King Saud University
College of Sciences
Department of Mathematics

MATH 201
MULTIVARIABLE CALCULUS

CLASS NOTES
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Chapter 1

Partial Derivatives

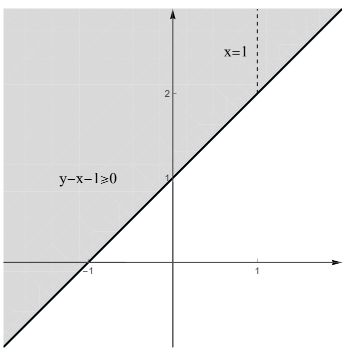
1.1 Functions of several variables

1.1.1 Functions of two variables

Definition: A function f of two variables is a map that assigns to each ordered pair of real numbers $(x, y) \in D \subseteq \mathbb{R}^2$ a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is $\{f(x, y) | (x, y) \in D\}$.

Example (1): If $f(x, y) = \dfrac{\sqrt{y - x - 1}}{x - 1}$, evaluate $f(2, 7)$, find the domain of f and sketch it.
Solution : $f(2, 7) = \dfrac{\sqrt{7 - 2 - 1}}{2 - 1} = \dfrac{\sqrt{4}}{1} = 2.$
The domain of f is the set $D = \big\{(x, y) \in \mathbb{R}^2 \mid y - x - 1 \geq 0 \ , \ x \neq 1\big\}$.
So, $D = \big\{(x, y) \in \mathbb{R}^2 \mid y \geq x + 1 \ , \ x \neq 1\big\}$.

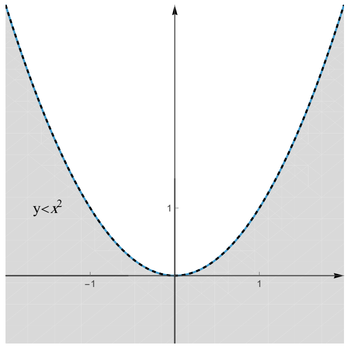
$y - x - 1 \geq 0 \implies y \geq x + 1$ represents the points on and above the line $y = x + 1$.
$x \neq 1$ means the point on the line $x = 1$ must be excluded from the domain .



Example (2): If $f(x, y) = \ln(x^2 - y)$, find the domain of f and sketch it.
Solution :

The domain of f is the set
$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y > 0\}$.
So, $D = \{(x, y) \in \mathbb{R}^2 \mid y < x^2\}$.

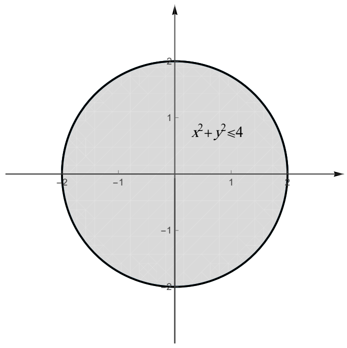
$x^2 - y > 0 \implies y < x^2$
represents the points below the parabola $y = x^2$.



Example (3): If $f(x, y) = \sqrt{4 - x^2 - y^2}$, find the domain and range of f.
Solution :
The domain of f is the set
$D = \{(x, y) \in \mathbb{R}^2 \mid 4 - x^2 - y^2 \geq 0\}$.
So, $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$.

$4 - x^2 - y^2 \geq 0 \implies x^2 + y^2 \leq 4$
represents the points on and inside the disk of center $(0, 0)$ and radius 2.

Note that $0 \leq \sqrt{4 - x^2 - y^2}$
and $\sqrt{4 - (x^2 + y^2)} \leq \sqrt{4} = 2$
So, the range of f is
$\{z \in \mathbb{R} \mid 0 \leq z \leq 2\} = [0, 2]$.



1.1.2 Graphs

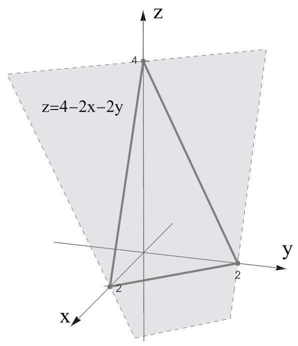
Definition: If f is a function of two variables with domain $D \subseteq \mathbb{R}^2$, then the graph of f is the set of all points $(x, y, z) \in \mathbb{R}^3$, such that $z = f(x, y)$ and $(x, y) \in D$.

Example (4): Sketch the graph of the function $f(x, y) = 4 - 2x - 2y$.
Solution:
$z = 4 - 2x - 2y \implies 2x + 2y + z = 4$
It represents a plain.

To find the x-intercept, put $y = z = 0$
$2x = 4 \implies x = 2$.
So the x-intercept is $(2, 0, 0)$.

To find the y-intercept, put $x = z = 0$
$2y = 4 \implies y = 2$.
So the y-intercept is $(0, 2, 0)$.

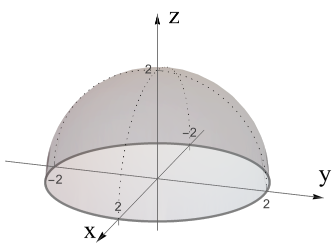
To find the z-intercept, put $x = y = 0$
$z = 4$, So the z-intercept is $(0, 0, 4)$.



Example (5): Sketch the graph of the function $f(x, y) = \sqrt{4 - x^2 - y^2}$.
Solution:

$z = \sqrt{4 - x^2 - y^2}$
$\implies z^2 = 4 - x^2 - y^2$
$\implies x^2 + y^2 + z^2 = 2^2$
Note that $z = \sqrt{4 - x^2 - y^2} \geq 0$.

It represents the upper half of the sphere centered at the origin and its radius is 2.

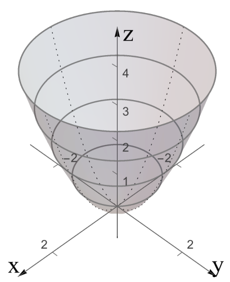


Example (6): Sketch the graph of the function $f(x, y) = x^2 + y^2$.
Solution:

The domain of f is $\mathbb{R}^2$.

$z = x^2 + y^2 \geq 0$
For each value of $z > 0$,
$x^2 + y^2 = z$ represents a circle centered at the origin and its radius is $\sqrt{z}$.

$f(x, y) = x^2 + y^2$ represents a paraboloid.



1.1.3 Level curves

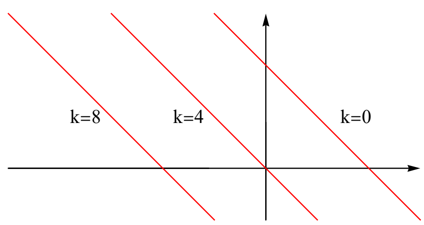
Definition: A level curve of a function $f(x, y)$ is the curve $f(x, y) = k$, where k is a constant (in the range of f).

Example (7): Sketch the level curves of the function $f(x, y) = 4 - 2x - 2y$ for the values $k = 0, 4, 8$.
Solution:
$4 - 2x - 2y = k$.
$\implies 2x + 2y = 4 - k$

For $k = 0 : 2y + 2x = 4$
$\implies y = -x + 2$

For $k = 4 : 2y + 2x = 0$
$\implies y = -x$

For $k = 8 : 2y + 2x = -4$
$\implies y = -x - 2$

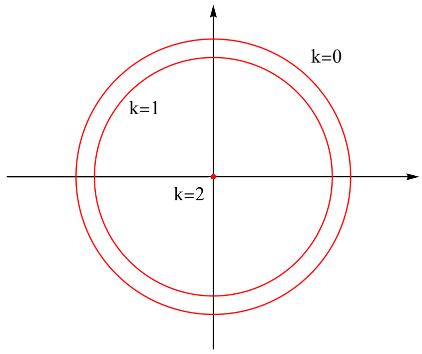


Example (8): Sketch the level curves of the function $f(x, y) = \sqrt{4 - x^2 - y^2}$ for the values $k = 0, 1, 2$.
Solution:
$\sqrt{4 - x^2 - y^2} = k$.
$\implies 4 - x^2 - y^2 = k^2$
$\implies x^2 + y^2 = 4 - k^2$

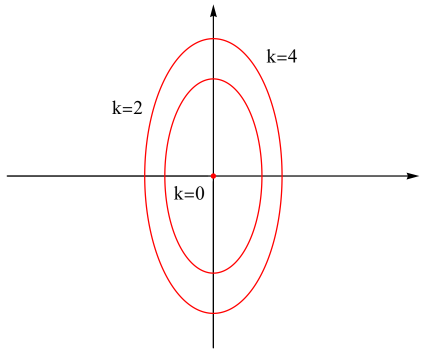
For $k = 0 : x^2 + y^2 = 4$
Circle: center is $(0, 0)$, radius $= 2$.

For $k = 1 : x^2 + y^2 = 3$
Circle: center is $(0, 0)$, radius $= \sqrt{3}$.

For $k = 2 : x^2 + y^2 = 0$
Just one point, the origin.



Example (9): Sketch the level curves of the function $f(x, y) = 4x^2 + y^2$ for the values $k = 0, 2, 4$.
Solution:



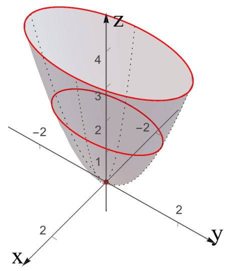
Note that the domain is $\mathbb{R}^2$.

$4x^2 + y^2 = k$.

For $k = 0 : 4x^2 + y^2 = 0$
$\implies x = 0 \ , \ y = 0$
The level curve is the origin.

For $k > 0 : 4x^2 + y^2 = k$
$\implies \frac{x^2}{\left(\frac{k}{4}\right)} + \frac{y^2}{k} = 1$
The level curve is an ellipse centered at the origin, the major axis lies on the y-axis and the minor axis lies on the x-axis.

Note that the graph of f is an elliptic paraboloid.



1.1.4 Functions of three variables

Definition: A function f of two variables is a map that assigns to each ordered pair of real numbers $(x, y, z) \in D \subseteq \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. The set D is the domain of f and its range is $\{f(x, y, z)|(x, y, z) \in D\}$.

Example (10): Find the domain of $f(x, y, z) = \ln(z - x) + yz \sin x$.
Solution : $f(x, y, z)$ is defined when $z - x > 0$.
Therefore, $D = \left\{(x, y, z) \in \mathbb{R}^3 \mid z > x\right\}$.

Example (11): Find the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$.
Solution : $x^2 + y^2 + z^2 = k$, where $k \geq 0$.
If $k = 0$, then the level surface is just the origin $(0, 0, 0) \in \mathbb{R}^3$.
If $k > 0$, then the level surface is $x^2 + y^2 + z^2 = \left(\sqrt{k}\right)^2$, which is a sphere centered at the origin, and its radius is $\sqrt{k}$.

1.1.5 EXERCISES

1. Let $f(x, y) = x^2 \ln(x + y)$

   (a). Evaluate $f(3, 1)$.

   (b). Find and sketch the domain of f.

   (c). Find the range of f.

2. Let $f(x, y, z) = \ln\left(z - \sqrt{x^2 + y^2}\right)$.

   (a). Evaluate $f(4, 23, 6)$.

   (b). Find and describe the domain of f.

3. Find and sketch the domain of the following:

   (a). $f(x, y) = \sqrt{x - 2} + \sqrt{y - 1}$.

   (b). $f(x, y) = \ln(9 - x^2 - y^2)$.

   (c). $f(x, y) = \dfrac{\ln(2 - x)}{1 - x^2 - y^2}$.

(*) Find and sketch the domain of the following:

   (a). $f(x, y) = \sqrt{25 - x^2 - y^2} + \ln(2 + x)$.

   (b). $f(x, y) = \cos(x + y) + \dfrac{x^2 - y^2 - 3}{\sqrt{x + y - 4}}$.

   (c). $f(x, y) = \sqrt{x + y^2} + \sqrt{y - x^2}$.

1.2 Limits and Continuity

1.2.1 Limits of Functions of Two Variables

Definition: Let f be a function of two variables whose domain $D \subseteq \mathbb{R}^2$ includes points arbitrarily close to (a, b). Then we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$,
if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$.

1.2.2 Showing That a Limit Does Not Exist

NOTE: If $f(x, y) \to L_1$ as $(x, y) \longrightarrow (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \longrightarrow (a, b)$ along a different path C_2, where $L_1 \neq L_2$, then $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Example (1): Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Solution: Note that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2} \qquad \left(\dfrac{0}{0}\right)$.
Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{x^2(1 - m^2)}{x^2(1 + m^2)} = \frac{1 - m^2}{1 + m^2} .$$

Note that the limit depends only on m.
Let C_1 be the path $y = 0$ (where m=0) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2} = \dfrac{1 - 0}{1 + 0} = 1$.

Let C_2 be the path $y = x$ (where m=1) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2} = \dfrac{1 - 1}{1 + 1} = 0$.

Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Example (2): Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$ does not exist.

Solution: Note that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2} \qquad \left(\dfrac{0}{0}\right)$.
Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = \lim_{x\to 0} \frac{mx^2}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{mx^2}{x^2(1 + m^2)} = \frac{m}{1 + m^2} .$$

Note that the limit depends only on m.
Let C_1 be the path $y = 0$ (where m=0) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2} = \dfrac{0}{1 + 0} = 0$.

Let C_2 be the path $y = x$ (where m=1) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2} = \dfrac{1}{1 + 1} = \dfrac{1}{2}$.

Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$ does not exist.

Example (3): Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^4}$ does not exist.

Solution: Note that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4} \quad \left(\dfrac{0}{0}\right)$.
Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{x(m^2x^2)}{x^2+m^4x^4} = \lim_{x\to 0} \frac{mx^3}{x^2(1+m^4\ x^2)} = \lim_{x\to 0} \frac{mx}{1+m^4\ x^2} = 0$$

.
Note that the limit here depends on x and m .
Let C_1 be the path $y = 0$ (where m=0) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4} = 0$.
Let C_2 be the path $x = y^2$ (the parabola with vertex (0,0) and opens to the right) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4} = \lim\limits_{y\to 0} \dfrac{y^2\ y^2}{(y^2)^2+y^4} = \lim\limits_{y\to 0} \dfrac{y^4}{2y^4} = \dfrac{1}{2}$.
Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ does not exist.

Example (4): Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2}$ does not exist.
Solution: Note that $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2} \quad \left(\dfrac{0}{0}\right)$.
Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2} = \lim_{x\to 0} \frac{\sin(mx^2)}{x^2+m^2x^2} = \lim_{x\to 0} \frac{\sin(mx^2)}{(1+m^2)x^2} = \frac{m}{1+m^2}.$$

Note that the limit depends only on m.
Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2}$ does not exist.

1.2.3 Properties of Limits

If $f(x,y)$, $g(x,y)$ are two functions defined on $D \setminus \{(a,b)\}$, where $D \subseteq \mathbb{R}^2$, and $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L_1$, $\lim\limits_{(x,y)\to(a,b)} g(x,y) = L_2$, where $L_1, L_2 \in \mathbb{R}$ then
(1). $\lim\limits_{(x,y)\to(a,b)} [f(x,y) + g(x,y)] = L_1 + L_2$.
(2). $\lim\limits_{(x,y)\to(a,b)} [f(x,y) - g(x,y)] = L_1 - L_2$.
(3). $\lim\limits_{(x,y)\to(a,b)} k\ f(x,y) = k\ L_1$, where $k \in \mathbb{R}$.
(4). $\lim\limits_{(x,y)\to(a,b)} [f(x,y).g(x,y)] = L_1L_2$.
(5). $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L_1}{L_2}$, where $L_2 \neq 0$.
(6). If $P(x,y)$ is a polynomial in x, y then $\lim\limits_{(x,y)\to(a,b)} P(x,y) = P(a,b)$.

