



King Saud University  
College of Sciences  
Department of Mathematics

**MATH 201**  
**MULTIVARIABLE CALCULUS**

**CLASS NOTES**  
**DRAFT - January, 2026**

*Dr. Tariq A. AlFadhel<sup>1</sup>*  
*Associate Professor*  
*Mathematics Department*

---

<sup>1</sup>E-mail : [alfadhel@ksu.edu.sa](mailto:alfadhel@ksu.edu.sa)



# Contents

|          |                                                             |          |
|----------|-------------------------------------------------------------|----------|
| <b>1</b> | <b>Partial Derivatives</b>                                  | <b>7</b> |
| 1.1      | Functions of several variables . . . . .                    | 7        |
| 1.1.1    | Functions of two variables . . . . .                        | 7        |
| 1.1.2    | Graphs . . . . .                                            | 8        |
| 1.1.3    | Level curves . . . . .                                      | 9        |
| 1.1.4    | Functions of three variables . . . . .                      | 11       |
| 1.1.5    | EXERCISES . . . . .                                         | 12       |
| 1.2      | Limits and Continuity . . . . .                             | 13       |
| 1.2.1    | Limits of Functions of Two Variables . . . . .              | 13       |
| 1.2.2    | Showing That a Limit Does Not Exist . . . . .               | 13       |
| 1.2.3    | Properties of Limits . . . . .                              | 14       |
| 1.2.4    | Continuity . . . . .                                        | 16       |
| 1.2.5    | Limit and Continuity of a function of three variables . . . | 17       |
| 1.2.6    | EXERCISES . . . . .                                         | 18       |
| 1.3      | Partial Derivatives . . . . .                               | 20       |
| 1.3.1    | Partial Derivatives of Functions of Two Variables . . . . . | 20       |
| 1.3.2    | Interpretations of Partial Derivatives . . . . .            | 20       |
| 1.3.3    | Functions of Three Variables . . . . .                      | 22       |
| 1.3.4    | Higher Derivatives . . . . .                                | 22       |
| 1.3.5    | Partial Differential Equations . . . . .                    | 23       |
| 1.3.6    | Differentiability . . . . .                                 | 23       |
| 1.3.7    | EXERCISES . . . . .                                         | 26       |
| 1.4      | Chain Rule . . . . .                                        | 27       |
| 1.4.1    | The Chain Rule (Case 1) . . . . .                           | 27       |
| 1.4.2    | The Chain Rule (Case 2) . . . . .                           | 27       |
| 1.4.3    | The Chain Rule (The General Case) . . . . .                 | 28       |
| 1.4.4    | Implicit Differentiation . . . . .                          | 29       |
| 1.4.5    | EXERCISES . . . . .                                         | 30       |
| 1.5      | Maximum and Minimum Values . . . . .                        | 31       |
| 1.5.1    | Local Maximum and Minimum Values . . . . .                  | 31       |
| 1.5.2    | Absolute Maximum and Minimum Values . . . . .               | 32       |
| 1.5.3    | EXERCISES . . . . .                                         | 36       |
| 1.6      | Lagrange Multipliers . . . . .                              | 37       |
| 1.6.1    | Lagrange Multipliers (One Constraint) . . . . .             | 37       |
| 1.6.2    | EXERCISES . . . . .                                         | 40       |

|          |                                                            |           |
|----------|------------------------------------------------------------|-----------|
| <b>2</b> | <b>Multiple Integrals</b>                                  | <b>41</b> |
| 2.1      | Double Integrals over Rectangles . . . . .                 | 41        |
| 2.1.1    | Iterated Integrals . . . . .                               | 41        |
| 2.1.2    | Volume . . . . .                                           | 42        |
| 2.1.3    | Average Value . . . . .                                    | 43        |
| 2.1.4    | EXERCISES . . . . .                                        | 44        |
| 2.2      | Double Integrals over General Regions . . . . .            | 45        |
| 2.2.1    | General Regions . . . . .                                  | 45        |
| 2.2.2    | Changing the Order of Integration . . . . .                | 47        |
| 2.2.3    | Properties of Double Integrals . . . . .                   | 48        |
| 2.2.4    | EXERCISES . . . . .                                        | 50        |
| 2.3      | Double Integrals in Polar Coordinates . . . . .            | 51        |
| 2.3.1    | Double Integrals in Polar Coordinates . . . . .            | 51        |
| 2.3.2    | EXERCISES . . . . .                                        | 54        |
| 2.4      | Triple Integrals . . . . .                                 | 55        |
| 2.4.1    | Triple Integrals over Rectangular Boxes . . . . .          | 55        |
| 2.4.2    | Triple Integrals over General Regions . . . . .            | 56        |
| 2.4.3    | EXERCISES . . . . .                                        | 61        |
| 2.5      | Triple Integrals in Cylindrical Coordinates . . . . .      | 62        |
| 2.5.1    | Cylindrical Coordinates . . . . .                          | 62        |
| 2.5.2    | Triple Integrals in Cylindrical Coordinates . . . . .      | 64        |
| 2.5.3    | EXERCISES . . . . .                                        | 68        |
| 2.6      | Triple Integrals in Spherical Coordinates . . . . .        | 69        |
| 2.6.1    | Spherical Coordinates . . . . .                            | 69        |
| 2.6.2    | Triple Integrals in Spherical Coordinates . . . . .        | 71        |
| 2.6.3    | EXERCISES . . . . .                                        | 75        |
| <b>3</b> | <b>Sequences, Series, and Power Series</b>                 | <b>77</b> |
| 3.1      | Sequences . . . . .                                        | 77        |
| 3.1.1    | Infinite Sequences . . . . .                               | 77        |
| 3.1.2    | The Limit of a Sequence . . . . .                          | 77        |
| 3.1.3    | Properties of Convergent Sequences . . . . .               | 78        |
| 3.1.4    | Monotonic and bounded sequences . . . . .                  | 79        |
| 3.2      | Series . . . . .                                           | 80        |
| 3.2.1    | Infinite Series . . . . .                                  | 80        |
| 3.2.2    | Sum of a Geometric Series . . . . .                        | 80        |
| 3.2.3    | Test for Divergence . . . . .                              | 81        |
| 3.2.4    | Properties of Convergent Series . . . . .                  | 82        |
| 3.2.5    | EXERCISES . . . . .                                        | 83        |
| 3.3      | The Integral Test . . . . .                                | 84        |
| 3.3.1    | The Integral Test . . . . .                                | 84        |
| 3.3.2    | EXERCISES . . . . .                                        | 87        |
| 3.4      | The Comparison Tests . . . . .                             | 88        |
| 3.4.1    | The Direct Comparison Test . . . . .                       | 88        |
| 3.4.2    | Limit Comparison Test . . . . .                            | 89        |
| 3.4.3    | EXERCISES . . . . .                                        | 91        |
| 3.5      | Alternating Series and Absolute Convergence . . . . .      | 92        |
| 3.5.1    | Alternating Series . . . . .                               | 92        |
| 3.5.2    | Absolute Convergence and Conditional Convergence . . . . . | 93        |
| 3.5.3    | EXERCISES . . . . .                                        | 96        |

|        |                                                               |     |
|--------|---------------------------------------------------------------|-----|
| 3.6    | The Ratio and Root Tests . . . . .                            | 97  |
| 3.6.1  | The Ratio Test . . . . .                                      | 97  |
| 3.6.2  | The Root Test . . . . .                                       | 98  |
| 3.6.3  | EXERCISES . . . . .                                           | 100 |
| 3.7    | Strategy for Testing Series . . . . .                         | 101 |
| 3.8    | Power Series . . . . .                                        | 103 |
| 3.8.1  | Power Series . . . . .                                        | 103 |
| 3.8.2  | Interval of Convergence . . . . .                             | 104 |
| 3.8.3  | EXERCISES . . . . .                                           | 106 |
| 3.9    | Representations of Functions as Power Series . . . . .        | 107 |
| 3.9.1  | Representations of Functions using Geometric Series . . . . . | 107 |
| 3.9.2  | Differentiation and Integration of Power Series . . . . .     | 108 |
| 3.9.3  | EXERCISES . . . . .                                           | 110 |
| 3.10   | Taylor and Maclaurin Series . . . . .                         | 111 |
| 3.10.1 | Definitions of Taylor Series and Maclaurin Series . . . . .   | 111 |
| 3.10.2 | Remainder of a Taylor series . . . . .                        | 112 |
| 3.10.3 | Taylor Series of Important Functions . . . . .                | 113 |
| 3.10.4 | New Taylor Series from Old . . . . .                          | 115 |
| 3.10.5 | EXERCISES . . . . .                                           | 117 |



# Chapter 1

## Partial Derivatives

### 1.1 Functions of several variables

#### 1.1.1 Functions of two variables

**Definition:** A function  $f$  of two variables is a map that assigns to each ordered pair of real numbers  $(x, y) \in D \subseteq \mathbb{R}^2$  a unique real number denoted by  $f(x, y)$ . The set  $D$  is the domain of  $f$  and its range is  $\{f(x, y) | (x, y) \in D\}$ .

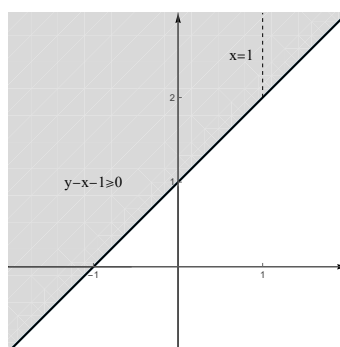
**Example (1):** If  $f(x, y) = \frac{\sqrt{y-x-1}}{x-1}$ , evaluate  $f(2, 7)$ , find the domain of  $f$  and sketch it.

Solution :  $f(2, 7) = \frac{\sqrt{7-2-1}}{2-1} = \frac{\sqrt{4}}{1} = 2$ .

The domain of  $f$  is the set  $D = \{(x, y) \in \mathbb{R}^2 \mid y - x - 1 \geq 0, x \neq 1\}$ .

So,  $D = \{(x, y) \in \mathbb{R}^2 \mid y \geq x + 1, x \neq 1\}$ .

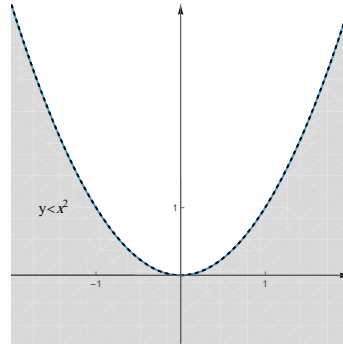
$y - x - 1 \geq 0 \implies y \geq x + 1$   
represents the points on and above the line  $y = x + 1$ .  
 $x \neq 1$  means the point on the line  $x = 1$  must be excluded from the domain .



**Example (2):** If  $f(x, y) = \ln(x^2 - y)$ , find the domain of  $f$  and sketch it.  
Solution :

The domain of  $f$  is the set  
 $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y > 0\}$ .  
 So,  $D = \{(x, y) \in \mathbb{R}^2 \mid y < x^2\}$ .

$x^2 - y > 0 \implies y < x^2$   
 represents the points below the  
 parabola  $y = x^2$ .



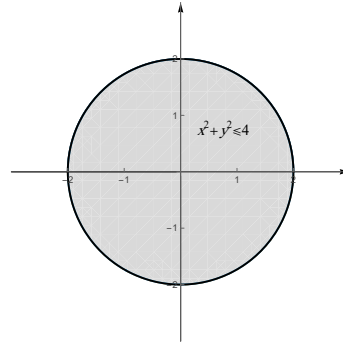
**Example (3):** If  $f(x, y) = \sqrt{4 - x^2 - y^2}$ , find the domain and range of  $f$ .

Solution :

The domain of  $f$  is the set  
 $D = \{(x, y) \in \mathbb{R}^2 \mid 4 - x^2 - y^2 \geq 0\}$ .  
 So,  $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$ .

$4 - x^2 - y^2 \geq 0 \implies x^2 + y^2 \leq 4$   
 represents the points on and inside the  
 disk of center  $(0, 0)$  and radius 2.

Note that  $0 \leq \sqrt{4 - x^2 - y^2}$   
 and  $\sqrt{4 - (x^2 + y^2)} \leq \sqrt{4} = 2$   
 So, the range of  $f$  is  
 $\{z \in \mathbb{R} \mid 0 \leq z \leq 2\} = [0, 2]$ .



### 1.1.2 Graphs

**Definition:** If  $f$  is a function of two variables with domain  $D \subseteq \mathbb{R}^2$ , then the graph of  $f$  is the set of all points  $(x, y, z) \in \mathbb{R}^3$ , such that  $z = f(x, y)$  and  $(x, y) \in D$ .

**Example (4):** Sketch the graph of the function  $f(x, y) = 4 - 2x - 2y$ .

Solution:

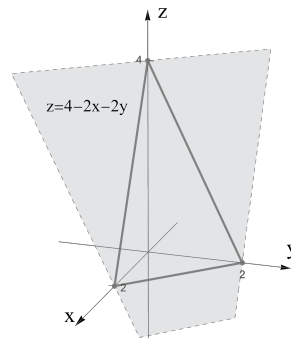
$$z = 4 - 2x - 2y \implies 2x + 2y + z = 4$$

It represents a plain.

To find the  $x$ -intercept, put  $y = z = 0$   
 $2x = 4 \implies x = 2$ .  
 So the  $x$ -intercept is  $(2, 0, 0)$ .

To find the  $y$ -intercept, put  $x = z = 0$   
 $2y = 4 \implies y = 2$ .  
 So the  $y$ -intercept is  $(0, 2, 0)$ .

To find the  $z$ -intercept, put  $x = y = 0$   
 $z = 4$ , So the  $z$ -intercept is  $(0, 0, 4)$ .



**Example (5):** Sketch the graph of the function  $f(x, y) = \sqrt{4 - x^2 - y^2}$ .

Solution:

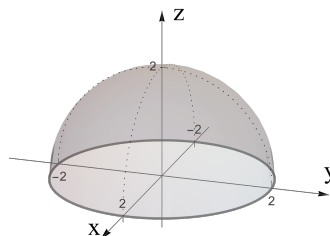
$$z = \sqrt{4 - x^2 - y^2}$$

$$\implies z^2 = 4 - x^2 - y^2$$

$$\implies x^2 + y^2 + z^2 = 2^2$$

$$\text{Note that } z = \sqrt{4 - x^2 - y^2} \geq 0.$$

It represents the upper half of the sphere centered at the origin and its radius is 2.



**Example (6):** Sketch the graph of the function  $f(x, y) = x^2 + y^2$ .

Solution:

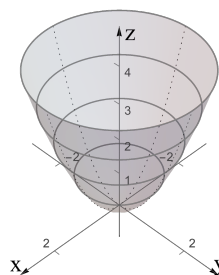
The domain of  $f$  is  $\mathbb{R}^2$ .

$$z = x^2 + y^2 \geq 0$$

For each value of  $z > 0$ ,

$x^2 + y^2 = z$  represents a circle centered at the origin and its radius is  $\sqrt{z}$ .

$f(x, y) = x^2 + y^2$  represents a paraboloid.



### 1.1.3 Level curves

**Definition:** A level curve of a function  $f(x, y)$  is the curve  $f(x, y) = k$ , where  $k$  is a constant (in the range of  $f$ ).

**Example (7):** Sketch the level curves of the function  $f(x, y) = 4 - 2x - 2y$  for the values  $k = 0, 4, 8$ .

Solution:

$$4 - 2x - 2y = k.$$

$$\implies 2x + 2y = 4 - k$$

$$\text{For } k = 0 : 2y + 2x = 4$$

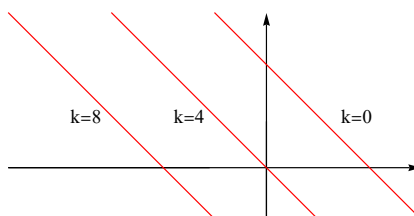
$$\implies y = -x + 2$$

$$\text{For } k = 4 : 2y + 2x = 0$$

$$\implies y = -x$$

$$\text{For } k = 8 : 2y + 2x = -4$$

$$\implies y = -x - 2$$



**Example (8):** Sketch the level curves of the function  $f(x, y) = \sqrt{4 - x^2 - y^2}$  for the values  $k = 0, 1, 2$ .

Solution:

$$\sqrt{4 - x^2 - y^2} = k.$$

$$\implies 4 - x^2 - y^2 = k^2$$

$$\implies x^2 + y^2 = 4 - k^2$$

For  $k = 0$  :  $x^2 + y^2 = 4$

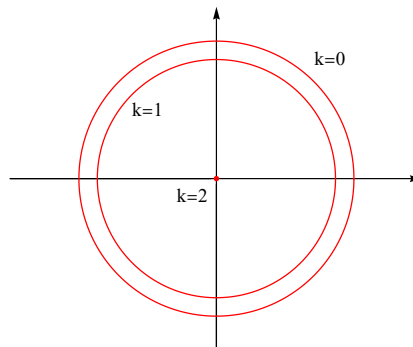
Circle: center is  $(0, 0)$ , radius = 2.

For  $k = 1$  :  $x^2 + y^2 = 3$

Circle: center is  $(0, 0)$ , radius =  $\sqrt{3}$ .

For  $k = 2$  :  $x^2 + y^2 = 0$

Just one point, the origin.



**Example (9):** Sketch the level curves of the function  $f(x, y) = 4x^2 + y^2$  for the values  $k = 0, 2, 4$ .

Solution:

Note that the domain is  $\mathbb{R}^2$ .

$$4x^2 + y^2 = k.$$

For  $k = 0$  :  $4x^2 + y^2 = 0$

$$\implies x = 0, y = 0$$

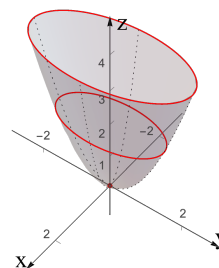
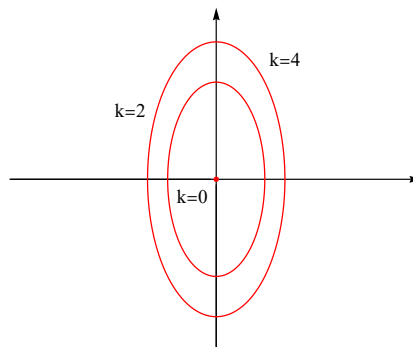
The level curve is the origin.

For  $k > 0$  :  $4x^2 + y^2 = k$

$$\implies \frac{x^2}{\left(\frac{k}{4}\right)} + \frac{y^2}{k} = 1$$

The level curve is an ellipse centered at the origin, the major axis lies on the  $y$ -axis and the minor axis lies on the  $x$ -axis.

Note that the graph of  $f$  is an elliptic paraboloid.



### 1.1.4 Functions of three variables

**Definition:** A function  $f$  of two variables is a map that assigns to each ordered pair of real numbers  $(x, y, z) \in D \subseteq \mathbb{R}^3$  a unique real number denoted by  $f(x, y, z)$ . The set  $D$  is the domain of  $f$  and its range is  $\{f(x, y, z) | (x, y, z) \in D\}$ .

**Example (10):** Find the domain of  $f(x, y, z) = \ln(z - x) + yz \sin x$ .

Solution :  $f(x, y, z)$  is defined when  $z - x > 0$ .

Therefore,  $D = \{(x, y, z) \in \mathbb{R}^3 \mid z > x\}$ .

**Example (11):** Find the level surfaces of  $f(x, y, z) = x^2 + y^2 + z^2$ .

Solution :  $x^2 + y^2 + z^2 = k$ , where  $k \geq 0$ .

If  $k = 0$ , then the level surface is just the origin  $(0, 0, 0) \in \mathbb{R}^3$ .

If  $k > 0$ , then the level surface is  $x^2 + y^2 + z^2 = (\sqrt{k})^2$ , which is a sphere centered at the origin, and its radius is  $\sqrt{k}$ .

**1.1.5 EXERCISES**

1. Let  $f(x, y) = x^2 \ln(x + y)$ 
  - (a). Evaluate  $f(3, 1)$ .
  - (b). Find and sketch the domain of  $f$ .
  - (c). Find the range of  $f$ .
  
2. Let  $f(x, y, z) = \ln(z - \sqrt{x^2 + y^2})$ .
  - (a). Evaluate  $f(4, 23, 6)$ .
  - (b). Find and describe the domain of  $f$ .
  
3. Find and sketch the domain of the following:
  - (a).  $f(x, y) = \sqrt{x - 2} + \sqrt{y - 1}$ .
  - (b).  $f(x, y) = \ln(9 - x^2 - y^2)$ .
  - (c).  $f(x, y) = \frac{\ln(2 - x)}{1 - x^2 - y^2}$ .
  
- (\*) Find and sketch the domain of the following:
  - (a).  $f(x, y) = \sqrt{25 - x^2 - y^2} + \ln(2 + x)$ .
  - (b).  $f(x, y) = \cos(x + y) + \frac{x^2 - y^2 - 3}{\sqrt{x + y - 4}}$ .
  - (c).  $f(x, y) = \sqrt{x + y^2} + \sqrt{y - x^2}$ .

## 1.2 Limits and Continuity

### 1.2.1 Limits of Functions of Two Variables

**Definition:** Let  $f$  be a function of two variables whose domain  $D \subseteq \mathbb{R}^2$  includes points arbitrarily close to  $(a, b)$ . Then we say that the limit of  $f(x, y)$  as  $(x, y)$  approaches  $(a, b)$  is  $L$  and we write  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$ ,

if for every number  $\epsilon > 0$  there is a corresponding number  $\delta > 0$  such that if  $(x, y) \in D$  and  $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$  then  $|f(x, y) - L| < \epsilon$ .

### 1.2.2 Showing That a Limit Does Not Exist

**NOTE:** If  $f(x, y) \rightarrow L_1$  as  $(x, y) \rightarrow (a, b)$  along a path  $C_1$  and  $f(x, y) \rightarrow L_2$  as  $(x, y) \rightarrow (a, b)$  along a different path  $C_2$ , where  $L_1 \neq L_2$ , then  $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$  does not exist.

**Example (1):** Show that  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$  does not exist.

Solution: Note that  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ .

Let  $y = mx$ , where  $m \in \mathbb{R}$ , note that  $x \rightarrow 0$  as  $y \rightarrow 0$ .

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{x^2 - m^2 x^2}{x^2 + m^2 x^2} = \lim_{x \rightarrow 0} \frac{x^2(1 - m^2)}{x^2(1 + m^2)} = \frac{1 - m^2}{1 + m^2}.$$

Note that the limit depends only on  $m$ .

Let  $C_1$  be the path  $y = 0$  (where  $m=0$ ) then  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{1 - 0}{1 + 0} = 1$ .

Let  $C_2$  be the path  $y = x$  (where  $m=1$ ) then  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{1 - 1}{1 + 1} = 0$ .

Therefore,  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$  does not exist.

**Example (2):** Show that  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$  does not exist.

Solution: Note that  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ .

Let  $y = mx$ , where  $m \in \mathbb{R}$ , note that  $x \rightarrow 0$  as  $y \rightarrow 0$ .

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{mx^2}{x^2 + m^2 x^2} = \lim_{x \rightarrow 0} \frac{mx^2}{x^2(1 + m^2)} = \frac{m}{1 + m^2}.$$

Note that the limit depends only on  $m$ .

Let  $C_1$  be the path  $y = 0$  (where  $m=0$ ) then  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = \frac{0}{1 + 0} = 0$ .

Let  $C_2$  be the path  $y = x$  (where  $m=1$ ) then  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = \frac{1}{1 + 1} = \frac{1}{2}$ .

Therefore,  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$  does not exist.

**Example (3):** Show that  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4}$  does not exist.

Solution: Note that  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ .

Let  $y = mx$ , where  $m \in \mathbb{R}$ , note that  $x \rightarrow 0$  as  $y \rightarrow 0$ .

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{x \rightarrow 0} \frac{x(m^2 x^2)}{x^2 + m^4 x^4} = \lim_{x \rightarrow 0} \frac{mx^3}{x^2(1 + m^4 x^2)} = \lim_{x \rightarrow 0} \frac{mx}{1 + m^4 x^2} = 0$$

Note that the limit here depends on  $x$  and  $m$ .

Let  $C_1$  be the path  $y = 0$  (where  $m=0$ ) then  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4} = 0$ .

Let  $C_2$  be the path  $x = y^2$  (the parabola with vertex  $(0,0)$  and opens to the right) then  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{y \rightarrow 0} \frac{y^2 y^2}{(y^2)^2 + y^4} = \lim_{y \rightarrow 0} \frac{y^4}{2y^4} = \frac{1}{2}$ .

Therefore,  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4}$  does not exist.

**Example (4):** Show that  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x^2 + y^2}$  does not exist.

Solution: Note that  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x^2 + y^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ .

Let  $y = mx$ , where  $m \in \mathbb{R}$ , note that  $x \rightarrow 0$  as  $y \rightarrow 0$ .

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{\sin(mx^2)}{x^2 + m^2 x^2} = \lim_{x \rightarrow 0} \frac{\sin(mx^2)}{(1 + m^2)x^2} = \frac{m}{1 + m^2}.$$

Note that the limit depends only on  $m$ .

Therefore,  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x^2 + y^2}$  does not exist.

### 1.2.3 Properties of Limits

If  $f(x, y)$ ,  $g(x, y)$  are two functions defined on  $D \setminus \{(a, b)\}$ , where  $D \subseteq \mathbb{R}^2$ , and  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L_1$ ,  $\lim_{(x,y) \rightarrow (a,b)} g(x, y) = L_2$ , where  $L_1, L_2 \in \mathbb{R}$  then

- (1).  $\lim_{(x,y) \rightarrow (a,b)} [f(x, y) + g(x, y)] = L_1 + L_2$ .
- (2).  $\lim_{(x,y) \rightarrow (a,b)} [f(x, y) - g(x, y)] = L_1 - L_2$ .
- (3).  $\lim_{(x,y) \rightarrow (a,b)} k f(x, y) = k L_1$ , where  $k \in \mathbb{R}$ .
- (4).  $\lim_{(x,y) \rightarrow (a,b)} [f(x, y) \cdot g(x, y)] = L_1 L_2$ .
- (5).  $\lim_{(x,y) \rightarrow (a,b)} \frac{f(x, y)}{g(x, y)} = \frac{L_1}{L_2}$ , where  $L_2 \neq 0$ .
- (6). If  $P(x, y)$  is a polynomial in  $x, y$  then  $\lim_{(x,y) \rightarrow (a,b)} P(x, y) = P(a, b)$ .

**Example (5):** Evaluate  $\lim_{(x,y) \rightarrow (2,1)} (x^2 y^2 - 2xy + x + y - 1)$ .

Solution: Note that  $P(x, y) = x^2 y^2 - 2xy + x + y - 1$  is a polynomial.

$$\text{So, } \lim_{(x,y) \rightarrow (2,1)} (x^2 y^2 - 2xy + x + y - 1) = P(2, 1)$$

$$= (2)^2(1)^2 - 2(2)(1) + 2 + 1 - 1 = 2.$$

**Example (6):** Evaluate  $\lim_{(x,y) \rightarrow (-1,3)} \frac{2x^2y+1}{xy^3-2x}$ .

Solution: Note that  $P(x,y) = 2x^2y+1$  and  $Q(x,y) = xy^3-2x$  are both polynomials.

$$\lim_{(x,y) \rightarrow (-1,3)} \frac{2x^2y+1}{xy^3-2x} = \frac{P(-1,3)}{Q(-1,3)} = \frac{2(-1)^2(3)+1}{(-1)(3)^3-2(-1)} = \frac{7}{-25}.$$

Note that  $Q(-1,3) \neq 0$ .

**Example (7):** Show that  $\lim_{(x,y) \rightarrow (0,0)} \frac{2xy^2}{x^2+y^2} = 0$ .

First Solution:  $\forall x, y \in \mathbb{R}^*$ ,

$$y^2 \leq x^2 + y^2 \implies \frac{y^2}{x^2 + y^2} \leq 1.$$

$$0 \leq \left| \frac{2xy^2}{x^2 + y^2} \right| = \frac{|2x| |y^2|}{|x^2 + y^2|} = 2|x| \frac{y^2}{x^2 + y^2} \leq 2|x|.$$

Note that  $(x,y) \rightarrow (0,0) \implies x \rightarrow 0$ .

Since  $\lim_{(x,y) \rightarrow (0,0)} 0 = 0$  and  $\lim_{(x,y) \rightarrow (0,0)} 2|x| = 2(0) = 0$ ,

By Squeeze Theorem  $\lim_{(x,y) \rightarrow (0,0)} \frac{2xy^2}{x^2 + y^2} = 0$ .

Second Solution: Using polar coordinates,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2xy^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{2r^3 \cos \theta \sin^2 \theta}{r^2} = \lim_{r \rightarrow 0} 2r \cos \theta \sin^2 \theta = 0.$$

Note that  $\lim_{r \rightarrow 0} 2r = 0$  and  $\cos \theta \sin^2 \theta$  is bounded.

**Example (8):** Discuss the existence of  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2}$ .

First Solution:  $\forall x, y \in \mathbb{R}^*$ ,

$$0 \leq \left| \frac{x^3 + y^3}{x^2 + y^2} \right| \leq \left| \frac{x^3}{x^2 + y^2} \right| + \left| \frac{y^3}{x^2 + y^2} \right| = |x| \left| \frac{x^2}{x^2 + y^2} \right| + |y| \left| \frac{y^2}{x^2 + y^2} \right| \leq |x| + |y|$$

Since  $\lim_{(x,y) \rightarrow (0,0)} |x| = 0$  and  $\lim_{(x,y) \rightarrow (0,0)} |y| = 0$ ,

By Squeeze Theorem  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2} = 0$ .

Second Solution: Using polar coordinates,

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2} &= \lim_{r \rightarrow 0} \frac{r^3 \cos^3 \theta + r^3 \sin^3 \theta}{r^2} = \lim_{r \rightarrow 0} \frac{r^3 (\cos^3 \theta + \sin^3 \theta)}{r^2} \\ &= \lim_{r \rightarrow 0} r (\cos^3 \theta + \sin^3 \theta) = 0. \end{aligned}$$

Note that  $\lim_{r \rightarrow 0} r = 0$  and  $(\cos^3 \theta + \sin^3 \theta)$  is bounded.

**Example (9):** Discuss the existence of  $\lim_{(x,y) \rightarrow (-1,0)} \frac{y(x+1)^2 + y^2 \sin(\pi x)}{(x+1)^2 + y^2}$ .

Solution:  $\forall (x,y) \neq (-1,0)$ ,

$$0 \leq \left| \frac{y(x+1)^2 + y^2 \sin(\pi x)}{(x+1)^2 + y^2} \right| \leq |y| \left| \frac{(x+1)^2}{(x+1)^2 + y^2} \right| + |\sin(\pi x)| \left| \frac{y^2}{(x+1)^2 + y^2} \right|$$

$$\text{Therefore, } 0 \leq \left| \frac{y(x+1)^2 + y^2 \sin(\pi x)}{(x+1)^2 + y^2} \right| \leq |y| + |\sin(\pi x)|$$

Since  $\lim_{(x,y) \rightarrow (-1,0)} |y| = 0$  and  $\lim_{(x,y) \rightarrow (-1,0)} |\sin(\pi x)| = \sin(-\pi) = 0$ ,

By Squeeze Theorem,  $\lim_{(x,y) \rightarrow (-1,0)} \frac{y(x+1)^2 + y^2 \sin(\pi x)}{(x+1)^2 + y^2} = 0$ .

### 1.2.4 Continuity

**Definition:** Let  $f(x, y)$  be a function of two variables defined on a set  $D \subseteq \mathbb{R}^2$  and  $(a, b) \in D$ . If  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$ , then  $f$  is continuous at  $(a, b)$ .

If  $f$  is continuous at every point in  $D$ , then  $f$  is continuous on  $D$ .

**Example (10):** Discuss the continuity of  $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ .

Solution:

$f$  is not defined at  $(0, 0)$ , so it is not continuous at  $(0, 0)$ .

$$\forall (a, b) \neq (0, 0), \lim_{(x,y) \rightarrow (a,b)} f(x, y) = \lim_{(x,y) \rightarrow (a,b)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{a^2 - b^2}{a^2 + b^2} = f(a, b).$$

So,  $f$  is continuous on  $D = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\} = \mathbb{R}^2 \setminus \{(0, 0)\}$ .

**Example (11):** Discuss the continuity of  $f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ .

Solution:

$$\forall (a, b) \neq (0, 0), \lim_{(x,y) \rightarrow (a,b)} f(x, y) = \lim_{(x,y) \rightarrow (a,b)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{a^2 - b^2}{a^2 + b^2} = f(a, b).$$

$f$  is defined at  $(0, 0)$  and  $f(0, 0) = 0$ , but  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  does not exist.

