function $f(x) = \dfrac{1}{1-x}$.

Solution :

$f(x) = \dfrac{1}{1-x} \implies f(0) = 1 = 0!$.

$f'(x) = \dfrac{1}{(1-x)^2} \implies f'(0) = 1 = 1!$.

$f''(x) = \dfrac{2}{(1-x)^3} \implies f''(0) = 2 = 2!$.

$f'''(x) = \dfrac{6}{(1-x)^4} \implies f'''(0) = 6 = 3!$.

$f^{(n)}(x) = \dfrac{1.2.3\cdots n}{(1-x)^{n+1}} \implies f^{(n)}(0) = n!$.

The Maclaurin series for $f(x) = \dfrac{1}{1-x}$ is $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} \dfrac{n!}{n!}x^n = \sum_{n=0}^{\infty} x^n$.

Let $a_n = x^n$, then $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{x^{n+1}}{x^n}\right| = |x|$.

Therefore, the Maclaurin series converges when $|x| < 1$. Hence $R = 1$.

Example (2): Find the Maclaurin Series and its radius of convergence for the function $f(x) = e^x$.

Solution :

Note that $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$.

The Maclaurin series for $f(x) = e^x$ is $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.
Let $a_n = \frac{x^n}{n!}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!}\ \frac{n!}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$.
Therefore, the Maclaurin series converges for all $x \in \mathbb{R}$. Hence $R = \infty$.

3.10.2 Remainder of a Taylor series

Definition :
Suppose $f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$ then $f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i + \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$,
$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$ is called the nth-degree polynomial of f at a, and
$R_n(x) = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$ is called the remainder of the Taylor series.

Theorem : If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the nth-degree polynomial of f at a , and If $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Theorem : (Taylor's Inequality)
Suppose $f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i + \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i = T_n(x) + R_n(x)$.
If $\left|f^{(i)}(x)\right| \le M$ for $|x-a| \le d$ and for all $i \ge n+1$, where M and d are positive real numbers, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality : $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$, for $|x-a| \le d$.
In this case, $\lim_{n\to\infty} |R_n(x)| = 0$, hence $\lim_{n\to\infty} R_n(x) = 0$ for all values of x.

Example (3): Show that e^x is equal to the sum of its Maclaurin series.
Solution :
$f(x) = e^x \implies f^{(i)}(x) = e^x$ for all $i \ge 1$.
If $|x| \le d$ where $d > 0$, then $\left|f^{(i)}(x)\right| = e^x \le e^d$, for all $i \ge n+1$.
So Taylor's inequality, with $a = 0$ and $M = e^d$ is
$|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1}$ for $|x| \le d$.
Therefore, $\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = 0$.
So, $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \cdots$, for all $x \in \mathbb{R}$.
Note : If $x = 1$ then $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$.

Example (4): Find the Taylor series of $f(x) = e^x$ at $a = 2$.
Solution :
$f(x) = e^x \implies f^{(n)}(x) = e^x \implies f^{(n)}(2) = e^2$, for all n.
So, $e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2(x-2)^n}{n!}$ for all for all $x \in \mathbb{R}$.
Another Solution : $e^{x-2} = \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} \implies \frac{e^x}{e^2} = \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$
$\implies e^x = e^2 \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^2(x-2)^n}{n!}$.

Example (5): Write the function $f(x) = e^{x^2-1}$ as a Taylor series of x.
Solution :
$f(x) = e^{x^2-1} = e^{-1}e^{x^2} = e^{-1} \sum_{n=0}^{\infty} \frac{\left(x^2\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{e} \frac{x^{2n}}{n!}$ for all x .

3.10.3 Taylor Series of Important Functions

Example (6): Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.
Solution :
$f(x) = \sin x \implies f(0) = \sin(0) = 0$.
$f'(x) = \cos x \implies f'(0) = \cos(0) = 1$.
$f''(x) = -\sin x \implies f''(0) = -\sin(0) = 0$.
$f'''(x) = -\cos x \implies f'(0) = -\cos(0) = -1$.
$f^{(4)}(x) = \sin x \implies f^{(4)}(0) = \sin(0) = 0$.
Therefore, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.
$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!} \frac{(2n+1)!}{(-1)^n x^{2n+1}}\right|$
$= \lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0 < 1.$
Therefore, the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.
Since $f^{(i)}(x) = \pm\sin x$ or $\pm\cos x$ for all $i \geq 1$, then $\left|f^{(i)}(x)\right| \leq 1$.
Taylor's inequality, with $a = 0$ and $M = 1$ is
$|R_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1}$ for $|x| \leq d$, where $d > 0$.
Therefore, $\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$.
So, $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for all x.

Example (7): Find the Maclaurin series for $\cos x$ and prove that it represents $\cos x$ for all x.

Solution :
$f(x) = \cos x \implies f(0) = \cos(0) = 1$.
$f'(x) = -\sin x \implies f'(0) = -\sin(0) = 0$.
$f''(x) = -\cos x \implies f''(0) = -\cos(0) = -1$.
$f'''(x) = \sin x \implies f'(0) = \sin(0) = 0$.
$f^{(4)}(x) = \cos x \implies f^{(4)}(0) = \cos(0) = 1$.
Therefore, $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.
$\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!}\frac{(2n)!}{(-1)^n x^{2n}}\right|$
$= \displaystyle\lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+1)} = 0 < 1.$
Therefore, the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.
Since $f^{(i)}(x) = \pm\sin x$ or $\pm\cos x$ for all $i \geq 1$, then $\left|f^{(i)}(x)\right| \leq 1$.
Taylor's inequality, with $a = 0$ and $M = 1$ is
$|R_n(x)| \leq \dfrac{1}{(n+1)!}|x|^{n+1}$ for $|x| \leq d$, where $d > 0$.
Therefore, $\displaystyle\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$.
So, $\cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for all x .