Example (5): Evaluate $\lim\limits_{(x,y)\to(2,1)} \left(x^2y^2 - 2xy + x + y - 1\right)$.
Solution: Note that $P(x,y) = x^2y^2 - 2xy + x + y - 1$ is a polynomial.
So, $\lim\limits_{(x,y)\to(2,1)} \left(x^2y^2 - 2xy + x + y - 1\right) = P(2,1)$
$= (2)^2(1)^2 - 2(2)(1) + 2 + 1 - 1 = 2$.

Example (6): Evaluate $\displaystyle\lim_{(x,y)\to(-1,3)} \frac{2x^2y+1}{xy^3-2x}$.

Solution: Note that $P(x,y) = 2x^2y+1$ and $Q(x,y) = xy^3-2x$ are both polynomials.

$$\lim_{(x,y)\to(-1,3)} \frac{2x^2y+1}{xy^3-2x} = \frac{P(-1,3)}{Q(-1,3)} = \frac{2(-1)^2(3)+1}{(-1)(3)^3-2(-1)} = \frac{7}{-25}.$$

Note that $Q(-1,3) \neq 0$.

Example (7): Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = 0$.

First Solution: $\forall\ x,y \in \mathbb{R}^*$,

$$y^2 \le x^2+y^2 \implies \frac{y^2}{x^2+y^2} \le 1 .$$

$$0 \le \left|\frac{2xy^2}{x^2+y^2}\right| = \frac{|2x|\ |y^2|}{|x^2+y^2|} = 2|x|\ \frac{y^2}{x^2+y^2} \le 2|x| .$$

Note that $(x,y)\to(0,0) \implies x \to 0$.

Since $\displaystyle\lim_{(x,y)\to(0,0)} 0 = 0$ and $\displaystyle\lim_{(x,y)\to(0,0)} 2|x| = 2(0) = 0$,

By Squeeze Theorem $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = 0$.

Second Solution: Using polar coordinates,

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = \lim_{r\to0} \frac{2r^3\cos\theta\ \sin^2\theta}{r^2} = \lim_{r\to0} 2r\cos\theta\sin^2\theta = 0.$$

Note that $\displaystyle\lim_{r\to0} 2r = 0$ and $\cos\theta\sin^2\theta$ is bounded .

1.2.4 Continuity

Definition: Let $f(x,y)$ be a function of two variables defined on a set $D \subseteq \mathbb{R}^2$ and $(a,b) \in D$, If $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, then f is continuous at (a,b).

If f is continuous at every point in D, then f is continuous on D .

Example (8): Discuss the continuity of $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$.

Solution:

f is not defined at $(0,0)$, so it is not continuous at $(0,0)$.

$$\forall\ (a,b) \neq (0,0),\ \lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} \frac{x^2-y^2}{x^2+y^2} = \frac{a^2-b^2}{a^2+b^2} = f(a,b).$$

So, f is continuous on $D = \big\{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\big\} = \mathbb{R}^2 \setminus \{(0,0)\}$.

Example (9): Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$.

Solution:

$$\forall\ (a,b) \neq (0,0),\ \lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} \frac{x^2-y^2}{x^2+y^2} = \frac{a^2-b^2}{a^2+b^2} = f(a,b).$$

f is defined at $(0,0)$ and $f(0,0) = 0$, but $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

So, f is continuous on $D = \{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\} = \mathbb{R}^2 \setminus \{(0,0)\}$.

Example (10): Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{2xy^2}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$.

Solution:

$\forall\ (a,b) \neq (0,0)$, $\lim\limits_{(x,y)\to(a,b)} f(x,y) = \lim\limits_{(x,y)\to(a,b)} \dfrac{2xy^2}{x^2+y^2} = \dfrac{2ab^2}{a^2+b^2} = f(a,b)$.

f is defined at $(0,0)$ and $f(0,0) = 0$, also $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

So, f is continuous on $\mathbb{R}^2$.

Example (11): Discuss the continuity of $f(x,y) = e^{-x^2-y^2}$.

Solution: $f(x,y) = e^{-x^2-y^2} = e^{-(x^2+y^2)}$.

f is defined on $\mathbb{R}^2$ and $\lim\limits_{(x,y)\to(a,b)} f(x,y) = e^{-(a^2+b^2)} = f(a,b)$.

Therefore, f is continuous on $\mathbb{R}^2$.

Example (12): Discuss the continuity of $f(x,y) = \tan^{-1}\left(\dfrac{y}{x}\right)$.

Solution:

f is not defined where $x = 0$.

$\forall\ (a,b)$ where $a \neq 0$, $\lim\limits_{(x,y)\to(a,b)} f(x,y) = \tan^{-1}\left(\dfrac{b}{a}\right) = f(a,b)$.

So, f is continuous on $D = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0\}$ which is $\mathbb{R}^2$ except the y-axis.

1.2.5 Limit and Continuity of a function of three variables

Definition: Let f be a function of three variables whose domain $D \subseteq \mathbb{R}^3$ includes points arbitrarily close to (a,b,c). Then we say that the limit of $f(x,y,z)$ as (x,y,z) approaches (a,b,c) is L and we write $\lim\limits_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$,

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

if $(x,y,z) \in D$ and $0 < \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} < \delta$

then $|f(x,y,z) - L| < \epsilon$.

Definition: Let $f(x,y,z)$ be a function of three variables defined on a set $D \subseteq \mathbb{R}^3$ and $(a,b,c) \in D$, If $\lim\limits_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c)$, then f is continuous at (a,b,c).

If f is continuous at every point in D, then f is continuous on D .

Example (13): Discuss the continuity of $f(x,y,z) = \dfrac{1}{1-x^2-y^2-z^2}$.

Solution:

f is not defined where $1 - x^2 - y^2 - z^2 = 0 \implies x^2+y^2+z^2 = 1$,

f is not continuous on the unit sphere.

$\forall\ (a,b,c) \in \mathbb{R}^3$ where $a^2+b^2+c^2 \neq 1$,

$\lim\limits_{(x,y,z)\to(a,b,c)} f(x,y,z) = \dfrac{1}{1-a^2+b^2+c^2} = f(a,b,c)$.

So, f is continuous on $D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \neq 1\}$ which is $\mathbb{R}^3$ except the unit sphere.

1.2.6 EXERCISES

1. Find the limit of the following:

$$(a).\ \lim_{(x,y)\to(3,2)} \left(x^2y^3 - 4y^2\right) \quad (b).\ \lim_{(x,y)\to(2,-1)} \frac{x^2y + yx^2}{x^2 - y^2}$$

2. Show that the following limits do not exist:

$$(a).\ \lim_{(x,y)\to(0,0)} \frac{y^2}{x^2 + y^2} \qquad (b).\ \lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2 + y^2}$$

$$(c).\ \lim_{(x,y)\to(1,1)} \frac{y - x}{1 - y + \ln x}$$

3. Discuss the existence of the following limits (and find its value if exists):

$$(a).\ \lim_{(x,y)\to(2,3)} \frac{3x - 2y}{4x^2 - y^2} \quad (b).\ \lim_{(x,y)\to(0,0)} \frac{x^2y\cos y}{x^2 + y^4}$$

4. Use the Squeeze Theorem to find the limit of the following :

$$(a).\ \lim_{(x,y)\to(0,0)} xy\ \sin\left(\frac{1}{x^2 + y^2}\right) \quad (b).\ \lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2 + y^2 + z^2}$$

5. Determine the set of points of continuity of the following:

$$\text{(a). } f(x,y) = \begin{cases} \dfrac{x^2y^3}{2x^2 + y^2} & if \quad (x,y) \neq (0,0) \\ 1 & if \quad (x,y) = (0,0) \end{cases}$$

$$\text{(b). } f(x,y) = \begin{cases} \dfrac{xy}{x^2 + xy + y^2} & if \quad (x,y) \neq (0,0) \\ 1 & if \quad (x,y) = (0,0) \end{cases}$$

6. Use polar coordinates to find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2}$.

(*) Show that the following limits do not exist:

$$(a).\ \lim_{(x,y)\to(0,0)} \frac{x^3+y^2}{x^2+y^2} \qquad (b).\ \lim_{(x,y)\to(0,0)} \frac{xy}{x^4+y^4}$$

(*) Use Squeeze theorem to find the following limits:

$$(a).\ \lim_{(x,y)\to(-1,0)} \frac{y(x+1)^2+y^2\sin(\pi x)}{(x+1)^2+y^2} \qquad (b).\ \lim_{(x,y)\to(0,0)} \frac{y^2\sin x+yx^2}{x^2+y^2}$$

(*) Show that the limit is zero in the following:

$$(a).\ \lim_{(x,y)\to(0,0)} \frac{x^5+y^5}{x^2+y^2} \qquad (b).\ \lim_{(x,y)\to(0,0)} \frac{x^6+y^6}{x^4+y^4}$$

$$(c).\ \lim_{(x,y)\to(0,0)} \frac{x^2+2xy}{\sqrt{x^2+y^2}} \qquad (d).\ \lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{x^2+y^2}$$

1.3 Partial Derivatives

1.3.1 Partial Derivatives of Functions of Two Variables

Definition: Let $f(x, y)$ be a function of two variables defined on a set $D \subseteq \mathbb{R}^2$,
If $\lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$ and $\lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}$ exist, then the partial derivatives of f are denoted by f_x and f_y and are defined as
(1). $f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$.
(2).$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}$.

Notations for Partial Derivatives: If $z = f(x, y)$ then
(1). $f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = D_x f$.
(2). $f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = D_y f$.

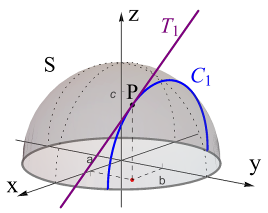
Rule for Finding Partial Derivatives
(1). To find f_x : differentiate $f(x, y)$ with respect to x regarding y as a constant.
(2). To find f_y : differentiate $f(x, y)$ with respect to y regarding x as a constant.

Example (1): If $f(x, y) = x^2 - xy^3 - 3y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$.
Solution:
(1). $f_x(x, y) = 2x - (1)y^3 - 0 = 2x - y^3$,
$f_x(1, 1) = 2(1) - (1)^3 = 1$.
(2). $f_y(x, y) = 0 - x(3y^2) - 3(2y) = -3xy^2 - 6y$,
$f_y(1, 1) = -3(1)(1)^2 - 6(1) = -9$.

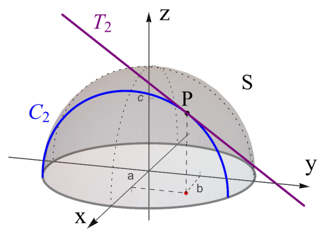
Example (2): If $f(x, y) = \sin\left(\frac{2x}{1+3y}\right)$, Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial x}$.
Solution:
(1). $\frac{\partial f}{\partial x} = \cos\left(\frac{2x}{1+3y}\right)\left(\frac{2}{1+3y}\right)$.
(2). $\frac{\partial f}{\partial y} = \cos\left(\frac{2x}{1+3y}\right)\left((2x)(-1)(1+3y)^{-2}(3)\right) = \cos\left(\frac{2x}{1+3y}\right)\left(\frac{-6x}{(1+3y)^2}\right)$.

1.3.2 Interpretations of Partial Derivatives

Let S be the surface represented by $z = f(x, y)$ and $f(a, b) = c$.
Then $P(a, b, c)$ lies on S.
Let C_1 be the curve where the plain $y = b$ intersects the surface S,
The slope of the tangent line T_1 to the curve C_1 at $P(a, b, c)$ is $f_x(a, b)$.

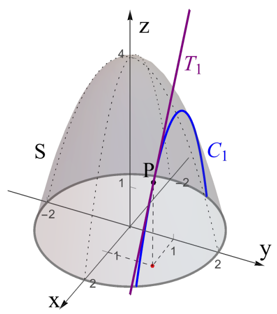


Let C_2 be the curve where the plain $x = a$ intersects the surface S,
The slope of the tangent line T_2 to the curve C_2 at $P(a, b, c)$ is $f_y(a, b)$.

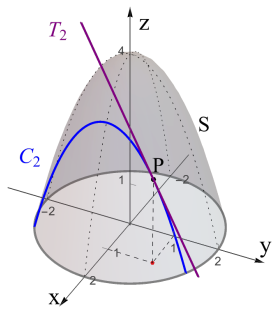


Example (3): If $f(x, y) = 4 - x^2 - y^2$, find $f_x(1, 1)$, $f_y(1, 1)$ and interpret them as slopes. Solution:

$z = 4 - x^2 - y^2$.
$f_x(x, y) = -2x$.
$f_x(1, 1) = -2(1) = -2.$
$f(1, 1) = 4 - 1 - 1 = 2$.
C_1 is the intersection of $z = 4 - x^2 - y^2$ and $y = 1$, and it is the parabola $z = 3 - x^2$.
The line T_1 is tangent to the curve C_1 at the point $P(1, 1, 2)$ and its slope is $f_x(1, 1) = -2$.



$z = 4 - x^2 - y^2$.
$f_y(x, y) = -2y$.
$f_y(1, 1) = -2(1) = -2.$
$f(1, 1) = 4 - 1 - 1 = 2$.
C_2 is the intersection of $z = 4 - x^2 - y^2$ and $x = 1$, and it is the parabola $z = 3 - y^2$.
The line T_2 is tangent to the curve C_2 at the point $P(1, 1, 2)$ and its slope is $f_y(1, 1) = -2$.



Example (4): If $x^3 + y^3 + z^3 + 6xyz + 4 = 0$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.
Solution:
Differentiating implicitly with respect to x .

$$3x^2 + 0 + 3z^2 \frac{\partial z}{\partial x} + 6y\left(z + x\frac{\partial z}{\partial x}\right) + 0 = 0$$

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}\left(3z^2 + 6xy\right) = -3x^2 - 6yz$$

$\frac{\partial z}{\partial x} = \frac{-3x^2 - 6yz}{3z^2 + 6xy} = \frac{-x^2 - 2yz}{z^2 + 2xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$.
Differentiating implicitly with respect to y .
$0 + 3y^2 + 3z^2 \frac{\partial z}{\partial y} + 6x\left(z + y\frac{\partial z}{\partial y}\right) + 0 = 0$
$3y^2 + 3z^2 \frac{\partial z}{\partial y} + 6xz + 6xy\frac{\partial z}{\partial y} = 0$
$\frac{\partial z}{\partial y}\left(3z^2 + 6xy\right) = -3y^2 - 6xz$
$\frac{\partial z}{\partial y} = \frac{-3y^2 - 6xz}{3z^2 + 6xy} = \frac{-y^2 - 2xz}{z^2 + 2xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$.