So,  $f$  is continuous on  $D = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\} = \mathbb{R}^2 \setminus \{(0, 0)\}$ .

**Example (12):** Discuss the continuity of  $f(x, y) = \begin{cases} \frac{2xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ .

Solution:

$$\forall (a, b) \neq (0, 0), \lim_{(x,y) \rightarrow (a,b)} f(x, y) = \lim_{(x,y) \rightarrow (a,b)} \frac{2xy^2}{x^2 + y^2} = \frac{2ab^2}{a^2 + b^2} = f(a, b).$$

$f$  is defined at  $(0, 0)$  and  $f(0, 0) = 0$ , also  $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$ .

So,  $f$  is continuous on  $\mathbb{R}^2$ .

**Example (13):** Discuss the continuity of  $f(x, y) = e^{-x^2 - y^2}$ .

Solution:  $f(x, y) = e^{-x^2 - y^2} = e^{-(x^2 + y^2)}$ .

$f$  is defined on  $\mathbb{R}^2$  and  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = e^{-(a^2 + b^2)} = f(a, b)$ .

Therefore,  $f$  is continuous on  $\mathbb{R}^2$ .

**Example (14):** Discuss the continuity of  $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ .

Solution:

$f$  is not defined where  $x = 0$ .

$\forall (a, b)$  where  $a \neq 0$ ,  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = \tan^{-1} \left( \frac{b}{a} \right) = f(a, b)$ .

So,  $f$  is continuous on  $D = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$  which is  $\mathbb{R}^2$  except the  $y$ -axis.

### 1.2.5 Limit and Continuity of a function of three variables

**Definition:** Let  $f$  be a function of three variables whose domain  $D \subseteq \mathbb{R}^3$  includes points arbitrarily close to  $(a, b, c)$ . Then we say that the limit of  $f(x, y, z)$  as  $(x, y, z)$  approaches  $(a, b, c)$  is  $L$  and we write  $\lim_{(x,y,z) \rightarrow (a,b,c)} f(x, y, z) = L$ , if for every number  $\epsilon > 0$  there is a corresponding number  $\delta > 0$  such that if  $(x, y, z) \in D$  and  $0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta$  then  $|f(x, y, z) - L| < \epsilon$ .

**Definition:** Let  $f(x, y, z)$  be a function of three variables defined on a set  $D \subseteq \mathbb{R}^3$  and  $(a, b, c) \in D$ . If  $\lim_{(x,y,z) \rightarrow (a,b,c)} f(x, y, z) = f(a, b, c)$ , then  $f$  is continuous at  $(a, b, c)$ .

If  $f$  is continuous at every point in  $D$ , then  $f$  is continuous on  $D$ .

**Example (15):** Discuss the continuity of  $f(x, y, z) = \frac{1}{1 - x^2 - y^2 - z^2}$ .

Solution:

$f$  is not defined where  $1 - x^2 - y^2 - z^2 = 0 \implies x^2 + y^2 + z^2 = 1$ ,

$f$  is not continuous on the unit sphere.

$\forall (a, b, c) \in \mathbb{R}^3$  where  $a^2 + b^2 + c^2 \neq 1$ ,

$\lim_{(x,y,z) \rightarrow (a,b,c)} f(x, y, z) = \frac{1}{1 - a^2 - b^2 - c^2} = f(a, b, c)$ .

So,  $f$  is continuous on  $D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \neq 1\}$  which is  $\mathbb{R}^3$  except the unit sphere.

## 1.2.6 EXERCISES

1. Find the limit of the following:

$$(a). \lim_{(x,y) \rightarrow (3,2)} (x^2y^3 - 4y^2) \quad (b). \lim_{(x,y) \rightarrow (2,-1)} \frac{x^2y + yx^2}{x^2 - y^2}$$

2. Show that the following limits do not exist:

$$(a). \lim_{(x,y) \rightarrow (0,0)} \frac{y^2}{x^2 + y^2} \quad (b). \lim_{(x,y) \rightarrow (0,0)} \frac{(x+y)^2}{x^2 + y^2}$$

$$(c). \lim_{(x,y) \rightarrow (1,1)} \frac{y-x}{1-y+\ln x}$$

3. Discuss the existence of the following limits (and find its value if exists):

$$(a). \lim_{(x,y) \rightarrow (2,3)} \frac{3x-2y}{4x^2-y^2} \quad (b). \lim_{(x,y) \rightarrow (0,0)} \frac{x^2y \cos y}{x^2 + y^4}$$

4. Use the Squeeze Theorem to find the limit of the following :

$$(a). \lim_{(x,y) \rightarrow (0,0)} xy \sin \left( \frac{1}{x^2 + y^2} \right) \quad (b). \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{x^2y^2z^2}{x^2 + y^2 + z^2}$$

5. Determine the set of points of continuity of the following:

$$(a). f(x, y) = \begin{cases} \frac{x^2y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

$$(b). f(x, y) = \begin{cases} \frac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

(\*) Show that the following limits do not exist:

$$(a). \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^2}{x^2 + y^2} \quad (b). \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^4 + y^4}$$

(\*) Use Squeeze theorem to find  $\lim_{(x,y) \rightarrow (0,0)} \frac{y^2 \sin x + yx^2}{x^2 + y^2}$

(\*) Show that the limit is zero in the following:

$$(a). \lim_{(x,y) \rightarrow (0,0)} \frac{x^5 + y^5}{x^2 + y^2} \quad (b). \lim_{(x,y) \rightarrow (0,0)} \frac{x^6 + y^6}{x^4 + y^4}$$
$$(c). \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + 2xy}{\sqrt{x^2 + y^2}} \quad (d). \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^3 + y^3)}{x^2 + y^2}$$

## 1.3 Partial Derivatives

### 1.3.1 Partial Derivatives of Functions of Two Variables

**Definition:** Let  $f(x, y)$  be a function of two variables defined on a set  $D \subseteq \mathbb{R}^2$ . If  $\lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$  and  $\lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$  exist, then the partial derivatives of  $f$  are denoted by  $f_x$  and  $f_y$  and are defined as

$$(1). f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

$$(2). f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

**Notations for Partial Derivatives:** If  $z = f(x, y)$  then

$$(1). f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = D_x f .$$

$$(2). f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = D_y f .$$

#### Rule for Finding Partial Derivatives

- (1). To find  $f_x$  : differentiate  $f(x, y)$  with respect to  $x$  regarding  $y$  as a constant.
- (2). To find  $f_y$  : differentiate  $f(x, y)$  with respect to  $y$  regarding  $x$  as a constant.

**Example (1):** If  $f(x, y) = x^2 - xy^3 - 3y^2$ , find  $f_x(1, 1)$  and  $f_y(1, 1)$ .

Solution:

$$(1). f_x(x, y) = 2x - (1)y^3 - 0 = 2x - y^3 ,$$

$$f_x(1, 1) = 2(1) - (1)^3 = 1.$$

$$(2). f_y(x, y) = 0 - x(3y^2) - 3(2y) = -3xy^2 - 6y ,$$

$$f_y(1, 1) = -3(1)(1)^2 - 6(1) = -9.$$

**Example (2):** If  $f(x, y) = \sin\left(\frac{2x}{1+3y}\right)$ , Evaluate  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ .

Solution:

$$(1). \frac{\partial f}{\partial x} = \cos\left(\frac{2x}{1+3y}\right) \left(\frac{2}{1+3y}\right) .$$

$$(2). \frac{\partial f}{\partial y} = \cos\left(\frac{2x}{1+3y}\right) ((2x)(-1)(1+3y)^{-2}(3)) = \cos\left(\frac{2x}{1+3y}\right) \left(\frac{-6x}{(1+3y)^2}\right) .$$

### 1.3.2 Interpretations of Partial Derivatives

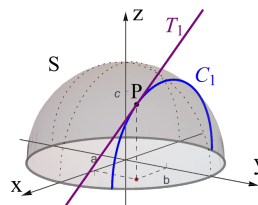
Let  $S$  be the surface represented by

$$z = f(x, y) \text{ and } f(a, b) = c.$$

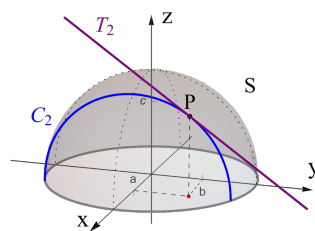
Then  $P(a, b, c)$  lies on  $S$ .

Let  $C_1$  be the curve where the plane  $y = b$  intersects the surface  $S$ ,

The slope of the tangent line  $T_1$  to the curve  $C_1$  at  $P(a, b, c)$  is  $f_x(a, b)$ .



Let  $C_2$  be the curve where the plane  $x = a$  intersects the surface  $S$ ,  
 The slope of the tangent line  $T_2$  to the curve  $C_2$  at  $P(a, b, c)$  is  $f_y(a, b)$ .



**Example (3):** If  $f(x, y) = 4 - x^2 - y^2$ , find  $f_x(1, 1)$ ,  $f_y(1, 1)$  and interpret them as slopes. Solution:

$$z = 4 - x^2 - y^2.$$

$$f_x(x, y) = -2x.$$

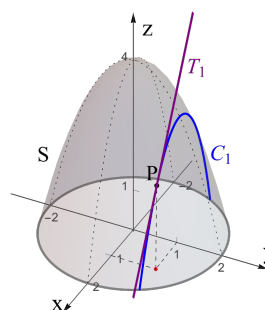
$$f_x(1, 1) = -2(1) = -2.$$

$$f(1, 1) = 4 - 1 - 1 = 2.$$

$C_1$  is the intersection of  $z = 4 - x^2 - y^2$  and  $y = 1$ , and it is the parabola

$$z = 3 - x^2.$$

The line  $T_1$  is tangent to the curve  $C_1$  at the point  $P(1, 1, 2)$  and its slope is  $f_x(1, 1) = -2$ .



$$z = 4 - x^2 - y^2.$$

$$f_y(x, y) = -2y.$$

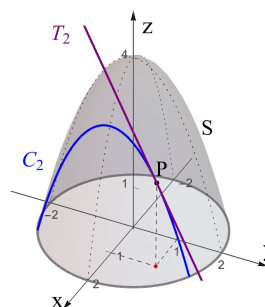
$$f_y(1, 1) = -2(1) = -2.$$

$$f(1, 1) = 4 - 1 - 1 = 2.$$

$C_2$  is the intersection of  $z = 4 - x^2 - y^2$  and  $x = 1$ , and it is the parabola

$$z = 3 - y^2.$$

The line  $T_2$  is tangent to the curve  $C_2$  at the point  $P(1, 1, 2)$  and its slope is  $f_y(1, 1) = -2$ .



**Example (4):** If  $x^3 + y^3 + z^3 + 6xyz + 4 = 0$ , find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$ .

Solution:

Differentiating implicitly with respect to  $x$ .

$$3x^2 + 0 + 3z^2 \frac{\partial z}{\partial x} + 6y \left( z + x \frac{\partial z}{\partial x} \right) + 0 = 0$$

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} (3z^2 + 6xy) = -3x^2 - 6yz$$

$$\frac{\partial z}{\partial x} = \frac{-3x^2 - 6yz}{3z^2 + 6xy} = \frac{-x^2 - 2yz}{z^2 + 2xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}.$$

Differentiating implicitly with respect to  $y$ .

$$0 + 3y^2 + 3z^2 \frac{\partial z}{\partial y} + 6x \left( z + y \frac{\partial z}{\partial y} \right) + 0 = 0$$

$$3y^2 + 3z^2 \frac{\partial z}{\partial y} + 6xz + 6xy \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} (3z^2 + 6xy) = -3y^2 - 6xz$$

$$\frac{\partial z}{\partial y} = \frac{-3y^2 - 6xz}{3z^2 + 6xy} = \frac{-y^2 - 2xz}{z^2 + 2xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}.$$

### 1.3.3 Functions of Three Variables

**Definition:** Let  $f(x, y, z)$  be a function of three variables defined on a set  $D \subseteq \mathbb{R}^3$ . If  $\lim_{h \rightarrow 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$ ,  $\lim_{h \rightarrow 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}$  and  $\lim_{h \rightarrow 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}$  exist, then the partial derivatives of  $f$  are denoted by  $f_x$ ,  $f_y$  and  $f_z$  and are defined as

$$(1). f_x(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}.$$

$$(2). f_y(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}.$$

$$(3). f_z(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}.$$

**Example (5):** If  $f(x, y, z) = e^{xy} \ln(z^2 + 1)$ , find  $f_x$ ,  $f_y$  and  $f_z$ .

Solution:

$$f_x(x, y, z) = (e^{xy} y) \ln(z^2 + 1) = ye^{xy} \ln(z^2 + 1).$$

$$f_y(x, y, z) = (e^{xy} x) \ln(z^2 + 1) = xe^{xy} \ln(z^2 + 1).$$

$$f_z(x, y, z) = e^{xy} \left( \frac{2z}{z^2 + 1} \right) = \frac{2ze^{xy}}{z^2 + 1}.$$

### 1.3.4 Higher Derivatives

**Definition:** If  $z = f(x, y)$ , then the second partial derivatives of  $f$  are

$$(1) f_{xx} = (f_x)_x = \frac{\partial}{\partial x} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}.$$

$$(2) f_{xy} = (f_x)_y = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}.$$

$$(3) f_{yx} = (f_y)_x = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}.$$

$$(y) f_{yy} = (f_y)_y = \frac{\partial}{\partial y} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}.$$

**Example (6):** Find the second partial derivatives of  $f(x, y) = x^3 + x^2y^2 - 2y^3$ .

Solution:

$$\begin{aligned}
f_x(x, y) &= 3x^2 + (2x)y^2 - 0 = 3x^2 + 2xy^2 . \\
f_{xx}(x, y) &= 3(2x) + 2y^2(1) = 6x + 2y^2 . \\
f_{xy}(x, y) &= 0 + 2x(2y) = 4xy . \\
f_y(x, y) &= 0 + x^2(2y) - 2(3y^2) = 2x^2y - 6y^2 . \\
f_{yx}(x, y) &= 2y(2x) - 0 = 4xy . \\
f_{yy} &= 2x^2(1) - 6(2y) = 2x^2 - 12y . \\
\text{NOTE : } f_{xy}(x, y) &= f_{yx}(x, y) .
\end{aligned}$$

**Clairaut's Theorem:** Suppose  $f$  is defined on a set  $D \subseteq \mathbb{R}^2$  that contains the point  $(a, b)$ . If the functions  $f_{xy}$  and  $f_{yx}$  are both continuous on  $D$ , then  $f_{xy}(a, b) = f_{yx}(a, b)$ .

**Example (7):** If  $f(x, y, z) = \sin(2x - yz)$ , find  $f_{xyz}(x, y)$ .

Solution:

$$\begin{aligned}
f_x(x, y, z) &= \cos(2x - yz)(2) = 2\cos(2x - yz) . \\
f_{xy}(x, y, z) &= 2[-\sin(2x - yz)(-z)] = 2z\sin(2x - yz) . \\
f_{xyz}(x, y, z) &= (2)\sin(2x - yz) + 2z[\cos(2x - yz)(-y)] \\
&= 2\sin(2x - yz) - 2yz\cos(2x - yz) .
\end{aligned}$$

### 1.3.5 Partial Differential Equations

(1). **Laplace's Equation:** If  $u(x, y)$  is a function of two variables and  $u_{xx} + u_{yy} = 0$ , then  $u$  satisfies the Laplace's equation, and  $u$  is called a harmonic function.

**Example (8):** Show that  $u(x, y) = e^x \cos y$  is a solution of Laplace's equation.

Solution:

$$\begin{aligned}
u_x(x, y) &= e^x \cos y \text{ and } u_{xx}(x, y) = e^x \cos y . \\
u_y(x, y) &= -e^x \sin y \text{ and } u_{yy}(x, y) = -e^x \cos y . \\
u_{xx} + u_{yy} &= e^x \cos y - e^x \cos y = 0 .
\end{aligned}$$

(2). **Wave Equation:** If  $u(x, t)$  is a function of two variables and  $u_{tt} = a^2 u_{xx}$ , then  $u$  satisfies the wave equation.

**Example (9):** Show that  $u(x, t) = \sin(x - at)$  is a solution of the wave equation.

Solution:

$$\begin{aligned}
u_x(x, t) &= \cos(x - at) \text{ and } u_{xx}(x, t) = -\sin(x - at) . \\
u_t(x, t) &= -a \cos(x - at) \text{ and } u_{tt}(x, t) = -a^2 \sin(x - at) . \\
u_{tt} &= a^2(-\sin(x - at)) = a^2 u_{xx} .
\end{aligned}$$

### 1.3.6 Differentiability

**Definition:** If  $z = f(x, y)$  and  $(a, b) \in D_f$ , if  $x$  changes from  $a$  to  $a + \Delta x$  and  $y$  changes from  $b$  to  $b + \Delta y$ , then the increment of  $z$  is

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b).$$

**Definition:** If  $z = f(x, y)$ , then  $f$  is differentiable at  $(a, b)$ , if

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \sqrt{(\Delta x)^2 + (\Delta y)^2} \epsilon(\Delta x, \Delta y),$$

where  $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \epsilon(\Delta x, \Delta y) = 0$ .

**NOTE:**

$$\text{If } \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{f(a + \Delta x, b + \Delta y) - f(a, b) - f_x(a, b)\Delta x - f_y(a, b)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0, \text{ then}$$

$f$  is differentiable at  $(a, b)$ .

**Theorem:** If  $f$  is differentiable at  $(a, b)$ , then  $f$  is continuous at  $(a, b)$ .

**NOTE:** If  $f$  is not continuous at  $(a, b)$ , then  $f$  is not differentiable at  $(a, b)$ .

**Example (10):** Show that  $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

is not differentiable at  $(0, 0)$ .

Solution:

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} f(x, y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r \cos \theta \ r \sin \theta}{r^2} \\ &= \lim_{r \rightarrow 0} \frac{r^2 \cos \theta \sin \theta}{r^2} = \cos \theta \sin \theta. \end{aligned}$$

The limit depends on  $\theta$ , so the limit does not exist.

$f$  is not continuous at  $(0, 0)$ , hence,  $f$  is not differentiable at  $(0, 0)$ .

**Example (11):** Show that  $f(x, y) = \begin{cases} \frac{xy^2}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

is differentiable at  $(0, 0)$ .

Solution:

$$f(0, 0) = 0.$$

$$f(0 + \Delta x, 0 + \Delta y) = f(\Delta x, \Delta y) = \frac{\Delta x (\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}.$$

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0.$$

$$f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0.$$

$$\begin{aligned} \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{f(\Delta x, \Delta y) - f(0, 0) - f_x(0, 0)\Delta x - f_y(0, 0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \\ = \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{\left( \frac{\Delta x (\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{\Delta x (\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} = 0. \end{aligned}$$

$$\text{Note that } 0 \leq \left| \frac{\Delta x (\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \right| = |\Delta x| \left| \frac{(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \right| \leq |\Delta x|$$

By the squeeze theorem  $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \left| \frac{\Delta x (\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \right| = 0$ ,

So,  $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{\Delta x (\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} = 0$ .

Therefore,  $f$  is differentiable at  $(0,0)$ .

**Example (12):** Show that  $f(x, y) = \begin{cases} \frac{y^2 \sin x + yx^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

is not differentiable at  $(0,0)$ .

Solution:

$f(0,0) = 0$ .

$$f(0 + \Delta x, 0 + \Delta y) = f(\Delta x, \Delta y) = \frac{(\Delta y)^2 \sin(\Delta x) + \Delta y (\Delta x)^2}{(\Delta x)^2 + (\Delta y)^2}.$$

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0.$$

$$f_y(0,0) = \lim_{h \rightarrow 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0.$$

$$\begin{aligned} & \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{f(\Delta x, \Delta y) - f(0,0) - f_x(0,0)\Delta x - f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \\ &= \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{\left( \frac{(\Delta y)^2 \sin(\Delta x) + \Delta y (\Delta x)^2}{(\Delta x)^2 + (\Delta y)^2} \right)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{(\Delta y)^2 \sin(\Delta x) + \Delta y (\Delta x)^2}{\left[ (\Delta x)^2 + (\Delta y)^2 \right]^{\frac{3}{2}}} \end{aligned}$$

Taking the path  $\Delta y = \Delta x$  :

$$\begin{aligned} & \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{(\Delta y)^2 \sin(\Delta x) + \Delta y (\Delta x)^2}{\left[ (\Delta x)^2 + (\Delta y)^2 \right]^{\frac{3}{2}}} = \lim_{\Delta x \rightarrow 0} \frac{(\Delta x)^2 \sin(\Delta x) + \Delta x (\Delta x)^2}{\left[ (\Delta x)^2 + (\Delta x)^2 \right]^{\frac{3}{2}}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(\Delta x)^2 [\sin(\Delta x) + \Delta x]}{\left[ 2(\Delta x)^2 \right]^{\frac{3}{2}}} = \lim_{\Delta x \rightarrow 0} \frac{(\Delta x)^2 [\sin(\Delta x) + \Delta x]}{\sqrt{8} (\Delta x)^3} = \lim_{\Delta x \rightarrow 0} \frac{\sin(\Delta x) + \Delta x}{\sqrt{8} \Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{8}} \left( \frac{\sin(\Delta x)}{\Delta x} + \frac{\Delta x}{\Delta x} \right) = \frac{1}{\sqrt{8}} (1 + 1) = \frac{2}{\sqrt{8}} = \frac{1}{\sqrt{2}} \neq 0. \end{aligned}$$

Therefore,  $f$  is not differentiable at  $(0,0)$ .

## 1.3.7 EXERCISES

1. Find the first partial derivatives of the function.

$$\begin{array}{ll} (a). f(x, y) = x^4 - 5xy^3 & (b). g(x, y) = x^3 \sin y \\ (c). w(u, v) = \frac{u}{v^2} & (d). u(r, \theta) = \sin(r \cos \theta) \\ (e). w(x, y, z) = y \tan(x + 2z) \end{array}$$

2. If  $f(x, y) = y \sin^{-1}(xy)$ , find  $f_y \left(1, \frac{1}{2}\right)$ .

3. Find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  :

$$(a). z = f(x) + g(y) \quad (b). z = f(x + y)$$

4. Find all the second partial derivatives of  $f(x, y) = x^4y - 2x^3y^2$ .

5. Verify that the conclusion of Clairaut's Theorem holds for

$$u(x, y) = x^4y^3 - y^4.$$

6. If  $f(x, y) = x^4y^2 - x^3y$ , find  $f_{xxx}$  and  $f_{xyx}$ .

- (\*) Discuss the differentiability of the following functions at the given points:

$$(a). f(x, y) = \begin{cases} \frac{x^2(y-2)}{x^2 + (y-2)^2} & \text{if } (x, y) \neq (0, 2) \\ 0 & \text{if } (x, y) = (0, 2) \end{cases} \quad \text{at } (0, 2).$$

$$(b). f(x, y) = \begin{cases} \frac{x^2y - xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \quad \text{at } (0, 0).$$

$$(c). f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \quad \text{at } (0, 0).$$

## 1.4 Chain Rule

### 1.4.1 The Chain Rule (Case 1)

**Theorem:** Suppose that  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ , where  $x = g(t)$  and  $y = h(t)$  are both differentiable functions of  $t$ . Then  $z$  is a differentiable function of  $t$  and  $\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$ .

**Example (1):** If  $f(x, y) = x^3y + 2xy^3$ , where  $x = \sin 2t$  and  $y = \cos t$ , find  $\frac{dz}{dt}$  at  $t = 0$ .

Solution:

$$\begin{aligned}\frac{\partial f}{\partial x} &= 3x^2y + 2y^3, \quad \frac{\partial f}{\partial y} = x^3 + 6xy^2. \\ \frac{dx}{dt} &= 2 \cos 2t, \quad \frac{dy}{dt} = -\sin t. \\ \frac{dz}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= (3x^2y + 2y^3)(2 \cos 2t) + (x^3 + 6xy^2)(-\sin t) \\ \left. \frac{dz}{dt} \right|_{t=0} &= (0 + 2)(2) + (0 + 0)(0) = 4.\end{aligned}$$

### 1.4.2 The Chain Rule (Case 2)

**Theorem:** Suppose that  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ , where  $x = g(s, t)$  and  $y = h(s, t)$  are both differentiable functions of  $s$  and  $t$ .

Then  $z$  is a differentiable function of  $t$  and  $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}$ ,

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$$

**Example (2):** If  $f(x, y) = e^x \sin y$ , where  $x = s^2 + t^2$  and  $y = s^2 t^2$ , find  $\frac{\partial f}{\partial s}$  and  $\frac{\partial f}{\partial t}$ .

Solution:

$$\begin{aligned}\frac{\partial f}{\partial x} &= e^x \sin y, \quad \frac{\partial f}{\partial y} = e^x \cos y. \\ \frac{\partial x}{\partial s} &= 2s, \quad \frac{\partial y}{\partial s} = 2st^2. \\ (1). \quad \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} = (e^x \sin y)(2s) + (e^x \cos y)(2st^2) \\ &= 2se^{s^2+t^2} \sin(s^2 t^2) + 2st^2 e^{s^2+t^2} \cos(s^2 t^2). \\ \text{Also, } \frac{\partial x}{\partial t} &= 2t, \quad \frac{\partial y}{\partial t} = 2ts^2. \\ (2). \quad \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = (e^x \sin y)(2t) + (e^x \cos y)(2ts^2) \\ &= 2te^{s^2+t^2} \sin(s^2 t^2) + 2ts^2 e^{s^2+t^2} \cos(s^2 t^2).\end{aligned}$$

### 1.4.3 The Chain Rule (The General Case)

**Theorem:** Suppose that  $u$  is a differentiable function of the  $n$  variables  $x_1, x_2, \dots, x_n$  and each  $x_j$  is a differentiable function of the  $m$  variables  $t_1, t_2, \dots, t_m$ . Then  $u$  is a function of  $t_1, t_2, \dots, t_m$  and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}, \text{ for each } i = 1, 2, \dots, m.$$

**Example (3):** If  $u(x, y, z) = x^2y^2 + yz^3$ , where  $x = r^2se^t$ ,  $y = rse^{-t}$  and  $z = rs^2 \sin t$ , find  $\frac{\partial u}{\partial s}$  when  $r = 1$ ,  $s = 1$  and  $t = 0$ .

Solution:

$$\frac{\partial u}{\partial x} = 2xy^2, \quad \frac{\partial u}{\partial y} = 2x^2y + z^3 \text{ and } \frac{\partial u}{\partial z} = 3yz^2.$$

$$\frac{\partial x}{\partial s} = r^2e^t, \quad \frac{\partial y}{\partial s} = re^{-t} \text{ and } \frac{\partial z}{\partial s} = 2rs \sin t.$$

$$\text{So, } \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s}$$

$$\frac{\partial u}{\partial s} = (2xy^2)(r^2e^t) + (2x^2y + z^3)(re^{-t}) + (3yz^2)(2rs \sin t)$$

$$\text{When } r = 1, s = 1, t = 0, \quad \frac{\partial u}{\partial s} = (2)(1) + (2)(1) + (0)(0) = 4.$$

**Example (4):** If  $w = f(x, y)$  is differentiable at  $(x, y)$  and  $x = s + t$ ,  $y = s - t$ .

$$\text{Show that } \frac{\partial w}{\partial s} \frac{\partial w}{\partial t} = \left( \frac{\partial w}{\partial x} \right)^2 - \left( \frac{\partial w}{\partial y} \right)^2.$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} = f_x(1) + f_y(1) = f_x + f_y.$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} = f_x(1) + f_y(-1) = f_x - f_y.$$

$$\frac{\partial w}{\partial s} \frac{\partial w}{\partial t} = (f_x + f_y)(f_x - f_y) = (f_x)^2 - (f_y)^2 = \left( \frac{\partial w}{\partial x} \right)^2 - \left( \frac{\partial w}{\partial y} \right)^2.$$

**Example (5):** If  $z = f(s^2 - t^2, t^2 - s^2)$  is differentiable at  $(s, t)$ ,

$$\text{Show that } t \frac{\partial z}{\partial s} + s \frac{\partial z}{\partial t} = 0.$$

Solution:

$$\text{Let } z = f(x, y), \text{ where } x = s^2 - t^2 \text{ and } y = t^2 - s^2.$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = f_x(2s) + f_y(-2s) = 2sf_x - 2sf_y.$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = f_x(-2t) + f_y(2t) = -2tf_x + 2tf_y.$$

$$t \frac{\partial z}{\partial s} + s \frac{\partial z}{\partial t} = t(2sf_x - 2sf_y) + s(-2tf_x + 2tf_y) \\ = 2stf_x - 2stf_y - 2stf_x + 2stf_y = 0.$$

**Example (6):** If  $z = f(x, y)$  has continuous second-order partial derivatives

$$\text{and } x = r^2 + s, \quad y = 3rs. \text{ Find } \frac{\partial^2 z}{\partial r^2}.$$

Solution:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = f_x(2r) + f_y(3s) = 2rf_x + 3sf_y.$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial r} \right) = \frac{\partial}{\partial r} (2rf_x + 3sf_y) = 2f_x + 2r \left( \frac{\partial f_x}{\partial r} \right) + 3s \left( \frac{\partial f_y}{\partial r} \right) \\
&= 2f_x + 2r \left( \frac{\partial f_x}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f_x}{\partial y} \frac{\partial y}{\partial r} \right) + 3s \left( \frac{\partial f_y}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f_y}{\partial y} \frac{\partial y}{\partial r} \right) \\
&= 2f_x + 2r (f_{xx}(2r) + f_{xy}(3s)) + 3s (f_{yx}(2r) + f_{yy}(3s)) \\
&= 2f_x + 4r^2 f_{xx} + 6rs f_{xy} + 6rs f_{yx} + 9s^2 f_{yy} .
\end{aligned}$$

#### 1.4.4 Implicit Differentiation

Let  $F(x, y) = 0$  where  $y = f(x)$ , using chain rule  $\frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0$

$$\implies F_x(1) + F_y \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{F_x}{F_y} .$$

**Example (7):** If  $x^2 + y^2 = 5xy$ , Find  $\frac{dy}{dx}$ .

Solution:

$$\begin{aligned}
x^2 + y^2 = 5xy &\implies x^2 + y^2 - 5xy = 0, \text{ Let } F(x, y) = x^2 + y^2 - 5xy \text{ then} \\
F(x, y) &= 0. \\
\frac{dy}{dx} &= -\frac{F_x}{F_y} = -\frac{2x - 5y}{2y - 5x}.
\end{aligned}$$

Let  $F(x, y, z) = 0$  where  $z = f(x, y)$ , then  $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$  and  $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$ .

**Example (8):** If  $x^3 + y^3 + z^2 + 3xyz - 1 = 0$ , Find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$ .

Solution:

$$\begin{aligned}
\text{Let } F(x, y, z) &= x^3 + y^3 + z^2 + 3xyz - 1 \text{ then } F(x, y, z) = 0. \\
\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = -\frac{3x^2 + 3yz}{2z + 3xy} . \\
\frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = -\frac{3y^2 + 3xz}{2z + 3xy} .
\end{aligned}$$

## 1.4.5 EXERCISES

1. Find  $\frac{dw}{dt}$  of the following:

(a).  $w = xy^3 - x^2y$ , where  $x = t^2 + 1$  and  $y = t^2 - 1$ .

(b).  $w = \sin x \cos y$ , where  $x = \sqrt{t}$  and  $y = \frac{1}{t}$ .

(c).  $w = xe^{\frac{y}{z}}$ , where  $x = t^2$ ,  $y = 1 - t$  and  $z = 1 + 2t$ .

(d).  $w = \ln \sqrt{x^2 + y^2 + z^2}$ , where  $x = \sin t$ ,  $y = \cos t$  and  $z = \tan t$ .

2. Find  $\frac{\partial z}{\partial s}$  and  $\frac{\partial z}{\partial t}$  of the following:

(a).  $z = (x - y)^5$ , where  $x = s^2t$  and  $y = st^2$ .

(b).  $z = \tan^{-1}(x^2 + y^2)$ , where  $x = s \ln t$  and  $y = t e^s$ .

(c).  $z = \frac{\sin \theta}{r}$ , where  $r = st$  and  $\theta = s^2 + t^2$ .

3. If  $z = x^4 + x^2y$ , where  $x = s + 2t - u$  and  $y = stu^2$ ,

Find  $\frac{\partial z}{\partial s}$ ,  $\frac{\partial z}{\partial t}$  and  $\frac{\partial z}{\partial u}$  at  $s = 4$ ,  $t = 2$ ,  $u = 1$ .