Example (8): Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.
Solution :
$f(x) = (1+x)^k \implies f(0) = 1$.
$f'(x) = k(1+x)^{k-1} \implies f(0) = k$.
$f''(x) = k(k-1)(1+x)^{k-2} \implies f(0) = k(k-1)$.
$f'''(x) = k(k-1)(k-2)(1+x)^{k-3} \implies f(0) = k(k-1)(k-2)$.
$f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n} \implies f(0) = k(k-1)\cdots(k-n+1)$.
So, $(1+x)^k = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n.$
Let $a_n = \dfrac{k(k-1)\cdots(k-n+1)}{n!}x^n$ for all $n \geq 1$.
$\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!}\dfrac{n!}{k(k-1)\cdots(k-n+1)x^n}\right|$
$= \left|\dfrac{k-n}{n+1}\,x\right| = \dfrac{\left|1-\frac{k}{n}\right|}{1+\frac{1}{n}}|x|.$
$\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\left|1-\frac{k}{n}\right|}{1+\frac{1}{n}}|x| = |x|$.
So, $(1+x)^k = \displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n$ converges when $|x| < 1$. Therefore, $R = 1$.
Note : $(1+x)^k = \displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n$ is called the binomial series.

Example (9): For $f(x) = \dfrac{1}{\sqrt{4-x}}$, find the Maclaurin series and its radius of convergence.
Solution :

$$f(x) = \frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} = \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\left(1+\left(-\frac{x}{4}\right)\right)^{-\frac{1}{2}}.$$

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1+\left(-\frac{x}{4}\right)\right)^{-\frac{1}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n \text{ , where } \left|-\frac{x}{4}\right| < 1.$$

$$\left|-\frac{x}{4}\right| < 1 \implies \frac{|x|}{4} < 1 \implies |x| < 4.$$

The radius of convergence of the Maclaurin series is $R = 4$.

Table of Important Series :

No.	Series	R
1	$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$	1
2	$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\dfrac{x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots$	1
3	$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots$	1
4	$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n} x^n = 1 + kx + \dfrac{k(k-1)}{2!}x^2 + \cdots$	1
5	$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	∞
6	$\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$	∞
7	$\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots$	∞

3.10.4 New Taylor Series from Old

Example (10): Find the Maclaurin series for $f(x) = x^2 \cos x$.
Solution :

$$f(x) = x^2 \cos x = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n)!} \text{ for all } x.$$

Example (11): Find the Maclaurin series for $f(x) = \ln(1+4x^2)$.
Solution :

$$f(x) = \ln(1+4x^2) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{(4x^2)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{4^n x^{2n}}{n} \text{ ,}$$

Where $\left|4x^2\right| < 1 \implies |x|^2 < \dfrac{1}{4} \implies |x| < \dfrac{1}{2}$.

Example (12): Find the function represented by the power series $\sum_{n=0}^{\infty}(-1)^n\frac{3^n x^n}{n!}$.

Solution :

$$\sum_{n=0}^{\infty}(-1)^n\frac{3^n x^n}{n!}=\sum_{n=0}^{\infty}\frac{(-3x)^n}{n!}=e^{-3x}\ .$$

Example (13): Find the sum of the series $\frac{1}{1\ (2)}-\frac{1}{2\ (2^2)}+\frac{1}{3\ (2^3)}-\cdots$.

Solution :

$$\frac{1}{1\ (2)}-\frac{1}{2\ (2^2)}+\frac{1}{3\ (2^3)}-\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n\ (2^n)}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\left(\frac{1}{2}\right)^n}{n}$$

$$=\ln\left(1+\frac{1}{2}\right)=\ln\left(\frac{3}{2}\right)\ .$$

Example (14): Use series to evaluate $\lim_{x\to 0}\frac{e^x-1-x}{x^2}$.

Solution : For all $x\neq 0$,

$$\frac{e^x-1-x}{x^2}=\frac{1}{x^2}\left[\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)-1-x\right]$$

$$=\frac{1}{x^2}\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)=\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots$$

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2}=\lim_{x\to 0}\left(\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots\right)=\frac{1}{2}.$$

3.10.5 EXERCISES

1. Use the definition of a Taylor series to find the first four nonzero terms of the series for $f(x)$ centered at the given value of a.

   (1). $f(x) = xe^x$, $a = 0$ (2). $f(x) = \dfrac{1}{1+x}$, $a = 2$

   (3). $f(x) = \ln x$, $a = 1$ (4). $f(x) = \cos^2 x$, $a = 0$

2. Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. Also find the associated radius of convergence.

   (1). $f(x) = (1-x)^{-2}$ (2). $f(x) = \ln(1+x)$ (3). $f(x) = \cos x$

   (4). $f(x) = 2x^4 - 3x^2 + 3$ (5). $f(x) = x\cos x$ (6). $f(x) = \sinh x$

3. Find the Taylor series for $f(x)$ centered at the given value of a. Also find the associated radius of convergence.

   (1). $f(x) = x^5 + 2x^3 + x$, $a = 2$ (2). $f(x) = \ln x$, $a = 2$

   (3). $f(x) = \sin x$, $a = \pi$ (4). $f(x) = \cos x$, $a = \dfrac{\pi}{2}$

4. Show that : (a). $\sinh x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ (b). $\cosh x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

   Hint : use $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$.