1.3.3 Functions of Three Variables

Definition: Let $f(x, y, z)$ be a function of three variables defined on a set $D \subseteq \mathbb{R}^3$, If $\lim_{h\to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$, $\lim_{h\to 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}$ and $\lim_{h\to 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}$ exist, then the partial derivatives of f are denoted by f_x f_y and f_z and are defined as

(1). $f_x(x, y, z) = \lim_{h\to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$.
(2). $f_y(x, y, z) = \lim_{h\to 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}$.
(3). $f_z(x, y, z) = \lim_{h\to 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}$.

Example (5): If $f(x, y, z) = e^{xy}\ln(z^2 + 1)$, find f_x, f_y and f_z.
Solution:
$f_x(x, y, z) = (e^{xy}y)\ln(z^2 + 1) = ye^{xy}\ln(z^2 + 1)$.
$f_y(x, y, z) = (e^{xy}x)\ln(z^2 + 1) = xe^{xy}\ln(z^2 + 1)$.
$f_z(x, y, z) = e^{xy}\left(\frac{2z}{z^2 + 1}\right) = \frac{2ze^{xy}}{z^2 + 1}$.

1.3.4 Higher Derivatives

Definition: If $z = f(x, y)$, then the second partial derivatives of f are
(1) $f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial f^2}{\partial x^2}$.
(2) $f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial f^2}{\partial y \partial x}$.
(3) $f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial f^2}{\partial x \partial y}$.
(y) $f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial f^2}{\partial y^2}$.

Example (6): Find the second partial derivatives of $f(x, y) = x^3 + x^2y^2 - 2y^3$.
Solution:

$f_x(x,y) = 3x^2 + (2x)y^2 - 0 = 3x^2 + 2xy^2$.
$f_{xx}(x,y) = 3(2x) + 2y^2(1) = 6x + 2y^2$.
$f_{xy}(x,y) = 0 + 2x(2y) = 4xy$.
$f_y(x,y) = 0 + x^2(2y) - 2(3y^2) = 2x^2y - 6y^2$.
$f_{yx}(x,y) = 2y(2x) - 0 = 4xy$.
$f_{yy} = 2x^2(1) - 6(2y) = 2x^2 - 12y$.
NOTE : $f_{xy}(x,y) = f_{yx}(x,y)$.

Clairaut's Theorem: Suppose f is defined on a set $D \subseteq \mathbb{R}^2$ that contains the point (a,b), If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$.

Example (7): If $f(x,y,z) = \sin(2x - yz)$, find $f_{xyz}(x,y)$.
Solution:
$f_x(x,y,z) = \cos(2x - yz)(2) = 2\cos(2x - yz)$.
$f_{xy}(x,y,z) = 2\left[-\sin(2x - yz)\ (-z)\right] = 2z\sin(2x - yz)$.
$f_{xyz}(x,y,z) = (2)\sin(2x - yz) + 2z\left[\cos(2x - yz)\ (-y)\right]$
$= 2\sin(2x - yz) - 2yz\cos(2x - yz)$.

1.3.5 Partial Differential Equations

(1). **Laplace's Equation:** If $u(x,y)$ is a function of two variables and $u_{xx} + u_{yy} = 0$, then u satisfies the Laplace's equation, and u is called a harmonic function.

Example (8): Show that $u(x,y) = e^x \cos y$ is a solution of Laplace's equation.
Solution:
$u_x(x,y) = e^x \cos y$ and $u_{xx}(x,y) = e^x \cos y$.
$u_y(x,y) = -e^x \sin y$ and $u_{yy}(x,y) = -e^x \cos y$.
$u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0$.

(2). **Wave Equation:** If $u(x,t)$ is a function of two variables and $u_{tt} = a^2 u_{xx}$, then u satisfies the wave equation.

Example (9): Show that $u(x,t) = \sin(x - at)$ is a solution of the wave equation.
Solution:
$u_x(x,t) = \cos(x - at)$ and $u_{xx}(x,t) = -\sin(x - at)$.
$u_t(x,t) = -a\cos(x - at)$ and $u_{tt}(x,t) = -a^2 \sin(x - at)$.
$u_{tt} = a^2(-\sin(x - at)) = a^2 u_{xx}$.

1.3.6 Differentiability

Definition: If $z = f(x,y)$ and $(a,b) \in D_f$, if x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$, then the increment of z is
$\Delta z = f\left(a + \Delta x, b + \Delta y\right) - f(a,b)$.

Definition: If $z = f(x, y)$, then f is differentiable at (a, b), if
$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \sqrt{(\Delta x)^2 + (\Delta y)^2}\ \epsilon(\Delta x, \Delta y)$,
where $\lim\limits_{(\Delta x, \Delta y)\to(0,0)} \epsilon(\Delta x, \Delta y) = 0$.

NOTE:
If $\lim\limits_{(\Delta x, \Delta y)\to(0,0)} \dfrac{f(a + \Delta x, b + \Delta y) - f(a, b) - f_x(a, b)\Delta x - f_y(a, b)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$, then
f is differentiable at (a, b).

Theorem: If f is differentiable at (a, b), then f is continuous at (a, b).

NOTE: If f is not continuous at (a, b), then f is not differentiable at (a, b).

Example (10): Show that $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0) \end{cases}$
is not differentiable at $(0, 0)$.
Solution:
$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = \lim_{r\to 0} \frac{r\cos\theta\ r\sin\theta}{r^2}$$
$$= \lim_{r\to 0} \frac{r^2\ \cos\theta\ \sin\theta}{r^2} = \cos\theta\ \sin\theta.$$
The limit depends on θ, so the limit does not exist.
f is not continuous at $(0, 0)$, hence, f is not differentiable at $(0, 0)$.

Example (11): Show that $f(x, y) = \begin{cases} \dfrac{xy^2}{\sqrt{x^2 + y^2}} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0) \end{cases}$
is differentiable at $(0, 0)$.
Solution:
$f(0, 0) = 0$.
$$f(0 + \Delta x, 0 + \Delta y) = f(\Delta x, \Delta y) = \frac{\Delta x\,(\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}.$$
$$f_x(0, 0) = \lim_{h\to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$
$$f_y(0, 0) = \lim_{h\to 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$
$$\lim_{(\Delta x, \Delta y)\to(0,0)} \frac{f(\Delta x, \Delta y) - f(0, 0) - f_x(0, 0)\Delta x - f_y(0, 0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$
$$= \lim_{(\Delta x, \Delta y)\to(0,0)} \frac{\left(\frac{\Delta x(\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}\right)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim_{(\Delta x, \Delta y)\to(0,0)} \frac{\Delta x\,(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} = 0.$$
Note that $0 \leq \left|\dfrac{\Delta x\,(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2}\right| = |\Delta x| \left|\dfrac{(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2}\right| \leq |\Delta x|$

By the squeeze theorem $\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\left|\dfrac{\Delta x\,(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2}\right|=0,$

So, $\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\dfrac{\Delta x\,(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2}=0.$

Therefore, f is differentiable at $(0,0)$.

Example (12): Show that $f(x,y)=\begin{cases}\dfrac{y^2\sin x+yx^2}{x^2+y^2} & \text{if} \quad (x,y)\neq(0,0)\\ 0 & \text{if} \quad (x,y)=(0,0)\end{cases}$

is not differentiable at $(0,0)$.

Solution:

$f(0,0)=0.$

$f(0+\Delta x,0+\Delta y)=f(\Delta x,\Delta y)=\dfrac{(\Delta y)^2\sin(\Delta x)+\Delta y\,(\Delta x)^2}{(\Delta x)^2+(\Delta y)^2}.$

$f_x(0,0)=\lim\limits_{h\to0}\dfrac{f(0+h,0)-f(0,0)}{h}=\lim\limits_{h\to0}\dfrac{0}{h}=0.$

$f_y(0,0)=\lim\limits_{h\to0}\dfrac{f(0,0+h)-f(0,0)}{h}=\lim\limits_{h\to0}\dfrac{0}{h}=0.$

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{f(\Delta x,\Delta y)-f(0,0)-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\left(\frac{(\Delta y)^2\sin(\Delta x)+\Delta y(\Delta x)^2}{(\Delta x)^2+(\Delta y)^2}\right)}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{(\Delta y)^2\sin(\Delta x)+\Delta y\,(\Delta x)^2}{\left[(\Delta x)^2+(\Delta y)^2\right]^{\frac{3}{2}}}$$

Taking the path $\Delta y=\Delta x$:

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{(\Delta y)^2\sin(\Delta x)+\Delta y\,(\Delta x)^2}{\left[(\Delta x)^2+(\Delta y)^2\right]^{\frac{3}{2}}}=\lim_{\Delta x\to0}\frac{(\Delta x)^2\sin(\Delta x)+\Delta x\,(\Delta x)^2}{\left[(\Delta x)^2+(\Delta x)^2\right]^{\frac{3}{2}}}$$

$$=\lim_{\Delta x\to0}\frac{(\Delta x)^2\left[\sin(\Delta x)+\Delta x\right]}{\left[2\,(\Delta x)^2\right]^{\frac{3}{2}}}=\lim_{\Delta x\to0}\frac{(\Delta x)^2\left[\sin(\Delta x)+\Delta x\right]}{\sqrt{8}\,(\Delta x)^3}=\lim_{\Delta x\to0}\frac{\sin(\Delta x)+\Delta x}{\sqrt{8}\Delta x}$$

$$=\lim_{\Delta x\to0}\frac{1}{\sqrt{8}}\left(\frac{\sin(\Delta x)}{\Delta x}+\frac{\Delta x}{\Delta x}\right)=\frac{1}{\sqrt{8}}(1+1)=\frac{2}{\sqrt{8}}=\frac{1}{\sqrt{2}}\neq0.$$

Therefore, f is not differentiable at $(0,0)$.

1.3.7 EXERCISES

1. Find the first partial derivatives of the function.

   $(a).\ f(x,y) = x^4 - 5xy^3$ $\quad(b).\ g(x,y) = x^3 \sin y$

   $(c).\ w(u,v) = \dfrac{u}{v^2}$ $\quad(d).\ u(r,\theta) = \sin(r\cos\theta)$

   $(e).\ w(x,y,z) = y\tan(x+2z)$

2. If $f(x,y) = y\sin^{-1}(xy)$, find $f_y\left(1, \dfrac{1}{2}\right)$.

3. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$:

   $(a).\ z = f(x) + g(y)$ $\quad(b).\ z = f(x+y)$

4. Find all the second partial derivatives of $f(x,y) = x^4y - 2x^3y^2$.

5. Verify that the conclusion of Clairaut's Theorem holds for

   $u(x,y) = x^4y^3 - y^4$.

6. If $f(x,y) = x^4y^2 - x^3y$, find f_{xxx} and f_{xyx}.

(*) Discuss the differentiability of the following functions at the given points:

$$(a).\ f(x,y) = \begin{cases} \dfrac{x^2(y-2)}{x^2+(y-2)^2} & \text{if} \quad (x,y) \neq (0,2) \\ 0 & \text{if} \quad (x,y) = (0,2) \end{cases} \quad \text{at } (0,2).$$

$$(b).\ f(x,y) = \begin{cases} \dfrac{x^2y - xy^2}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases} \quad \text{at } (0,0).$$

$$(c).\ f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases} \quad \text{at } (0,0).$$

1.4 Chain Rule

1.4.1 The Chain Rule (Case 1)

Theorem: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and $\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\,\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\,\dfrac{dy}{dt}$.

Example (1): If $f(x, y) = x^3y + 2xy^3$, where $x = \sin 2t$ and $y = \cos t$, find $\dfrac{dz}{dt}$ at $t = 0$.

Solution:

$\dfrac{\partial f}{\partial x} = 3x^2y + 2y^3$, $\dfrac{\partial f}{\partial y} = x^3 + 6xy^2$.

$\dfrac{dx}{dt} = 2\cos 2t$, $\dfrac{dy}{dt} = -\sin t$.

$\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\,\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\,\dfrac{dy}{dt}$

$= (3x^2y + 2y^3)(2\cos 2t) + (x^3 + 6xy^2)(-\sin t)$

$\left.\dfrac{dz}{dt}\right|_{t=0} = (0 + 2)(2) + (0 + 0)(0) = 4$.

1.4.2 The Chain Rule (Case 2)

Theorem: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t. Then z is a differentiable function of t and $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x}\,\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\,\dfrac{\partial y}{\partial s}$,

$\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\,\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\,\dfrac{\partial y}{\partial t}$

Example (2): If $f(x, y) = e^x \sin y$, where $x = s^2 + t^2$ and $y = s^2t^2$, find $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$.

Solution:

$\dfrac{\partial f}{\partial x} = e^x \sin y$, $\dfrac{\partial f}{\partial y} = e^x \cos y$.

$\dfrac{\partial x}{\partial s} = 2s$, $\dfrac{\partial y}{\partial s} = 2st^2$.

(1). $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x}\,\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\,\dfrac{\partial y}{\partial s} = (e^x \sin y)\,(2s) + (e^x \cos y)\,(2st^2)$

$= 2se^{s^2+t^2}\sin(s^2t^2) + 2st^2e^{s^2+t^2}\cos(s^2t^2)$.

Also, $\dfrac{\partial x}{\partial t} = 2t$, $\dfrac{\partial y}{\partial s} = 2ts^2$.

(2). $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\,\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\,\dfrac{\partial y}{\partial t} = (e^x \sin y)\,(2t) + (e^x \cos y)\,(2ts^2)$

$= 2te^{s^2+t^2}\sin(s^2t^2) + 2ts^2e^{s^2+t^2}\cos(s^2t^2)$.

1.4.3 The Chain Rule (The General Case)

Theorem: Suppose that u is a differentiable function of the n variables $x_1, x_2, \cdots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \cdots, t_m$. Then u is a function of $t_1, t_2, \cdots, t_m$ and
$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$, for each $i = 1, 2, \cdots, m$.

Example (3): If $u(x, y, z) = x^2y^2 + yz^3$, where $x = r^2se^t$, $y = rse^{-t}$ and $z = rs^2 \sin t$, find $\frac{\partial u}{\partial s}$ when $r = 1$, $s = 1$ and $t = 0$.
Solution:
$\frac{\partial u}{\partial x} = 2xy^2$, $\frac{\partial u}{\partial y} = 2x^2y + z^3$ and $\frac{\partial u}{\partial z} = 3yz^2$.
$\frac{\partial x}{\partial s} = r^2e^t$, $\frac{\partial y}{\partial s} = re^{-t}$ and $\frac{\partial z}{\partial s} = 2rs \sin t$.
So, $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$
$\frac{\partial u}{\partial s} = (2xy^2)(r^2e^t) + (2x^2y + z^3)(re^{-t}) + (3yz^2)(2rs \sin t)$
When $r = 1, s = 1, t = 0$, $\frac{\partial u}{\partial s} = (2)(1) + (2)(1) + (0)(0) = 4$.

Example (4): If $w = f(x, y)$ is differentiable at (x, y) and $x = s + t$, $y = s - t$.
Show that $\frac{\partial w}{\partial s}\frac{\partial w}{\partial t} = \left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{\partial w}{\partial y}\right)^2$.
$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = f_x(1) + f_y(1) = f_x + f_y$.
$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = f_x(1) + f_y(-1) = f_x - f_y$.
$\frac{\partial w}{\partial s}\frac{\partial w}{\partial t} = (f_x + f_y)(f_x - f_y) = (f_x)^2 - (f_y)^2 = \left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{\partial w}{\partial y}\right)^2$.