4. If  $z = f(x, y)$ , where  $x = r \cos \theta$  and  $y = r \sin \theta$ , Show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2.$$

5. If  $z = f(x + at) + g(x - at)$ , Show that  $\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$ .

(\*) If  $xe^{yz} - 2ye^{xz} + 3ze^{xy} = 1$ , find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$ .

## 1.5 Maximum and Minimum Values

### 1.5.1 Local Maximum and Minimum Values

**Definition:** If  $f$  is a differentiable function of two variables  $x$  and  $y$ , then the gradient of  $f$  is the vector function  $\nabla f$  and it is defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}.$$

**Definition:** If  $f$  is a differentiable function of three variables  $x, y$  and  $z$ , then the gradient of  $f$  is the vector function  $\nabla f$  and it is defined by

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}.$$

**Definition (Critical point):** If  $f(x, y)$  is a function of two variables then  $(a, b) \in D_f$  is a critical point if both  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$ , or  $f$  is not differentiable at  $(a, b)$ .

**Definition (Local maximum and Local minimum):**

- (i) A function  $f$  of two variables has a local maximum at  $(a, b) \in D_f$  if  $f(x, y) \leq f(a, b)$ , for all points  $(x, y)$  in some disk with center  $(a, b)$ .
- (ii) A function  $f$  of two variables has a local minimum at  $(a, b) \in D_f$  if  $f(x, y) \geq f(a, b)$ , for all points  $(x, y)$  in some disk with center  $(a, b)$ .

**Theorem:** If  $f(x, y)$  has a local maximum or minimum at  $(a, b) \in D_f$  and the first-order partial derivatives of  $f$  exist there, then  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$ .

**Second Derivatives Test:** Suppose the second partial derivatives of  $f(x, y)$  are continuous on a disk with center  $(a, b) \in D_f$ , and suppose that  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$  [so  $(a, b)$  is a critical point of  $f$ ].

Let  $D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$ .

- (1). If  $D > 0$  and  $f_{xx}(a, b) > 0$ , then  $f(a, b)$  is a local minimum.
- (2). If  $D > 0$  and  $f_{xx}(a, b) < 0$ , then  $f(a, b)$  is a local maximum.
- (3). If  $D < 0$ , then  $(a, b)$  is a saddle point.

**NOTES:**

- (1). If  $D = 0$ , then the test gives no information.

$$(2). D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}f_{yx} = f_{xx}f_{yy} - [f_{xy}]^2.$$

**Example (1):** Find the local maximum and minimum values and saddle points of  $f(x, y) = 6xy - 2x^3 + y^2$ .

Solution:

$$f_x(x, y) = 6y - 6x^2 \text{ and } f_y(x, y) = 6x + 2y.$$

$$f_x(x, y) = 0 \implies 6y - 6x^2 = 0 \implies y = x^2.$$

$$f_y(x, y) = 0 \implies 6x + 2y = 0 \implies y = -3x.$$

$$f_x(x, y) = f_y(x, y) \implies x^2 = -3x \implies x^2 + 3x = 0$$

$$\implies x(x + 3) = 0 \implies x = 0, x = -3 \implies y = 0, y = 9.$$

So, the critical points are  $(0, 0)$  and  $(-3, 9)$ .

$$f_{xx}(x, y) = -12x, f_{xy}(x, y) = 6 \text{ and } f_{yy}(x, y) = 2 .$$

First- At the point  $(0, 0)$  :

$$D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = (0)(2) - (6)^2 = -36 < 0.$$

Therefore,  $(0, 0)$  is a saddle point.

Second- At the point  $(-3, 9)$  :

$$D(-3, 9) = f_{xx}(-3, 9)f_{yy}(-3, 9) - [f_{xy}(-3, 9)]^2 = (36)(2) - (6)^2 = 36 > 0.$$

Since  $f_{xx}(-3, 9) = 36 > 0$ , then  $f$  attains a local minimum at  $(-3, 9)$ ,

and its value is  $f(-3, 9) = 6(-3)(9) - 2(-3)^3 + (-9)^2 = -162 + 54 + 81 = -27$ .

**Example (2):** Find the local maximum and minimum values and saddle points of  $f(x, y) = x^3 - y^3 - 3x + 3y + 5$  .

Solution:

$$f_x(x, y) = 3x^2 - 3 \text{ and } f_y(x, y) = -3y^2 + 3 .$$

$$f_x(x, y) = 0 \implies 3x^2 - 3 = 0 \implies x^2 - 1 = 0 \implies x = \pm 1 .$$

$$f_y(x, y) = 0 \implies -3y^2 + 3 = 0 \implies y^2 - 1 = 0 \implies y = \pm 1 .$$

So, the critical points are  $(1, 1)$ ,  $(1, -1)$ ,  $(-1, 1)$  and  $(-1, -1)$  .

$$f_{xx}(x, y) = 6x, f_{xy}(x, y) = 0 \text{ and } f_{yy}(x, y) = -6y .$$

First- At the point  $(1, 1)$  :

$$D(1, 1) = f_{xx}(1, 1)f_{yy}(1, 1) - [f_{xy}(1, 1)]^2 = (6)(-6) - (0)^2 = -36 < 0.$$

Therefore,  $(1, 1)$  is a saddle point.

Second- At the point  $(1, -1)$  :

$$D(1, -1) = f_{xx}(1, -1)f_{yy}(1, -1) - [f_{xy}(1, -1)]^2 = (6)(6) - (0)^2 = 36 > 0.$$

Since  $f_{xx}(1, -1) = 6 > 0$ , then  $f$  attains a local minimum at  $(1, -1)$ ,

and its value is  $f(1, -1) = 1 + 1 - 3 - 3 + 5 = 1$ .

Third- At the point  $(-1, 1)$  :

$$D(-1, 1) = f_{xx}(-1, 1)f_{yy}(-1, 1) - [f_{xy}(-1, 1)]^2 = (-6)(-6) - (0)^2 = 36 > 0.$$

Since  $f_{xx}(-1, 1) = -6 < 0$ , then  $f$  attains a local maximum at  $(-1, 1)$ ,

and its value is  $f(-1, 1) = -1 - 1 + 3 + 3 + 5 = 9$ .

Fourth- At the point  $(-1, -1)$  :

$$\begin{aligned} D(-1, -1) &= f_{xx}(-1, -1)f_{yy}(-1, -1) - [f_{xy}(-1, -1)]^2 \\ &= (-6)(6) - (0)^2 = -36 < 0. \end{aligned}$$

Therefore,  $(-1, -1)$  is a saddle point.

### 1.5.2 Absolute Maximum and Minimum Values

**Definition:** Let  $(a, b)$  be a point in the domain  $D$  of a function  $f$  of two variables. Then  $f(a, b)$  is the

- (1). absolute maximum value of  $f$  on  $D$  if  $f(a, b) \geq f(x, y)$  for all  $(x, y)$  in  $D$ .
- (2). absolute minimum value of  $f$  on  $D$  if  $f(a, b) \leq f(x, y)$  for all  $(x, y)$  in  $D$ .

**Theorem:** If  $f$  is continuous on a closed, bounded set  $D$  in  $\mathbb{R}^2$ , then  $f$  attains an absolute maximum value  $f(x_1, y_1)$  and an absolute minimum value  $f(x_2, y_2)$  at some points  $(x_1, y_1)$  and  $(x_2, y_2)$  in  $D$ .

**NOTES:** To find the absolute maximum and minimum values of a continuous function  $f(x, y)$  on a closed, bounded set  $D \subseteq \mathbb{R}^2$ :

1. Find the values of  $f$  at the critical points of  $f$  in the interior of  $D$ .
2. Find the extreme values of  $f$  on the boundary of  $D$ .
3. The largest of the values from steps (1) and (2) is the absolute maximum value of  $f$  on  $D$ , and the smallest of these values is the absolute minimum value of  $f$  on  $D$ .

**Example (3):** Find the absolute maximum and minimum values of the function  $f(x, y) = xy + 7$  on the plain region bounded by the graphs of the lines  $x = 0$ ,  $y = 0$  and  $y + x = 2$ .

Solution:

$$f_x(x, y) = y \text{ and } f_y(x, y) = x .$$

$$f_x(x, y) = 0 \implies y = 0 .$$

$$f_y(x, y) = 0 \implies x = 0 .$$

The critical point is  $(0, 0)$ .

Note that the critical point is not inside the given region.

Let  $L_1$  be the line  $x = 0$ :

$$\text{then } f(x, y) = f(0, y) = 7 .$$

$$f(x, y) = 7 \text{ for all } (x, y) \in L_1 .$$

Let  $L_2$  be the line  $y = 0$ :

$$\text{then } f(x, y) = f(x, 0) = 7 .$$

$$f(x, y) = 7 \text{ for all } (x, y) \in L_2 .$$

Let  $L_3$  be the line  $y + x = 2$ :

$$\text{then } y = -x + 2 \text{ where } 0 \leq x \leq 2 .$$

$$f(x, y) = f(x, -x + 2) \\ = x(-x + 2) + 7 = -x^2 + 2x + 7 .$$

$$f'(x) = -2x + 2 ,$$

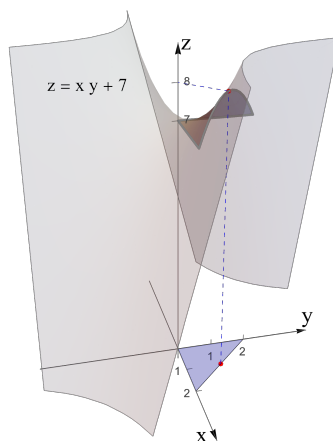
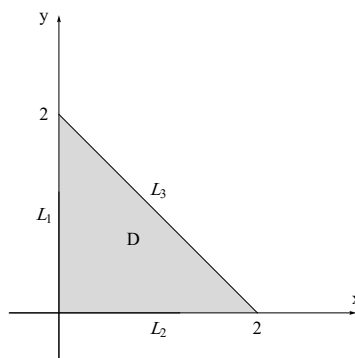
$$f'(x) = 0 \implies -2x + 2 = 0$$

$$\implies x = 1 .$$

$$\text{So, } y = -2 + 1 = 1 .$$

$$f(1, 1) = (1)(1) + 7 = 8 .$$

$$\text{Also, } f(0, 2) = 7 \text{ and } f(2, 0) = 7 .$$



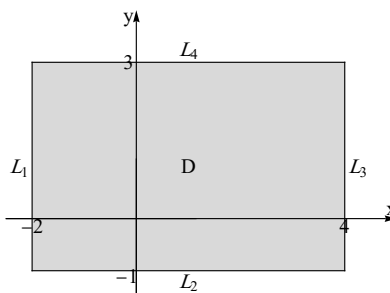
The absolute maximum is 8, and  $f$  takes it at  $(1, 1)$ .

The absolute minimum is 7, and  $f$  takes it at any point on  $L_1 \cup L_2$ .

**Example (4):** Find the absolute maximum and minimum values of the function  $f(x, y) = x^2 + 2xy + 3y^2$  on the closed and bounded region  $D = \{(x, y) \in \mathbb{R}^2 : -2 \leq x \leq 4, -1 \leq y \leq 3\}$ .

Solution:

$$\begin{aligned} f_x(x, y) &= 2x + 2y. \\ f_y(x, y) &= 2x + 6y. \\ f_x(x, y) = 0 &\implies x = -y. \\ f_y(x, y) = 0 &\implies x = -3y. \\ -y = -3y &\implies y = 0 \implies x = 0. \\ \text{The critical point is } (0, 0). \\ f(0, 0) &= (0)^2 + 2(0)(0) + 3(0)^3 = 0. \end{aligned}$$



Let  $L_1$  be the line between  $(-2, -1)$  and  $(-2, 3)$ .

On  $L_1$  :  $x = -2$  and  $-1 \leq y \leq 3$ .

$$f(x, y) = f(-2, y) = 4 - 4y + 3y^2 \implies f'(y) = -4 + 6y.$$

$$f'(y) = 0 \implies 6y = 4 \implies y = \frac{2}{3}.$$

Note that  $\left(-2, \frac{2}{3}\right) \in L_1$ .

$$f\left(-2, \frac{2}{3}\right) = (-2)^2 + 2(-2)\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)^2 = 4 - \frac{8}{3} + \frac{4}{3} = \frac{8}{3}.$$

Let  $L_2$  be the line between  $(-2, -1)$  and  $(4, -1)$ .

On  $L_2$  :  $y = -1$  and  $-2 \leq x \leq 4$ .

$$f(x, y) = f(x, -1) = x^2 - 2x + 3 \implies f'(x) = 2x - 2.$$

$$f'(x) = 0 \implies 2x = 2 \implies x = 1.$$

Note that  $(1, -1) \in L_2$ .

$$f(1, -1) = (1)^2 + 2(1)(-1) + 3(-1)^2 = 1 - 2 + 3 = 2.$$

Let  $L_3$  be the line between  $(4, -1)$  and  $(4, 3)$ .

On  $L_3$  :  $x = 4$  and  $-1 \leq y \leq 3$ .

$$f(x, y) = f(4, y) = 16 + 8y + 3y^2 \implies f'(y) = 8 + 6y.$$

$$f'(y) = 0 \implies 6y = -8 \implies y = -\frac{4}{3}.$$

Note that  $\left(4, -\frac{4}{3}\right) \notin L_3$ .

Let  $L_4$  be the line between  $(-2, 3)$  and  $(4, 3)$ .

On  $L_4$  :  $y = 3$  and  $-2 \leq x \leq 4$ .

$$f(x, y) = f(x, 3) = x^2 + 6x + 27 \implies f'(x) = 2x + 6.$$

$$f'(x) = 0 \implies 2x = -6 \implies x = -3.$$

Note that  $(-3, 3) \notin L_4$ .

Evaluating  $f(x, y)$  at the four corners of  $D$  :

$$f(-2, -1) = (-2)^2 + 2(-2)(-1) + 3(-1)^2 = 4 + 4 + 3 = 11.$$

$$f(-2, 3) = (-2)^2 + 2(-2)(3) + 3(3)^2 = 4 - 12 + 27 = 19.$$

$$f(4, -1) = (4)^2 + 2(4)(-1) + 3(-1)^2 = 16 - 8 + 3 = 11.$$

$$f(4, 3) = (4)^2 + 2(4)(3) + 3(3)^2 = 16 + 24 + 27 = 67.$$

The absolute maximum is 67, and  $f$  takes it at  $(4, 3)$ .

The absolute minimum is 0, and  $f$  takes it at  $(0, 0)$ .

**Example (5):** Find the absolute maximum and minimum values of the function  $f(x, y) = x^2 + y^2 - 2x + 2$  on the closed region with vertices  $(0, 0)$ ,  $(2, 1)$  and  $(2, -2)$ .

Solution:

$$f_x(x, y) = 2x - 2 \text{ and } f_y(x, y) = 2y .$$

$$f_x(x, y) = 0 \implies x = 1 .$$

$$f_y(x, y) = 0 \implies y = 0 .$$

The critical point is  $(1, 0)$ .

$$f(1, 0) = 1 + 0 - 2 + 2 = 1 .$$

Let  $L_1$  be the line passing through  $(0, 0)$  and  $(2, 1)$  then  $y = \frac{x}{2}$ .

$$f\left(x, \frac{x}{2}\right) = x^2 + \frac{x^2}{4} - 2x + 2 .$$

$$f(x) = \frac{5}{4}x^2 - 2x + 2 .$$

$$f'(x) = \frac{5}{2}x - 2 .$$

$$f'(x) = 0 \implies x = \frac{4}{5} \text{ and } y = \frac{2}{5} .$$

$$\begin{aligned} f\left(\frac{4}{5}, \frac{2}{5}\right) &= \frac{16}{25} + \frac{4}{25} - \frac{8}{5} + 2 \\ &= \frac{16}{25} + \frac{4}{25} - \frac{40}{25} + \frac{50}{25} = \frac{30}{25} = \frac{6}{5} . \end{aligned}$$

Let  $L_2$  be the line passing through  $(0, 0)$  and  $(2, -2)$  then  $y = -x$ .

$$f(x, -x) = x^2 + x^2 - 2x + 2$$

$$f(x) = 2x^2 - 2x + 2 .$$

$$f'(x) = 4x - 2 .$$

$$f'(x) = 0 \implies x = \frac{1}{2} \text{ and } y = -\frac{1}{2} .$$

$$f\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} - 1 + 2 = \frac{3}{2} .$$

Let  $L_3$  be the line passing through  $(2, -2)$  and  $(2, 1)$  then  $x = 2$ .

$$f(2, y) = 4 + y^2 - 4 + 2 = y^2 + 2, \text{ so } f(y) = y^2 + 2 \implies f'(y) = 2y .$$

$$f'(y) = 0 \implies y = 0 \text{ and } x = 2 .$$

$$f(2, 0) = 4 + 0 - 4 + 2 = 2 .$$

Evaluating  $f(x, y)$  at the three corners of  $D$  :

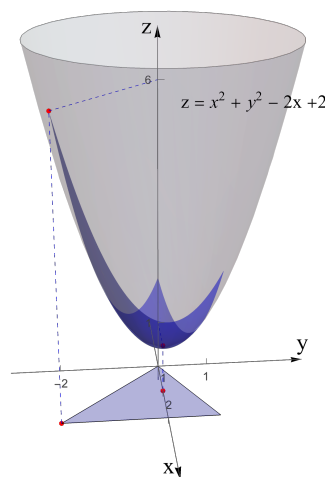
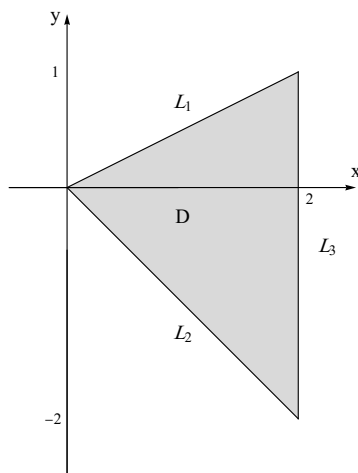
$$f(0, 0) = 0 + 0 - 0 + 2 = 2 .$$

$$f(2, 1) = 4 + 1 - 4 + 2 = 3 .$$

$$f(2, -2) = 4 + 4 - 4 + 2 = 6 .$$

The absolute maximum is 6, and  $f$  takes it at  $(2, -2)$ .

The absolute minimum is 1, and  $f$  takes it at  $(1, 0)$ .



**1.5.3 EXERCISES**

1. Find the local maximum and minimum values and saddle point(s) of the functions:

(a).  $f(x, y) = x^2 + xy + y^2 + y$  .

(b).  $f(x, y) = x^3 + y^3 + 3xy$  .

(c).  $f(x, y) = 2 - x^4 + 2x^2 - y^2$  .

(d).  $f(x, y) = x^4 - 2x^2 + y^3 - 3y$  .

2. Find the absolute maximum and minimum values of  $f(x, y) = x^2 + y^2 - 2x$  on the closed region with vertices  $(2, 0)$ ,  $(0, 2)$  and  $(0, -2)$ .

- (\*) Find the local maximum and minimum values and saddle point(s) of the functions:

(a).  $f(x, y) = 4x^3 - 2x^2y + y^2$  .

(b).  $f(x, y) = 2x^3 - 3x^2 + 3y^2 - 6xy + 2$  .

(c).  $f(x, y) = x^4 + y^3 + 32x - 3y$  .

## 1.6 Lagrange Multipliers

### 1.6.1 Lagrange Multipliers (One Constraint)

To find the extreme values of  $f(x, y)$  subject to a constraint  $g(x, y) = k$ .

Let  $z = f(x, y)$  be the gray surface, and  $g(x, y)$  is the blue curve representing the constraint in the  $xy$ -plane.

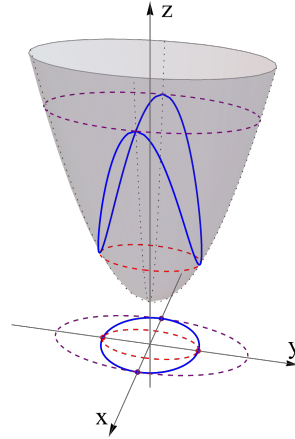
Note that the level curves  $f(x, y)$  touches  $g(x, y)$  at the points where  $f(x, y)$  have minimum and maximum values.

This means that  $\nabla f(x, y)$  is parallel to  $\nabla g(x, y)$  at these point.

Find these points by solving the equation:  $\nabla f(x, y) = \lambda \nabla g(x, y)$ ,

where  $\lambda \in \mathbb{R}$  and  $\nabla g(x, y) \neq 0$ .

Evaluate  $f(x, y)$  at these points, The largest is the maximum value of  $f$  and the smallest is the minimum value of  $f$ .



**Example (1):** Find the extreme values of the function  $f(x, y) = 1 + xy$  on the circle  $x^2 + y^2 = 1$ .

Solution:

$$f(x, y) = 1 + xy \text{ and } g(x, y) = x^2 + y^2 - 1.$$

$$\nabla f(x, y) = \lambda \nabla g(x, y) \implies \langle y, x \rangle = \lambda \langle 2x, 2y \rangle = \langle 2\lambda x, 2\lambda y \rangle$$

$$\implies \begin{cases} y = 2\lambda x \\ x = 2\lambda y \end{cases} \implies x = 2\lambda(2\lambda x) = 4\lambda^2 x \implies x(1 - 4\lambda^2) = 0$$

$$\implies x = 0, \lambda = \pm \frac{1}{2}.$$

If  $x = 0$  then  $y = 2\lambda(0) = 0$ , but  $(0, 0)$  does not lie on the unit circle, so  $(0, 0)$  is excluded.

$$\text{If } \lambda = \pm \frac{1}{2} \implies y = 2\left(\pm \frac{1}{2}\right)x \implies y = \pm x.$$

$$\text{From the constraint : } x^2 + y^2 = 1 \implies 2x^2 = 1 \implies x = \pm \frac{1}{\sqrt{2}}.$$

$$\text{This means } \frac{1}{2} + y^2 = 1 \implies y = \pm \frac{1}{\sqrt{2}}.$$

So, There are 4 points  $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$  and  $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ .

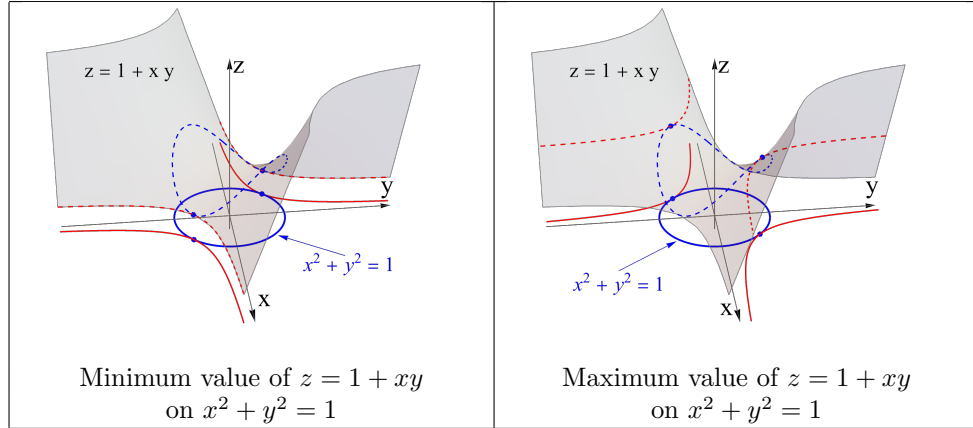
$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 1 + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = 1 + \frac{1}{2} = \frac{3}{2}.$$

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 1 + \frac{1}{\sqrt{2}} \frac{-1}{\sqrt{2}} = 1 - \frac{1}{2} = \frac{1}{2}.$$

$$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 1 + \frac{-1}{\sqrt{2}} \frac{1}{\sqrt{2}} = 1 - \frac{1}{2} = \frac{1}{2}.$$

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 1 + \frac{-1}{\sqrt{2}} \frac{-1}{\sqrt{2}} = 1 + \frac{1}{2} = \frac{3}{2}.$$

The maximum value is  $\frac{3}{2}$ , and  $f$  takes it at  $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$  and  $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ .  
 The minimum value is  $\frac{1}{2}$ , and  $f$  takes it at  $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$  and  $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ .



**Note:** to find the extreme values of  $f(x, y, z)$  subject to a constraint  $g(x, y, z) = k$ , where  $k \in \mathbb{R}$ .

- (1). Find the points that satisfy the equation  $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$   
 $f_x(x, y, z) = \lambda g_x(x, y, z)$ ,  $f_y(x, y, z) = \lambda g_y(x, y, z)$  and  $f_z(x, y, z) = \lambda g_z(x, y, z)$ .
- (2). Evaluate  $f(x, y, z)$  at these points, The largest is the maximum value of  $f$  and the smallest is the minimum value of  $f$ .

**Example (2):** Find the points on the sphere  $x^2 + y^2 + z^2 = 1$  that are closest to and farthest from the point  $(2, 2, 2)$ .

Solution:

Let  $f(x, y, z)$  be the function of the square of the distance between any point in the sphere and the point  $(2, 2, 2)$ , then  $f(x, y, z) = (x-2)^2 + (y-2)^2 + (z-2)^2$ .  
 Let  $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$  be the constraint.

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \implies \langle 2(x-2), 2(y-2), 2(z-2) \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

$$\begin{cases} 2(x-2) = 2\lambda x \\ 2(y-2) = 2\lambda y \\ 2(z-2) = 2\lambda z \end{cases} \implies \begin{cases} x-2 = \lambda x \\ y-2 = \lambda y \\ z-2 = \lambda z \end{cases} \implies \begin{cases} x(1-\lambda) = 2 \\ y(1-\lambda) = 2 \\ z(1-\lambda) = 2 \end{cases}$$

If  $\lambda = 1$  then  $x-2 = x \implies -2 = 0$ , so  $\lambda \neq 1$ .

Therefore,  $x(1-\lambda) = y(1-\lambda) = z(1-\lambda) \implies x = y = z$ .

$$\text{So, } x^2 + y^2 + z^2 = 1 \implies 3x^2 = 1 \implies x = \pm \frac{1}{\sqrt{3}}$$

Hence,  $x = y = z = \pm \frac{1}{\sqrt{3}}$ .

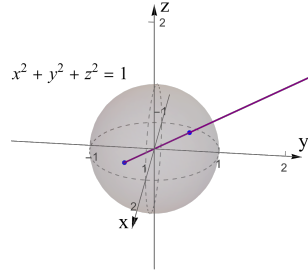
The required points are  $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$  and  $\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$ .

$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}} - 2\right)^2 + \left(\frac{1}{\sqrt{3}} - 2\right)^2 + \left(\frac{1}{\sqrt{3}} - 2\right)^2 = 3\left(2 - \frac{1}{\sqrt{3}}\right)^2.$$

$$f\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) = \left(\frac{-1}{\sqrt{3}} - 2\right)^2 + \left(\frac{-1}{\sqrt{3}} - 2\right)^2 + \left(\frac{-1}{\sqrt{3}} - 2\right)^2 = 3\left(2 + \frac{1}{\sqrt{3}}\right)^2.$$

The point  $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$   
is the closest to  $(2, 2, 2)$ .

The point  $\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$   
is the farthest to  $(2, 2, 2)$ .



**Example (3):** Find the maximum volume of a rectangular box without a lid, where its surface area equals  $12 \text{ cm}^2$ .

Solution:

Suppose the sides of the rectangular box are  $x$ ,  $y$  and  $z$ .

The volume of the rectangular box is  $V(x, y, z) = xyz$ , subject to the constraint  $2xz + 2yz + xy = 12$ .

The constraint is  $g(x, y, z) = 2xz + 2yz + xy - 12 = 0$ .

$$\nabla V(x, y, z) = \lambda \nabla g(x, y, z) \implies \langle yz, xz, xy \rangle = \lambda \langle 2z + y, 2z + x, 2x + 2z \rangle$$

$$\begin{cases} yz = \lambda(2z + y) \\ xz = \lambda(2z + x) \\ xy = \lambda(2x + 2y) \end{cases} \implies \begin{cases} xyz = \lambda(2xz + xy) & \longrightarrow (1) \\ xyz = \lambda(2yz + xy) & \longrightarrow (2) \\ xyz = \lambda(2xz + 2yz) & \longrightarrow (3) \end{cases}$$

If  $\lambda = 0$  then  $V(x, y, z) = 0$ .

If  $x = 0$ ,  $y = 0$  or  $z = 0$ , then  $V(x, y, z) = 0$ .

If  $\lambda \neq 0$ , From equations (1) and (2) :

$$\lambda(2xz + xy) = \lambda(2yz + xy) \implies 2xz + xy = 2yz + xy \implies 2xz = 2yz \implies x = y.$$

From equations (2) and (3), and  $x = y$  :

$$\begin{aligned} \lambda(2yz + xy) &= \lambda(2xz + 2yz) \implies 2xz + x^2 = 2xz + 2xz \\ \implies x^2 &= 2xz \implies x^2 - 2xz = 0 \implies x(x - 2z) = 0 \implies x = 2z. \end{aligned}$$

From the equation of the constraint and  $x = y = 2z$  :

$$4z^2 + 4z^2 + 4z^2 = 12 \implies z^2 = 1 \implies z = 1 \text{ and } x = y = 2z = 2.$$

The sides of the rectangular box are 2, 2 and 1, and its maximum volume is  $4 \text{ cm}^3$ .

**1.6.2 EXERCISES**

1. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint:

(a).  $f(x, y) = x^2 - y^2$  ,  $x^2 + y^2 = 1$  .

(b).  $f(x, y) = xye^{-x^2-y^2}$  ,  $2x - y = 0$  .

(c).  $f(x, y, z) = xy^2z$  ,  $x^2 + y^2 + z^2 = 4$  .

(d).  $f(x, y, z) = x^4 + y^4 + z^4$  ,  $x^2 + y^2 + z^2 = 1$  .

2. Find the extreme values of  $f$  on the region described by the inequality:

(a).  $f(x, y) = x^2 + y^2 + 4x - 4y$  ,  $x^2 + y^2 \leq 9$  .

(b).  $f(x, y) = e^{-xy}$  ,  $x^2 + 4y^2 \leq 1$  .

3. Show that the problem of finding the minimum value of  $f$  subject to the given constraint can be solved using Lagrange multipliers, but  $f$  does not have a maximum value with that constraint:

(a).  $f(x, y) = x^2 + y^2$  ,  $xy = 1$  .

(b).  $f(x, y, z) = x^2 + 2y^2 + 3z^2$  ,  $x + 2y + 3z = 10$  .

## Chapter 2

# Multiple Integrals

### 2.1 Double Integrals over Rectangles

#### 2.1.1 Iterated Integrals

Suppose that  $f(x, y)$  is integrable on the rectangle  $R = [a, b] \times [c, d]$ , then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_a^b \left( \int_c^d f(x, y) \, dy \right) dx.$$

**Example (1):** Evaluate the integral:

(a).  $\int_0^2 \int_1^2 x^2 y \, dy \, dx$  , (b).  $\int_1^2 \int_0^2 x^2 y \, dx \, dy$ .

Solution:

(a).  $\int_0^2 \int_1^2 x^2 y \, dy \, dx = \int_0^2 \left( \int_1^2 x^2 y \, dy \right) dx = \int_0^2 \left( x^2 \int_1^2 y \, dy \right) dx$   
 $= \int_0^2 x^2 \left[ \frac{y^2}{2} \right]_1^2 dx = \int_0^2 x^2 \left[ \frac{4}{2} - \frac{1}{2} \right] dx$   
 $= \frac{3}{2} \int_0^2 x^2 \, dx = \frac{3}{2} \left[ \frac{x^3}{3} \right]_0^2 = \frac{3}{2} \left[ \frac{8}{3} - 0 \right] = 4.$   
(b).  $\int_1^2 \int_0^2 x^2 y \, dx \, dy = \int_1^2 \left( \int_0^2 x^2 y \, dx \right) dy = \int_1^2 \left( y \int_0^2 x^2 \, dx \right) dy$   
 $= \int_1^2 y \left[ \frac{x^3}{3} \right]_0^2 dy = \int_1^2 y \left[ \frac{8}{3} - 0 \right] dy = \frac{8}{3} \int_1^2 y \, dy$   
 $= \frac{8}{3} \left[ \frac{y^2}{2} \right]_1^2 = \frac{8}{3} \left[ \frac{4}{2} - \frac{1}{2} \right] = \frac{8}{3} \cdot \frac{3}{2} = 4.$

**Fubini's Theorem:** If  $f$  is continuous on the rectangle  $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$ ,

Then  $\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$ .

**Example (2):** Evaluate the integral  $\iint_R y \cos(xy) \, dA$  ,

where  $R = [0, 1] \times \left[0, \frac{\pi}{2}\right]$ .

Solution: Using Fubini's Theorem.

$$\begin{aligned}\iint_R y \cos(xy) \, dA &= \int_0^{\frac{\pi}{2}} \int_0^1 y \cos(xy) \, dx dy = \int_0^{\frac{\pi}{2}} \left( \int_0^1 y \cos(xy) \, dx \right) dy \\ &= \int_0^{\frac{\pi}{2}} [\sin(xy)]_0^1 dy = \int_0^{\frac{\pi}{2}} [\sin y - \sin(0)] dy = \int_0^{\frac{\pi}{2}} \sin y \, dy \\ &= [-\cos y]_0^{\frac{\pi}{2}} = -\cos\left(\frac{\pi}{2}\right) - (-\cos(0)) = 0 + 1 = 1 .\end{aligned}$$

Note : Solving  $\iint_R y \cos(x, y) \, dA = \int_0^1 \int_0^{\frac{\pi}{2}} y \cos(xy) \, dy dx$  is hard, it needs integration by parts.

### 2.1.2 Volume

If  $f(x, y) \geq 0$ , then the volume  $V$  of the solid that lies above the rectangle  $R = [a, b] \times [c, d]$  and below the surface  $z = f(x, y)$  is

$$V = \iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy dx = \int_c^d \int_a^b f(x, y) \, dx dy .$$

**Example (3):** Find the volume of the solid  $S$  that is bounded by  $z = x^2 + y^2 + 1$ , the planes  $x = 1$  and  $y = 3$ , and the three coordinate planes.

Solution:

Note that  $S$  is the solid that lies under the surface  $z = x^2 + y^2 + 1$  and above the square  $R = [0, 1] \times [0, 3]$ .

$$\begin{aligned}V &= \iint_R (x^2 + y^2 + 1) \, dA = \int_0^1 \int_0^3 (x^2 + y^2 + 1) \, dy dx \\ &= \int_0^1 \left[ x^2 y + \frac{y^3}{3} + y \right]_0^3 dx = \int_0^1 [(3x^2 + 9 + 3) - (0 + 0 + 0)] dx \\ &= \int_0^1 (3x^2 + 12) dx = [x^3 + 12x]_0^1 = (1 + 12) - (0 + 0) = 13 .\end{aligned}$$

**Corollary:** If  $f(x, y) = g(x)h(y)$  is continuous on the rectangle  $R = [a, b] \times [c, d]$ ,

$$\text{then } \iint_R f(x, y) \, dA = \iint_R g(x)h(y) \, dA = \int_a^b g(x) \, dx \int_c^d h(y) \, dy .$$

**Example (4):** Evaluate  $\iint_R x \cos y \, dA$ , where  $R = [0, 2] \times [0, \frac{\pi}{2}]$ .

Solution:

$$\begin{aligned}\iint_R x \cos y \, dA &= \int_0^2 \int_0^{\frac{\pi}{2}} x \cos y \, dy dx = \left( \int_0^2 x \, dx \right) \left( \int_0^{\frac{\pi}{2}} \cos y \, dy \right) \\ &= \left[ \frac{x^2}{2} \right]_0^2 [\sin y]_0^{\frac{\pi}{2}} = [2 - 0][1 - 0] = 2 .\end{aligned}$$

**Example (5):** Evaluate  $\int_0^1 \int_1^2 (e^y e^{-x} + y^2) \, dx dy$ .

Solution:

$$\int_0^1 \int_1^2 (e^y e^{-x} + y^2) \, dx dy = \int_0^1 \int_1^2 e^y e^{-x} \, dx dy + \int_0^1 \int_1^2 y^2 \, dx dy$$

$$\begin{aligned}
&= \left( \int_1^2 e^{-x} dx \right) \left( \int_0^1 e^y dy \right) + \left( \int_1^2 dx \right) \left( \int_0^1 y^2 dy \right) \\
&= [-e^{-x}]_1^2 [e^y]_0^1 + [x]_1^2 \left[ \frac{y^3}{3} \right]_0^1 = (-e^{-2} - (-e^{-1})) (e - 1) + (2 - 1) \left( \frac{1}{3} - 0 \right) \\
&= (e - 1) \left( \frac{1}{e} - \frac{1}{e^2} \right) + \frac{1}{3} .
\end{aligned}$$

### 2.1.3 Average Value

If  $f(x, y)$  is defined on a rectangle  $R$  then its average value is  $f_{avg} = \frac{1}{A(R)} \iint_R f(x, y) dA$ , where  $A(R)$  is the area of the rectangle  $R$ .

**Example (6):** Evaluate  $f_{avg}$  of  $f(x, y) = x \cos y$  on  $R = [0, 2] \times \left[0, \frac{\pi}{2}\right]$ .

Solution:

$$f_{avg} = \frac{1}{A(R)} \iint_R x \cos y dA = \frac{2}{(2 - 0) \left(\frac{\pi}{2} - 0\right)} = \frac{2}{\pi} .$$

## 2.1.4 EXERCISES

1. Calculate the iterated integrals :

$$(a). \int_1^4 \int_0^2 (6x^2y - 2x) \, dy \, dx .$$

$$(b). \int_{-3}^1 \int_1^2 (x^2 + y^{-2}) \, dy \, dx .$$

$$(c). \int_{-3}^3 \int_0^{\frac{\pi}{2}} (y + y^2 \cos x) \, dx \, dy .$$

$$(d). \int_1^3 \int_1^5 \frac{\ln y}{xy} \, dy \, dx .$$

$$(e). \int_0^1 \int_0^1 xy\sqrt{x^2 + y^2} \, dy \, dx .$$

$$(f). \int_0^1 \int_0^1 v(u + v^2)^4 \, du \, dv .$$

2. Calculate the double integrals :

$$(a). \iint_R x \sec^2 y \, dA, \text{ where } R = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq \frac{\pi}{4}\} .$$

$$(b). \iint_R \frac{xy^2}{x^2 + 1} \, dA, \text{ where } R = \{(x, y) \mid 0 \leq x \leq 1, -3 \leq y \leq 3\} .$$

$$(c). \iint_R \frac{1}{1 + x + y} \, dA, \text{ where } R = [1, 3] \times [1, 2] .$$

3. Find the volume of the solid that lies under the plane  $4x + 6y - 2z + 15 = 0$  and above the rectangle  $R = \{(x, y) \mid -1 \leq x \leq 2, -1 \leq y \leq 1\}$  .
4. Find the volume of the solid that lies under  $z = 3y^2 - x^2 + 2$  and above the rectangle  $R = [-1, 1] \times [1, 2]$  .

## 2.2 Double Integrals over General Regions

### 2.2.1 General Regions

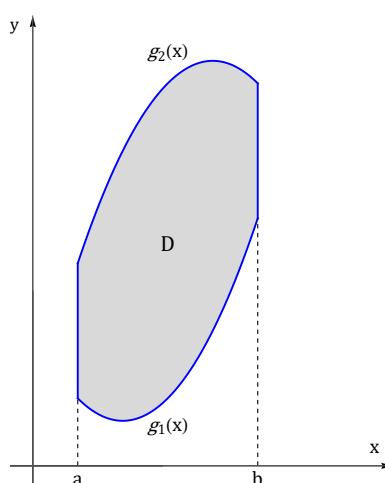
First - Regions of Type I:

Let  $D$  be the region

$$\{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

If  $f$  is continuous on  $D$ , then

$$\begin{aligned} & \iint_D f(x, y) \, dA \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx \\ &= \int_a^b \left( \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \right) dx \end{aligned}$$



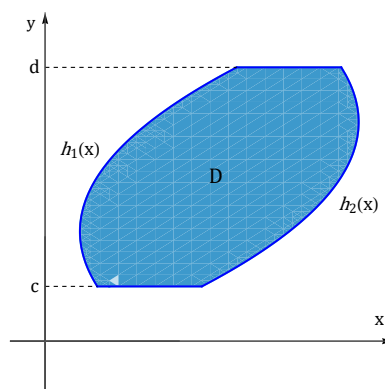
Second - Regions of Type II:

Let  $D$  be the region

$$\{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

If  $f$  is continuous on  $D$ , then

$$\begin{aligned} & \iint_D f(x, y) \, dA \\ &= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy \\ &= \int_c^d \left( \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \right) dy \end{aligned}$$



**Example (1):** Evaluate  $\iint_D 2xy \, dA$ , where  $D$  is the region bounded by the graphs of  $y = 2x^2$  and  $y = x^2 + 1$ .  
Solution:

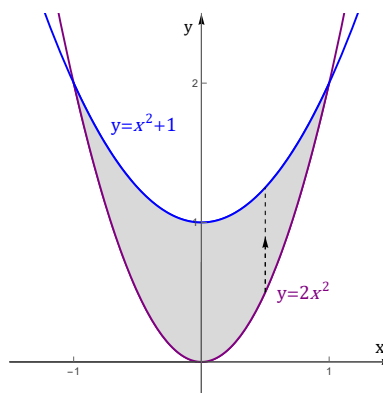
Points of intersection:

$$2x^2 = x^2 + 1 \implies x^2 = 1$$

$$\implies x = \pm 1$$

So,  $D$  is the region where  $-1 \leq x \leq 1$   
and  $2x^2 \leq y \leq x^2 + 1$

$$\begin{aligned} \iint_D 2xy \, dA &= \int_{-1}^1 \int_{2x^2}^{x^2+1} 2xy \, dy \, dx \\ &= \int_{-1}^1 x [y^2]_{2x^2}^{x^2+1} \, dx \\ &= \int_{-1}^1 x [(x^2+1)^2 - (2x^2)^2] \, dx \\ &= \int_{-1}^1 x (x^4 + 2x^2 + 1 - 4x^4) \, dx \\ &= \int_{-1}^1 x (-3x^4 + 2x^2 + 1) \, dx = \int_{-1}^1 (-3x^5 + 2x^3 + x) \, dx \\ &= \left[ -\frac{x^6}{2} + \frac{x^4}{2} + \frac{x^2}{2} \right]_{-1}^1 \\ &= \left( -\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) - \left( -\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) = 0. \end{aligned}$$



**Example (2):** Evaluate  $\iint_D (x^2 + y^2) \, dA$ , where  $D$  is the region bounded by the graphs of  $y = x^2$  and  $y = x$ .

Solution:

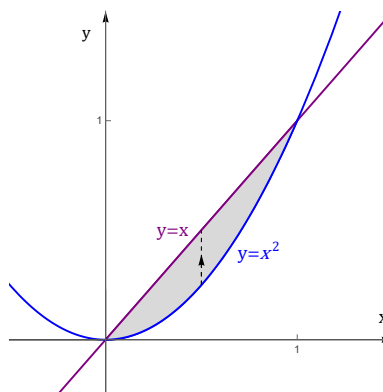
Points of intersection:

$$x^2 = x \implies x^2 - x = 0$$

$$\implies x(x-1) = 0 \implies x = 0, x = 1$$

So,  $D$  is the region where  $0 \leq x \leq 1$   
and  $x^2 \leq y \leq x$

$$\begin{aligned} \iint_D (x^2 + y^2) \, dA &= \int_0^1 \int_{x^2}^x (x^2 + y^2) \, dy \, dx \\ &= \int_0^1 \left[ x^2 y + \frac{y^3}{3} \right]_{x^2}^x \, dx \\ &= \int_0^1 \left[ \left( x^3 + \frac{x^3}{3} \right) - \left( x^4 - \frac{x^6}{3} \right) \right] \, dx \\ &= \int_0^1 \left( -\frac{x^6}{3} - x^4 + \frac{4x^3}{3} \right) \, dx = \left[ -\frac{x^7}{21} - \frac{x^5}{5} + \frac{x^4}{3} \right]_0^1 \\ &= \left( -\frac{1}{21} - \frac{1}{5} + \frac{1}{3} \right) - (0 + 0 + 0) = \frac{-5 - 21 + 35}{105} = \frac{9}{105} = \frac{3}{35}. \end{aligned}$$



**Example (3):** Evaluate  $\iint_D xy \, dA$ , where  $D$  is the region bounded by the graphs of  $x = y^2$  and  $x = y + 2$ .

Solution:

Points of intersection:

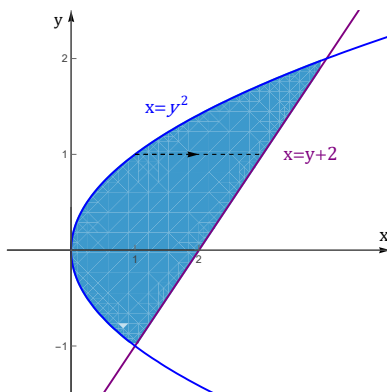
$$y^2 = y + 2 \implies y^2 - y - 2 = 0$$

$$\implies (y - 2)(y + 1) = 0$$

$$\implies y = -1, y = 2$$

So,  $D$  is the region where  $-1 \leq y \leq 2$   
and  $y^2 \leq x \leq y + 2$

$$\begin{aligned} \iint_D xy \, dA &= \int_{-1}^2 \int_{y^2}^{y+2} xy \, dx \, dy \\ &= \int_{-1}^2 y \left[ \frac{x^2}{2} \right]_{y^2}^{y+2} dy \\ &= \frac{1}{2} \int_{-1}^2 y [(y+2)^2 - (y^2)^2] dy \\ &= \frac{1}{2} \int_{-1}^2 y (y^2 + 4y + 4 - y^4) dy \\ &= \frac{1}{2} \int_{-1}^2 (-y^5 + y^3 + 4y^2 + 4y) dy = \frac{1}{2} \left[ -\frac{y^6}{6} + \frac{y^4}{4} + \frac{4y^3}{3} + 2y^2 \right]_{-1}^2 \\ &= \frac{1}{2} \left[ \left( -\frac{64}{6} + \frac{16}{4} + \frac{32}{3} + 8 \right) - \left( -\frac{1}{6} + \frac{1}{4} - \frac{4}{3} + 2 \right) \right] = \frac{45}{8}. \end{aligned}$$



### 2.2.2 Changing the Order of Integration

**Example (4):** Change the order of integration to evaluate  $\int_0^2 \int_x^2 e^{5+y^2} dy dx$ .

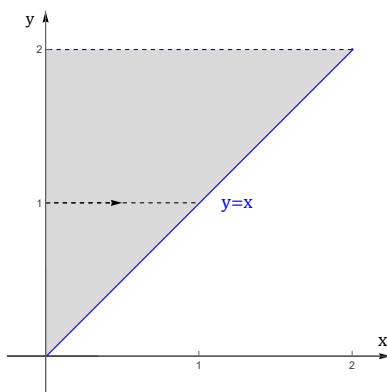
Solution:

$D$  is the region where  $0 \leq x \leq 2$  and

$x \leq y \leq 2$ .

Or,  $0 \leq y \leq 2$  and  $0 \leq x \leq y$ .

$$\begin{aligned} &\int_0^2 \int_x^2 e^{5+y^2} dy dx \\ &= \int_0^2 \int_0^y e^{5+y^2} dx dy \\ &= \int_0^2 e^{5+y^2} [x]_0^y dy = \int_0^2 ye^{5+y^2} dy \\ &= \frac{1}{2} \int_0^2 e^{5+y^2} (2y) dy \\ &= \frac{1}{2} [e^{5+y^2}]_0^2 = \frac{1}{2} (e^9 - e^5). \end{aligned}$$



**Example (5):** Change the order of integration to evaluate  $\int_0^2 \int_{y^2}^4 \cos\left(x^{\frac{3}{2}}\right) dx dy$ .

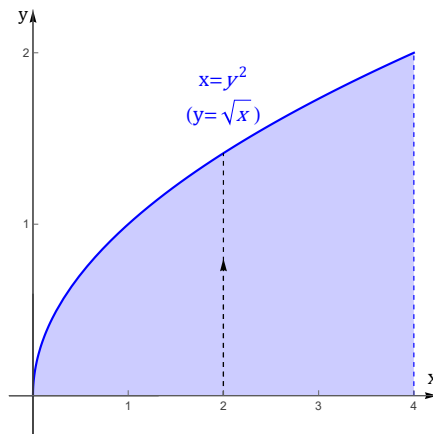
Solution:

$D$  is the region where  $0 \leq y \leq 2$  and  $y^2 \leq x \leq 4$ .

Or,  $0 \leq x \leq 4$  and  $0 \leq y \leq \sqrt{x}$ .

$$\begin{aligned} & \int_0^2 \int_{y^2}^4 \cos\left(x^{\frac{3}{2}}\right) dx dy \\ &= \int_0^4 \int_0^{\sqrt{x}} \cos\left(x^{\frac{3}{2}}\right) dy dx \\ &= \int_0^4 \cos\left(x^{\frac{3}{2}}\right) [y]_0^{\sqrt{x}} dx \\ &= \frac{2}{3} \int_0^4 \cos\left(x^{\frac{3}{2}}\right) \left(\frac{3}{2}x^{\frac{1}{2}}\right) dx \end{aligned}$$

$$= \frac{2}{3} \left[ \sin\left(x^{\frac{3}{2}}\right) \right]_0^4 = \frac{2}{3} [\sin(8) - \sin(0)] = \frac{2}{3} \sin(8) .$$



**Example (6):** Change the order of integration to evaluate  $\int_0^1 \int_{\sqrt{y}}^1 \sin(x^3) dx dy$ .

Solution:

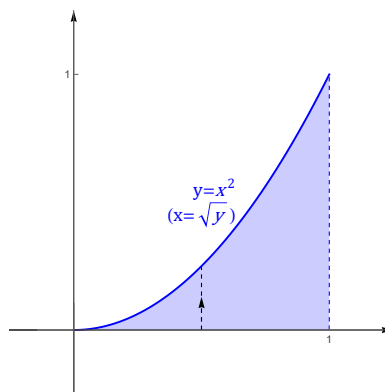
$D$  is the region where  $0 \leq y \leq 1$  and

$\sqrt{y} \leq x \leq 1$ .

Or,  $0 \leq x \leq 1$  and  $0 \leq y \leq x^2$ .

$$\begin{aligned} & \int_0^1 \int_{\sqrt{y}}^1 \sin(x^3) dx dy \\ &= \int_0^1 \int_0^{x^2} \sin(x^3) dy dx \\ &= \int_0^1 \sin(x^3) [y]_0^{x^2} dx \\ &= \int_0^1 \sin(x^3) x^2 dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{3} \int_0^1 \sin(x^3) (3x^2) dx = \frac{1}{3} [-\cos(x^3)]_0^1 . \\ &= \frac{1}{3} [-\cos(1) - (-\cos(0))] = \frac{1 - \cos(1)}{3} . \end{aligned}$$



### 2.2.3 Properties of Double Integrals

If  $f(x, y)$  and  $g(x, y)$  are both integrable on  $D \subseteq \mathbb{R}^2$ , then

- (1).  $\iint_D [f(x, y) + g(x, y)] dA = \iint_D f(x, y) dA + \iint_D g(x, y) dA$ .
- (2).  $\iint_D k f(x, y) dA = k \iint_D f(x, y) dA$ , where  $k \in \mathbb{R}$ .
- (3). If  $f(x, y) \geq g(x, y)$  on  $D$ , then  $\iint_D f(x, y) dA \geq \iint_D g(x, y) dA$ .

(4). If  $D = D_1 \cup D_2$ , where  $D_1 \cap D_2 = \phi$ , then

$$\iint_D f(x, y) \, dA = \iint_{D_1} f(x, y) \, dA + \iint_{D_2} f(x, y) \, dA.$$

(5).  $\iint_D 1 \, dA = A(D)$ , where  $A(D)$  is the area of the region  $D$ .

(6). If  $m \leq f(x, y) \leq M$  on  $D$ , then  $m \, A(D) \leq \iint_D f(x, y) \, dA \leq M \, A(D)$ .

## 2.2.4 EXERCISES

1. Evaluate the iterated integrals :

$$(a). \int_1^5 \int_0^x (8x - 2y) \, dy \, dx \quad (b). \int_0^2 \int_0^{y^2} x^2 y \, dx \, dy$$

$$(c). \int_0^1 \int_0^{e^x} \sqrt{1 + e^x} \, dy \, dx$$

2. Evaluate
- $\iint_D 2y \, dA$
- , where
- $D$
- is the region bounded by the graphs of
- $y = 3x - x^2$
- and
- $y = x$
- .

3. Evaluate the double integrals:

$$(a). \iint_D \frac{y}{x^2 + 1} \, dA, \, D = \{(x, y) \mid 0 \leq x \leq 4, \, 0 \leq y \leq \sqrt{x}\}.$$

$$(b). \iint_D (2x + y) \, dA, \, D = \{(x, y) \mid 1 \leq y \leq 2, \, y - 1 \leq x \leq 1\}.$$

$$(c). \iint_D x \, dA, \, D \text{ is enclosed by the lines } y = x, \, y = 0 \text{ and } x = 1.$$

$$(d). \iint_D xy \, dA, \, D \text{ is enclosed by the curves } y = x^2 \text{ and } y = 3x.$$

$$(e). \iint_D x \cos y \, dA, \, D \text{ is bounded by the } y = 0, \, y = x^2 \text{ and } x = 1.$$

$$(f). \iint_D y^2 \, dA, \, D \text{ is the triangular region with vertices } (0, 1), \, (1, 2) \text{ and } (4, 1).$$

4. Evaluate the integral by reversing the order of integration:

$$(a). \int_0^1 \int_{3y}^3 e^{x^2} \, dx \, dy \quad (b). \int_0^1 \int_{x^2}^1 \sqrt{y} \sin y \, dy \, dx$$

$$(c). \int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1} \, dy \, dx \quad (d). \int_0^2 \int_{\frac{y}{2}}^1 y \cos(x^3 - 1) \, dx \, dy$$

## 2.3 Double Integrals in Polar Coordinates

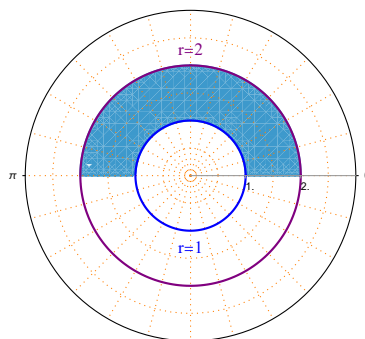
### 2.3.1 Double Integrals in Polar Coordinates

If  $f$  is continuous on the polar region  $R$  given by  $a \leq r \leq b$  and  $\theta_1 \leq \theta \leq \theta_2$ , where  $0 \leq \theta_2 - \theta_1 \leq 2\pi$ , then  $\iint_R f(x, y) \, dA = \int_{\theta_1}^{\theta_2} \int_a^b f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta$ .

**Example (1):** Evaluate  $\iint_R (4x^2 + 3y) \, dA$ , where  $R$  is the region in the upper half-plane bounded by the circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$ .  
Solution:

$R$  is the region where  $1 \leq r \leq 2$  and  $0 \leq \theta \leq \pi$ .

$$\begin{aligned} & \iint_R (4x^2 + 3y) \, dA \\ &= \int_0^\pi \int_1^2 (4r^2 \cos^2 \theta + 3r \sin \theta) \, r \, dr \, d\theta \\ &= \int_0^\pi \int_1^2 (4r^3 \cos^2 \theta + 3r^2 \sin \theta) \, dr \, d\theta \\ &= \int_0^\pi [r^4 \cos^2 \theta + r^3 \sin \theta]_1^2 \, d\theta \end{aligned}$$



$$\begin{aligned} &= \int_0^\pi [(16 \cos^2 \theta + 8 \sin \theta) - (\cos^2 \theta + \sin \theta)] \, d\theta = \int_0^\pi (15 \cos^2 \theta + 7 \sin \theta) \, d\theta \\ &= \int_0^\pi \left[ 15 \left( \frac{1 + \cos 2\theta}{2} \right) + 7 \sin \theta \right] \, d\theta = \frac{15}{2} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^\pi + 7 [-\cos \theta]_0^\pi \\ &= \frac{15}{2} [(\pi + 0) - (0 + 0)] + 7[(-(-1)) - (-1)] = \frac{15\pi}{2} + 14. \end{aligned}$$

**Example (2):** Evaluate  $\int_{-1}^1 \int_0^{\sqrt{1-y^2}} \frac{8(x^2 + y^2)}{9 + (x^2 + y^2)^2} \, dx \, dy$

Solution:

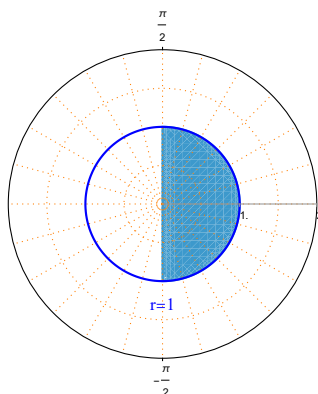
$R$  is the region where  $-1 \leq y \leq 1$  and

$0 \leq x \leq \sqrt{1 - y^2}$

In polar coordinates.  $0 \leq r \leq 1$  and

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ .

$$\begin{aligned} & \int_{-1}^1 \int_0^{\sqrt{1-y^2}} \frac{8(x^2 + y^2)}{9 + (x^2 + y^2)^2} \, dx \, dy \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \frac{8r^2}{9 + r^4} r \, dr \, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 2 \left( \frac{4r^3}{9 + r^4} \right) \, dr \, d\theta \\ &= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\ln(9 + r^4)]_0^1 \, d\theta = 2(\ln 10 - \ln 9) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \end{aligned}$$



$$= 2 \ln \left( \frac{10}{9} \right) [\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2 \ln \left( \frac{10}{9} \right) \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = 2\pi \ln \left( \frac{10}{9} \right).$$

**Example (3):** Find the volume of the solid bounded by  $z = 4 - x^2 - y^2$  and  $z = 0$

Solution:

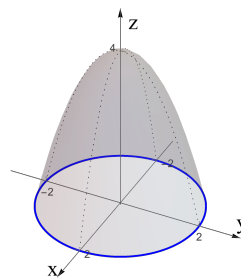
The surface of intersection is :

$$4 - x^2 - y^2 = 0 \implies x^2 + y^2 = 4.$$

$R$  is the region inside the circle centered at the origin with radius 2.

In polar coordinates,  $0 \leq r \leq 2$  and  $0 \leq \theta \leq 2\pi$ .

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^2 (4 - r^2) r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta \\ &= \int_0^{2\pi} \left[ 2r^2 - \frac{r^4}{4} \right]_0^2 \, d\theta = \int_0^{2\pi} [(8 - 4) - (0 - 0)] \, d\theta \\ &= 4 \int_0^{2\pi} d\theta = 4 [\theta]_0^{2\pi} = 4(2\pi - 0) = 8\pi. \end{aligned}$$



**NOTE:** If  $f$  is continuous on the polar region  $R = \{(r, \theta) | \theta_1 \leq \theta \leq \theta_2, g_1(\theta) \leq r \leq g_2(\theta)\}$ ,

$$\text{then } \iint_R f(x, y) \, dA = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.$$

**Example (4):** Evaluate  $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2 + y^2)^{\frac{1}{2}} \, dy \, dx$ .

Solution:

$R$  is the region where  $0 \leq x \leq 2$  and

$$0 \leq y \leq \sqrt{2x - x^2}.$$

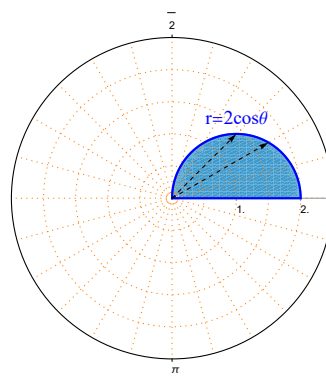
$$y = \sqrt{2x - x^2} \implies y^2 = 2x - x^2$$

$$\implies x^2 - 2x + y^2 = 0$$

$$\implies (x^2 - 2x + 1) + y^2 = 1$$

$$\implies (x - 1)^2 + y^2 = 1.$$

$y = \sqrt{2x - x^2}$  is the upper-half of the circle centered at  $(1, 0)$  with radius 1.



$$\begin{aligned} \text{Note that : } y &= \sqrt{2x - x^2} \implies y^2 = 2x - x^2 \implies x^2 + y^2 = 2x \\ \implies r^2 &= 2r \cos \theta \implies r = 2 \cos \theta. \end{aligned}$$

In polar coordinates,  $0 \leq r \leq 2 \cos \theta$  and  $0 \leq \theta \leq \frac{\pi}{2}$ .

$$\begin{aligned} \int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2 + y^2)^{\frac{1}{2}} dy dx &= \int_0^{\frac{\pi}{2}} \int_0^{2 \cos \theta} r r dr d\theta = \int_0^{\frac{\pi}{2}} \int_0^{2 \cos \theta} r^2 dr d\theta \\ &= \int_0^{\frac{\pi}{2}} \left[ \frac{r^3}{3} \right]_0^{2 \cos \theta} d\theta = \frac{8}{3} \int_0^{\frac{\pi}{2}} \cos^3 \theta d\theta = \frac{8}{3} \int_0^{\frac{\pi}{2}} \cos^2 \theta \cos \theta d\theta \\ &= \frac{8}{3} \int_0^{\frac{\pi}{2}} (1 - \sin^2 \theta) \cos \theta d\theta = \frac{8}{3} \int_0^{\frac{\pi}{2}} (\cos \theta - \sin^2 \theta \cos \theta) d\theta \\ &= \frac{8}{3} \left[ \sin \theta - \frac{\sin^3 \theta}{3} \right]_0^{\frac{\pi}{2}} = \frac{8}{3} \left[ \left( 1 - \frac{1}{3} \right) - (0 - 0) \right] = \frac{16}{9}. \end{aligned}$$

## 2.3.2 EXERCISES

1. Evaluate the given integral by changing to polar coordinates.

(a).  $\iint_R (2x - y) \, dA$ , where  $R$  is the region in the first quadrant enclosed by the circle  $x^2 + y^2 = 4$  and the lines  $x = 0$  and  $y = x$ .

(b).  $\iint_R e^{-x^2-y^2} \, dA$ , where  $R$  is the region bounded by the semicircle  $x = \sqrt{4 - y^2}$  and the  $y$ -axis.

(c).  $\iint_R \cos \sqrt{x^2 + y^2} \, dA$ , where  $R$  is the disk with center the origin and radius 2.

2. Use polar coordinates to find the volume of the given solid.

(a). Under the paraboloid  $z = x^2 + y^2$  and above the disk  $x^2 + y^2 \leq 25$ .

(b). Below the cone  $z = \sqrt{x^2 + y^2}$  and above the ring  $1 \leq x^2 + y^2 \leq 4$ .

(c). Below the plane  $2x + y + z = 4$  and above the disk  $x^2 + y^2 \leq 1$ .

(d). Inside the sphere  $x^2 + y^2 + z^2 = 16$  and outside the cylinder  $x^2 + y^2 = 4$ .

(e). Above the cone  $z = \sqrt{x^2 + y^2}$  and below the sphere  $x^2 + y^2 + z^2 = 1$ .

(f). Bounded by the paraboloids  $z = 6 - x^2 - y^2$  and  $z = 2x^2 + 2y^2$ .

3. Evaluate the iterated integral by converting to polar coordinates.

(a).  $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2} \, dy \, dx$ .

(b).  $\int_0^{\frac{1}{2}} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2 \, dx \, dy$ .

## 2.4 Triple Integrals

### 2.4.1 Triple Integrals over Rectangular Boxes

#### Fubini's Theorem:

If  $f$  is continuous on the rectangular box  $E = [a, b] \times [c, d] \times [r, s]$ , then

$$\begin{aligned} \iiint_E f(x, y, z) \, dV &= \int_r^s \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz = \int_r^s \int_a^b \int_c^d f(x, y, z) \, dy \, dx \, dz \\ &= \int_c^d \int_r^s \int_a^b f(x, y, z) \, dx \, dz \, dy = \int_c^d \int_a^b \int_r^s f(x, y, z) \, dz \, dx \, dy \\ &= \int_a^b \int_r^s \int_c^d f(x, y, z) \, dy \, dz \, dx = \int_a^b \int_c^d \int_r^s f(x, y, z) \, dz \, dy \, dx . \end{aligned}$$

**Example (1):** Evaluate  $\iiint_E xyz^2 \, dV$ , where  $E = [0, 1] \times [0, 2] \times [0, 3]$ .

Solution:

$$\begin{aligned} \iiint_E xyz^2 \, dV &= \int_0^1 \int_0^2 \int_0^3 xyz^2 \, dz \, dy \, dx = \int_0^1 \int_0^2 xy \left[ \frac{z^3}{3} \right]_0^3 \, dy \, dx \\ &= \left[ \frac{27}{3} - \frac{0}{3} \right] \int_0^1 \int_0^2 xy \, dy \, dx = 9 \int_0^1 x \left[ \frac{y^2}{2} \right]_0^2 \, dx = 9 \int_0^1 x \left[ \frac{2^2}{2} - \frac{0}{2} \right] \, dx \\ &= 9 \int_0^1 2x \, dx = 9 [x^2]_0^1 = 9[1 - 0] = 9 . \end{aligned}$$

**Example (2):** Evaluate  $\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^1 (2xz \cos(y)e^{x^2} + 3x^2 \sin(y)e^z) \, dx \, dy \, dz$ .