Example (5): If $z = f(s^2 - t^2, t^2 - s^2)$ is differentiable at (s, t),
Show that $t\frac{\partial z}{\partial s} + s\frac{\partial z}{\partial t} = 0$.
Solution:
Let $z = f(x, y)$, where $x = s^2 - t^2$ and $y = t^2 - s^2$.
$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = f_x(2s) + f_y(-2s) = 2sf_x - 2sf_y$.
$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = f_x(-2t) + f_y(2t) = -2tf_x + 2tf_y$.
$t\frac{\partial z}{\partial s} + s\frac{\partial z}{\partial t} = t(2sf_x - 2sf_y) + s(-2tf_x + 2tf_y)$
$= 2stf_x - 2stf_y - 2stf_x + 2stf_y = 0$.

Example (6): If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s$, $y = 3rs$. Find $\frac{\partial^2 z}{\partial r^2}$.
Solution:
$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = f_x(2r) + f_y(3s) = 2rf_x + 3sf_y$.

$\dfrac{\partial^2 z}{\partial r^2} = \dfrac{\partial}{\partial r}\left(\dfrac{\partial z}{\partial r}\right) = \dfrac{\partial}{\partial r}(2rf_x + 3sf_y) = 2f_x + 2r\left(\dfrac{\partial f_x}{\partial r}\right) + 3s\left(\dfrac{\partial f_y}{\partial r}\right)$
$= 2f_x + 2r\left(\dfrac{\partial f_x}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f_x}{\partial y}\dfrac{\partial y}{\partial r}\right) + 3s\left(\dfrac{\partial f_y}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f_y}{\partial y}\dfrac{\partial y}{\partial r}\right)$
$= 2f_x + 2r\left(f_{xx}(2r) + f_{xy}(3s)\right) + 3s\left(f_{yx}(2r) + f_{yy}(3s)\right)$
$= 2f_x + 4r^2 f_{xx} + 6rsf_{xy} + 6rsf_{yx} + 9s^2 f_{yy}$.

1.4.4 Implicit Differentiation

Let $F(x, y) = 0$ where $y = f(x)$, using chain rule $\dfrac{\partial F}{\partial x}\dfrac{dx}{dx} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dx} = 0$
$\implies F_x(1) + F_y\dfrac{dy}{dx} = 0 \implies \dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.

Example (7): If $x^2 + y^2 = 5xy$, Find $\dfrac{dy}{dx}$.
Solution:
$x^2 + y^2 = 5xy \implies x^2 + y^2 - 5xy = 0$, Let $F(x, y) = x^2 + y^2 - 5xy$ then $F(x, y) = 0$.
$\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{2x - 5y}{2y - 5x}$.

Let $F(x, y, z) = 0$ where $z = f(x, y)$, then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$.

Example (8): If $x^3 + y^3 + z^2 + 3xyz - 1 = 0$, Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.
Solution:
Let $F(x, y, z) = x^3 + y^3 + z^2 + 3xyz - 1$ then $F(x, y, z) = 0$.
$\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{3x^2 + 3yz}{2z + 3xy}$.
$\dfrac{\partial z}{\partial x} = -\dfrac{F_y}{F_z} = -\dfrac{3y^2 + 3xz}{2z + 3xy}$.

1.4.5 EXERCISES

1. Find $\dfrac{dw}{dt}$ of the following:

   (a). $w = xy^3 - x^2y$, where $x = t^2 + 1$ and $y = t^2 - 1$.

   (b). $w = \sin x \cos y$, where $x = \sqrt{t}$ and $y = \dfrac{1}{t}$.

   (c). $w = xe^{\frac{y}{z}}$, where $x = t^2$, $y = 1 - t$ and $z = 1 + 2t$.

   (d). $w = \ln \sqrt{x^2 + y^2 + z^2}$, where $x = \sin t$, $y = \cos t$ and $z = \tan t$.

2. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ of the following:

   (a). $z = (x - y)^5$, where $x = s^2t$ and $y = st^2$.

   (b). $z = \tan^{-1}\left(x^2 + y^2\right)$, where $x = s\ \ln t$ and $y = t\ e^s$.

   (c). $z = \dfrac{\sin\theta}{r}$, where $r = st$ and $\theta = s^2 + t^2$.

3. If $z = x^4 + x^2y$, where $x = s + 2t - u$ and $y = stu^2$,

   Find $\dfrac{\partial z}{\partial s}, \dfrac{\partial z}{\partial t}$ and $\dfrac{\partial z}{\partial u}$ at $s = 4$, $t = 2$, $u = 1$.

4. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, Show that

   $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

5. If $z = f(x + at) + g(x - at)$, Show that $\dfrac{\partial^2 z}{\partial t^2} = a^2\ \dfrac{\partial^2 z}{\partial x^2}$.

(*) If $xe^{yz} - 2ye^{xz} + 3ze^{xy} = 1$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

1.5 Maximum and Minimum Values

1.5.1 Local Maximum and Minimum Values

Definition: If f is a differentiable function of two variables x and y, then the gradient of f is the vector function ∇f and it is defined by
$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j}$.

Definition: If f is a differentiable function of three variables x,y and z, then the gradient of f is the vector function ∇f and it is defined by
$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j} + \dfrac{\partial f}{\partial z}\vec{k}$.

Definition (Critical point): If $f(x,y)$ is a function of two variables then $(a,b) \in D_f$ is a critical point if both $f_x(a,b) = 0$ and $f_y(a,b) = 0$, or f is not differentiable at (a,b).

Definition (Local maximum and Local minimum):
(i) A function f of two variables has a local maximum at $(a,b) \in D_f$ if $f(x,y) \leq f(a,b)$, for all points (x,y) in some disk with center (a,b).
(ii) A function f of two variables has a local minimum at $(a,b) \in D_f$ if $f(x,y) \geq f(a,b)$, for all points (x,y) in some disk with center (a,b).

Theorem: If $f(x,y)$ has a local maximum or minimum at $(a,b) \in D_f$ and the first-order partial derivatives of f exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Second Derivatives Test: Suppose the second partial derivatives of $f(x,y)$ are continuous on a disk with center $(a,b) \in D_f$, and suppose that
$f_x(a,b) = 0$ and $f_y(a,b) = 0$ [so (a,b) is a critical point of f].
Let $D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$.
(1). If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
(2). If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
(3). If $D < 0$, then (a,b) is a saddle point.

NOTES:
(1). If $D = 0$, then the test gives no information.
(2). $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}f_{yx} = f_{xx}f_{yy} - [f_{xy}]^2$.

Example (1): Find the local maximum and minimum values and saddle points of $f(x,y) = 6xy - 2x^3 + y^2$.
Solution:
$f_x(x,y) = 6y - 6x^2$ and $f_y(x,y) = 6x + 2y$.
$f_x(x,y) = 0 \implies 6y - 6x^2 = 0 \implies y = x^2$.
$f_y(x,y) = 0 \implies 6x + 2y = 0 \implies y = -3x$.
$f_x(x,y) = f_y(x,y) \implies x^2 = -3x \implies x^2 + 3x = 0$
$\implies x(x+3) = 0 \implies x = 0\ ,\ x = -3 \implies y = 0\ ,\ y = 9$.
So, the critical points are $(0,0)$ and $(-3,9)$.

$f_{xx}(x, y) = -12x,\ f_{xy}(x, y) = 6$ and $f_{yy}(x, y) = 2$.

First- At the point $(0, 0)$:
$D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{x,y}(0, 0)]^2 = (0)(2) - (6)^2 = -36 < 0.$
Therefore, $(0, 0)$ is a saddle point.

Second- At the point $(-3, 9)$:
$D(-3, 9) = f_{xx}(-3, 9)f_{yy}(-3, 9) - [f_{x,y}(-3, 9)]^2 = (36)(2) - (6)^2 = 36 > 0.$
Since $f_{xx}(-3, 9) = 36 > 0$, then f attains a local minimum at $(-3, 9)$,
and its value is $f(-3, 9) = 6(-3)(9) - 2(-3)^3 + (-9)^2 = -162 + 54 + 81 = -27.$

Example (2): Find the local maximum and minimum values and saddle points of $f(x, y) = x^3 - y^3 - 3x + 3y + 5$.
Solution:
$f_x(x, y) = 3x^2 - 3$ and $f_y(x, y) = -3y^2 + 3$.
$f_x(x, y) = 0 \implies 3x^2 - 3 = 0 \implies x^2 - 1 = 0 \implies x = \pm 1$.
$f_y(x, y) = 0 \implies -3y^2 + 3 = 0 \implies y^2 - 1 = 0 \implies y = \pm 1$.
So, the critical points are $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$.

$f_{xx}(x, y) = 6x,\ f_{xy}(x, y) = 0$ and $f_{yy}(x, y) = -6y$.

First- At the point $(1, 1)$:
$D(1, 1) = f_{xx}(1, 1)f_{yy}(1, 1) - [f_{x,y}(1, 1)]^2 = (6)(-6) - (0)^2 = -36 < 0.$
Therefore, $(1, 1)$ is a saddle point.

Second- At the point $(1, -1)$:
$D(1, -1) = f_{xx}(1, -1)f_{yy}(1, -1) - [f_{x,y}(1, -1)]^2 = (6)(6) - (0)^2 = 36 > 0.$
Since $f_{xx}(1, -1) = 6 > 0$, then f attains a local minimum at $(1, -1)$,
and its value is $f(1, -1) = 1 + 1 - 3 - 3 + 5 = 1.$

Third- At the point $(-1, 1)$:
$D(-1, 1) = f_{xx}(-1, 1)f_{yy}(-1, 1) - [f_{x,y}(-1, 1)]^2 = (-6)(-6) - (0)^2 = 36 > 0.$
Since $f_{xx}(-1, 1) = -6 < 0$, then f attains a local maximum at $(-1, 1)$,
and its value is $f(-1, 1) = -1 - 1 + 3 + 3 + 5 = 9.$

Fourth- At the point $(-1, -1)$:
$D(-1, -1) = f_{xx}(-1, -1)f_{yy}(-1, -1) - [f_{x,y}(-1, -1)]^2$
$= (-6)(6) - (0)^2 = -36 < 0.$
Therefore, $(-1, -1)$ is a saddle point.

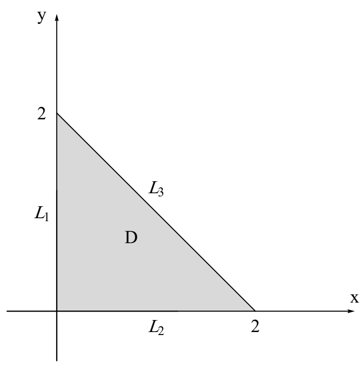
1.5.2 Absolute Maximum and Minimum Values

Definition: Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the
(1). absolute maximum value of f on D if $f(a, b) \geq f(x, y)$ for all (x, y) in D.
(2). absolute minimum value of f on D if $f(a, b) \leq f(x, y)$ for all (x, y) in D.

Theorem: If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

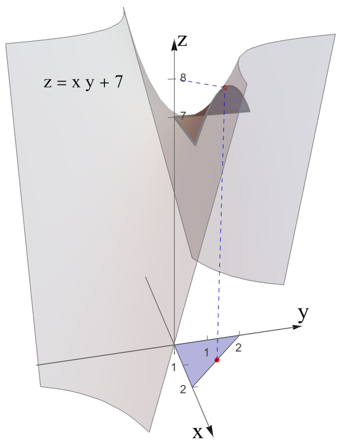
NOTES: To find the absolute maximum and minimum values of a continuous function $f(x,y)$ on a closed, bounded set $D \subseteq \mathbb{R}^2$:
1. Find the values of f at the critical points of f in the interior of D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps (1) and (2) is the absolute maximum value of f on D, and the smallest of these values is the absolute minimum value of f on D.

Example (3): Find the absolute maximum and minimum values of the function $f(x,y) = xy + 7$ on the plain region bounded by the graphs of the lines $x = 0$, $y = 0$ and $y + x = 2$.
Solution:



$f_x(x,y) = y$ and $f_y(x,y) = x$.
$f_x(x,y) = 0 \implies y = 0$.
$f_y(x,y) = 0 \implies x = 0$.
The critical point is $(0,0)$.
Note that the critical point is not inside the given region.

Let L_1 be the line $x = 0$:
then $f(x,y) = f(0,y) = 7$.
$f(x,y) = 7$ for all $(x,y) \in L_1$.



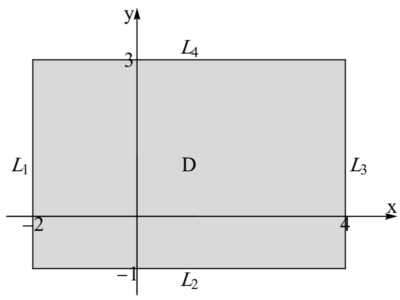
Let L_2 be the line $y = 0$:
then $f(x,y) = f(x,0) = 7$.
$f(x,y) = 7$ for all $(x,y) \in L_2$.

Let L_3 be the line $y + x = 2$:
then $y = -x + 2$ where $0 \leq x \leq 2$.
$f(x,y) = f(x, -x+2)$
$= x(-x+2) + 7 = -x^2 + 2x + 7$.
$f'(x) = -2x + 2$,
$f'(x) = 0 \implies -2x + 2 = 0$
$\implies x = 1$.
So, $y = -2 + 1 = 1$.
$f(1,1) = (1)(1) + 7 = 8$.
Also, $f(0,2) = 7$ and $f(2,0) = 7$.

The absolute maximum is 8, and f takes it at $(1,1)$.
The absolute minimum is 7, and f takes it at any point on $L_1 \cup L_2$.

Example (4): Find the absolute maximum and minimum values of the function $f(x,y) = x^2 + 2xy + 3y^2$ on the closed and bounded region
$D = \{(x,y) \in \mathbb{R}^2 : -2 \leq x \leq 4 \ , \ -1 \leq y \leq 3\}$.

Solution:



$f_x(x,y) = 2x + 2y$.
$f_y(x,y) = 2x + 6y$.
$f_x(x,y) = 0 \implies x = -y$.
$f_y(x,y) = 0 \implies x = -3y$.
$-y = -3y \implies y = 0 \implies x = 0$.
The critical point is $(0,0)$.
$f(0,0) = (0)^2 + 2(0)(0) + 3(0)^3 = 0$.

Let L_1 be the line between $(-2,-1)$ and $(-2,3)$.
On $L_1 : x = -2$ and $-1 \le y \le 3$.
$f(x,y) = f(-2,y) = 4 - 4y + 3y^2 \implies f'(y) = -4 + 6y$.
$f'(y) = 0 \implies 6y = 4 \implies y = \dfrac{2}{3}$.
Note that $\left(-2, \dfrac{2}{3}\right) \in L_1$.
$f\left(-2, \dfrac{2}{3}\right) = (-2)^2 + 2(-2)\left(\dfrac{2}{3}\right) + 3\left(\dfrac{2}{3}\right)^2 = 4 - \dfrac{8}{3} + \dfrac{4}{3} = \dfrac{8}{3}$.