Solution:

$$\begin{aligned} &\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^1 (2xz \cos(y)e^{x^2} + 3x^2 \sin(y)e^z) \, dx \, dy \, dz \\ &= \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^1 (2xz \cos(y)e^{x^2}) \, dx \, dy \, dz + \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^1 (3x^2 \sin(y)e^z) \, dx \, dy \, dz \\ &= \left( \int_0^1 2xe^{x^2} \, dx \right) \left( \int_0^{\frac{\pi}{2}} \cos(y) \, dy \right) \left( \int_0^1 z \, dz \right) \\ &\quad + \left( \int_0^1 3x^2 \, dx \right) \left( \int_0^{\frac{\pi}{2}} \sin(y) \, dy \right) \left( \int_0^1 e^z \, dz \right) \\ &= [e^{x^2}]_0^1 [\sin(y)]_0^{\frac{\pi}{2}} \left[ \frac{z^2}{2} \right]_0^1 + [x^3]_0^1 [-\cos(y)]_0^{\frac{\pi}{2}} [e^z]_0^1 \\ &= (e - e^0) \left( \sin\left(\frac{\pi}{2}\right) - \sin(0) \right) \left( \frac{1}{2} - 0 \right) + (1 - 0) \left( -\cos\left(\frac{\pi}{2}\right) - (-\cos(0)) \right) (e - e^0) \\ &= (e - 1)(1 - 0) \left( \frac{1}{2} \right) + (1)(0 + 1)(e - 1) = \frac{e - 1}{2} + (e - 1) = \frac{3(e - 1)}{2} . \end{aligned}$$

### 2.4.2 Triple Integrals over General Regions

First - Regions of Type I:

Case (1) :

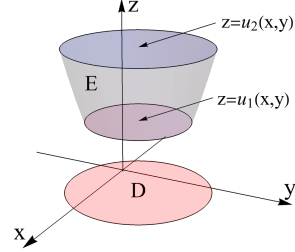
Let  $E$  be the region where

$(x, y) \in D$  and

$u_1(x, y) \leq z \leq u_2(x, y)$ .

If  $f$  is continuous on  $E$ , then

$$\begin{aligned} & \iiint_E f(x, y, z) \, dV \\ &= \iint_D \left( \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \right) \, dA. \end{aligned}$$



Case (2) :

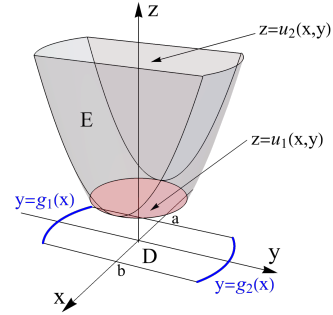
Let  $E$  be the region where

$a \leq x \leq b$ ,  $g_1(x) \leq y \leq g_2(x)$

and  $u_1(x, y) \leq z \leq u_2(x, y)$ .

If  $f$  is continuous on  $E$ , then

$$\begin{aligned} & \iiint_E f(x, y, z) \, dV \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \, dy \, dx. \end{aligned}$$



Case (3) :

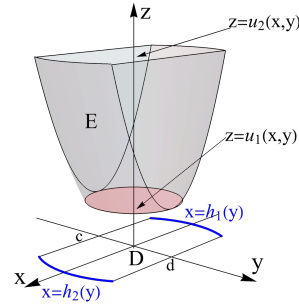
Let  $E$  be the region where

$c \leq y \leq d$ ,  $h_1(y) \leq x \leq h_2(y)$

and  $u_1(x, y) \leq z \leq u_2(x, y)$ .

If  $f$  is continuous on  $E$ , then

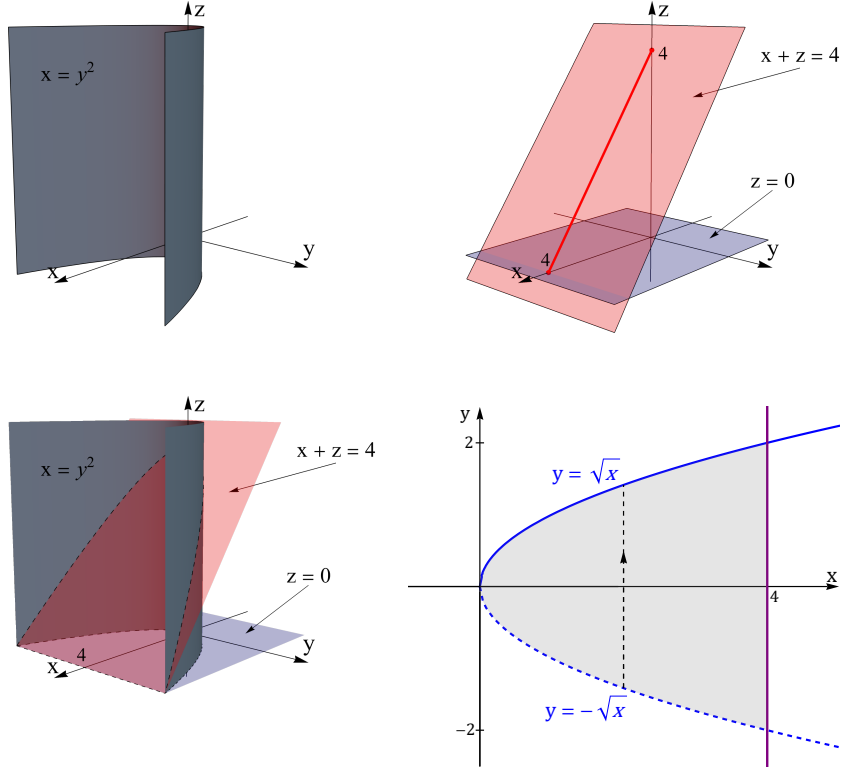
$$\begin{aligned} & \iiint_E f(x, y, z) \, dV \\ &= \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \, dx \, dy. \end{aligned}$$



**NOTE:**  $\iiint_E 1 \, dV = V(E)$ , where  $V(E)$  is the volume of the region  $E$ .

**Example (3):** Find the volume of the solid bounded by the cylinder  $x = y^2$  and the planes  $z = 0$  and  $x + z = 4$ .

Solution:



Let  $E$  be the region bounded by the cylinder  $x = y^2$  and the planes  $z = 0$  and  $x + z = 4$ .

Note that  $z = 0$  intersects  $x + z = 4$  at the line where  $x = 4$  and  $z = 4$ .

On  $E$  :  $0 \leq x \leq 4$  ,  $-\sqrt{x} \leq y \leq \sqrt{x}$  ,  $0 \leq z \leq 4 - x$  .

$$\begin{aligned} V(E) &= \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{4-x} 1 \, dz \, dy \, dx = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} [z]_0^{4-x} \, dy \, dx = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} (4-x) \, dy \, dx \\ &= \int_0^4 (4-x) [y]_{-\sqrt{x}}^{\sqrt{x}} \, dx = \int_0^4 2\sqrt{x} (4-x) \, dx = 2 \int_0^4 \left( 4x^{\frac{1}{2}} - x^{\frac{3}{2}} \right) \, dx \\ &= 2 \left[ \frac{8}{3} x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}} \right]_0^4 = 2 \left[ \frac{8}{3} (4)^{\frac{3}{2}} - \frac{2}{5} (4)^{\frac{5}{2}} \right] = 2 \left[ \frac{8}{3} (2^3) - \frac{2}{5} (2^5) \right] \\ &= 2 \left[ \frac{2^6}{3} - \frac{2^6}{5} \right] = 2^7 \left( \frac{1}{3} - \frac{1}{5} \right) = 2^7 \left( \frac{2}{15} \right) = \frac{2^8}{15} = \frac{256}{15}. \end{aligned}$$

Another Solution :

On  $E$  :  $-2 \leq y \leq 2$  ,  $y^2 \leq x \leq 4$  ,  $0 \leq z \leq 4 - x$ .

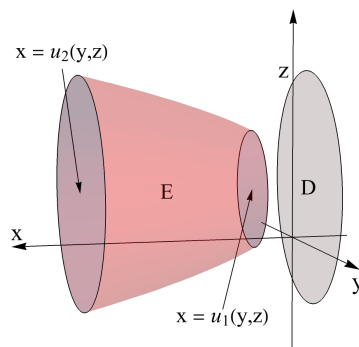
$$V(E) = \int_{-2}^2 \int_{y^2}^4 \int_0^{4-x} 1 \, dz \, dx \, dy .$$

Second - Regions of Type II:

Let  $E$  be the region where  
 $(y, z) \in D$  and  
 $u_1(y, z) \leq x \leq u_2(y, z)$ .

If  $f$  is continuous on  $E$ , then

$$\begin{aligned} & \iiint_E f(x, y, z) \, dV \\ &= \iint_D \left( \int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z) \, dx \right) dA. \end{aligned}$$

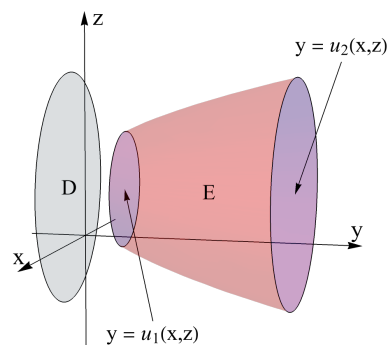


Third - Regions of Type III:

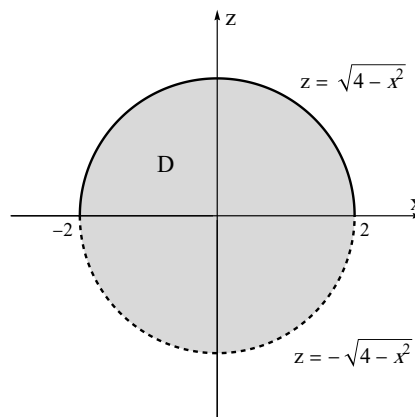
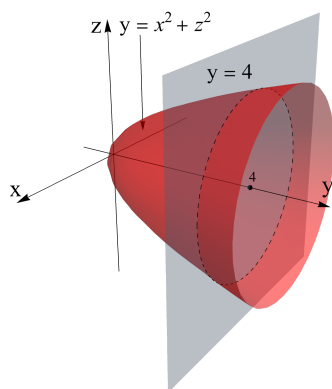
Let  $E$  be the region where  
 $(x, z) \in D$  and  
 $u_1(x, z) \leq y \leq u_2(x, z)$ .

If  $f$  is continuous on  $E$ , then

$$\begin{aligned} & \iiint_E f(x, y, z) \, dV \\ &= \iint_D \left( \int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z) \, dy \right) dA. \end{aligned}$$



**Example (4):** Evaluate  $\iiint_E \sqrt{x^2 + z^2} \, dV$ , where  $E$  is the region bounded by the paraboloid  $y = x^2 + z^2$  and the plane  $y = 4$ .  
 Solution:



Note that  $y = x^2 + z^2$  intersects  $y = 4$  at  $x^2 + z^2 = 4$ .

So,  $E$  is the region where  $x^2 + z^2 \leq y \leq 4$ ,  $-\sqrt{4 - x^2} \leq z \leq \sqrt{4 - x^2}$  and

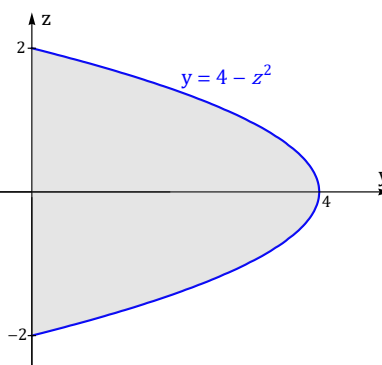
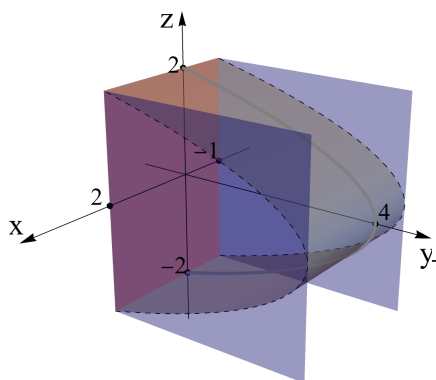
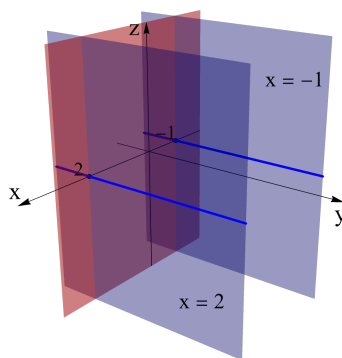
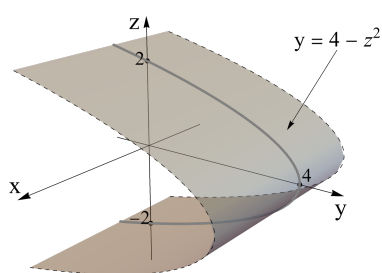
$$-2 \leq x \leq 2.$$

$$\begin{aligned} \iiint_E \sqrt{x^2 + z^2} \, dV &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+z^2}^4 \sqrt{x^2 + z^2} \, dy \, dz \, dx \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [y]_{x^2+z^2}^4 \sqrt{x^2 + z^2} \, dz \, dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4 - (x^2 + z^2)] \sqrt{x^2 + z^2} \, dz \, dx \\ \text{Using the polar coordinates } x &= r \cos \theta \text{ and } z = r \sin \theta, \\ \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4 - (x^2 + z^2)] \sqrt{x^2 + z^2} \, dz \, dx &= \int_0^{2\pi} \int_0^2 (4 - r^2) r \, r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 (4r^2 - r^4) \, dr \, d\theta = \int_0^{2\pi} \left[ 4\frac{r^3}{3} - \frac{r^5}{5} \right]_0^2 \, d\theta \\ &= \left( 4\frac{2^3}{3} - \frac{2^5}{5} \right) (2\pi - 0) = 2\pi \left( \frac{2^5}{3} - \frac{2^5}{5} \right) = 2^6 \left( \frac{1}{3} - \frac{1}{5} \right) \pi = \frac{128\pi}{15}. \end{aligned}$$

**Example (5):** Find the volume of the solid bounded by the surfaces  $y = 4 - z^2$ ,  $x = -1$ ,  $x = 2$  and  $y = 0$ .

Solution:

Let  $E$  be the region bounded by the surfaces  $y = 4 - z^2$ ,  $x = -1$ ,  $x = 2$  and  $y = 0$ .



The surface  $y = 4 - z^2$  intersects the plane  $y = 0$  at the two lines passing through  $z = \pm 2$ .

On  $E$  :  $-1 \leq x \leq 2$  ,  $0 \leq y \leq 4 - z^2$  and  $-2 \leq z \leq 2$ .

$$\begin{aligned}
 V(E) &= \iiint_E dV = \int_{-1}^2 \int_{-2}^2 \int_0^{4-z^2} dy \, dz \, dx \\
 &= \int_{-1}^2 \int_{-2}^2 [y]_0^{4-z^2} dz \, dx = \int_{-1}^2 \int_{-2}^2 (4 - z^2) dz \, dx \\
 &= \int_{-1}^2 \left[ 4z - \frac{z^3}{3} \right]_{-2}^2 dx = \left[ \left( 8 - \frac{8}{3} \right) - \left( -8 + \frac{8}{3} \right) \right] \int_{-1}^2 dx \\
 &= \left( 16 - \frac{16}{3} \right) (2 - (-1)) = 3 \left( 16 - \frac{16}{3} \right) = 48 - 16 = 32 .
 \end{aligned}$$

**2.4.3 EXERCISES**

1. Evaluate the iterated integral.

$$(a). \int_0^1 \int_y^{2y} \int_0^{x+y} 6xyz \, dz \, dx \, dy \quad (b). \int_1^2 \int_0^{2z} \int_0^{\ln x} x e^{-y} \, dy \, dx \, dz$$

2. Evaluate the triple integral.

$$(a). \iiint_E y \, dV, \text{ where}$$

$$E = \{(x, y, z) \mid 0 \leq x \leq 3, 0 \leq y \leq x, x - y \leq z \leq x + y\}.$$

$$(b). \iiint_E \frac{1}{x^3} \, dV, \text{ where}$$

$$E = \{(x, y, z) \mid 0 \leq y \leq 1, 0 \leq z \leq y^2, 1 \leq x \leq z + 1\}.$$

$$(c). \iiint_E 6xy \, dV, \text{ where } E \text{ lies under the plane } z = 1 + x + y \text{ and above the region in the } xy\text{-plane bounded by the curves } y = \sqrt{x}, y = 0, \text{ and } x = 1.$$

3. Use a triple integral to find the volume of the given solid.

$$(a). \text{ The tetrahedron enclosed by the coordinate planes and the plane } 2x + y + z = 4.$$

$$(b). \text{ The solid enclosed by the paraboloids } y = x^2 + z^2 \text{ and } y = 8 - x^2 - z^2.$$

## 2.5 Triple Integrals in Cylindrical Coordinates

### 2.5.1 Cylindrical Coordinates

If  $P(x, y, z)$  is a point in  $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ , then its cylindrical coordinates

$P(r, \theta, z)$  are :

$$r = \sqrt{x^2 + y^2},$$

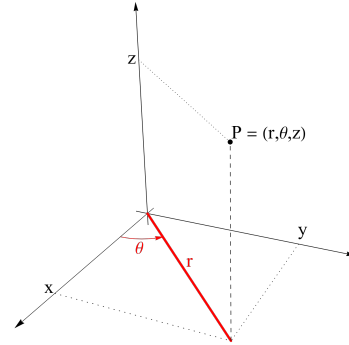
$$\theta = \tan^{-1} \left( \frac{y}{x} \right), \text{ where } x \neq 0,$$

and  $z = z$ .

Note:  $r \in \mathbb{R}$  and  $\theta \in [0, 2\pi]$ .

If  $P(r, \theta, z)$  is given, the the Cartesian coordinates are :

$$x = r \cos \theta, y = r \sin \theta \text{ and } z = z.$$

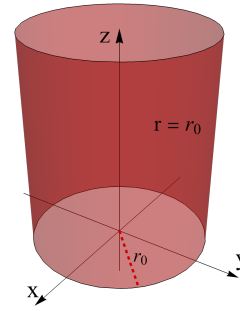


**Important equations in Cylindrical coordinates**

(1).  $r = r_0$ , where  $r_0 \neq 0$ .

$$r = r_0 \implies r^2 = r_0^2 \\ \implies x^2 + y^2 = r_0^2.$$

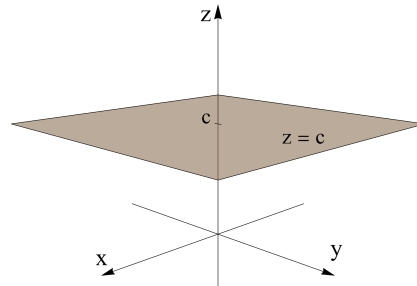
A cylinder,  
centered at the origin,  
and its radius is  $r_0$ .



(2).  $z = c$ , where  $c \in \mathbb{R}$ .

All the points  $(x, y, c)$  in  $\mathbb{R}^3$ ,  
where  $x, y \in \mathbb{R}$ .

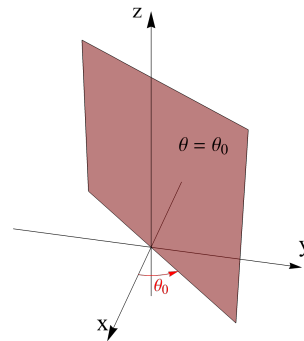
A horizontal plain,  
parallel to the  $xy$ -plain,  
and passes through  $(0, 0, c)$ .



(3).  $\theta = \theta_0$ , where  $\theta_0 \in [0, 2\pi]$ .

$$\begin{aligned}\theta = \theta_0 &\implies \tan(\theta) = \tan(\theta_0), \\ \implies \frac{y}{x} &= \tan(\theta_0) \\ \implies y &= \tan(\theta_0)x.\end{aligned}$$

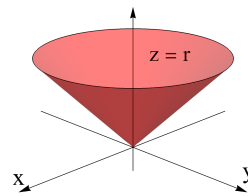
A vertical plane,  
passes through the origin.



(4).  $z = r$ , where  $r \neq 0$ .

$$z = r \implies z = \sqrt{x^2 + y^2}.$$

A Cone.



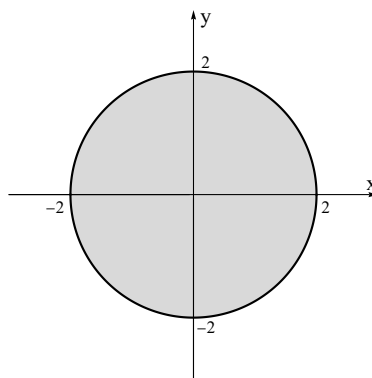
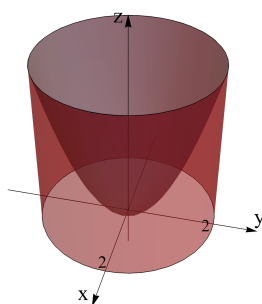
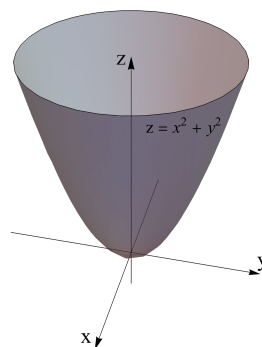
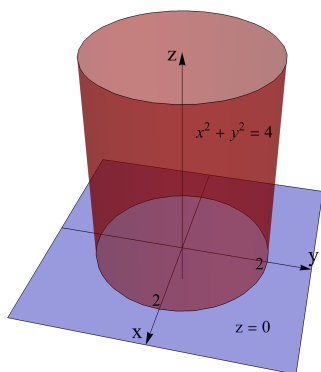
### 2.5.2 Triple Integrals in Cylindrical Coordinates

Suppose  $f$  is continuous on  $E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$ , where  $D$  is given in polar coordinates by  $D = \{(r, \theta) \mid \theta_1 \leq \theta \leq \theta_2, r_1(\theta) \leq r \leq r_2(\theta)\}$ , then

$$\iiint_E f(x, y, z) dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{u_1(r, \theta)}^{u_2(r, \theta)} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta.$$

**Example (1):** Find the volume of the solid within  $x^2 + y^2 = 4$ , bounded above by  $z = x^2 + y^2$  and below by  $z = 0$ .

Solution:



$$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2, 0 \leq z \leq x^2 + y^2 = r^2\}.$$

$$\begin{aligned} \text{Volume} &= \iiint_E dV = \int_0^{2\pi} \int_0^2 \int_0^{x^2+y^2} r dz dr d\theta = \int_0^{2\pi} \int_0^2 \int_0^{r^2} r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^2 [z]_0^{r^2} r dr d\theta = \int_0^{2\pi} \int_0^2 (r^2 - 0)r dr d\theta = \int_0^{2\pi} \int_0^2 r^3 dr d\theta \\ &= \int_0^{2\pi} \left[ \frac{r^4}{4} \right]_0^2 d\theta = \left[ \frac{2^4}{4} - 0 \right] \int_0^{2\pi} d\theta = 4(2\pi - 0) = 8\pi. \end{aligned}$$

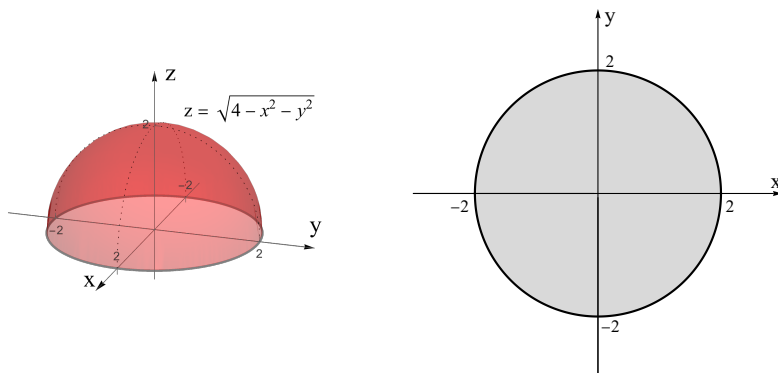
**Example (2):** Evaluate  $\iiint_E z \, dx \, dy \, dz$ , where  $E = \{(x, y, z) \mid 0 \leq z \leq \sqrt{4 - x^2 - y^2}\}$ .

Solution:

$z = \sqrt{4 - x^2 - y^2}$  is the upper-half of the sphere centered at the origin with radius 2.

$\sqrt{4 - x^2 - y^2} = 0 \implies x^2 + y^2 = 4$ , the upper-half of the sphere intersects the plane  $z = 0$  at the circle centered at the origin with radius 2.

$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2, 0 \leq z \leq \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}\}$ .



$$\begin{aligned} \iiint_E z \, dx \, dy \, dz &= \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{4-r^2}} z \, dz \, r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^2 [z^2]_0^{\sqrt{4-r^2}} r \, dr \, d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^2 (4 - r^2) r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} \left[ 2r^2 - \frac{r^4}{4} \right]_0^2 d\theta \\ &= \frac{1}{2} (8 - 4) \int_0^{2\pi} d\theta = 2(2\pi - 0) = 4\pi. \end{aligned}$$

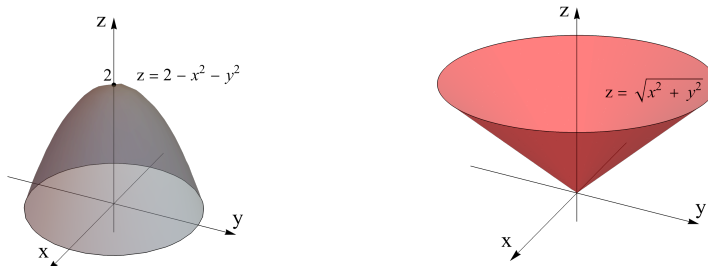
**Example (3):** Find the volume of the solid bounded above by  $z = 2 - x^2 - y^2$  and below by  $z = \sqrt{x^2 + y^2}$ .

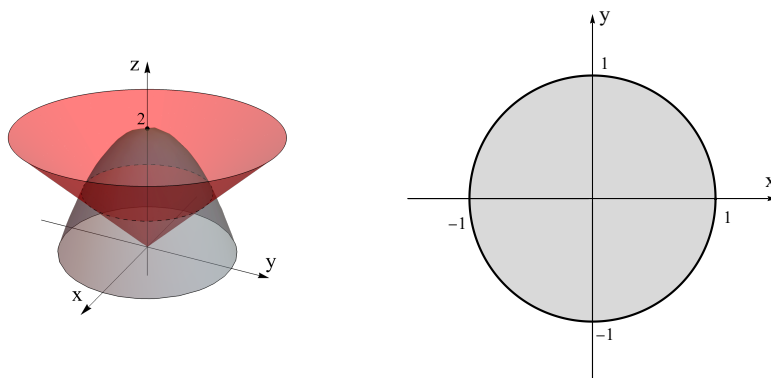
Solution:

$z = 2 - x^2 - y^2 = 2 - r^2$  intersects  $z = \sqrt{x^2 + y^2} = r$  at :

$2 - r^2 = r \implies r^2 + r - 2 = 0 \implies (r + 2)(r - 1) = 0 \implies r = 1$ .

(Note that  $r = -2$  is excluded because  $r \geq 0$ ).





$$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, r \leq z \leq 2 - r^2\}.$$

$$\begin{aligned} \text{Volume} &= \iiint_E dV = \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 [z]_r^{2-r^2} r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 (2 - r^2 - r) r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (-r^3 - r^2 + 2r) \, dr \, d\theta \\ &= \int_0^{2\pi} \left[ -\frac{r^4}{4} - \frac{r^3}{3} + r^2 \right]_0^1 d\theta = \left( -\frac{1}{4} - \frac{1}{3} + 1 \right) \int_0^{2\pi} d\theta = \frac{5}{12} (2\pi - 0) = \frac{5\pi}{6}. \end{aligned}$$

**Example (4):** Evaluate the integral  $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz \, dx \, dy$

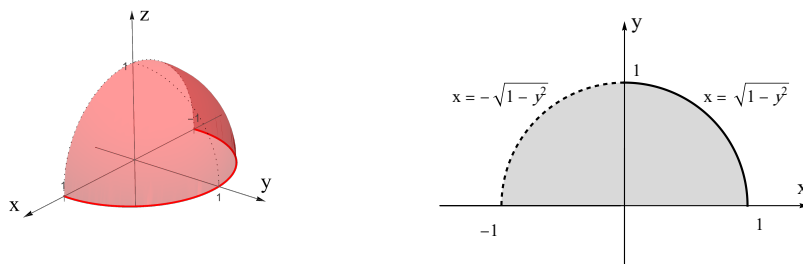
Solution:

Note that  $0 \leq y \leq 1$ ,  $-\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}$  and  $0 \leq z \leq \sqrt{1-x^2-y^2}$ .

$z = \sqrt{1-x^2-y^2} \implies x^2 + y^2 + z^2 = 1$  represents the upper half of the unit sphere.

$x = \sqrt{1-y^2} \implies x^2 + y^2 = 1$  represents the right half of the unit circle.

$x = -\sqrt{1-y^2} \implies x^2 + y^2 = 1$  represents the left half of the unit circle.



In cylindrical coordinates:  $0 \leq \theta \leq \pi$ ,  $0 \leq r \leq 1$  and  $0 \leq z \leq \sqrt{1-r^2}$ .

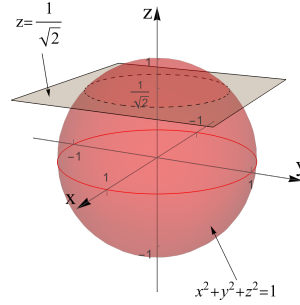
$$\begin{aligned} \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz \, dx \, dy &= \int_0^\pi \int_0^1 \int_0^{\sqrt{1-r^2}} dz \, r \, dr \, d\theta \\ &= \int_0^\pi \int_0^1 \sqrt{1-r^2} \, r \, dr \, d\theta = -\frac{1}{2} \int_0^\pi \int_0^1 (1-r^2)^{\frac{1}{2}} (-2r) \, dr \, d\theta \\ &= -\frac{1}{2} \left[ \frac{2}{3} (1-r^2)^{\frac{3}{2}} \right]_0^1 \int_0^\pi d\theta = -\frac{1}{2} \left( 0 - \frac{2}{3} \right) (\pi - 0) = \frac{\pi}{3}. \end{aligned}$$

**Example (5):** Find the volume of the solid inside the sphere  $x^2 + y^2 + z^2 = 1$  and above the plain  $z = \frac{1}{\sqrt{2}}$ .

Solution:

Note that  $x^2 + y^2 + z^2 = 1$  intersects  $z = \frac{1}{\sqrt{2}}$  at  $x^2 + y^2 = \frac{1}{2}$  which is a circle centered at the origin and its radius is  $\frac{1}{\sqrt{2}}$ .

The solid is bounded above by the upper-half of the sphere, and below by the plain.



$$x^2 + y^2 + z^2 = 1$$

$$\Rightarrow z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - r^2}.$$

The cylindrical coordinates of the solid :  $0 \leq r \leq \frac{1}{\sqrt{2}}$  ,  $0 \leq \theta \leq 2\pi$

and  $\frac{1}{\sqrt{2}} \leq z \leq \sqrt{1 - r^2}$  .