Let L_2 be the line between $(-2,-1)$ and $(4,-1)$.
On $L_2 : y = -1$ and $-2 \le x \le 4$.
$f(x,y) = f(x,-1) = x^2 - 2x + 3 \implies f'(x) = 2x - 2$.
$f'(x) = 0 \implies 2x = 2 \implies x = 1$.
Note that $(1,-1) \in L_2$.
$f(1,-1) = (1)^2 + 2(1)(-1) + 3(-1)^2 = 1 - 2 + 3 = 2$.

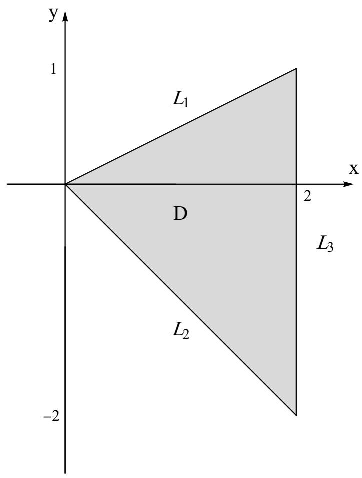
Let L_3 be the line between $(4,-1)$ and $(4,3)$.
On $L_3 : x = 4$ and $-1 \le y \le 3$.
$f(x,y) = f(4,y) = 16 + 8y + 3y^2 \implies f'(y) = 8 + 6y$.
$f'(y) = 0 \implies 6y = -8 \implies y = -\dfrac{4}{3}$.
Note that $\left(4, -\dfrac{4}{3}\right) \notin L_3$.

Let L_4 be the line between $(-2,3)$ and $(4,3)$.
On $L_4 : y = 3$ and $-2 \le x \le 4$.
$f(x,y) = f(x,3) = x^2 + 6x + 27 \implies f'(x) = 2x + 6$.
$f'(x) = 0 \implies 2x = 2 \implies x = -3$.
Note that $(-3,) \notin L_4$.

Evaluating $f(x,y)$ at the four corners of D :
$f(-2,-1) = (-2)^2 + 2(-2)(-1) + 3(-1)^2 = 4 + 4 + 3 = 11$.
$f(-2,3) = (-2)^2 + 2(-2)(3) + 3(3)^2 = 4 - 12 + 27 = 19$.
$f(4,-1) = (4)^2 + 2(4)(-1) + 3(-1)^2 = 16 - 8 + 3 = 11$.
$f(4,3) = (4)^2 + 2(4)(3) + 3(3)^2 = 16 + 24 + 27 = 67$.

The absolute maximum is 67, and f takes it at $(4,3)$.
The absolute minimum is 0, and f takes it at $(0,0)$.

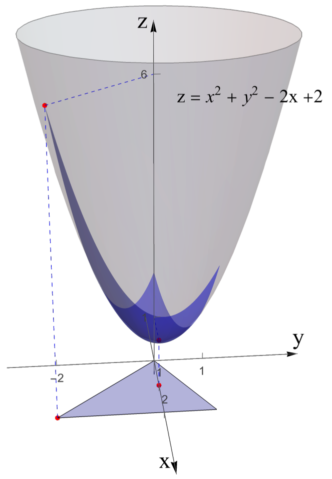
Example (5): Find the absolute maximum and minimum values of the function $f(x,y) = x^2 + y^2 - 2x + 2$ on the closed region with vertices $(0,0)$, $(2,1)$ and $(2,-2)$.
Solution:



$f_x(x,y) = 2x - 2$ and $f_y(x,y) = 2y$.
$f_x(x,y) = 0 \implies x = 1$.
$f_y(x,y) = 0 \implies y = 0$.
The critical point is $(1,0)$.
$f(1,0) = 1 + 0 - 2 + 2 = 1$.

Let L_1 be the line passing through $(0,0)$ and $(2,1)$ then $y = \frac{x}{2}$.
$f\left(x, \frac{x}{2}\right) = x^2 + \frac{x^2}{4} - 2x + 2$.
$f(x) = \frac{5}{4}x^2 - 2x + 2$.
$f'(x) = \frac{5}{2}x - 2$.
$f'(x) = 0 \implies x = \frac{4}{5}$ and $y = \frac{2}{5}$.
$f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{16}{25} + \frac{4}{25} - \frac{8}{5} + 2$
$= \frac{16}{25} + \frac{4}{25} - \frac{40}{25} + \frac{50}{25} = \frac{30}{25} = \frac{6}{5}$.

Let L_2 be the line passing through $(0,0)$ and $(2,-2)$ then $y = -x$.
$f(x,-x) = x^2 + x^2 - 2x + 2$
$f(x) = 2x^2 - 2x + 2$.
$f'(x) = 4x - 2$.
$f'(x) = 0 \implies x = \frac{1}{2}$ and $y = -\frac{1}{2}$.
$f\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} + 1 + 2 = \frac{7}{2}$.



Let L_3 be the line passing through $(2,-2)$ and $(2,1)$ then $x = 2$.
$f(2,y) = 4 + y^2 - 4 + 2 = y^2 + 2$, so $f(y) = y^2 + 2 \implies f'(y) = 2y$.
$f'(y) = 0 \implies y = 0$ and $x = 2$.
$f(2,0) = 4 + 0 - 4 + 2 = 2$.

Evaluating $f(x,y)$ at the three corners of D :
$f(0,0) = 0 + 0 - 0 + 2 = 2$.
$f(2,1) = 4 + 1 - 4 + 2 = 3$.
$f(2,-2) = 4 + 4 - 4 + 2 = 6$.

The absolute maximum is 6, and f takes it at $(2,-2)$.
The absolute minimum is 1, and f takes it at $(1,0)$.

1.5.3 EXERCISES

1. Find the local maximum and minimum values and saddle point(s) of the functions:

   (a). $f(x, y) = x^2 + xy + y^2 + y$.

   (b). $f(x, y) = x^3 + y^3 + 3xy$.

   (c). $f(x, y) = 2 - x^4 + 2x^2 - y^2$.

   (d). $f(x, y) = x^4 - 2x^2 + y^3 - 3y$.

2. Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 - 2x$ on the closed region with vertices $(2, 0)$, $(0, 2)$ and $(0, -2)$.

(*) Find the local maximum and minimum values and saddle point(s) of the functions:

   (a). $f(x, y) = 4x^3 - 2x^2y + y^2$.

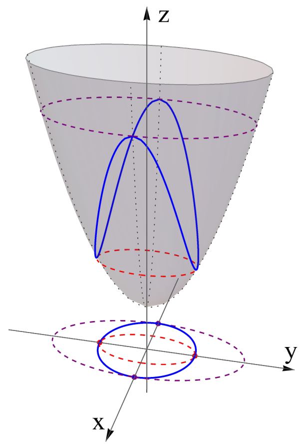
   (b). $f(x, y) = 2x^3 - 3x^2 + 3y^2 - 6xy + 2$.

   (c). $f(x, y) = x^4 + y^3 + 32x - 3y$.

1.6 Lagrange Multipliers

1.6.1 Lagrange Multipliers (One Constraint)

To find the extreme values of $f(x, y)$ subject to a constraint $g(x, y) = k$.
Let $z = f(x, y)$ be the gray surface, and $g(x, y)$ is the blue curve representing the constraint in the xy-plain.
Note that the level curves $f(x, y)$ touches $g(x, y)$ at the points where $f(x, y)$ have minimum and maximum values.
This means that $\nabla f(x, y)$ is parallel to $\nabla g(x, y)$ at these point.
Find these points by solving the equation: $\nabla f(x, y) = \lambda \nabla g(x, y)$,
where $\lambda \in \mathbb{R}$ and $\nabla g(x, y) \neq 0$.
Evaluate $f(x, y)$ at these points, The largest is the maximum value of f and the smallest is the minimum value of f.



Example (1): Find the extreme values of the function $f(x, y) = 1 + xy$ on the circle $x^2 + y^2 = 1$.
Solution:
$f(x, y) = 1 + xy$ and $g(x, y) = x^2 + y^2 - 1$.
$\nabla f(x, y) = \lambda \nabla g(x, y) \implies \langle y, x\rangle = \lambda\langle 2x, 2y\rangle = \langle 2\lambda x, 2\lambda y\rangle$

$$\implies \begin{cases} y = 2\lambda x \\ x = 2\lambda y \end{cases} \implies x = 2\lambda(2\lambda x) = 4\lambda^2 x \implies x\left(1 - 4\lambda^2\right) = 0$$

$$\implies x = 0\ ,\ \lambda = \pm\frac{1}{2}\ .$$

If $x = 0$ then $y = 2\lambda(0) = 0$, but $(0, 0)$ does not lie on the unit circle, so $(0, 0)$ is excluded.
If $\lambda = \pm\frac{1}{2} \implies y = 2\left(\pm\frac{1}{2}\right)x \implies y = \pm x$.

From the constraint : $x^2 + y^2 = 1 \implies 2x^2 = 1 \implies x = \pm\frac{1}{\sqrt{2}}$.

This means $\frac{1}{2} + y^2 = 1 \implies y = \pm\frac{1}{\sqrt{2}}$.

So, There are 4 points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

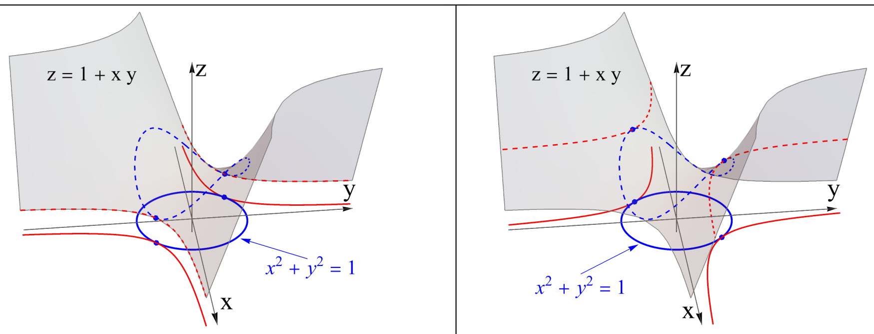
$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 1 + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 1 + \frac{1}{2} = \frac{3}{2}.$$

$$f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = 1 + \frac{1}{\sqrt{2}}\frac{-1}{\sqrt{2}} = 1 - \frac{1}{2} = \frac{1}{2}.$$

$$f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 1 + \frac{-1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 1 - \frac{1}{2} = \frac{1}{2}.$$

$$f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = 1 + \frac{-1}{\sqrt{2}}\frac{-1}{\sqrt{2}} = 1 + \frac{1}{2} = \frac{3}{2}.$$

The maximum value is $\frac{3}{2}$, and f takes it at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.
The minimum value is $\frac{1}{2}$, and f takes it at $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.



Minimum value of $z = 1 + xy$ on $x^2 + y^2 = 1$

Maximum value of $z = 1 + xy$ on $x^2 + y^2 = 1$

Note: to find the extreme values of $f(x, y, z)$ subject to a constraint $g(x, y, z) = k$, where $k \in \mathbb{R}$.
(1). Find the points that satisfy the equation $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$
$f_x(x, y, z) = \lambda g_x(x, y, z)$, $f_y(x, y, z) = \lambda g_y(x, y, z)$ and $f_z(x, y, z) = \lambda g_z(x, y, z)$.
(2). Evaluate $f(x, y, z)$ at these points, The largest is the maximum value of f and the smallest is the minimum value of f.

Example (2): Find the points on the sphere $x^2 + y^2 + z^2 = 1$ that are closest to and farthest from the point $(2, 2, 2)$.
Solution:
Let $f(x, y, z)$ be the function of the square of the distance between any point in the sphere and the point $(2, 2, 2)$, then $f(x, y, z) = (x-2)^2 + (y-2)^2 + (z-2)^2$.
Let $g(x, y, z) = x^2 + z^2 + y^2 - 1 = 0$ be the constraint.

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \implies \langle 2(x-2), 2(y-2), 2(z-2)\rangle = \lambda \langle 2x, 2y, 2z\rangle$$

$$\begin{cases} 2(x-2) = 2\lambda x \\ 2(y-2) = 2\lambda y \\ 2(z-2) = 2\lambda z \end{cases} \implies \begin{cases} x-2 = \lambda x \\ y-2 = \lambda y \\ z-2 = \lambda z \end{cases} \implies \begin{cases} x(1-\lambda) = 2 \\ y(1-\lambda) = 2 \\ z(1-\lambda) = 2 \end{cases}$$

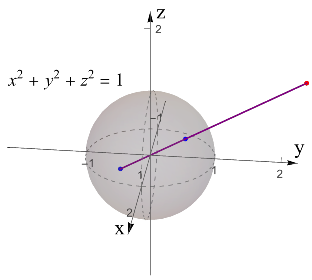
If $\lambda = 1$ then $x - 2 = x \implies -2 = 0$, so $\lambda \neq 1$.
Therefore, $x(1-\lambda) = y(1-\lambda) = z(1-\lambda) \implies x = y = z$.
So, $x^2 + y^2 + z^2 = 1 \implies 3x^2 = 1 \implies x = \pm\frac{1}{\sqrt{3}}$
Hence, $x = y = z = \pm\frac{1}{\sqrt{3}}$.
The required points are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$.

$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}} - 2\right)^2 + \left(\frac{1}{\sqrt{3}} - 2\right)^2 + \left(\frac{1}{\sqrt{3}} - 2\right)^2 = 3\left(2 - \frac{1}{\sqrt{3}}\right)^2.$$

$$f\left(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)=\left(\frac{-1}{\sqrt{3}}-2\right)^2+\left(\frac{-1}{\sqrt{3}}-2\right)^2+\left(\frac{-1}{\sqrt{3}}-2\right)^2=3\left(2+\frac{1}{\sqrt{3}}\right)^2.$$

The point $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ is the closest to $(2,2,2)$.

The point $\left(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)$ is the farthest to $(2,2,2)$.



Example (3): Find the maximum volume of a rectangular box without a lid, where its surface area equals 12 cm^2.
Solution:
Suppose the sides of the rectangular box are x, y and z.
The volume of the rectangular box is $V(x,y,z)=xyz$, subject to the constraint $2xz+2yz+yz=12$.
The constraint is $g(x,y,z)=2xz+2yz+xy-12=0$.
$\nabla V(x,y,z)=\lambda\nabla g(x,y,z) \implies \langle yz,xz,xy\rangle=\lambda\langle 2z+y,2z+x+2x+2z\rangle$

$$\begin{cases} yz=\lambda(2z+y) \\ xz=\lambda(2z+x) \\ xy=\lambda(2x+2y)\end{cases} \implies \begin{cases} xyz=\lambda(2xz+xy) & \longrightarrow (1) \\ xyz=\lambda(2yz+xy) & \longrightarrow (2) \\ xyz=\lambda(2xz+2yz) & \longrightarrow (3)\end{cases}$$

If $\lambda=0$ then $V(x,y,z)=0$.
If $x=0$, $y=0$ or $z=0$, then $V(x,y,z)=0$.