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \int_{\frac{1}{\sqrt{2}}}^{\sqrt{1-r^2}} dz \, r \, dr \, d\theta = \left( \int_0^{2\pi} d\theta \right) \left( \int_0^{\frac{1}{\sqrt{2}}} \int_{\frac{1}{\sqrt{2}}}^{\sqrt{1-r^2}} dz \, r \, dr \right) \\ &= [\theta]_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} [z]_{\frac{1}{\sqrt{2}}}^{\sqrt{1-r^2}} r \, dr = (2\pi - 0) \int_0^{\frac{1}{\sqrt{2}}} \left( \sqrt{1-r^2} - \frac{1}{\sqrt{2}} \right) r \, dr \\ &= 2\pi \int_0^{\frac{1}{\sqrt{2}}} \left( r\sqrt{1-r^2} - \frac{r}{\sqrt{2}} \right) dr = 2\pi \left[ -\frac{1}{2} \frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{r^2}{2\sqrt{2}} \right]_0^{\frac{1}{\sqrt{2}}} \\ &= 2\pi \left[ \left( -\frac{1}{2} \frac{2}{3} \left( 1 - \frac{1}{2} \right)^{\frac{3}{2}} - \frac{1}{4\sqrt{2}} \right) - \left( -\frac{1}{2} \frac{2}{3} - 0 \right) \right] \\ &= 2\pi \left( -\frac{1}{3} \frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{2}} + \frac{1}{3} \right) = \frac{2\pi}{3} \left( -\frac{1}{2\sqrt{2}} - \frac{3}{4\sqrt{2}} + 1 \right) \\ &= \frac{2\pi}{3} \left( 1 - \frac{5}{4\sqrt{2}} \right). \end{aligned}$$

**2.5.3 EXERCISES**

1. Evaluate  $\iiint_E \sqrt{x^2 + y^2} \, dV$ , where  $E$  is the region that lies inside the cylinder  $x^2 + y^2 = 16$  and between the planes  $z = -5$  and  $z = 4$ .
2. Evaluate  $\iiint_E z \, dV$ , where  $E$  is enclosed by the paraboloid  $z = x^2 + y^2$  and the plane  $z = 4$ .
3. Evaluate  $\iiint_E (x + y + z) \, dV$ , where  $E$  is the solid in the first octant that lies under the paraboloid  $z = 4 - x^2 - y^2$ .
4. Find the volume of the solid that lies within both the cylinder  $x^2 + y^2 = 1$  and the sphere  $x^2 + y^2 + z^2 = 4$ .
5. Find the volume of the solid that is enclosed by the cone  $z = \sqrt{x^2 + y^2}$  and the sphere  $x^2 + y^2 + z^2 = 2$ .
6. Find the volume of the solid that lies between the paraboloid  $z = x^2 + y^2$  and the sphere  $x^2 + y^2 + z^2 = 2$ .
7. Evaluate  $\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^2 xz \, dz \, dx \, dy$ .
8. Evaluate  $\int_{-3}^3 \int_0^{\sqrt{9-y^2}} \int_0^{9-x^2-y^2} \sqrt{x^2 + y^2} \, dz \, dx \, dy$ .

## 2.6 Triple Integrals in Spherical Coordinates

### 2.6.1 Spherical Coordinates

If  $P(x, y, z)$  is a point in  $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ , then its spherical coordinates

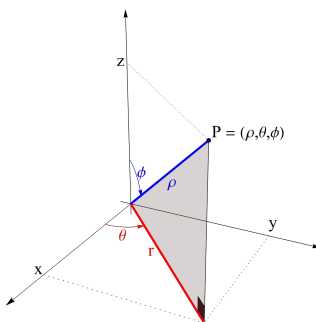
$P(\rho, \theta, \phi)$  are :

$$\rho = \sqrt{x^2 + y^2 + z^2},$$

$$\theta = \tan^{-1} \left( \frac{y}{x} \right), \text{ where } x \neq 0,$$

$$\text{and } \phi = \cos^{-1} \left( \frac{z}{\rho} \right), \text{ where } \rho \neq 0.$$

Note:  $\rho \geq 0$ ,  $\theta \in [0, 2\pi]$  and  $\phi \in [0, \pi]$ .



Note that if  $P(\rho, \theta, \phi)$  is given then :

$$\sin \phi = \frac{r}{\rho} \implies r = \rho \sin \phi \text{ and } \cos \phi = \frac{z}{\rho} \implies z = \rho \cos \phi.$$

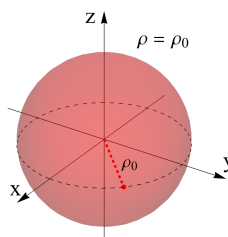
$$\text{So, } x = r \cos \theta = \rho \cos \theta \sin \phi \text{ and } y = r \sin \theta = \rho \sin \theta \sin \phi.$$

**Important equations in Spherical coordinates**

(1).  $\rho = \rho_0$ , where  $\rho_0 > 0$ .

$$\rho = \rho_0 \implies \rho^2 = \rho_0^2 \\ \implies x^2 + y^2 + z^2 = \rho_0^2.$$

A sphere,  
centered at the origin,  
and its radius is  $\rho_0$ .



(2).  $\theta = \theta_0$ , where  $\theta_0 \in [0, 2\pi]$ .

Since  $\phi \in [0, \pi]$  then

$$\sin \phi \geq 0.$$

If  $\theta \in [0, \pi]$  then :

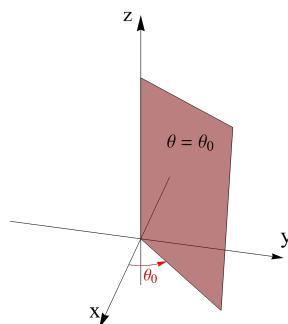
$$y = \rho \sin \theta \sin \phi \geq 0$$

$\theta = \theta_0$  represents a half-plane.

If  $\theta \in [\pi, 2\pi]$  then :

$$y = \rho \sin \theta \sin \phi \leq 0$$

$\theta = \theta_0$  represents the other half-plane.



(3).  $\phi = \phi_0$ , where  $0 < \phi_0 < \frac{\pi}{2}$ .

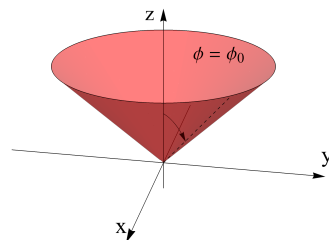
$$\cot \phi = \frac{z}{r} \implies z = r \cot \phi,$$

$$\implies z = \cot \phi \sqrt{x^2 + y^2}$$

Since  $\phi \in \left(0, \frac{\pi}{2}\right)$  then

$\cot \phi > 0$ .

$\phi = \phi_0$  represents an upper cone.



(4).  $\phi = \phi_0$ , where  $\frac{\pi}{2} < \phi_0 < \pi$ .

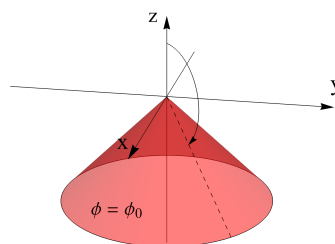
$$\cot \phi = \frac{z}{r} \implies z = r \cot \phi,$$

$$\implies z = \cot \phi \sqrt{x^2 + y^2}$$

Since  $\phi \in \left(\frac{\pi}{2}, \pi\right)$  then

$\cot \phi < 0$ .

$\phi = \phi_0$  represents a lower cone.



### 2.6.2 Triple Integrals in Spherical Coordinates

$$\iiint_E f(x, y, z) dV = \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \int_a^b f(\rho, \theta, \phi) \rho^2 \sin \phi d\rho d\theta d\phi ,$$

where  $E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \theta_1 \leq \theta \leq \theta_2, \phi_1 \leq \phi \leq \phi_2\}$ .

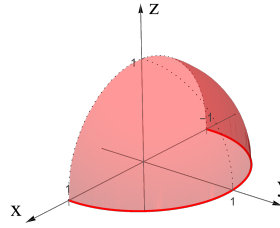
**Example (1):** Evaluate the integral  $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz dx dy$

Solution: Referring to Example(4) in section (2.5).

In Spherical coordinates :

$$0 \leq \rho \leq 1, 0 \leq \theta \leq \pi, \text{ and } 0 \leq \phi \leq \frac{\pi}{2} .$$

$$\begin{aligned} & \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz dx dy \\ &= \int_0^{\frac{\pi}{2}} \int_0^{\pi} \int_0^1 \rho^2 \sin \phi d\rho d\theta d\phi \\ &= \left( \int_0^1 \rho^2 d\rho \right) \left( \int_0^{\frac{\pi}{2}} \sin \phi d\phi \right) \left( \int_0^{\pi} d\theta \right) \\ &= \left[ \frac{\rho^3}{3} \right]_0^1 [-\cos \phi]_0^{\frac{\pi}{2}} [\theta]_0^{\pi} = \left( \frac{1}{3} - 0 \right) \left( -\cos \left( \frac{\pi}{2} \right) + \cos(0) \right) (\pi - 0) = \frac{\pi}{3} . \end{aligned}$$



**Example (2):** Evaluate the integral  $\iiint_E 4z dV$ , where  $E$  is the region bounded above by  $x^2 + y^2 + z^2 = 4$  and below by  $z = 0$ .

Solution:

Note that  $x^2 + y^2 + z^2 = 4$  intersects  $z = 0$  at  $x^2 + y^2 = 4$  which is a circle centered at the origin and its radius is 2.

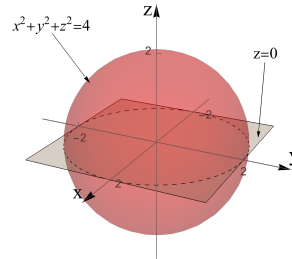
$E$  in spherical coordinates:

$$0 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi \text{ and } 0 \leq \phi \leq \frac{\pi}{2} .$$

$$z = \rho \cos \phi \implies 4z = 4\rho \cos \phi .$$

$$\iiint_E 4z dV$$

$$\begin{aligned} &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 (4\rho \cos \phi) \rho^2 \sin \phi d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 (4\rho^3) (\cos \phi \sin \phi) d\rho d\phi d\theta \\ &= \left( \int_0^2 4\rho^3 d\rho \right) \left( \int_0^{\frac{\pi}{2}} \cos \phi \sin \phi d\phi \right) \left( \int_0^{2\pi} d\theta \right) = [\rho^4]_0^2 \left[ \frac{\sin^2 \phi}{2} \right]_0^{\frac{\pi}{2}} [\theta]_0^{2\pi} \\ &= (16 - 0) \left( \frac{1}{2} - 0 \right) (2\pi - 0) = 16\pi . \end{aligned}$$

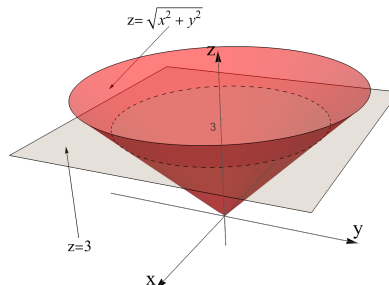


**Example (3):** Evaluate  $\iiint_E \sqrt{x^2 + y^2 + z^2} dV$ , where  $E$  is the region bounded above by the plane  $z = 3$  and below by the cone  $z = \sqrt{x^2 + y^2}$ .

Solution:

Note that  $z = \sqrt{x^2 + y^2}$  intersects  $z = 3$  at  $x^2 + y^2 = 9$  which is a circle centered at the origin and its radius is 3.

$$\begin{aligned} z &= \sqrt{x^2 + y^2} = r \\ \Rightarrow \rho \cos \phi &= \rho \sin \phi \\ \Rightarrow \cos \phi &= \sin \phi \\ \Rightarrow \phi &= \frac{\pi}{4}. \end{aligned}$$



$$z = 3 \Rightarrow \rho \cos \phi = 3 \Rightarrow \rho = \frac{3}{\cos \phi}.$$

The spherical coordinates of  $E$  :  $0 \leq \rho \leq \frac{3}{\cos \phi}$ ,  $0 \leq \theta \leq 2\pi$  and  $0 \leq \phi \leq \frac{\pi}{4}$ .

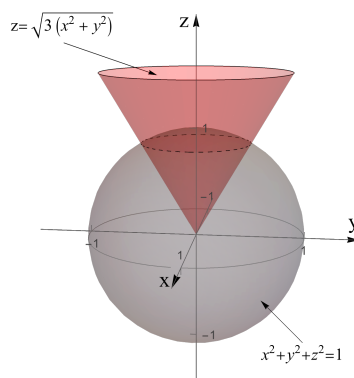
$$\begin{aligned} \iiint_E \sqrt{x^2 + y^2 + z^2} dV &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\frac{3}{\cos \phi}} \rho^2 \sin \phi d\rho d\phi d\theta \\ &= \left( \int_0^{2\pi} d\theta \right) \left( \int_0^{\frac{\pi}{4}} \int_0^{\frac{3}{\cos \phi}} \rho^3 \sin \phi d\rho d\phi \right) = [\theta]_0^{2\pi} \int_0^{\frac{\pi}{4}} \left[ \frac{\rho^4}{4} \right]_0^{\frac{3}{\cos \phi}} \sin \phi d\phi \\ &= (2\pi - 0) \frac{3^4}{4} \int_0^{\frac{\pi}{4}} (\cos \phi)^{-4} \sin \phi d\phi = \frac{81\pi}{2} \left[ \frac{-(\cos \phi)^{-3}}{-3} \right]_0^{\frac{\pi}{4}} = \frac{27\pi}{2} (2\sqrt{2} - 1). \end{aligned}$$

**Example (4):** Find the volume of the solid inside the sphere  $x^2 + y^2 + z^2 = 1$  and inside the cone  $z = \sqrt{3(x^2 + y^2)}$ .

Solution:

Note that  $x^2 + y^2 + z^2 = 1$  intersects  $z = \sqrt{3(x^2 + y^2)}$  at  $3x^2 + 3y^2 = 1 - x^2 - y^2$   
 $\Rightarrow 4x^2 + 4y^2 = 1$   
 $\Rightarrow x^2 + y^2 = \frac{1}{4}$   
 which is a circle centered at the origin and its radius is  $\frac{1}{2}$ .

$$\begin{aligned} z &= \sqrt{3(x^2 + y^2)} = \sqrt{3}r \Rightarrow \\ \rho \cos \phi &= \sqrt{3}\rho \sin \phi \Rightarrow \tan \phi = \frac{1}{\sqrt{3}} \\ \Rightarrow \phi &= \frac{\pi}{6}. \end{aligned}$$



The spherical coordinates of the solid :  $0 \leq \rho \leq 1$ ,  $0 \leq \theta \leq 2\pi$  and  $0 \leq \phi \leq \frac{\pi}{6}$ .

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
&= \left( \int_0^1 \rho^2 \, d\rho \right) \left( \int_0^{\frac{\pi}{6}} \sin \phi \, d\phi \right) \left( \int_0^{2\pi} d\theta \right) \\
&= \left[ \frac{\rho^3}{3} \right]_0^1 [-\cos \phi]_0^{\frac{\pi}{6}} [\theta]_0^{2\pi} = \left( \frac{1}{3} - 0 \right) \left( -\frac{\sqrt{3}}{2} - (-1) \right) (2\pi - 0) \\
&= \frac{2\pi}{3} \left( 1 - \frac{\sqrt{3}}{2} \right) = \frac{\pi}{3} (2 - \sqrt{3}) .
\end{aligned}$$

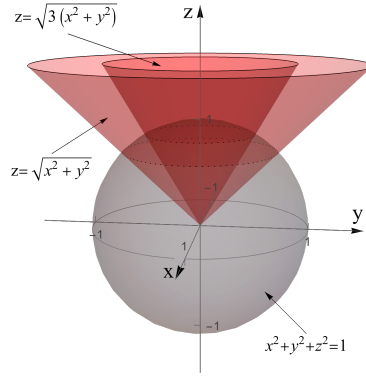
**Example (5)** Find the volume of the solid bounded by the surfaces

$$x^2 + y^2 + z^2 = 1, \quad z = \sqrt{3(x^2 + y^2)} \quad \text{and} \quad z = \sqrt{x^2 + y^2}.$$

Solution:

Note that  $x^2 + y^2 + z^2 = 1$  is the unit sphere.

$$\begin{aligned}
z &= \sqrt{3(x^2 + y^2)} = \sqrt{3}r \implies \\
\rho \cos \phi &= \sqrt{3}\rho \sin \phi \implies \tan \phi = \frac{1}{\sqrt{3}} \\
\implies \phi &= \frac{\pi}{6}. \\
z &= \sqrt{x^2 + y^2} = r \implies \\
\rho \cos \phi &= \rho \sin \phi \implies \tan \phi = 1 \\
\implies \phi &= \frac{\pi}{4}.
\end{aligned}$$



The spherical coordinates of the solid :

$$0 \leq \rho \leq 1, \quad 0 \leq \theta \leq 2\pi \quad \text{and} \quad \frac{\pi}{6} \leq \phi \leq \frac{\pi}{4}.$$

$$\begin{aligned}
V &= \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
&= \left( \int_0^1 \rho^2 \, d\rho \right) \left( \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin \phi \, d\phi \right) \left( \int_0^{2\pi} d\theta \right) \\
&= \left[ \frac{\rho^3}{3} \right]_0^1 [-\cos \phi]_{\frac{\pi}{6}}^{\frac{\pi}{4}} [\theta]_0^{2\pi} = \left( \frac{1}{3} - 0 \right) \left( -\frac{\sqrt{2}}{2} - \left( -\frac{\sqrt{3}}{2} \right) \right) (2\pi - 0) \\
&= \frac{2\pi}{3} \left( \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \right) = \frac{\pi}{3} (\sqrt{3} - \sqrt{2}) .
\end{aligned}$$

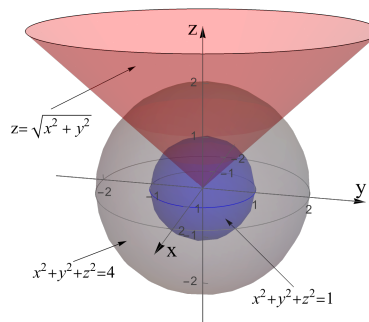
**Example (6):** Evaluate  $\iiint_E z \, dV$ , where  $E$  is the region bounded by the surfaces  $x^2 + y^2 + z^2 = 1$ ,  $x^2 + y^2 + z^2 = 4$  and  $z = \sqrt{x^2 + y^2}$ .

Solution:

Note that  $x^2 + y^2 + z^2 = 1$  is the unit sphere.

$x^2 + y^2 + z^2 = 4$  is a sphere centered at the origin and its radius equals 2.

$$\begin{aligned} z &= \sqrt{x^2 + y^2} = r \implies \\ \rho \cos \phi &= \rho \sin \phi \implies \tan \phi = 1 \\ \implies \phi &= \frac{\pi}{4}. \end{aligned}$$



The spherical coordinates of  $E$  :

$$1 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi \text{ and } 0 \leq \phi \leq \frac{\pi}{4}.$$

$$\begin{aligned} \iiint_E z \, dV &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_1^2 (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_1^2 \rho^3 \sin \phi \cos \phi \, d\rho \, d\phi \, d\theta = \left( \int_1^2 \rho^3 \, d\rho \right) \left( \int_0^{\frac{\pi}{4}} \sin \phi \cos \phi \, d\phi \right) \left( \int_0^{2\pi} d\theta \right) \\ &= \left[ \frac{\rho^4}{4} \right]_1^2 \left[ \frac{(\sin \phi)^2}{2} \right]_0^{\frac{\pi}{4}} [\theta]_0^{2\pi} = \left( \frac{16}{4} - \frac{1}{4} \right) \left[ \frac{\left( \frac{1}{\sqrt{2}} \right)^2}{2} - \frac{0}{2} \right] (2\pi - 0) \\ &= \frac{15}{4} \cdot \frac{1}{4} \cdot 2\pi = \frac{15\pi}{8}. \end{aligned}$$

**2.6.3 EXERCISES**

1. Change from rectangular to spherical coordinates:

(a).  $(3, 3, 0)$  .                      (b).  $(1, -\sqrt{3}, 2\sqrt{3})$ .

2. Identify the surface whose equation is  $\rho = \cos \phi$ .

3. Sketch the solid described by the given inequalities:

(a).  $\rho \leq 1$  ,  $0 \leq \theta \leq \pi$  ,  $0 \leq \phi \leq \frac{\pi}{6}$ .                      (b).  $\rho \leq 2$  ,  $\rho \leq \csc \phi$  .

4. Evaluate  $\iiint_E (x^2 + y^2 + z^2)^2 dV$  , where  $E$  is the ball with center the origin and radius 5.

5. Evaluate  $\iiint_E y^2 z^2 dV$  , where  $E$  lies above the cone  $\phi = \frac{\pi}{3}$  and below the sphere  $\rho = 1$ .

6. Evaluate  $\iiint_E x e^{x^2+y^2+z^2} dV$  , where  $E$  is the portion of the unit ball  $x^2 + y^2 + z^2 \leq 1$  that lies in the first octant.

7. Find the volume of the part of the ball  $\rho \leq a$  that lies between the cones  $\phi = \frac{\pi}{6}$  and  $\phi = \frac{\pi}{3}$ .



## Chapter 3

# Sequences, Series, and Power Series

### 3.1 Sequences

#### 3.1.1 Infinite Sequences

**Definition:** An infinite sequence is a function  $f : \mathbb{N} \longrightarrow \mathbb{R}$ , where  $f(n) = a_n$ .

**Notation:** The sequence  $\{a_1, a_2, \dots\}$  is also denoted by  $\{a_n\}_{n=1}^{\infty}$  or  $\{a_n\}$ .

**Example (1):** Some sequences can be defined by giving a formula for the  $n^{\text{th}}$  term.

(a).  $a_n = \frac{1}{2^n}$  is the  $n^{\text{th}}$  term of the sequence  $\left\{\frac{1}{2^n}\right\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right\}$ .

(b).  $\left\{\frac{(-1)^n(n+2)}{5^n}\right\} = \left\{-\frac{3}{5}, \frac{4}{25}, -\frac{5}{125}, \dots\right\}$ .

**Example (2):** The Fibonacci sequences is defined recursively by

$f_1 = 1$  ,  $f_2 = 1$  ,  $f_n = f_{n-1} + f_{n-2}$  ,  $n \geq 3$ .

$\{f_n\} = \{1, 1, 2, 3, 5, 8, 13, \dots\}$  .

#### 3.1.2 The Limit of a Sequence

**Definition:** A sequence  $\{a_n\}$  has the limit  $L \in \mathbb{R}$ , if for every  $\epsilon > 0$  there exists an  $N \in \mathbb{N}$  such that : if  $n \geq N$  then  $|a_n - L| < \epsilon$ , and we write  $\lim_{n \rightarrow \infty} a_n = L$  .

**Definition:** A sequence  $\{a_n\}$  goes to  $\infty$ , if for every positive real number  $M$  there exists an  $N \in \mathbb{N}$  such that : if  $n \geq N$  then  $a_n \geq M$ , and we write  $\lim_{n \rightarrow \infty} a_n = \infty$ .

### 3.1.3 Properties of Convergent Sequences

**Theorem:** If  $\lim_{x \rightarrow \infty} f(x) = L$  and  $f(n) = a_n$  where  $n \in \mathbb{N}$  then  $\lim_{x \rightarrow \infty} a_n = L$ .

**Example (3):** Calculate  $\lim_{n \rightarrow \infty} \frac{\ln n}{n}$ .

Solution : Let  $f(x) = \frac{\ln x}{x}$ .

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0 \text{ (By L'Hôpital's Rule).}$$

Therefore,  $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$ .

**Theorem:** Suppose  $\lim_{n \rightarrow \infty} a_n = L_1$ ,  $\lim_{n \rightarrow \infty} b_n = L_2$  and  $c \in \mathbb{R}$ , then:

- (1).  $\lim_{n \rightarrow \infty} c = c$ .
- (2).  $\lim_{n \rightarrow \infty} a_n \pm b_n = L_1 \pm L_2$ .
- (3).  $\lim_{n \rightarrow \infty} c a_n = c L_1$ .
- (4).  $\lim_{n \rightarrow \infty} a_n \cdot b_n = L_1 \cdot L_2$ .
- (5).  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L_1}{L_2}$ , where  $L_2 \neq 0$ .

**Power Law:**  $\lim_{n \rightarrow \infty} a_n^p = \left[ \lim_{n \rightarrow \infty} a_n \right]^p$ , if  $p > 0$  and  $a_n > 0$ .

**Squeeze Theorem for sequences:** If  $a_n \leq b_n \leq c_n$  for  $n \geq N_0$  and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ , then  $\lim_{n \rightarrow \infty} b_n = L$ .

**Example (4):** Calculate  $\lim_{n \rightarrow \infty} \frac{n!}{n^n}$ .

Solution:

$$0 \leq \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \frac{1}{n} \left( \frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n} \right) \leq \frac{1}{n} (1) = \frac{1}{n}$$

Since  $\lim_{n \rightarrow \infty} 0 = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , then  $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$  (By Squeeze Theorem).

**Theorem:** If  $\lim_{n \rightarrow \infty} |a_n| = 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Example (5):** Calculate  $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$ .

Solution: Let  $a_n = \frac{(-1)^n}{n}$ .

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \rightarrow \infty} \frac{|(-1)^n|}{|n|} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Therefore  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$ .

**Theorem:** If  $\lim_{n \rightarrow \infty} a_n = L$ , and the function  $f$  is continuous at  $L$ , then  $\lim_{n \rightarrow \infty} f(a_n) = f(L)$ .

**Example (6):** Calculate  $\lim_{n \rightarrow \infty} \cos\left(\frac{\pi}{n}\right)$ .

Solution:

$$\lim_{n \rightarrow \infty} \frac{\pi}{n} = \lim_{n \rightarrow \infty} \pi \frac{1}{n} = \pi(0) = 0.$$

Since the cosine function is continuous at 0, then  $\lim_{n \rightarrow \infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$ .

**Important Note:** The sequence  $\{r^n\}$  is convergent if  $-1 < r \leq 1$ , and divergent for all other values of  $r$ .

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}.$$

### 3.1.4 Monotonic and bounded sequences

**Definition:**

- (1). A sequence  $\{a_n\}$  is called increasing if  $a_{n+1} \geq a_n$  for all  $n \geq 1$ .
- (2). A sequence  $\{a_n\}$  is called decreasing if  $a_{n+1} \leq a_n$  for all  $n \geq 1$ .
- (3). A sequence is called monotonic if it is either increasing or decreasing.

**Definition:**

- (1). A sequence  $\{a_n\}$  is bounded above if there exists a real number  $M$  such that  $a_n \leq M$  for all  $n \geq 1$ .
- (2). A sequence  $\{a_n\}$  is bounded below if there exists a real number  $m$  such that  $a_n \geq m$  for all  $n \geq 1$ .
- (3). If a sequence is bounded above and below, then it is called a bounded sequence.

#### Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent. In particular :

- (1). A sequence that is increasing and bounded above converges.
- (2). A sequence that is decreasing and bounded below converges.

**Important Notes :**

- (1). If  $a_n \leq b_n$  for some  $n \geq N_0$ ,  $\{a_n\}$  is an increasing sequence and  $\{b_n\}$  is convergent, then  $\{a_n\}$  is convergent.
- (2). If  $a_n \leq b_n$  for some  $n \geq N_0$ , if  $\lim_{n \rightarrow \infty} a_n = \infty$ , then  $\lim_{n \rightarrow \infty} b_n = \infty$ .

## 3.2 Series

### 3.2.1 Infinite Series

**Definition:**

(1). The sum of the terms of the sequence  $\{a_n\}$  is called an infinite series (or a series) and it is denoted by  $\sum_{n=1}^{\infty} a_n$  or  $\sum a_n$ .

(2).  $s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$  is called the  $n^{\text{th}}$  partial sum of the series.

**Definition:**

If the sequence  $\{s_n\}$  of the partial sums of the series  $\sum_{n=1}^{\infty} a_n$  is convergent, and

$\lim_{n \rightarrow \infty} s_n = s \in \mathbb{R}$ , then the series  $\sum_{n=1}^{\infty} a_n$  is convergent and  $\sum_{n=1}^{\infty} a_n = s$ .

The number  $s$  is called the sum of the series.

If the sequence  $\{s_n\}$  is divergent, then the series is divergent.

**Example (1):** Show that  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$  is convergent and find its sum.

Solution : Note that  $\frac{1}{n} - \frac{1}{n+1} = \frac{(n+1) - n}{n(n+1)} = \frac{1}{n(n+1)}$ .

$$\begin{aligned} s_n &= \sum_{i=1}^n \frac{1}{i(i+1)} = \sum_{i=1}^n \left( \frac{1}{i} - \frac{1}{i+1} \right) \\ &= \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots + \left( \frac{1}{n} - \frac{1}{n+1} \right) \end{aligned}$$

$$\text{So, } s_n = \sum_{i=1}^n \frac{1}{i(i+1)} = 1 - \frac{1}{n+1}.$$

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left( 1 - \frac{1}{n+1} \right) = 1 - 0 = 1.$$

Therefore,  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$  is convergent, and  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ .

### 3.2.2 Sum of a Geometric Series

**Definition :**  $\sum_{n=0}^{\infty} ar^n$  (where  $a \neq 0$  and  $r \in \mathbb{R}$ ) is called a geometric series.

Note that  $s_n = a + ar + ar^2 + \cdots + ar^n$ ,  $rs_n = ra + ar^2 + ar^3 + \cdots + ar^n + ar^{n+1}$ .  
 $s_n - rs_n = a - ar^{n+1} \implies (1-r)s_n = a(1 - a^{n+1}) \implies s_n = \frac{a(1 - a^{n+1})}{1-r}$   
 where  $r \neq 1$ .

**Sum of a geometric series :**

- (1). If  $|r| < 1$  , then  $\sum_{n=0}^{\infty} ar^n$  is convergent and its sum is  $\frac{a}{1-r}$ .
- (2). If  $|r| \geq 1$  , then  $\sum_{n=0}^{\infty} ar^n$  is divergent.

**Example (2):** Find the sum of the geometric series  $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \cdots$ .

Solution :  $a = 3$  .

$$r = \frac{a_2}{a_1} = \frac{-\frac{3}{2}}{3} = -\frac{1}{2}.$$

the sum of the geometric series is  $\frac{a}{1-r} = \frac{3}{1-(-\frac{1}{2})} = \frac{3}{(\frac{3}{2})} = 2$  .

**Example (3):** Does the series  $\sum_{n=1}^{\infty} 3^{2n} 5^{1-n}$  converge or diverge?.

Solution :

$$\begin{aligned} \sum_{n=1}^{\infty} 3^{2n} 5^{1-n} &= \sum_{n=1}^{\infty} 9^n 5^{1-n} = \sum_{n=1}^{\infty} \frac{9^n}{5^{n-1}} \\ &= \sum_{n=1}^{\infty} \frac{9 (9^{n-1})}{5^{n-1}} = \sum_{n=1}^{\infty} 9 \left(\frac{9}{5}\right)^{n-1} = \sum_{n=0}^{\infty} 9 \left(\frac{9}{5}\right)^n . \end{aligned}$$

Since  $|r| = \left|\frac{9}{5}\right| = \frac{9}{5} > 1$  , then the geometric series diverges.

**3.2.3 Test for Divergence**

**Example (4):** Show that the harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent.

Solution :

$$s_2 = s_{2^1} = 1 + \frac{1}{2} .$$

$$s_4 = s_{2^2} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{3}{4} .$$

$$\begin{aligned} s_8 &= s_{2^3} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2} . \end{aligned}$$

So,  $s_{2^n} > 1 + \frac{n}{2}$  (By induction).

$$\lim_{n \rightarrow \infty} s_{2^n} \geq \lim_{n \rightarrow \infty} \left(1 + \frac{n}{2}\right) = \infty .$$

Therefore, The harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent.

**Theorem :** If the series  $\sum_{n=1}^{\infty} a_n$  is convergent, then  $\lim_{n \rightarrow \infty} a_n = 0$  .

**Important Note :** The converse of the last theorem is not true.

In other words, if  $\lim_{n \rightarrow \infty} a_n = 0$  then this does not mean that  $\sum_{n=1}^{\infty} a_n$  is convergent.

For example  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , while  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent.

**Test of divergence :** If  $\lim_{n \rightarrow \infty} a_n$  does not exist, or  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then the series

$\sum_{n=1}^{\infty} a_n$  is divergent.

**Example (5):** Show that the series  $\sum_{n=1}^{\infty} \frac{n^3}{2n^3 + 1}$  is divergent.

Solution :

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^3}{2n^3 + 1} = \frac{1}{2} \neq 0 .$$

Therefore, the series  $\sum_{n=1}^{\infty} \frac{n^3}{2n^3 + 1}$  is divergent.

### 3.2.4 Properties of Convergent Series

**Theorem :** If  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are both convergent series and  $c \in \mathbb{R}$  then :

- (1). The series  $\sum_{n=1}^{\infty} (ca_n)$  is convergent, and  $\sum_{n=1}^{\infty} (ca_n) = c \sum_{n=1}^{\infty} a_n$ .
- (2). The series  $\sum_{n=1}^{\infty} (a_n + b_n)$  is convergent, and  $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ .
- (3). The series  $\sum_{n=1}^{\infty} (a_n - b_n)$  is convergent, and  $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$ .