If $\lambda\neq 0$, From equations (1) and (2) :
$\lambda(2xz+xy)=\lambda(2yz+xy) \implies 2xz+xy=2yz+xy \implies 2xz=2yz \implies x=y.$

From equations (2) and (3), and $x=y$:
$\lambda(2yz+xy)=\lambda(2xz+2yz) \implies 2xz+x^2=2xz+2xz$
$\implies x^2=2xz \implies x^2-2xz=0 \implies x(x-2z)=0 \implies x=2z.$

From the equation of the constraint and $x=y=2z$:
$4z^2+4z^2+4z^2=12 \implies z^2=1 \implies z=1$ and $x=y=2z=2$.

The sides of the rectangular box are 2, 2 and 1, and its maximum volume is 4 cm^3.

1.6.2 EXERCISES

1. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint:

   (a). $f(x,y) = x^2 - y^2$, $x^2 + y^2 = 1$.

   (b). $f(x,y) = xye^{-x^2-y^2}$, $2x - y = 0$.

   (c). $f(x,y,z) = xy^2z$, $x^2 + y^2 + z^2 = 4$.

   (d). $f(x,y,z) = x^4 + y^4 + z^4$, $x^2 + y^2 + z^2 = 1$.

2. Find the extreme values of f on the region described by the inequality:

   (a). $f(x,y) = x^2 + y^2 + 4x - 4y$, $x^2 + y^2 \leq 9$.

   (b). $f(x,y) = e^{-xy}$, $x^2 + 4y^2 \leq 1$.

3. Show that the problem of finding the minimum value of f subject to the given constraint can be solved using Lagrange multipliers, but f does not have a maximum value with that constraint:

   (a). $f(x,y) = x^2 + y^2$, $xy = 1$.

   (b). $f(x,y,z) = x^2 + 2y^2 + 3z^2$, $x + 2y + 3z = 10$.

Chapter 2

Multiple Integrals

2.1 Double Integrals over Rectangles

2.1.1 Iterated Integrals

Suppose that $f(x, y)$ is integrable on the rectangle $R = [a, b] \times [c, d]$, then
$$\iint_R f(x, y)\ dA = \int_a^b \int_c^d f(x, y)\ dydx = \int_a^b \left(\int_c^d f(x, y)\ dy \right)\ dx.$$

Example (1): Evaluate the integral:
(a). $\displaystyle\int_0^2 \int_1^2 x^2 y\ dydx$, (b). $\displaystyle\int_1^2 \int_0^2 x^2 y\ dxdy.$
Solution:
(a). $\displaystyle\int_0^2 \int_1^2 x^2 y\ dydx = \int_0^2 \left(\int_1^2 x^2 y\ dy \right)\ dx = \int_0^2 \left(x^2 \int_1^2 y\ dy \right)\ dx$
$\displaystyle= \int_0^2 x^2 \left[\frac{y^2}{2} \right]_1^2\ dx = \int_0^2 x^2 \left[\frac{4}{2} - \frac{1}{2} \right]\ dx$
$\displaystyle= \frac{3}{2} \int_0^2 x^2\ dx = \frac{3}{2} \left[\frac{x^3}{3} \right]_0^2 = \frac{3}{2} \left[\frac{8}{3} - 0 \right] = 4.$
(b). $\displaystyle\int_1^2 \int_0^2 x^2 y\ dxdy = \int_1^2 \left(\int_0^2 x^2 y\ dx \right) dy = \int_1^2 \left(y \int_0^2 x^2\ dx \right) dy.$
$\displaystyle= \int_1^2 y \left[\frac{x^3}{3} \right]_0^2 dy = \int_1^2 y \left[\frac{8}{3} - 0 \right]_0^2 dy = \frac{8}{3} \int_1^2 y\ dy$
$\displaystyle= \frac{8}{3} \left[\frac{y^2}{2} \right]_1 2 = \frac{8}{3} \left[\frac{4}{2} - \frac{1}{2} \right] = \frac{8}{3}\ \frac{3}{2} = 4.$

Fubini's Theorem: If f is continuous on the rectangle $R = \{(x, y) \mid a \leq x \leq b\ ,\ c \leq y \leq d\}$,
Then $\displaystyle\iint_R f(x, y)\ dA = \int_a^b \int_c^d f(x, y)\ dydx = \int_c^d \int_a^b f(x, y)\ dxdy.$

Example (2): Evaluate the integral $\displaystyle\iint_R y \cos(xy)\ dA$,
where $R = [0, 1] \times \left[0, \frac{\pi}{2}\right]$.

Solution: Using Fubini's Theorem.

$$\iint_R y\cos(xy)\ dA = \int_0^{\frac{\pi}{2}} \int_0^1 y\cos(xy)\ dxdy = \int_0^{\frac{\pi}{2}} \left(\int_0^1 y\cos(xy)\ dx \right)\ dy$$

$$= \int_0^{\frac{\pi}{2}} [\sin(xy)]_0^1\ dy = \int_0^{\frac{\pi}{2}} [\sin y - \sin(0)]\ dy = \int_0^{\frac{\pi}{2}} \sin y\ dy$$

$$= [-\cos y]_0^{\frac{\pi}{2}} = -\cos\left(\frac{\pi}{2}\right) - (-\cos(0)) = 0 + 1 = 1\ .$$

Note : Solving $\iint_R y\cos(x,y)\ dA = \int_0^1 \int_0^{\frac{\pi}{2}} y\cos(xy)\ dydx$ is hard, it needs integration by parts.

2.1.2 Volume

If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle $R = [a,b] \times [c,d]$ and below the surface $z = f(x,y)$ is

$$\text{V} = \iint_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dydx = \int_c^d \int_a^b f(x,y)\ dxdy\ .$$

Example (3): Find the volume of the solid S that is bounded by $z = x^2+y^2+1$, the plains $x = 1$ and $y = 3$, and the three coordinate plains.

Solution:

Note that S is the solid that lies under the surface $z = x^2 + y^2 + 1$ and above the square $R = [0,1] \times [0,3]$.

$$\text{V} = \iint_R (x^2 + y^2 + 1)\ dA = \int_0^1 \int_0^3 (x^2 + y^2 + 1)\ dydx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} + y\right]_0^3\ dx = \int_0^1 \left[(3x^2 + 9 + 3) - (0 + 0 + 0)\right]_0^3\ dx$$

$$= \int_0^1 (3x^2 + 12)\ dx = \left[x^3 + 12x\right]_0^1 = (1 + 12) - (0 + 0) = 13\ .$$

Corollary: If $f(x,y) = g(x)h(y)$ is continuous on the rectangle $R = [a,b]\times[c,d]$, then $\iint_R f(x,y)\ dA = \iint_R g(x)h(y)\ dA = \int_a^b g(x)\ dx\ \int_c^d h(y)\ dy$.

Example (4): Evaluate $\iint_R x\cos y\ dA$, where $R = [0,2] \times \left[0, \frac{\pi}{2}\right]$.

Solution:

$$\iint_R x\cos y\ dA = \int_0^2 \int_0^{\frac{\pi}{2}} x\cos y\ dydx = \left(\int_0^2 x\ dx\right) \left(\int_0^{\frac{\pi}{2}} \cos y\ dy\right)$$

$$= \left[\frac{x^2}{2}\right]_0^2 [\sin y]_0^{\frac{\pi}{2}} = [2 - 0][1 - 0] = 2\ .$$

2.1.3 Average Value

If $f(x, y)$ is defined on a rectangle R then its average value is $f_{avg} = \dfrac{1}{A(R)} \displaystyle\iint_R f(x, y)\ dA$, where $A(R)$ is the area of the rectangle R.

Example (5): Evaluate f_{avg} of $f(x, y) = x \cos y$ on $R = [0, 2] \times \left[0, \dfrac{\pi}{2}\right]$.
Solution:
$$f_{avg} = \frac{1}{A(R)} \iint_R x \cos y\ dA = \frac{2}{(2 - 0)\left(\frac{\pi}{2} - 0\right)} = \frac{2}{\pi}\ .$$

2.1.4 EXERCISES

1. Calculate the iterated integrals :

   (a). $\int_1^4 \int_0^2 (6x^2y - 2x)\ dy\ dx$.

   (b). $\int_{-3}^1 \int_1^2 (x^2 + y^{-2})\ dy\ dx$.

   (c). $\int_{-3}^3 \int_0^{\frac{\pi}{2}} (y + y^2 \cos x)\ dx\ dy$.

   (d). $\int_1^3 \int_1^5 \frac{\ln y}{xy}\ dy\ dx$.

   (e). $\int_0^1 \int_0^1 xy\sqrt{x^2 + y^2}\ dy\ dx$.

   (f). $\int_0^1 \int_0^1 v(u + v^2)^4\ du\ dv$.

2. Calculate the double integrals :

   (a). $\iint_R x \sec^2 y\ dA$, where $R = \left\{(x, y)|\ 0 \le x \le 2\ ,\ 0 \le y \le \frac{\pi}{4}\right\}$.

   (b). $\iint_R \frac{xy^2}{x^2 + 1}\ dA$, where $R = \{(x, y)|\ 0 \le x \le 1\ ,\ \ -3 \le y \le 3\}$.

   (c). $\iint_R \frac{1}{1 + x + y}\ dA$, where $R = [1, 3] \times [1, 2]$.

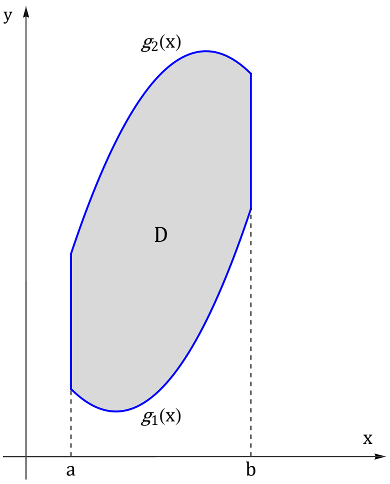
3. Find the volume of the solid that lies under the plain $4x+6y-2z+15 = 0$ and above the rectangle $R = \{(x, y)|\ -1 \le x \le 2\ ,\ \ -1 \le y \le 1\}$.

4. Find the volume of the solid that lies under $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

2.2 Double Integrals over General Regions

2.2.1 General Regions

First - Regions of Type I:

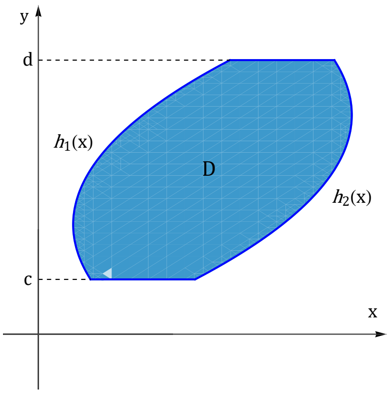


Let D be the region
$\{(x,y) \mid a \le x \le b\ ,\ g_1(x) \le y \le g_2(x)\}$

If f is continuous on D, then

$$\iint_D f(x,y)\ dA$$
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dydx$$
$$= \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\ dy \right)\ dx$$

Second - Regions of Type II:



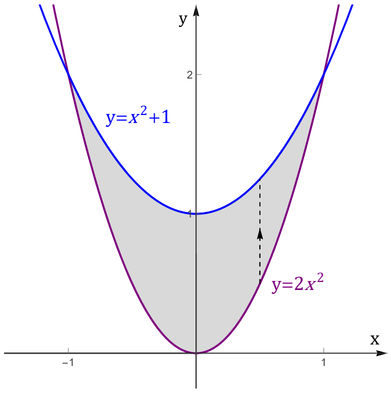
Let D be the region
$\{(x,y) \mid c \le y \le d\ ,\ h_1(y) \le x \le h_2(y)\}$

If f is continuous on D, then

$$\iint_D f(x,y)\ dA$$
$$= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dxdy$$
$$= \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\ dx \right)\ dy$$

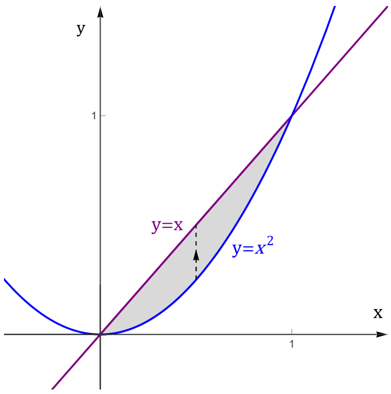
Example (1): Evaluate $\iint_D 2xy\ dA$, where D is the region bounded by the graphs of $y = 2x^2$ and $y = x^2 + 1$.
Solution:

Points of intersection:
$2x^2 = x^2 + 1 \implies x^2 = 1$
$\implies x = \pm 1$
So, D is the region where $-1 \leq x \leq 1$ and $2x^2 \leq y \leq x^2 + 1$



$$\iint_D 2xy\ dA = \int_{-1}^{1} \int_{2x^2}^{x^2+1} 2xy\ dy\ dx$$
$$= \int_{-1}^{1} x\left[y^2\right]_{2x^2}^{x^2+1}\ dx$$
$$= \int_{-1}^{1} x\left[(x^2+1)^2 - (2x^2)^2\right]\ dx$$
$$= \int_{-1}^{1} x\left(x^4 + 2x^2 + 1 - 4x^4\right)\ dx$$
$$= \int_{-1}^{1} x\left(-3x^4 + 2x^2 + 1\right)\ dx = \int_{-1}^{1} \left(-3x^5 + 2x^3 + x\right) dx$$
$$= \left[-\frac{x^6}{2} + \frac{x^4}{2} + \frac{x^2}{2}\right]_{-1}^{1}$$
$$= \left(-\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) - \left(-\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) = 0\ .$$

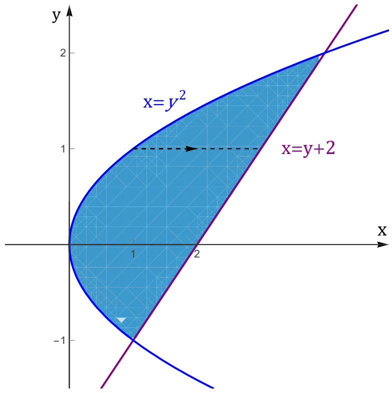
Example (2): Evaluate $\iint_D (x^2 + y^2)\ dA$, where D is the region bounded by the graphs of $y = x^2$ and $y = x$.
Solution:
Points of intersection:
$x^2 = x \implies x^2 - x = 0$
$\implies x(x-1) = 0 \implies x = 0\ ,\ x = 1$
So, D is the region where $0 \leq x \leq 1$ and $x^2 \leq y \leq x$



$$\iint_D (x^2 + y^2)\ dA$$
$$= \int_0^1 \int_{x^2}^{x} (x^2 + y^2)\ dy\ dx$$
$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_{x^2}^{x}\ dx$$
$$= \int_0^1 \left[\left(x^3 + \frac{x^3}{3}\right) - \left(x^4 - \frac{x^6}{3}\right)\right]\ dx$$
$$= \int_0^1 \left(-\frac{x^6}{3} - x^4 + \frac{4x^3}{3}\right)\ dx = \left[-\frac{x^7}{21} - \frac{x^5}{5} + \frac{x^4}{3}\right]_0^1$$
$$= \left(-\frac{1}{21} - \frac{1}{5} + \frac{1}{3}\right) - (0 + 0 + 0) = \frac{-5 - 21 + 35}{105} = \frac{9}{105} = \frac{3}{35}\ .$$

Example (3): Evaluate $\iint_D xy\ dA$, where D is the region bounded by the graphs of $x = y^2$ and $x = y + 2$.