**Example (6):** Find the sum of the series  $\sum_{n=1}^{\infty} \left( \frac{4}{n(n+1)} + \frac{2}{3^n} \right)$ .

Solution :

From Example (1),  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ .

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = \sum_{n=1}^{\infty} \frac{2}{3} \left( \frac{1}{3} \right)^{n-1} = \sum_{n=0}^{\infty} \frac{2}{3} \left( \frac{1}{3} \right)^n = \frac{2}{3} \frac{1}{1 - \frac{1}{3}} = \frac{2}{3} \frac{3}{2} = 1 .$$

$$\begin{aligned} \sum_{n=1}^{\infty} \left( \frac{4}{n(n+1)} + \frac{2}{3^n} \right) &= \sum_{n=1}^{\infty} \frac{4}{n(n+1)} + \sum_{n=1}^{\infty} \frac{2}{3^n} \\ &= 4 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{2}{3^n} = 4(1) + 1 = 5 . \end{aligned}$$

**3.2.5 EXERCISES**

1. Let  $a_n = \frac{2n}{3n+1}$ .
  - (a). Determine whether  $\{a_n\}$  is convergent.
  - (b). Determine whether  $\sum_{n=1}^{\infty} a_n$  is convergent.
2. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.
  - (1).  $\sum_{n=1}^{\infty} \frac{5}{\pi^n}$
  - (2).  $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$
  - (3).  $\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n}$
  - (4).  $\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$
  - (5).  $\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n}$
3. Determine whether the series is convergent or divergent. If it is convergent, find its sum.
  - (1).  $\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$
  - (2).  $\sum_{n=1}^{\infty} \frac{n^2}{n^2-2n+5}$
  - (3).  $\sum_{n=1}^{\infty} \frac{1}{4+e^{-n}}$
  - (4).  $\sum_{n=1}^{\infty} \frac{2^n+4^n}{e^n}$
  - (5).  $\sum_{n=1}^{\infty} (\sin 100)^n$
  - (6).  $\sum_{n=1}^{\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n}$
  - (7).  $\sum_{n=1}^{\infty} \ln \left( \frac{n^2+1}{2n^2+1} \right)$

### 3.3 The Integral Test

#### 3.3.1 The Integral Test

**Theorem (The Integral Test) :**

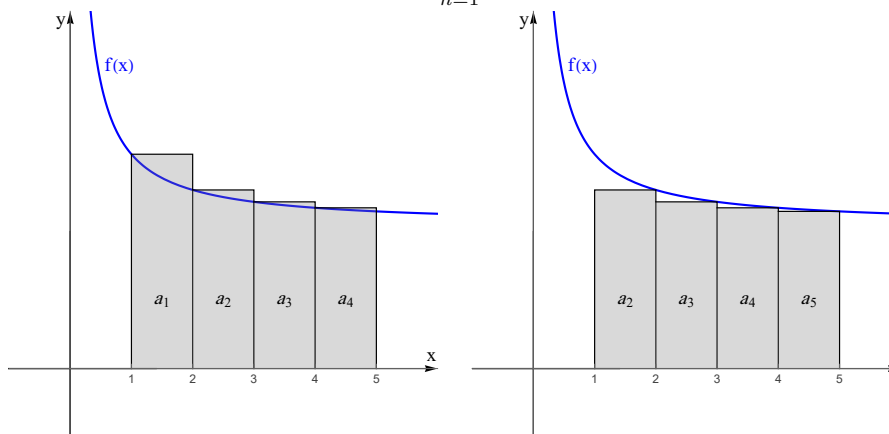
Suppose  $f$  is a positive, decreasing continuous function defined on  $[1, \infty)$ , and

let  $a_n = f(n)$ , then the series  $\sum_{n=1}^{\infty} a_n$  is convergent if and only if the improper

integral  $\int_1^{\infty} f(x) dx$  is convergent. In other words:

(1). If  $\int_1^{\infty} f(x) dx$  is convergent, then  $\sum_{n=1}^{\infty} a_n$  is convergent .

(2). If  $\int_1^{\infty} f(x) dx$  is divergent, then  $\sum_{n=1}^{\infty} a_n$  is divergent .



**Example (1):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$  .

Solution :

Let  $f(x) = \frac{1}{x^2 + 1}$  , the  $f$  is a positive continuous function on  $[1, \infty)$  .

$f'(x) = \frac{-2x}{(x^2 + 1)^2} < 0$  on the interval  $[1, \infty)$ , then  $f$  is decreasing on  $[1, \infty)$  .

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2 + 1} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2 + 1} dx = \lim_{t \rightarrow \infty} [\tan^{-1} x]_1^t \\ &= \lim_{t \rightarrow \infty} [\tan^{-1}(t) - \tan^{-1}(1)] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \end{aligned}$$

So,  $\int_1^{\infty} \frac{1}{x^2 + 1} dx$  is convergent, then  $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$  is convergent.

**Example (2):** Discuss the convergence of the series  $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$  .

Solution :

Let  $f(x) = \frac{1}{x \ln x}$  , the  $f$  is a positive continuous function on  $[2, \infty)$ .

$f'(x) = \frac{-(1 + \ln x)}{(x \ln x)^2} < 0$  on  $[2, \infty)$ , then  $f$  is decreasing on  $[2, \infty)$ .

$$\begin{aligned} \int_2^\infty \frac{1}{x \ln x} dx &= \lim_{t \rightarrow \infty} \int_2^t \frac{\left(\frac{1}{x}\right)}{\ln x} dx = \lim_{t \rightarrow \infty} [\ln(\ln x)]_2^t \\ &= \lim_{t \rightarrow \infty} [\ln(\ln t) - \ln(\ln 2)] = \infty. \quad (\text{Note that } \lim_{t \rightarrow \infty} \ln(t) = \infty). \end{aligned}$$

So,  $\int_2^\infty \frac{1}{x \ln x} dx$  is divergent, then  $\sum_{n=2}^\infty \frac{1}{n \ln n}$  is divergent.

**Definition :** (The  $p$ -series)

The series  $\sum_{n=1}^\infty \frac{1}{n^p}$  where  $p \in \mathbb{R}$  is called a  $p$ -series.

**Theorem :**

The  $p$ -series  $\sum_{n=1}^\infty \frac{1}{n^p}$  is convergent if  $p > 1$ , and divergent if  $p \leq 1$ .

Proof:

(1). If  $p < 0$ , then  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n^p} = \lim_{n \rightarrow \infty} n^{-p} = \infty$ . (Note:  $-p > 0$ ).

Since  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then  $\sum_{n=1}^\infty \frac{1}{n^p}$  is divergent.

(2). If  $p = 0$ , then  $a_n = \frac{1}{n^0} = 1$ , hence  $\lim_{n \rightarrow \infty} a_n = 1 \neq 0$ .

Therefore,  $\sum_{n=1}^\infty \frac{1}{n^p}$  is divergent.

(3). If  $0 < p < 1$ , then using the integral test:

Note that  $f(x) = \frac{1}{x^p}$  is a positive, decreasing continuous function on  $[1, \infty)$ .

$$\begin{aligned} \int_1^\infty \frac{1}{x^p} dx &= \lim_{t \rightarrow \infty} \int_1^t x^{-p} dx = \lim_{t \rightarrow \infty} \left[ \frac{x^{-p+1}}{-p+1} \right]_1^t = \lim_{t \rightarrow \infty} \left[ \frac{x^{1-p}}{1-p} \right]_1^t \\ &= \lim_{t \rightarrow \infty} \left[ \frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right] = \infty. \quad (\text{Note : } 1-p > 0). \end{aligned}$$

Since  $\int_1^\infty \frac{1}{x^p} dx$  is divergent, then  $\sum_{n=1}^\infty \frac{1}{n^p}$  is divergent.

(4). If  $p = 1$ , then using the integral test:

Note that  $f(x) = \frac{1}{x}$  is a positive, decreasing continuous function on  $[1, \infty)$ .

$$\int_1^\infty \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx = \lim_{t \rightarrow \infty} [\ln x]_1^t = \lim_{t \rightarrow \infty} [\ln(t) - \ln(1)] = \infty.$$

Since  $\int_1^\infty \frac{1}{x} dx$  is divergent, then  $\sum_{n=1}^\infty \frac{1}{n}$  is divergent.

(5). If  $p > 1$ , then using the integral test:

Note that  $f(x) = \frac{1}{x^p}$  is a positive, decreasing continuous function on  $[1, \infty)$ .

$$\begin{aligned} \int_1^\infty \frac{1}{x^p} dx &= \lim_{t \rightarrow \infty} \int_1^t x^{-p} dx = \lim_{t \rightarrow \infty} \left[ \frac{x^{-p+1}}{-p+1} \right]_1^t = \lim_{t \rightarrow \infty} \left[ \frac{1}{(1-p)x^{p-1}} \right]_1^t \\ &= \lim_{t \rightarrow \infty} \left[ \frac{1}{(1-p)t^{p-1}} - \frac{1}{1-p} \right] = 0 - \frac{1}{1-p} = \frac{1}{p-1}. \end{aligned}$$

Since  $\int_1^\infty \frac{1}{x^p} dx$  is convergent, then  $\sum_{n=1}^\infty \frac{1}{n^p}$  is convergent.

**Example (3):** Discuss the convergence of the series  $\sum_{n=1}^\infty \frac{1}{n^4}$ .

Solution :

$\sum_{n=1}^\infty \frac{1}{n^4}$  is a  $p$ -series, where  $p = 4 > 1$ , then  $\sum_{n=1}^\infty \frac{1}{n^4}$  is convergent.

**Example (4):** Discuss the convergence of the series  $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ .

Solution :

$\sum_{n=1}^\infty \frac{1}{\sqrt{n}} = \sum_{n=1}^\infty \frac{1}{n^{\frac{1}{2}}}$  is a  $p$ -series, where  $p = \frac{1}{2} < 1$ , then  $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$  is divergent.

**Example (5):** Discuss the convergence of the series  $\sum_{n=1}^\infty n e^{-n^2}$ .

Solution : Let  $f(x) = x e^{-x^2}$  then  $f$  is a positive continuous function on  $[1, \infty)$ .

$f'(x) = e^{-x^2} - 2x^2 e^{-x^2} = (1 - 2x^2) e^{-x^2} < 0$  on  $[1, \infty)$ .

Therefore,  $f$  is decreasing on  $[1, \infty)$ . Using the integral test :

$$\begin{aligned} \int_1^\infty x e^{-x^2} dx &= \lim_{t \rightarrow \infty} \left( -\frac{1}{2} \int_1^t e^{-x^2} (-2x) dx \right) = \lim_{t \rightarrow \infty} \left( -\frac{1}{2} [e^{-x^2}]_1^t \right) \\ &= \lim_{t \rightarrow \infty} \left( -\frac{1}{2} [e^{-t^2} - e^{-1}] \right) = \lim_{t \rightarrow \infty} \left( -\frac{1}{2} \left[ \frac{1}{e^{t^2}} - \frac{1}{e} \right] \right) = -\frac{1}{2} \left( 0 - \frac{1}{e} \right) = \frac{1}{2e}. \end{aligned}$$

Since  $\int_1^\infty x e^{-x^2} dx$  is convergent, then  $\sum_{n=1}^\infty n e^{-n^2}$  is convergent.

**3.3.2 EXERCISES**

1. Use the Integral Test to determine whether the series is convergent or divergent.

$$(1). \sum_{n=1}^{\infty} n^{-3} \quad (2). \sum_{n=1}^{\infty} \frac{2}{5n-1} \quad (3). \sum_{n=2}^{\infty} \frac{n^2}{n^3+1}$$

$$(4). \sum_{n=1}^{\infty} n^2 e^{-n^3} \quad (5). \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$$

2. Determine whether the series is convergent or divergent.

$$(1). \sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}} \quad (2). \sum_{n=3}^{\infty} n^{-0.9999} \quad (3). \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots$$

$$(4). \sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2} \quad (5). \sum_{n=1}^{\infty} \frac{1}{n^2+4} \quad (6). 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots$$

$$(7). \sum_{n=2}^{\infty} \frac{1}{n \ln n} \quad (8). \sum_{n=2}^{\infty} \frac{\ln n}{n^2} \quad (9). \sum_{n=1}^{\infty} \frac{1}{n^2+n^3}$$

## 3.4 The Comparison Tests

### 3.4.1 The Direct Comparison Test

**Theorem :** (The Direct Comparison Test)

Suppose that  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are series with positive terms.

- (1). If  $\sum_{n=1}^{\infty} b_n$  is convergent and  $a_n \leq b_n$  for all  $n$ , then  $\sum_{n=1}^{\infty} a_n$  is also convergent.
- (2). If  $\sum_{n=1}^{\infty} b_n$  is divergent and  $a_n \geq b_n$  for all  $n$ , then  $\sum_{n=1}^{\infty} a_n$  is also divergent.

**Example (1):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} \frac{\sin n}{2+3^n}$ .

Solution :

For all  $n \geq 1$  :  $\frac{\sin n}{2+3^n} \leq \frac{1}{2+3^n} < \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$ .

Since  $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$  is convergent (a geometric series with  $r = \frac{1}{3} < 1$ ), then

$\sum_{n=1}^{\infty} \frac{\sin n}{2+3^n}$  is also convergent (by The direct comparison test).

**Example (2):** Discuss the convergence of the series  $\sum_{n=2}^{\infty} \frac{2}{\sqrt{n}-1}$ .

Solution :

For all  $n \geq 2$  :  $\sqrt{n}-1 < \sqrt{n} \implies \frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$

$\implies \frac{2}{\sqrt{n}-1} > \frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$ .

Since  $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}}}$  is divergent ( a  $p$ -series, with  $p = \frac{1}{2} \leq 1$ ), then

$\sum_{n=2}^{\infty} \frac{2}{\sqrt{n}-1}$  is also divergent (by The direct comparison test).

**Example (3):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^4+1}$ .

Solution :

For all  $n \geq 1$  :  $e^{-n} = \frac{1}{e^n} < 1 \implies \frac{e^{-n}}{n^4+1} < \frac{1}{n^4+1} < \frac{1}{n^4}$ .

Since  $\sum_{n=1}^{\infty} \frac{1}{n^4}$  is convergent( a  $p$ -series, with  $p = 4 > 1$ ), then  $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^4+1}$  is also convergent.

**Example (4):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ .

Solution :

Since the function  $f(x) = \ln(x)$  is increasing on  $[1, \infty)$ , then for all  $n \geq 3$  :

$$\ln(n) \geq \ln(3) > \ln(e) = 1 \implies \frac{\ln n}{n} > \frac{1}{n} .$$

Since  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent ( a  $p$ -series, with  $p = 1 \leq 1$ ), then  $\sum_{n=1}^{\infty} \frac{\ln n}{n}$  is also divergent.

### 3.4.2 Limit Comparison Test

**Theorem:** (Limit comparison test)

Suppose that  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are series with positive terms.

If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$ , where  $c \in \mathbb{R}$  and  $c > 0$ , then either both series converge or both series diverge.

**Example (1):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ .

Solution :

Note that  $a_n = \frac{1}{2^n - 1}$ . Let  $b_n = \frac{1}{2^n}$ .

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{2^n - 1}\right)}{\left(\frac{1}{2^n}\right)} = \lim_{n \rightarrow \infty} \frac{2^n}{2^n - 1} = 1 > 0 .$$

Since  $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$  is convergent (a geometric series with  $r = \frac{1}{2} < 1$ ),

then  $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$  is convergent (by limit comparison test).

**Example (2):** Discuss the convergence of the series  $\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2 - 1}}$ .

Solution :

Note that  $a_n = \frac{1}{\sqrt[3]{n^2 - 1}}$ . Let  $b_n = \frac{1}{\sqrt[3]{n^2}} = \frac{1}{n^{\frac{2}{3}}}$ .

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sqrt[3]{n^2}}{\sqrt[3]{n^2 - 1}} = \lim_{n \rightarrow \infty} \sqrt[3]{\frac{n^2}{n^2 - 1}} = \sqrt[3]{1} = 1 > 0 .$$

Since  $\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{2}{3}}}$  is divergent (a  $p$ -series, with  $p = \frac{2}{3} \leq 1$ ), then

$\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2 - 1}}$  is divergent (by limit comparison test).

**Example (3):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} \frac{5n^2 + 3n}{3^n(n^2 + 2)}$ .

Solution :

Note that  $a_n = \frac{5n^2 + 3n}{3^n(n^2 + 2)}$ . Let  $b_n = \frac{5n^2}{3^n n^2} = \frac{5}{3^n}$ .

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left( \frac{5n^2 + 3n}{3^n(n^2 + 2)} \cdot \frac{3^n}{5} \right) = \lim_{n \rightarrow \infty} \frac{5n^2 + 3n}{5n^2 + 10} = 1 > 0.$$

Since  $\sum_{n=1}^{\infty} \frac{5}{3^n} = \sum_{n=1}^{\infty} 5 \left( \frac{1}{3} \right)^n$  is convergent (a geometric series with  $r = \frac{1}{3} < 1$ ),

then  $\sum_{n=1}^{\infty} \frac{5n^2 + 3n}{3^n(n^2 + 2)}$  is convergent.

**Example (4):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} \frac{n + 3n^2}{\sqrt{1 + n^5}}$ .

Solution :

Note that  $a_n = \frac{3n^2 + n}{\sqrt{n^5 + 1}}$ . Let  $b_n = \frac{3n^2}{\sqrt{n^5}} = \frac{3n^2}{n^{\frac{5}{2}}} = \frac{3}{n^{\frac{1}{2}}}$ .

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left( \frac{3n^2 + n}{\sqrt{n^5 + 1}} \cdot \frac{n^{\frac{1}{2}}}{3} \right) = \lim_{n \rightarrow \infty} \frac{3n^{\frac{5}{2}} + n^{\frac{3}{2}}}{3\sqrt{n^5 + 1}} = 1 > 0.$$

Since  $\sum_{n=1}^{\infty} \frac{3}{n^{\frac{1}{2}}}$  is divergent (a  $p$ -series, with  $p = \frac{1}{2} < 1$ ), then  $\sum_{n=1}^{\infty} \frac{n + 3n^2}{\sqrt{1 + n^5}}$  is divergent.

## 3.4.3 EXERCISES

1. (a). Use the Direct Comparison Test to show that the first series converges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n}{n^3 + 5}, \quad \sum_{n=2}^{\infty} \frac{1}{n^2}.$$

- (b). Use the Limit Comparison Test to show that the first series converges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n}{n^3 - 5}, \quad \sum_{n=2}^{\infty} \frac{1}{n^2}.$$

2. (a). Use the Direct Comparison Test to show that the first series diverges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n^2 + n}{n^3 - 2}, \quad \sum_{n=2}^{\infty} \frac{1}{n}.$$

- (b). Use the Limit Comparison Test to show that the first series diverges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n^2 - n}{n^3 + 2}, \quad \sum_{n=2}^{\infty} \frac{1}{n}.$$

3. Which of the following inequalities can be used to show that  $\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$  converges?

$$(a). \frac{n}{n^3 + 1} \geq \frac{1}{n^3 + 1} \quad (b). \frac{n}{n^3 + 1} \leq \frac{1}{n} \quad (c). \frac{n}{n^3 + 1} \leq \frac{1}{n^2}.$$

4. Determine whether the series converges or diverges.

$$\begin{array}{lll} (1). \sum_{n=1}^{\infty} \frac{1}{n^3 + 8} & (2). \sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1} & (3). \sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}} \\ (4). \sum_{n=1}^{\infty} \frac{n-1}{n^3 + 1} & (5). \sum_{n=1}^{\infty} \frac{9^n}{3 + 10^n} & (6). \sum_{n=1}^{\infty} \frac{6^n}{5^n - 1} \\ (7). \sum_{n=2}^{\infty} \frac{1}{\ln n} & (8). \sum_{n=1}^{\infty} \frac{1 + \cos n}{e^n} & (9). \sum_{n=1}^{\infty} \frac{1}{n^n} \\ (10). \sum_{n=1}^{\infty} \frac{n+1}{n^3 + n} & (11). \sum_{n=1}^{\infty} \frac{2 + \sin n}{n^2} & (12). \sum_{n=1}^{\infty} \frac{n!}{n^n} \end{array}$$

## 3.5 Alternating Series and Absolute Convergence

### 3.5.1 Alternating Series

**Definition :** If  $a_n \geq 0$ , the the series  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  is called an alternating series.

**Theorem :** (Alternating Series Test)

If  $a_n \geq 0$  and the alternating series  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  satisfies :

- (1). The sequence  $\{a_n\}$  is decreasing, i.e.  $a_{n+1} \leq a_n$  for all  $n \geq 1$ .
- (2).  $\lim_{n \rightarrow \infty} a_n = 0$ .

Then the alternating series  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  is convergent.

**Example (1):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ .

Solution :

- (1). For all  $n \geq 1$  :  $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$ .

The sequence  $\{a_n\} = \left\{ \frac{1}{n} \right\}$  is decreasing.

- (2).  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .

By the alternating series test,  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  is convergent.

**Example (2):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{4n^2 - 1}$ .

Solution :

- (1). Let  $f(x) = \frac{2x}{4x^2 - 1}$ , then for  $x \geq 1$  :

$$f'(x) = \frac{2(4x^2 - 1) - 2x(8x)}{(4x^2 - 1)^2} = \frac{8x^2 - 2 - 16x^2}{(4x^2 - 1)^2} = \frac{-2 - 8x^2}{(4x^2 - 1)^2} < 0.$$

The sequence  $\{a_n\} = \left\{ \frac{2n}{4n^2 - 1} \right\}$  is decreasing.

- (2).  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2n}{4n^2 - 1} = 0$ .

By the alternating series test,  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{4n^2 - 1}$  is convergent.

**Example (3):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n+1}$ .

Solution :

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0.$$

So, the alternating series test does not apply.

Note that if  $b_n = (-1)^{n+1} \frac{n}{2n+1}$ , then  $\lim_{n \rightarrow \infty} b_{2n} = \frac{1}{2}$  and  $\lim_{n \rightarrow \infty} b_{2n+1} = -\frac{1}{2}$ .

So,  $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} (-1)^{n+1} \frac{n}{2n+1}$  does not exist.

Therefore,  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n+1}$  is divergent (by test of divergence).

### 3.5.2 Absolute Convergence and Conditional Convergence

**Definition :** (Absolute Convergence)

A series  $\sum_{n=1}^{\infty} a_n$  is called absolutely convergent if the series of absolute values

$\sum_{n=1}^{\infty} |a_n|$  is convergent .

**Example (4):** The series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ , where  $p > 1$  is absolutely convergent.

Solution :

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^p} \right| = \sum_{n=1}^{\infty} \frac{1}{n^p}$  is convergent (a  $p$ -series, with  $p > 1$ ).

Therefore,  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$  is absolutely convergent.

**Definition :** (Conditional Convergence)

A series  $\sum_{n=1}^{\infty} a_n$  is called conditionally convergent if it is convergent but not

absolutely convergent. That is, if  $\sum_{n=1}^{\infty} a_n$  is convergent but  $\sum_{n=1}^{\infty} |a_n|$  is divergent.

**Example (5):** The series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  is conditionally convergent.

Solution :

Note that  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  is convergent (see Example (1)).

But  $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$  is divergent.

Therefore,  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  is conditionally convergent.

**Theorem :**

If a series is absolutely convergent then it is convergent.

**Example (6):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ .

Solution :

$$\left| \frac{\cos n}{n^2} \right| = \frac{|\cos n|}{|n^2|} \leq \frac{1}{n^2}.$$

Since  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  is convergent then by direct comparison test  $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$  is convergent.

Therefore,  $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$  is absolutely convergent and hence convergent.

**Example (7):** Discuss the convergence of the following series :

$$(a). \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1} \quad (b). \sum_{n=1}^{\infty} (-1)^n \frac{1}{1+n\sqrt{n}} \quad (c). \sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{(2n-3)^2}.$$

Solution :

$$(a). \sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{n+1}{n^2+1} \right| = \sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$$

Let  $a_n = \frac{n+1}{n^2+1}$ , put  $b_n = \frac{1}{n}$ , then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left( \frac{n+1}{n^2+1} \cdot n \right) = \lim_{n \rightarrow \infty} \frac{n^2+n}{n^2+1} = 1 > 0.$$

Since  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent, then  $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{n+1}{n^2+1} \right|$  is divergent.

(1). Let  $f(x) = \frac{x+1}{x^2+1}$ , then for  $x \geq 1$  :

$$\begin{aligned} f'(x) &= \frac{(1)(x^2+1) - (x+1)(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2-2x}{(x^2+1)^2} \\ &= \frac{-x^2-2x+1}{(x^2+1)^2} = \frac{1-(x^2+2x)}{(x^2+1)^2} < 0. \end{aligned}$$

(Note that  $1-(x^2+2x) < 0$  when  $x \geq 1$ ).

Therefore, the sequence  $\left\{ \frac{n+1}{n^2+1} \right\}$  is decreasing.

$$(2). \lim_{n \rightarrow \infty} \frac{n+1}{n^2+1} = 0.$$

From (1) and (2), the alternating series  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1}$  is convergent.

Hence,  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1}$  is conditionally convergent.

$$(b). \sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{1+n\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{1+n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{1+n^{\frac{3}{2}}}.$$

Let  $a_n = \frac{1}{1+n^{\frac{3}{2}}}$ , put  $b_n = \frac{1}{n^{\frac{3}{2}}}$ , then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^{\frac{3}{2}}}{n^{\frac{3}{2}}+1} = 1 > 0.$$

Since  $\sum_{n=1}^{\infty} \frac{1}{1+n^{\frac{3}{2}}}$  is convergent, then  $\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{1+n\sqrt{n}} \right|$  is convergent.

Therefore,  $\sum_{n=1}^{\infty} (-1)^n \frac{1}{1+n\sqrt{n}}$  is absolutely convergent.

(c). Let  $a_n = (-1)^n \frac{n^2+1}{(2n-3)^2} = (-1)^n \frac{n^2+1}{4n^2-12n+9}$

$\lim_{n \rightarrow \infty} a_{2n} = \frac{1}{4}$  and  $\lim_{n \rightarrow \infty} a_{2n+1} = -\frac{1}{4}$ .

Hence  $\lim_{n \rightarrow \infty} a_n$  does not exist.

Therefore,  $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{(2n-3)^2}$  is divergent.

## 3.5.3 EXERCISES

1. Test the series for convergence or divergence.

$$(1). \frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \cdots \quad (2). \frac{1}{\ln 3} - \frac{1}{\ln 4} + \frac{1}{\ln 5} - \frac{1}{\ln 6} + \frac{1}{\ln 7} - \cdots$$

$$(3). \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n} \quad (4). \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}} \quad (5). \sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$$

$$(6). \sum_{n=1}^{\infty} (-1)^n e^{-n} \quad (7). \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4} \quad (8). \sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$$

$$(9). \sum_{n=1}^{\infty} \frac{n \cos(n\pi)}{2^n} \quad (10). \sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right) \quad (11). \sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$$

2. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$(1). \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \quad (2). \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[3]{n^2}} \quad (3). \sum_{n=0}^{\infty} (-1)^{n+1} \frac{n^2}{n^2+1}$$

$$(4). \sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1} \quad (5). \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \quad (6). \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

$$(7). \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

## 3.6 The Ratio and Root Tests

### 3.6.1 The Ratio Test

**Theorem :** (the Ratio Test)

- (1). If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent (and therefore convergent).
- (2). If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$  or  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ , then the series  $\sum_{n=1}^{\infty} a_n$  is divergent.
- (3). If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ , the ratio test is inconclusive.

**Example (1):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} (-1)^n \frac{n^4}{2^n}$ .

Solution :

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^4}{2^{n+1}} \cdot \frac{2^n}{n^4} \right| = \lim_{n \rightarrow \infty} \frac{1}{2} \left( \frac{n+1}{n} \right)^4 = \frac{1}{2}(1)^4 = \frac{1}{2} < 1.$$

Therefore,  $\sum_{n=1}^{\infty} (-1)^n \frac{n^4}{2^n}$  is absolutely convergent.

**Example (2):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{n!}$ .

Solution :

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{5^{n+1}}{n! (n+1)} \cdot \frac{n!}{5^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{5}{n+1} = 0 < 1. \end{aligned}$$

Therefore,  $\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{n!}$  is absolutely convergent.

**Example (3):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ .

Solution :

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left[ \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right] = \lim_{n \rightarrow \infty} \left[ \frac{(n+1)^n (n+1)}{n! (n+1)} \cdot \frac{n!}{n^n} \right] \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^n}{n^n} = \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right)^n = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n = e > 1. \end{aligned}$$

Therefore,  $\sum_{n=1}^{\infty} \frac{n^n}{n!}$  is divergent.

Note that  $\lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n = e$ , using L'Hôpital's rule.

**Example (4):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{3^n}$ .

Solution : For  $n \geq 2$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\ln(n+1)}{3^{n+1}} \cdot \frac{3^n}{\ln(n)} \right| = \lim_{n \rightarrow \infty} \frac{1}{3} \frac{\ln(n+1)}{\ln(n)} \\ &= \lim_{n \rightarrow \infty} \frac{1}{3} \left( \frac{\frac{1}{n+1}}{\frac{1}{n}} \right) = \lim_{n \rightarrow \infty} \frac{1}{3} \frac{n}{n+1} = \frac{1}{3}(1) = \frac{1}{3} < 1. \end{aligned}$$

Therefore,  $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{3^n}$  is convergent.

**Notes :**

(1). The harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left( \frac{\frac{1}{n+1}}{\frac{1}{n}} \right) = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1.$$

(2). The series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  is convergent.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left( \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \right) = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^2 = (1)^2 = 1.$$

### 3.6.2 The Root Test

**Theorem :** (the Root Test)

(1). If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent (and therefore convergent).

(2). If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$  or  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$ , then the series  $\sum_{n=1}^{\infty} a_n$  is divergent.

(3). If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$ , the root test is inconclusive.

**Example (5):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} \left( \frac{n+3}{3n+2} \right)^n$ .

Solution :

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left( \frac{n+3}{3n+2} \right)^n \right|} = \lim_{n \rightarrow \infty} \left[ \left( \frac{n+3}{3n+2} \right)^n \right]^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{n+3}{3n+2} = \frac{1}{3} < 1. \end{aligned}$$

Therefore,  $\sum_{n=1}^{\infty} \left( \frac{n+3}{3n+2} \right)^n$  is absolutely convergent.

**Example (6):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} \left(\frac{2n+1}{n+3}\right)^n$ .

Solution :

$$\begin{aligned}\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{2n+1}{n+3}\right)^n\right|} = \lim_{n \rightarrow \infty} \left[\left(\frac{2n+1}{n+3}\right)^n\right]^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{2n+1}{n+3} = 2 > 1.\end{aligned}$$

Therefore,  $\sum_{n=1}^{\infty} \left(\frac{2n+1}{n+3}\right)^n$  is divergent.

**Example (7):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n}$ .

Solution :

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\frac{2^{3n+1}}{n^n}\right|} = \lim_{n \rightarrow \infty} \left(\frac{2^{3n+1}}{n^n}\right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{2^{3+\frac{1}{n}}}{n} = 0 < 1.$$

Therefore,  $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n}$  is absolutely convergent.

Another solution :

$$\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n} = \sum_{n=1}^{\infty} \frac{(2) 2^{3n}}{n^n} = \sum_{n=1}^{\infty} \frac{(2) (2^3)^n}{n^n} = 2 \sum_{n=1}^{\infty} \left(\frac{2^3}{n}\right)^n = 2 \sum_{n=1}^{\infty} \left(\frac{8}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{8}{n}\right)^n\right|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{8}{n}\right)^n\right|} = \lim_{n \rightarrow \infty} \frac{8}{n} = 0 < 1.$$

Therefore,  $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n}$  is absolutely convergent.

**Example (8):** Discuss the convergence of the series  $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$ .

Solution :

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{n+1}{n}\right)^n\right|} = \lim_{n \rightarrow \infty} \left[\left(\frac{n+1}{n}\right)^n\right]^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1.$$

So, The root test is inconclusive.

$$\text{But } \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0.$$

Therefore,  $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$  is divergent (by test of divergence).