Solution:
Points of intersection:
$y^2 = y + 2 \implies y^2 - y - 2 = 0$
$\implies (y-2)(y+1) = 0 \implies y = -1\ ,\ y = 2$
So, D is the region where $-1 \le y \le 2$ and $y^2 \le x \le y + 2$

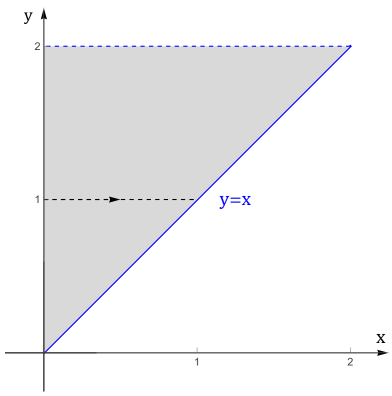


$$\iint_D xy\ dA = \int_{-1}^{2}\int_{y^2}^{y+2} xy\ dx\ dy$$
$$= \int_{-1}^{2} y\left[\frac{x^2}{2}\right]_{y^2}^{y+2} dy$$
$$= \frac{1}{2}\int_{-1}^{2} y\left[(y+2)^2 - (y^2)^2\right]\ dy$$
$$= \frac{1}{2}\int_{-1}^{2} y\left(y^2 + 4y + 4 - y^4\right)\ dy$$
$$= \frac{1}{2}\int_{-1}^{2}\left(-y^5 + y^3 + 4y^2 + 4y\right)\ dy = \frac{1}{2}\left[-\frac{y^6}{6} + \frac{y^4}{4} + \frac{4y^3}{3} + 2y^2\right]_{-1}^{2}$$
$$= \frac{1}{2}\left[\left(-\frac{64}{6} + \frac{16}{4} + \frac{32}{3} + 8\right) - \left(-\frac{1}{6} + \frac{1}{4} - \frac{4}{3} + 2\right)\right] = \frac{45}{8}\ .$$

2.2.2 Changing the Order of Integration

Example (4): Change the order of integration to evaluate $\displaystyle\int_0^2\int_x^2 e^{5+y^2}\ dy\ dx.$

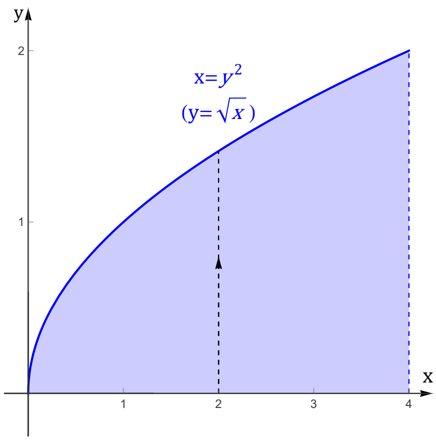
Solution:
D is the region where $0 \le x \le 2$ and $x \le y \le 2$.
Or, $0 \le y \le 2$ and $0 \le x \le y$.



$$\int_0^2\int_x^2 e^{5+y^2}\ dy\ dx$$
$$= \int_0^2\int_0^y e^{5+y^2}\ dx\ dy$$
$$= \int_0^2 e^{5+y^2}[x]_0^y\ dy = \int_0^2 ye^{5+y^2}\ dy$$
$$= \frac{1}{2}\int_0^2 e^{5+y^2}(2y)\ dy$$
$$= \frac{1}{2}\left[e^{5+y^2}\right]_0^2 = \frac{1}{2}\left(e^9 - e^5\right).$$

Example (5): Change the order of integration to evaluate $\int_0^2 \int_{y^2}^4 \cos\left(x^{\frac{3}{2}}\right)\ dx\ dy$.

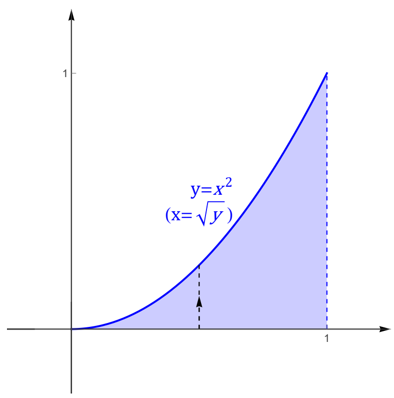
Solution:
D is the region where $0 \le y \le 2$ and $y^2 \le x \le 4$.
Or, $0 \le x \le 4$ and $0 \le y \le \sqrt{x}$.



$$\int_0^2 \int_{y^2}^4 \cos\left(x^{\frac{3}{2}}\right)\ dx\ dy$$
$$= \int_0^4 \int_0^{\sqrt{x}} \cos\left(x^{\frac{3}{2}}\right)\ dy\ dx$$
$$= \int_0^4 \cos\left(x^{\frac{3}{2}}\right)\ [y]_0^{\sqrt{x}}\ dx$$
$$= \frac{2}{3}\int_0^4 \cos\left(x^{\frac{3}{2}}\right)\ \left(\frac{3}{2}x^{\frac{1}{2}}\right)\ dx$$
$$= \frac{2}{3}\left[\sin\left(x^{\frac{3}{2}}\right)\right]_0^4 = \frac{2}{3}\left[\sin(8) - \sin(0)\right] = \frac{2}{3}\sin(8)\ .$$

Example (6): Change the order of integration to evaluate $\int_0^1 \int_{\sqrt{y}}^1 \sin\left(x^3\right)\ dx\ dy$.

Solution:
D is the region where $0 \le y \le 1$ and $\sqrt{y} \le x \le 1$.
Or, $0 \le x \le 1$ and $0 \le y \le x^2$.



$$\int_0^1 \int_{\sqrt{y}}^1 \sin\left(x^3\right)\ dx\ dy$$
$$= \int_0^1 \int_0^{x^2} \sin\left(x^3\right)\ dy\ dx$$
$$= \int_0^1 \sin\left(x^3\right)\ [y]_0^{x^2}\ dx$$
$$= \int_0^1 \sin\left(x^3\right)\ x^2\ dx$$
$$= \frac{1}{3}\int_0^1 \sin\left(x^3\right)\ (3x^2)\ dx = \frac{1}{3}\left[-\cos(x^3)\right]_0^1\ .$$
$$= \frac{1}{3}\left[-\cos(1) - (-\cos(0))\right] = \frac{1-\cos(1)}{3}\ .$$

2.2.3 Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are both integrable on $D \subseteq \mathbb{R}^2$, then

(1). $\iint_D [f(x, y) + g(x, y)]\ dA = \iint_D f(x, y)\ dA + \iint_D g(x, y)\ dA$.

(2). $\iint_D k\ f(x, y)\ dA = k \iint_D f(x, y)\ dA$, where $k \in \mathbb{R}$.

(3). If $f(x, y) \ge g(x, y)$ on D, then $\iint_D f(x, y)\ dA \ge \iint_D g(x, y)\ dA$.

(4). If $D = D_1 \cup D_2$, where $D_1 \cap D_2 = \phi$, then
$\iint_D f(x,y)\ dA = \iint_{D_1} f(x,y)\ dA + \iint_{D_2} f(x,y)\ dA.$

(5). $\iint_D 1\ dA = A(D)$, where $A(D)$ is the area of the region D.

(6). If $m \leq f(x,y) \leq M$ on D, then $m\ A(D) \leq \iint_D f(x,y)\ dA \leq M\ A(D)$.

2.2.4 EXERCISES

1. Evaluate the iterated integrals :

   (a). $\displaystyle\int_1^5 \int_0^x (8x - 2y)\ dy\ dx$ (b). $\displaystyle\int_0^2 \int_0^{y^2} x^2 y\ dx\ dy$

   (c). $\displaystyle\int_0^1 \int_0^{e^x} \sqrt{1 + e^x}\ dy\ dx$

2. Evaluate $\displaystyle\iint_D 2y\ dA$, where D is the region bounded by the graphs of $y = 3x - x^2$ and $y = x$.

3. Evaluate the double integrals:

   (a). $\displaystyle\iint_D \frac{y}{x^2+1}\ dA$, $D = \left\{(x,y) \mid 0 \le x \le 4\ ,\ 0 \le y \le \sqrt{x}\right\}$.

   (b). $\displaystyle\iint_D (2x + y)\ dA$, $D = \{(x,y) \mid 1 \le y \le 2\ ,\ y - 1 \le x \le 1\}$.

   (c). $\displaystyle\iint_D x\ dA$, D is enclosed by the lines $y = x$, $y = 0$ and $x = 1$.

   (d). $\displaystyle\iint_D xy\ dA$, D is enclosed by the curves $y = x^2$ and $y = 3x$.

   (e). $\displaystyle\iint_D x\cos y\ dA$, D is bounded by the $y = 0$, $y = x^2$ and $x = 1$.

   (f). $\displaystyle\iint_D y^2\ dA$, D is the triangular region with vertices $(0,1)$, $(1,2)$ and $(4,1)$.

4. Evaluate the integral by reversing the order of integration:

   (a). $\displaystyle\int_0^1 \int_{3y}^3 e^{x^2}\ dx\ dy$ (b). $\displaystyle\int_0^1 \int_{x^2}^1 \sqrt{y}\ \sin y\ dy\ dx$.

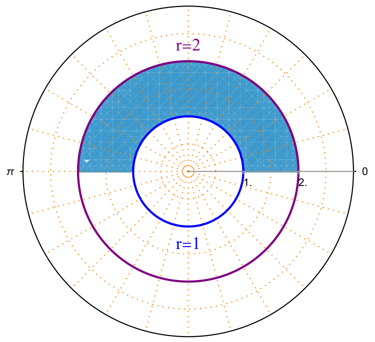
   (c). $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1}\ dy\ dx$ (d). $\displaystyle\int_0^2 \int_{\frac{y}{2}}^1 y\ \cos(x^3 - 1)\ dx\ dy$

2.3 Double Integrals in Polar Coordinates

2.3.1 ouble Integrals in Polar Coordinates

If f is continuous on the polar region R given by $a \leq r \leq b$ and $\theta_1 \leq \theta \leq \theta_2$, where $0 \leq \theta_2 - \theta_1 \leq 2\pi$, then $\displaystyle\iint_R f(x,y)\ dA = \int_{\theta_1}^{\theta_2}\int_a^b f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta$.

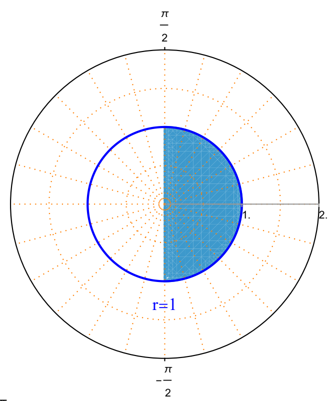
Example (1): Evaluate $\displaystyle\iint_R (4x^2+3y)\ dA$, where R is the region in the upper half-plain bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.
Solution:
R is the region where $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi$.



$$\iint_R (4x^2+3y)\ dA$$
$$= \int_0^\pi\int_1^2 (4r^2\cos^2\theta + 3r\sin\theta)\ r\ dr\ d\theta$$
$$= \int_0^\pi\int_1^2 (4r^3\cos^2\theta + 3r^2\sin\theta)\ dr\ d\theta$$
$$= \int_0^\pi \left[r^4\cos^2\theta + r^3\sin\theta\right]_1^2\ d\theta$$
$$= \int_0^\pi \left[(16\cos^2\theta + 8\sin\theta) - (\cos^2\theta + \sin\theta)\right]\ d\theta = \int_0^\pi (15\cos^2\theta + 7\sin\theta)\ d\theta$$
$$= \int_0^\pi \left[15\left(\frac{1+\cos 2\theta}{2}\right) + 7\sin\theta\right]\ d\theta = \frac{15}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^\pi + 7\left[-\cos\theta\right]_0^\pi$$
$$= \frac{15}{2}\left[(\pi+0)-(0+0)\right] + 7[-(-1)-(-1)] = \frac{15\pi}{2} + 14.$$

Example (2): Evaluate $\displaystyle\int_{-1}^1\int_0^{\sqrt{1-y^2}} \frac{8(x^2+y^2)}{9+(x^2+y^2)^2}\ dx\ dy$

Solution:
R is the region where $-1 \leq y \leq 1$ and $0 \leq x \leq \sqrt{1-y^2}$
In polar coordinates. $0 \leq r \leq 1$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.



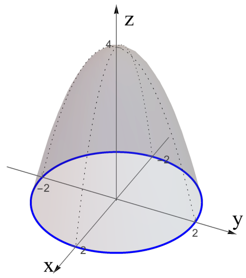
$$\int_{-1}^1\int_0^{\sqrt{1-y^2}} \frac{8(x^2+y^2)}{9+(x^2+y^2)^2}\ dx\ dy$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 \frac{8r^2}{9+r^4} r\ dr\ d\theta$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 2\left(\frac{4r^3}{9+r^4}\right)\ dr\ d\theta$$
$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\ln(9+r^4)\right]_0^1\ d\theta = 2(\ln 10 - \ln 9)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\ d\theta$$

$$= 2\ln\left(\frac{10}{9}\right)[\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\ln\left(\frac{10}{9}\right)\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 2\pi\ln\left(\frac{10}{9}\right).$$

Example (3): Find the volume of the solid bounded by $z = 4 - x^2 - y^2$ and $z = 0$

Solution:

The surface of intersection is :
$4 - x^2 - y^2 = 0 \implies x^2 + y^2 = 4.$
R is the region inside the circle centered at the origin with radius 2.
In polar coordinates, $0 \le r \le 2$ and $0 \le \theta \le 2\pi$.



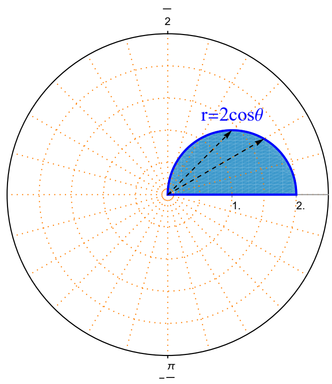
$$\text{V} = \int_0^{2\pi}\int_0^2 (4 - r^2)\ r\ dr\ d\theta$$
$$= \int_0^{2\pi}\int_0^2 (4r - r^3)\ dr\ d\theta$$
$$= \int_0^{2\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi}[(8-4)-(0-0)]\ d\theta$$
$$= 4\int_0^{2\pi} d\theta = 4\,[\theta]_0^{2\pi} = 4\,(2\pi - 0) = 8\pi\ .$$

NOTE: If f is continuous on the polar region $R = \{(r,\theta)|\theta_1 \le \theta \le \theta_2\ ,\ g_1(\theta) \le r \le g_2(\theta)\}$, then $\displaystyle\iint_R f(x,y)\ dA = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta.$

Example (4): Evaluate $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)^{\frac{1}{2}}\ dy\ dx.$

Solution:

R is the region where $0 \le x \le 2$ and $0 \le y \le \sqrt{2x - x^2}$.
$y = \sqrt{2x - x^2} \implies y^2 = 2x - x^2$
$\implies x^2 - 2x + y^2 = 0$
$\implies (x^2 - 2x + 1) + y^2 = 1$
$\implies (x-1)^2 + y^2 = 1.$



$y = \sqrt{2x - x^2}$ is the upper-half of the circle centered at $(1,0)$ with radius 1.