**3.6.3 EXERCISES**

1. What can you say about the series  $\sum_{n=1}^{\infty} a_n$  in each of the following cases?

(a).  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 8$       (b).  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.8$       (c).  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$

2. Use the Ratio Test to determine whether the series is convergent or divergent.

(1).  $\sum_{n=1}^{\infty} \frac{n}{5^n}$       (2).  $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$       (3).  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{2^n n^3}$

(4).  $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$       (5).  $\sum_{n=1}^{\infty} \frac{1}{n!}$       (6).  $\sum_{n=1}^{\infty} \frac{n}{100^n}$

(7).  $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

3. Use the Root Test to determine whether the series is convergent or divergent.

(1).  $\sum_{n=1}^{\infty} \left( \frac{n^2+1}{2n^2+1} \right)^n$       (2).  $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$       (3).  $\sum_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right)^{n^2}$

### 3.7 Strategy for Testing Series

(1). Divergence test :

If  $\lim_{n \rightarrow \infty} a_n \neq 0$ , or  $\lim_{n \rightarrow \infty} a_n$  does not exist, then  $\sum_{n=1}^{\infty} a_n$  is divergent.

$\sum_{n=1}^{\infty} \frac{n^2 + 2n}{n^2 + 4}$  is divergent, because  $\lim_{n \rightarrow \infty} \frac{n^2 + 2n}{n^2 + 4} = 1 \neq 0$ .

$\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{n+2}$  is divergent, because  $\lim_{n \rightarrow \infty} (-1)^n \frac{n-1}{n+2}$  does not exist.

(2).  $p$ -series : It has the form  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  :

If  $p > 1$  then  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is convergent, like  $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ .

If  $p \leq 1$  then  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is divergent, like  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ .

(3). Geometric series : It has the form  $\sum_{n=0}^{\infty} a r^n$  where  $a \neq 0$  and  $r \in \mathbb{R}$  :

If  $|r| < 1$  then  $\sum_{n=0}^{\infty} a r^n$  is convergent, like  $\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ .

If  $|r| \geq 1$  then  $\sum_{n=0}^{\infty} a r^n$  is divergent, like  $\sum_{n=1}^{\infty} \frac{3^n}{2}$ .

(4). Comparison test :

It is used when the series has a form that is similar to a  $p$ -series or a geometric series, it can be used also when  $a_n$  is a rational function or an algebraic function of  $n$ .

It is used on series of positive terms, it can be used on  $\sum_{n=1}^{\infty} |a_n|$  to test for absolute convergence.

$\sum_{n=1}^{\infty} \frac{3}{n+2}$  is divergent, comparing it with  $\sum_{n=1}^{\infty} \frac{1}{n}$ .

$\sum_{n=1}^{\infty} \frac{2}{7^n + 1}$  is convergent, comparing it with  $\sum_{n=1}^{\infty} \frac{1}{7^n}$ .

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 5}$  is convergent, comparing it with  $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ .

(5). Alternating series test :

It is used on the series  $\sum_{n=1}^{\infty} (-1)^n a_n$ , where the sequence  $\{a_n\}$  is decreasing and

$\lim_{n \rightarrow \infty} a_n = 0$ . Note that if  $\sum_{n=1}^{\infty} a_n$  is convergent, then  $\sum_{n=1}^{\infty} (-1)^n a_n$  is absolutely convergent.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 + 2}$  satisfies all the conditions of the alternating series test, hence it is absolutely convergent. Also,  $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3 + 2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3 + 2}$  is convergent (using comparison test), hence  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 + 2}$  is absolutely convergent.

(6). The Ratio test :

it is used on series involving factorial or other products like a constant raised to a power  $n$ .

It is not used on  $p$ -series or rational functions or algebraic functions of  $n$ .

$\sum_{n=1}^{\infty} \frac{2^n}{n!}$  is convergent since  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0 < 1$ .

(7). The Root test :

It is used on series of the form  $\sum_{n=1}^{\infty} (a_n)^n$ .

$\sum_{n=1}^{\infty} \left( \frac{3n^2 + 1}{4n^2 + 5} \right)^n$  is convergent, since  $\lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{3n^2 + 1}{4n^2 + 5} \right)^n} = \frac{3}{4} < 1$ .

(8). The Integral test :

It is used when  $a_n = f(n)$ , where  $f$  is a decreasing function on  $[1, \infty)$  and  $\int_1^{\infty} f(x) dx$  is easy to calculate.

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$  is convergent, since  $f(x) = \frac{1}{x(\ln x)^2}$  is decreasing on  $[2, \infty)$  and

$$\begin{aligned} \int_2^{\infty} \frac{1}{x(\ln x)^2} &= \lim_{t \rightarrow \infty} \int_2^t (\ln x)^{-2} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \left[ \frac{(\ln x)^{-1}}{-1} \right]_2^t = \lim_{t \rightarrow \infty} \left[ \frac{-1}{\ln x} \right]_2^t \\ &= \lim_{t \rightarrow \infty} \left[ \frac{-1}{\ln t} - \left( \frac{-1}{\ln 2} \right) \right] = 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2} . \end{aligned}$$

## 3.8 Power Series

### 3.8.1 Power Series

**Definition :** A power series is a series of the form  $\sum_{n=1}^{\infty} c_n x^n$ , where  $x$  is a variable and  $c_n$ 's are real constants called the coefficients.

**Notes :**

(1). If  $c_n = 1$  for all  $n \geq 1$ , then  $\sum_{n=1}^{\infty} c_n x^n = \sum_{n=1}^{\infty} x^n$  is the geometric series and it converges when  $|x| < 1$  and diverges when  $|x| > 1$ .

If  $x = \frac{1}{2}$ , then  $\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$  converges to 2.

If  $x = 2$ , then  $\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} 2^n$  diverges.

(2).  $\sum_{n=1}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$  is called a power series in  $(x-a)$  or a power series centered at  $a$  or a power series about  $a$ .

**Example (1):** For what values of  $x$  does the series  $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$  converge?

Solution :

Let  $a_n = \frac{(x-2)^n}{n}$ , using the ratio test :

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{n+1} \cdot \frac{n}{(x-2)^n} \right| = \lim_{n \rightarrow \infty} \frac{n}{n+1} |x-2| = |x-2|.$$

The series converges when  $|x-2| < 1 \implies -1 < x-2 < 1 \implies 1 < x < 3$ .

The series diverges when  $|x-2| > 1 \implies x-2 < -1$  or  $x-2 > 1$

$\implies x < 1$  or  $x > 3$ .

If  $x = 1$  then  $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  is the convergent alternating harmonic series.

If  $x = 3$  then  $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$  is divergent.

Therefore,  $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$  converges when  $x \in [1, 3)$  or  $1 \leq x < 3$ .

**Example (2):** For what values of  $x$  does the series  $\sum_{n=1}^{\infty} n! x^n$  converge?

Solution :

Let  $a_n = n! x^n$ , using the ratio test :

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = \lim_{n \rightarrow \infty} (n+1) |x|$$

If  $x \neq 0$ , then  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ . Therefore the series diverges when  $x \neq 0$ .

If  $x = 0$ , then  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$ . Therefore the series converges when  $x = 0$ .

**Example (3):** For what values of  $x$  does the series  $\sum_{n=1}^{\infty} \frac{x^n}{n!}$  converge?

Solution :

Let  $a_n = \frac{x^n}{n!}$ , using the ratio test :

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0 < 1.$$

Therefore, the series converges for all  $x \in \mathbb{R}$ .

### 3.8.2 Interval of Convergence

**Theorem :** For a power series  $\sum_{n=1}^{\infty} c_n(x-a)^n$ , there are only three possibilities:

- (1). The series converges only when  $x = a$ .
- (2). The series converges for all  $x \in \mathbb{R}$ .
- (3). There is a positive real number  $R$  such that the series converges when  $|x-a| < R$  and diverges when  $|x-a| > R$ .

**Notes :**

(1). The positive real number  $R$  is called the radius of convergence. If the series converges only when  $x = a$  then  $R = 0$ , and if the series converges for all  $x \in \mathbb{R}$  then  $R = \infty$ .

(2). The interval of convergence of a power series is the interval that consists of all values of  $x$  for which the series converges.

If the series converges only when  $x = a$  then the interval of convergence consists of only one point  $a$ .

If the series converges for all  $x \in \mathbb{R}$  then the interval of convergence is  $(-\infty, \infty)$ .

If the radius of convergence is  $R$ , then interval of convergence is  $(a-R, a+R)$ .

**Example (4):** Find the radius of convergence and interval of convergence of

the series  $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$ .

Solution :

Let  $a_n = \frac{(x-2)^n}{n^2+1}$ , using the ratio test :

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x-2)^n} \right| = \lim_{n \rightarrow \infty} \frac{n^2+1}{(n+1)^2+1} |x-2|$$

$$= \lim_{n \rightarrow \infty} \frac{n^2+1}{n^2+2n+2} |x-2| = (1) |x-2| = |x-2|.$$

The series converges when  $|x-2| < 1$ , so the radius of convergence is  $R = 1$ .

Therefore, the series converges on  $(1, 3)$ .

If  $x = 1$ , then  $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$  which is convergent.

If  $x = 3$ , then  $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1} = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$  which is convergent.

Therefore, the interval of convergence is  $[1, 3]$ .

**Example (5):** Find the radius of convergence and interval of convergence of the series  $\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n 5^n}$ .

Solution :

Let  $a_n = \frac{(2x-6)^n}{n 5^n}$ , using the ratio test :

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(2x-6)^{n+1}}{(n+1) 5^{n+1}} \cdot \frac{n 5^n}{(2x-6)^n} \right| = \lim_{n \rightarrow \infty} \frac{n}{5(n+1)} |2x-6| \\ &= \lim_{n \rightarrow \infty} \frac{n}{5n+5} |2x-6| = \frac{1}{5} |2x-6| = \frac{|2x-6|}{5}. \end{aligned}$$

The series converges when  $\frac{|2x-6|}{5} < 1 \implies 2|x-3| < 5 \implies |x-3| < \frac{5}{2}$ , so

the radius of convergence is  $R = \frac{5}{2}$ .

Therefore, the series converges on  $\left(3 - \frac{5}{2}, 3 + \frac{5}{2}\right) = \left(\frac{1}{2}, \frac{11}{2}\right)$ .

If  $x = \frac{1}{2}$ , then  $\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n 5^n} = \sum_{n=1}^{\infty} \frac{(-5)^n}{n 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  which is convergent.

If  $x = \frac{11}{2}$ , then  $\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n 5^n} = \sum_{n=1}^{\infty} \frac{5^n}{n 5^n} = \sum_{n=1}^{\infty} \frac{1}{n}$  which is divergent.

Therefore, the interval of convergence is  $\left[\frac{1}{2}, \frac{11}{2}\right)$ .

**Example (6):** Find the radius of convergence and interval of convergence of the series  $\sum_{n=1}^{\infty} \frac{n}{7^n} (x+3)^n$ .

Solution :

Let  $a_n = \frac{n}{7^n} (x+3)^n$ , using the ratio test :

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x+3)^{n+1}}{7^{n+1}} \cdot \frac{7^n}{n(x+3)^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{7n} |x+3| = \frac{1}{7} |x+3| = \frac{|x+3|}{7}. \end{aligned}$$

The series converges when  $\frac{|x+3|}{7} < 1 \implies |x+3| < 7$ , so the radius of convergence is  $R = 7$ .

Therefore, the series converges on  $(-3-7, -3+7) = (-10, 4)$ .

If  $x = -10$ , then  $\sum_{n=1}^{\infty} \frac{n}{7^n} (x+3)^n = \sum_{n=1}^{\infty} (-1)^n n$  which is divergent.

If  $x = 4$ , then  $\sum_{n=1}^{\infty} \frac{n}{7^n} (x+3)^n = \sum_{n=1}^{\infty} n$  which is divergent.

Therefore, the interval of convergence is  $(-10, 4)$ .

**3.8.3 EXERCISES**

1. Find the radius of convergence and interval of convergence of the power series.

$$(1). \sum_{n=1}^{\infty} \frac{x^n}{n} \quad (2). \sum_{n=1}^{\infty} (-1)^n n x^n \quad (3). \sum_{n=1}^{\infty} \sqrt{n} x^n$$

$$(4). \sum_{n=1}^{\infty} \frac{n}{5^n} x^n \quad (5). \sum_{n=1}^{\infty} \frac{5^n}{n} x^n \quad (6). \sum_{n=1}^{\infty} \frac{x^n}{n 3^n}$$

$$(7). \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2} \quad (8). \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (9). \sum_{n=1}^{\infty} n^n x^n$$

$$(10). \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 5^n} x^n \quad (11). \sum_{n=1}^{\infty} \frac{x^{2n}}{n!} \quad (12). \sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2 + 1}$$

$$(13). \sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n} \quad (14). \sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$$

## 3.9 Representations of Functions as Power Series

### 3.9.1 Representations of Functions using Geometric Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \text{ when } |x| < 1.$$

**Example (1):** Express  $\frac{1}{1+x^2}$  as the sum of a power series and find the interval of convergence.

Solution :

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \text{ where } |-x^2| < 1.$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - \cdots, \text{ where } |-x^2| < 1.$$

$$|-x^2| < 1 \implies |x^2| < 1 \implies |x| < 1.$$

If  $x = \pm 1$ , the series  $\sum_{n=0}^{\infty} (-1)^n x^{2n}$  diverge.

Therefore, The interval of convergence is  $(-1, 1)$ .

**Example (2):** Express  $\frac{x^2}{1-x^6}$  as the sum of a power series and find the interval of convergence.

Solution :

$$\frac{x^2}{1-x^6} = x^2 \frac{1}{1-x^6} = x^2 \sum_{n=0}^{\infty} (x^6)^n = x^2 \sum_{n=0}^{\infty} x^{6n} = \sum_{n=0}^{\infty} x^{6n+2}, \text{ where } |x^6| < 1.$$

$$|x^6| < 1 \implies |x| < 1.$$

If  $x = \pm 1$ , the series  $\sum_{n=0}^{\infty} x^{6n+2}$  diverge.

Therefore, The interval of convergence is  $(-1, 1)$ .

**Example (3):** Express  $\frac{x}{5-x}$  as the sum of a power series and find the interval of convergence.

Solution :

$$\frac{x}{5-x} = \frac{x}{5(1-\frac{x}{5})} = \frac{x}{5} \frac{1}{1-\frac{x}{5}} = \frac{x}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n = \frac{x}{5} \sum_{n=0}^{\infty} \frac{x^n}{5^n} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}},$$

where  $\left|\frac{x}{5}\right| < 1$ .

$$\left|\frac{x}{5}\right| < 1 \implies \frac{|x|}{5} < 1 \implies |x| < 5.$$

If  $x = \pm 5$ , the series  $\sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}}$  diverge.

Therefore, The interval of convergence is  $(-5, 5)$ .

**Example (4):** Express  $\frac{1}{2-x}$  as power series in  $x-1$ , and find the interval of convergence.

Solution :

$$\frac{1}{2-x} = \frac{1}{1-x+1} = \frac{1}{1-(x-1)} = \sum_{n=0}^{\infty} (x-1)^n, \text{ where } |x-1| < 1.$$

$$|x-1| < 1 \implies -1 < x-1 < 1 \implies 0 < x < 2.$$

If  $x=0$  or  $x=2$ , the series  $\sum_{n=0}^{\infty} (x-1)^n$  diverge.

Therefore, The interval of convergence is  $(0, 2)$ .

### 3.9.2 Differentiation and Integration of Power Series

**Theorem :** If the power series  $\sum_{n=0}^{\infty} c_n(x-a)^n$  has radius of convergence  $R > 0$ , then the function  $f$  defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval  $(a-R, a+R)$  and

$$(1). f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$$

$$(2). \int f(x) dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \cdots$$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}, \text{ where } C \text{ is a constant.}$$

The radii of convergence of the power series in (1) and (2) are both  $R$ .

**Notes :** Equations (1) and (2) in the last theorem can be rewritten as :

$$(1). \frac{d}{dx} \left[ \sum_{n=0}^{\infty} c_n(x-a)^n \right] = \sum_{n=0}^{\infty} \left[ \frac{d}{dx} c_n(x-a)^n \right].$$

$$(2). \int \left[ \sum_{n=0}^{\infty} c_n(x-a)^n \right] dx = \sum_{n=0}^{\infty} \left[ \int c_n(x-a)^n dx \right].$$

**Example (5):** Express  $\frac{1}{(1-x)^2}$  as a power series, and find its radius of convergence.

Solution :

$$\frac{d}{dx} \left[ \frac{1}{1-x} \right] = \frac{d}{dx} [(1-x)^{-1}] = (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}.$$

Note that  $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$ , where  $|x| < 1$ .

$$\frac{1}{(1-x)^2} = \frac{d}{dx} \left[ \frac{1}{1-x} \right] = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{d}{dx} [x^n] = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}.$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Since the radius of convergence of  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$  is  $R = 1$  then the radius of convergence of  $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$  is also  $R = 1$ .

**Example (6):** Express  $\ln(1+x)$  as a power series, and find its radius of convergence.

Solution :

$$\int \frac{1}{1+x} dx = \ln(1+x) + c$$

Note that  $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$ , where  $|x| < 1$ .

$$\begin{aligned} \ln(1+x) &= \int \frac{1}{1+x} dx = \int \left[ \sum_{n=0}^{\infty} (-1)^n x^n \right] dx = \sum_{n=0}^{\infty} \left[ \int (-1)^n x^n dx \right] \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C, \text{ where } |x| < 1. \end{aligned}$$

Put  $x = 0$  in the last equation :  $\ln(1+0) = 0 + C \implies C = 0$ .

$$\text{Therefore, } \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

The radius of convergence of  $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$  is  $R = 1$ .

**Example (7):** Express  $\tan^{-1} x$  as a power series, and find its radius of convergence.

Solution :

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$$

From Example (1),  $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ , where  $|x| < 1$ .

$$\begin{aligned} \tan^{-1} x &= \int \frac{1}{1+x^2} dx = \int \left[ \sum_{n=0}^{\infty} (-1)^n x^{2n} \right] dx = \sum_{n=0}^{\infty} \int [(-1)^n x^{2n}] dx \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C, \text{ where } |x| < 1. \end{aligned}$$

Put  $x = 0$  in the last equation :  $\tan^{-1}(0) = 0 + C \implies C = 0$ .

$$\text{Therefore, } \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

The radius of convergence of  $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$  is  $R = 1$ .

**3.9.3 EXERCISES**

1. Find a power series representation for the function and determine the interval of convergence.

$$(1). f(x) = \frac{x}{x+1} \quad (2). f(x) = \frac{1}{1-x^2} \quad (3). f(x) = \frac{5}{1-4x^2}$$

$$(4). f(x) = \frac{2}{3-x} \quad (5). f(x) = \frac{4}{2x+3} \quad (6). f(x) = \frac{x^2}{x^4+16}$$

$$(7). f(x) = \frac{x}{2x^2+1} \quad (8). f(x) = \frac{x-1}{x+2}$$

2. (a). Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(x+1)^2}, \text{ What is the radius of convergence?}$$

(b). Use part (a) to find a power series for  $f(x) = \frac{1}{(x+1)^3}$ .

(c). Use part (b) to find a power series for  $f(x) = \frac{x^2}{(x+1)^3}$ .

3. Find a power series representation for the function and determine the radius of convergence.

$$(1). f(x) = \ln(5-x)$$

$$(2). f(x) = x^2 \tan^{-1}(x^3).$$

## 3.10 Taylor and Maclaurin Series

### 3.10.1 Definitions of Taylor Series and Maclaurin Series

**Theorem :** If  $f$  has a power series representation at  $a$ , that is  $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ ,

where  $|x-a| < R$ , then the coefficient  $c_n$  is given by  $c_n = \frac{f^{(n)}(a)}{n!}$ .

Therefore,  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ ,

or  $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$ .

**Definition :**  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$  is called the Taylor series of the function  $f$  at  $a$  ( or about  $a$ , or centered at  $a$ ).

**Definition :** If  $a = 0$  in the Taylor series of  $f$ , then  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$  is

called the Maclaurin series of  $f$ . In this case :

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$ .

**Example (1):** Find the Maclaurin Series and its radius of convergence for the function  $f(x) = \frac{1}{1-x}$ .

Solution :

$$f(x) = \frac{1}{1-x} \implies f(0) = 1 = 0!.$$

$$f'(x) = \frac{1}{(1-x)^2} \implies f'(0) = 1 = 1!.$$

$$f''(x) = \frac{2}{(1-x)^3} \implies f''(0) = 2 = 2!.$$

$$f'''(x) = \frac{6}{(1-x)^4} \implies f'''(0) = 6 = 3!.$$

$$f^{(n)}(x) = \frac{1.2.3 \cdots n}{(1-x)^{n+1}} \implies f^{(n)}(0) = n!.$$

The Maclaurin series for  $f(x) = \frac{1}{1-x}$  is  $\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{n!}{n!}x^n = \sum_{n=0}^{\infty} x^n$ .

Let  $a_n = x^n$ , then  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{x^n} \right| = |x|$ .

Therefore, the Maclaurin series converges when  $|x| < 1$ . Hence  $R = 1$ .

**Example (2):** Find the Maclaurin Series and its radius of convergence for the function  $f(x) = e^x$ .

Solution :

Note that  $f^{(n)}(x) = e^x$ , so  $f^{(n)}(0) = e^0 = 1$ .

The Maclaurin series for  $f(x) = e^x$  is  $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ .

Let  $a_n = \frac{x^n}{n!}$ , then  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0 < 1$ .

Therefore, the Maclaurin series converges for all  $x \in \mathbb{R}$ . Hence  $R = \infty$ .

### 3.10.2 Remainder of a Taylor series

**Definition :**

Suppose  $f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!} (x-a)^i$  then  $f(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i + \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!} (x-a)^i$ ,

$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i$  is called the  $n^{th}$ -degree polynomial of  $f$  at  $a$ , and

$R_n(x) = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!} (x-a)^i$  is called the remainder of the Taylor series.

**Theorem :** If  $f(x) = T_n(x) + R_n(x)$ , where  $T_n(x)$  is the  $n^{th}$ -degree polynomial of  $f$  at  $a$ , and If  $\lim_{n \rightarrow \infty} R_n(x) = 0$  for  $|x-a| < R$ , then  $f$  is equal to the sum of its Taylor series on the interval  $|x-a| < R$ .

**Theorem :** (Taylor's Inequality)

Suppose  $f(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i + \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!} (x-a)^i = T_n(x) + R_n(x)$ .

If  $|f^{(i)}(x)| \leq M$  for  $|x-a| \leq d$  and for all  $i \geq n+1$ , where  $M$  and  $d$  are positive real numbers, then the remainder  $R_n(x)$  of the Taylor series satisfies the inequality :  $|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$ , for  $|x-a| \leq d$ .

In this case,  $\lim_{n \rightarrow \infty} |R_n(x)| = 0$ , hence  $\lim_{n \rightarrow \infty} R_n(x) = 0$  for all values of  $x$ .

**Example (3):** Show that  $e^x$  is equal to the sum of its Maclaurin series.

Solution :

$f(x) = e^x \implies f^{(i)}(x) = e^x$  for all  $i \geq 1$ .

If  $|x| \leq d$  where  $d > 0$ , then  $|f^{(i)}(x)| = e^x \leq e^d$ , for all  $i \geq n+1$ .

So Taylor's inequality, with  $a = 0$  and  $M = e^d$  is

$$|R_n(x)| \leq \frac{e^d}{(n+1)!} |x|^{n+1} \text{ for } |x| \leq d.$$

Therefore,  $\lim_{n \rightarrow \infty} |R_n(x)| = \lim_{n \rightarrow \infty} \frac{e^d}{(n+1)!} |x|^{n+1} = 0$ .

So,  $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \dots$ , for all  $x \in \mathbb{R}$ .

Note : If  $x = 1$  then  $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$ .

**Example (4):** Find the Taylor series of  $f(x) = e^x$  at  $a = 2$ .

Solution :

$$f(x) = e^x \implies f^{(n)}(x) = e^x \implies f^{(n)}(2) = e^2, \text{ for all } n.$$

$$\text{So, } e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n = \sum_{n=0}^{\infty} \frac{e^2 (x-2)^n}{n!} \text{ for all } x \in \mathbb{R}.$$

$$\text{Another Solution : } e^{x-2} = \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} \implies \frac{e^x}{e^2} = \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$$

$$\implies e^x = e^2 \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^2 (x-2)^n}{n!}.$$

**Example (5):** Write the function  $f(x) = e^{x^2-1}$  as a Taylor series of  $x$ .

Solution :

$$f(x) = e^{x^2-1} = e^{-1} e^{x^2} = e^{-1} \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{e} \frac{x^{2n}}{n!} \text{ for all } x.$$

### 3.10.3 Taylor Series of Important Functions

**Example (6):** Find the Maclaurin series for  $\sin x$  and prove that it represents  $\sin x$  for all  $x$ .

Solution :

$$f(x) = \sin x \implies f(0) = \sin(0) = 0.$$

$$f'(x) = \cos x \implies f'(0) = \cos(0) = 1.$$

$$f''(x) = -\sin x \implies f''(0) = -\sin(0) = 0.$$

$$f'''(x) = -\cos x \implies f'''(0) = -\cos(0) = -1.$$

$$f^{(4)}(x) = \sin x \implies f^{(4)}(0) = \sin(0) = 0.$$

$$\text{Therefore, } \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0 < 1.$$

Therefore, the radius of convergence is  $R = \infty$  and the interval of convergence is  $(-\infty, \infty)$ .

Since  $f^{(i)}(x) = \pm \sin x$  or  $\pm \cos x$  for all  $i \geq 1$ , then  $|f^{(i)}(x)| \leq 1$ .

Taylor's inequality, with  $a = 0$  and  $M = 1$  is

$$|R_n(x)| \leq \frac{1}{(n+1)!} |x|^{n+1} \text{ for } |x| \leq d, \text{ where } d > 0.$$

$$\text{Therefore, } \lim_{n \rightarrow \infty} |R_n(x)| = \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

$$\text{So, } \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \text{ for all } x.$$

**Example (7):** Find the Maclaurin series for  $\cos x$  and prove that it represents  $\cos x$  for all  $x$ .

Solution :

$$f(x) = \cos x \implies f(0) = \cos(0) = 1 .$$

$$f'(x) = -\sin x \implies f'(0) = -\sin(0) = 0 .$$

$$f''(x) = -\cos x \implies f''(0) = -\cos(0) = -1 .$$

$$f'''(x) = \sin x \implies f'''(0) = \sin(0) = 0 .$$

$$f^{(4)}(x) = \cos x \implies f^{(4)}(0) = \cos(0) = 1 .$$

$$\text{Therefore, } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} .$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!} \frac{(2n)!}{(-1)^n x^{2n}} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{|x|^2}{(2n+2)(2n+1)} = 0 < 1 .$$

Therefore, the radius of convergence is  $R = \infty$  and the interval of convergence is  $(-\infty, \infty)$ .

Since  $f^{(i)}(x) = \pm \sin x$  or  $\pm \cos x$  for all  $i \geq 1$ , then  $|f^{(i)}(x)| \leq 1$ .

Taylor's inequality, with  $a = 0$  and  $M = 1$  is

$$|R_n(x)| \leq \frac{1}{(n+1)!} |x|^{n+1} \text{ for } |x| \leq d, \text{ where } d > 0 .$$

$$\text{Therefore, } \lim_{n \rightarrow \infty} |R_n(x)| = \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0 .$$

$$\text{So, } \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \text{ for all } x .$$

**Example (8):** Find the Maclaurin series for  $f(x) = (1+x)^k$ , where  $k$  is any real number.

Solution :

$$f(x) = (1+x)^k \implies f(0) = 1 .$$

$$f'(x) = k(1+x)^{k-1} \implies f'(0) = k .$$

$$f''(x) = k(k-1)(1+x)^{k-2} \implies f''(0) = k(k-1) .$$

$$f'''(x) = k(k-1)(k-2)(1+x)^{k-3} \implies f'''(0) = k(k-1)(k-2) .$$

$$f^{(n)}(x) = k(k-1) \cdots (k-n+1)(1+x)^{k-n} \implies f^{(n)}(0) = k(k-1) \cdots (k-n+1) .$$

$$\text{So, } (1+x)^k = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1) \cdots (k-n+1)}{n!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n .$$

$$\text{Let } a_n = \frac{k(k-1) \cdots (k-n+1)}{n!} x^n \text{ for all } n \geq 1 .$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{k(k-1) \cdots (k-n+1)(k-n)x^{n+1}}{(n+1)!} \frac{n!}{k(k-1) \cdots (k-n+1)x^n} \right|$$

$$= \left| \frac{k-n}{n+1} x \right| = \frac{|1 - \frac{k}{n}|}{1 + \frac{1}{n}} |x| .$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{|1 - \frac{k}{n}|}{1 + \frac{1}{n}} |x| = |x| .$$

$$\text{So, } (1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \text{ converges when } |x| < 1 . \text{ Therefore, } R = 1 .$$

$$\text{Note : } (1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \text{ is called the binomial series.}$$

**Example (9):** For  $f(x) = \frac{1}{\sqrt{4-x}}$ , find the Maclaurin series and its radius of convergence.

Solution :

$$f(x) = \frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4(1-\frac{x}{4})}} = \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2} \left(1 + \left(-\frac{x}{4}\right)\right)^{-\frac{1}{2}}.$$

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2} \left(1 + \left(-\frac{x}{4}\right)\right)^{-\frac{1}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(-\frac{x}{4}\right)^n, \text{ where } \left|-\frac{x}{4}\right| < 1.$$

$$\left|-\frac{x}{4}\right| < 1 \implies \frac{|x|}{4} < 1 \implies |x| < 4.$$

The radius of convergence of the Maclaurin series is  $R = 4$ .

**Table of Important Series :**

| No. | Series                                                                                                       | R        |
|-----|--------------------------------------------------------------------------------------------------------------|----------|
| 1   | $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$                                              | 1        |
| 2   | $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$        | 1        |
| 3   | $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$ | 1        |
| 4   | $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \dots$                    | 1        |
| 5   | $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$                 | $\infty$ |
| 6   | $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ | $\infty$ |
| 7   | $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$     | $\infty$ |

### 3.10.4 New Taylor Series from Old

**Example (10):** Find the Maclaurin series for  $f(x) = x^2 \cos x$ .

Solution :

$$f(x) = x^2 \cos x = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n)!} \text{ for all } x.$$

**Example (11):** Find the Maclaurin series for  $f(x) = \ln(1+4x^2)$ .

Solution :

$$f(x) = \ln(1+4x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(4x^2)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4^n x^{2n}}{n},$$

$$\text{Where } |4x^2| < 1 \implies |x|^2 < \frac{1}{4} \implies |x| < \frac{1}{2}.$$

**Example (12):** Find the function represented by the power series  $\sum_{n=0}^{\infty} (-1)^n \frac{3^n x^n}{n!}$ .

Solution :

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = e^{-3x}.$$

**Example (13):** Find the sum of the series  $\frac{1}{1(2)} - \frac{1}{2(2^2)} + \frac{1}{3(2^3)} - \cdots$ .

Solution :

$$\begin{aligned} \frac{1}{1(2)} - \frac{1}{2(2^2)} + \frac{1}{3(2^3)} - \cdots &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n(2^n)} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n} \\ &= \ln \left(1 + \frac{1}{2}\right) = \ln \left(\frac{3}{2}\right). \end{aligned}$$

**Example (14):** Use series to evaluate  $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$ .

Solution : For all  $x \neq 0$ ,

$$\begin{aligned} \frac{e^x - 1 - x}{x^2} &= \frac{1}{x^2} \left[ \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right) - 1 - x \right] \\ &= \frac{1}{x^2} \left( \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right) = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \\ \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x \rightarrow 0} \left( \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \right) = \frac{1}{2}. \end{aligned}$$

**3.10.5 EXERCISES**

1. Use the definition of a Taylor series to find the first four nonzero terms of the series for  $f(x)$  centered at the given value of  $a$ .

$$(1). f(x) = xe^x, \quad a = 0 \quad (2). f(x) = \frac{1}{1+x}, \quad a = 2$$

$$(3). f(x) = \ln x, \quad a = 1 \quad (4). f(x) = \cos^2 x, \quad a = 0$$

2. Find the Maclaurin series for  $f(x)$  using the definition of a Maclaurin series. Also find the associated radius of convergence.

$$(1). f(x) = (1-x)^{-2} \quad (2). f(x) = \ln(1+x) \quad (3). f(x) = \cos x$$

$$(4). f(x) = 2x^4 - 3x^2 + 3 \quad (5). f(x) = x \cos x \quad (6). f(x) = \sinh x$$

3. Find the Taylor series for  $f(x)$  centered at the given value of  $a$ . Also find the associated radius of convergence.

$$(1). f(x) = x^5 + 2x^3 + x, \quad a = 2 \quad (2). f(x) = \ln x, \quad a = 2$$

$$(3). f(x) = \sin x, \quad a = \pi \quad (4). f(x) = \cos x, \quad a = \frac{\pi}{2}$$

4. Show that : (a).  $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$  (b).  $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

$$\text{Hint : use } \sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2} .$$