Note that : $y = \sqrt{2x - x^2} \implies y^2 = 2x - x^2 \implies x^2 + y^2 = 2x$
$\implies r^2 = 2r\cos\theta \implies r = 2\cos\theta.$

In polar coordinates, $0 \le r \le 2\cos\theta$ and $0 \le \theta \le \dfrac{\pi}{2}$.

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)^{\frac{1}{2}}\ dy\ dx = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r\ r\ dr\ d\theta = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\ dr\ d\theta$$
$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \cos^3\theta\ d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \cos^2\theta\ \cos\theta\ d\theta$$
$$= \frac{8}{3}\int_0^{\frac{\pi}{2}} \left(1-\sin^2\theta\right)\cos\theta\ d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \left(\cos\theta - \sin^2\theta\ \cos\theta\right)\ d\theta$$
$$= \frac{8}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{\frac{\pi}{2}} = \frac{8}{3}\left[\left(1-\frac{1}{3}\right)-(0-0)\right] = \frac{16}{9}.$$

2.3.2 EXERCISES

1. Evaluate the given integral by changing to polar coordinates.

   (a). $\iint_R (2x - y)\, dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.

   (b). $\iint_R e^{-x^2-y^2}\, dA$, where R is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis.

   (c). $\iint_R \cos\sqrt{x^2 + y^2}\, dA$, where R is the disk with center the origin and radius 2.

2. Use polar coordinates to find the volume of the given solid.

   (a). Under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leq 25$.

   (b). Below the cone $z = \sqrt{x^2 + y^2}$ and above the ring $1 \leq x^2 + y^2 \leq 4$.

   (c). Below the plain $2x + y + z = 4$ and above the disk $x^2 + y^2 \leq 1$.

   (d). Inside the sphere $x^2+y^2+z^2 = 16$ and outside the cylinder $x^2+y^2 = 4$.

   (e). Above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

   (f). Bounded by the paraboloids $z = 6 - x^2 - y^2$ and $z = 2x^2 + 2y^2$.

3. Evaluate the iterated integral by converting to polar coordinates.

   (a). $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\, dy\, dx$.

   (b). $\int_0^{\frac{1}{2}} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\, dx\, dy$.

2.4 Triple Integrals

2.4.1 Triple Integrals over Rectangular Boxes

Fubini's Theorem:
If f is continuous on the rectangular box $E = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_E f(x,y,z)\ dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\ dx\ dy\ dz = \int_r^s \int_a^b \int_c^d f(x,y,z)\ dy\ dx\ dz$$
$$= \int_c^d \int_r^s \int_a^b f(x,y,z)\ dx\ dz\ dy = \int_c^d \int_a^b \int_r^s f(x,y,z)\ dz\ dx\ dy$$
$$= \int_a^b \int_r^s \int_c^d f(x,y,z)\ dy\ dz\ dx = \int_a^b \int_c^d \int_r^s f(x,y,z)\ dz\ dy\ dx\ .$$

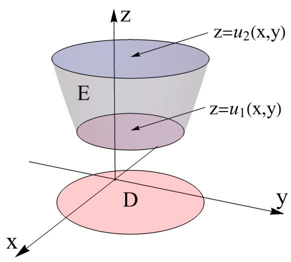
Example (1): Evaluate $\iiint_E xyz^2\ dV$, where $E = [0,1] \times [0,2] \times [0,3]$.

Solution:

$$\iiint_E xyz^2\ dV = \int_0^1 \int_0^2 \int_0^3 xyz^2\ dz\ dy\ dx = \int_0^1 \int_0^2 xy \left[\frac{z^3}{3}\right]_0^3 dy\ dx$$
$$= \left[\frac{27}{3} - \frac{0}{3}\right] \int_0^1 \int_0^2 xy\ dy\ dx = 9 \int_0^1 x \left[\frac{y^2}{2}\right]_0^2 dx = 9 \int_0^1 x \left[\frac{2^2}{2} - \frac{0}{2}\right] dx$$
$$= 9 \int_0^1 2x\ dx = 9\left[x^2\right]_0^1 = 9[1-0] = 9\ .$$

2.4.2 Triple Integrals over General Regions

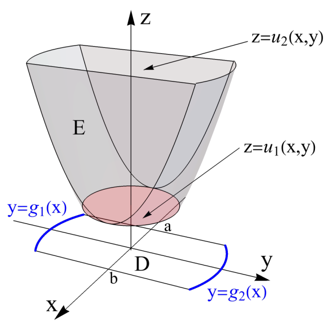
First - Regions of Type I:
Case (1) :
Let E be the region where
$(x,y) \in D$ and
$u_1(x,y) \leq z \leq u_2(x,y)$.



If f is continuous on E, then

$$\iiint_E f(x,y,z)\ dV$$
$$= \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz \right) dA.$$

Case (2) :
Let E be the region where
$a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$
and $u_1(x,y) \leq z \leq u_2(x,y)$.



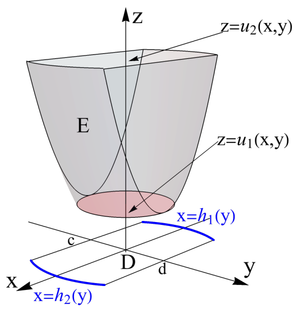
If f is continuous on E, then

$$\iiint_E f(x,y,z)\ dV$$
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\ dy\ dx.$$

Case (3) :
Let E be the region where
$c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$
and $u_1(x, y) \leq z \leq u_2(x, y)$.

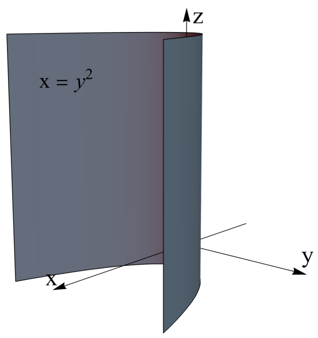
If f is continuous on E, then

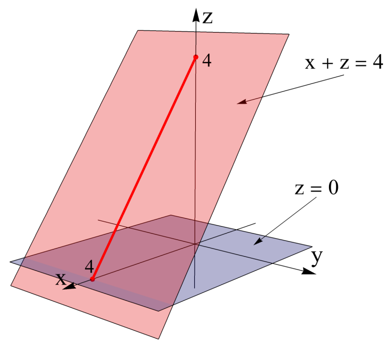
$$\iiint_E f(x, y, z)\ dV$$
$$= \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\ dz\ dx\ dy.$$

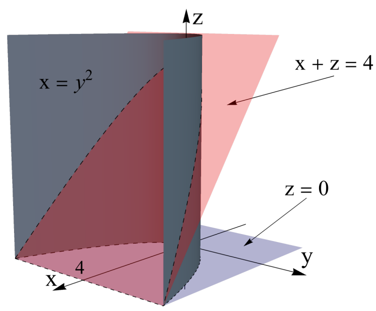


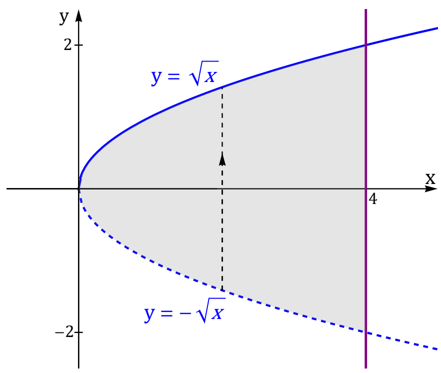
NOTE: $\iiint_E 1\ dV =$ V(E), where V(E) is the volume of the region E.

Example (2): Find the volume of the solid bounded by the cylinder $x = y^2$ and the plains $z = 0$ and $x + z = 4$.
Solution:









Let E be the region bounded by the cylinder $x = y^2$ and the plains $z = 0$ and $x + z = 4$.
Note that $z = 0$ intersects $x + z = 4$ at the line where $x = 4$ and $z = 4$.
On E : $0 \leq x \leq 4$, $-\sqrt{x} \leq y \leq \sqrt{x}$, $0 \leq z \leq 4 - x$.

$$\text{V(E)} = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{4-x} 1\ dz\ dy\ dx = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} [z]_0^{4-x}\ dy\ dx = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} (4 - x)\ dy\ dx$$

$= \int_0^4 (4-x)\ [y]_{-\sqrt{x}}^{\sqrt{x}}\ dx = \int_0^4 2\sqrt{x}\ (4-x)\ dx = 2\int_0^4 \left(4x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)\ dx$

$= 2\left[\frac{8}{3}\ x^{\frac{3}{2}} - \frac{2}{5}\ x^{\frac{5}{2}}\right]_0^4 = 2\left[\frac{8}{3}\ (4)^{\frac{3}{2}} - \frac{2}{5}\ (4)^{\frac{5}{2}}\right] = 2\left[\frac{8}{3}\ (2^3) - \frac{2}{5}(2^5)\right]$

$2\left[\frac{2^6}{3} - \frac{2^6}{5}\right] = 2^7\ \left(\frac{1}{3} - \frac{1}{5}\right) = 2^7\ \left(\frac{2}{15}\right) = \frac{2^8}{15} = \frac{256}{15}.$

Another Solution :

On E : $-2 \le y \le 2$, $y^2 \le x \le 4$, $0 \le z \le 4-x$.

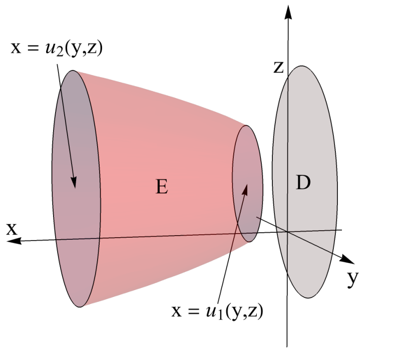
V(E) $= \int_{-2}^{2}\int_{y^2}^{4}\int_{0}^{4-x} 1\ dz\ dx\ dy$.

Second - Regions of Type II:

Let E be the region where $(y,z) \in D$ and $u_1(y,z) \le x \le u_2(y,z)$.

If f is continuous on E, then

$\iiint_E f(x,y,z)\ dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\ dx\right)\ dA.$

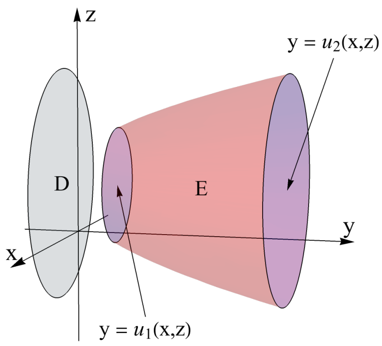


Third - Regions of Type III:

Let E be the region where $(x,z) \in D$ and $u_1(x,z) \le y \le u_2(x,z)$.

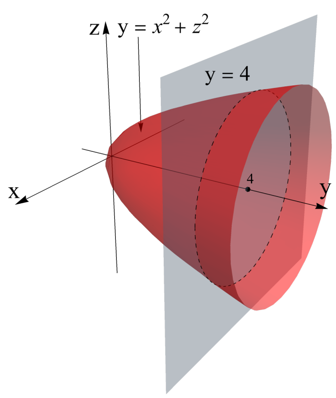
If f is continuous on E, then

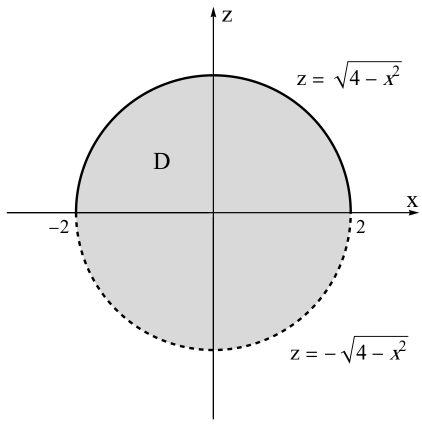
$\iiint_E f(x,y,z)\ dV = \iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\ dy\right)\ dA.$



Example (3): Evaluate $\iiint_E \sqrt{x^2+z^2}\ dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plain $y = 4$.

Solution:





Note that $y = x^2 + z^2$ intersects $y = 4$ at $x^z + z^2 = 4$.
So, E is the region where $x^2 + z^2 \leq y \leq 4$, $-\sqrt{4-x^2} \leq z \leq \sqrt{4-x^2}$ and $-2 \leq x \leq 2$.

$$\iiint_E \sqrt{x^2+z^2}\; dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\; dy\; dz\; dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [y]_{x^2+z^2}^{4} \sqrt{x^2+z^2}\; dz\; dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4-(x^2+z^2)\right] \sqrt{x^2+z^2}\; dz\; dx$$

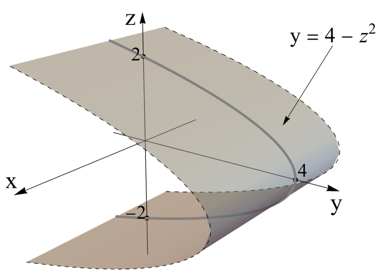
Using the polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$,

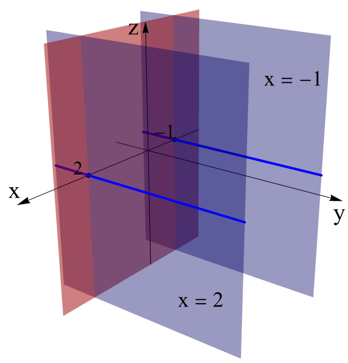
$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4-(x^2+z^2)\right] \sqrt{x^2+z^2}\; dz\; dx = \int_{0}^{2\pi}\int_{0}^{2} (4-r^2)\; r\; r\; dr\; d\theta$$

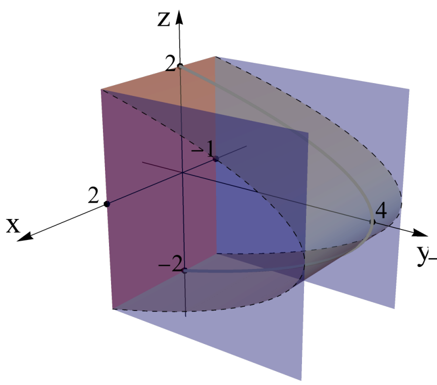
$$= \int_{0}^{2\pi}\int_{0}^{2} (4-r^2)\; r^2\; dr\; d\theta = \int_{0}^{2\pi}\int_{0}^{2} (4r^2-r^4)\; dr\; d\theta = \int_{0}^{2\pi} \left[4\frac{r^3}{3} - \frac{r^5}{5}\right]_0^2 d\theta$$

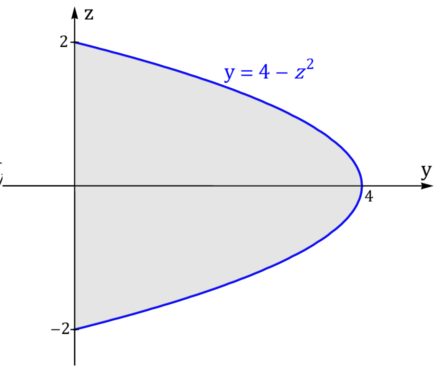
$$= \left(4\;\frac{2^3}{3} - \frac{2^5}{5}\right)\; (2\pi - 0) = 2\pi\left(\frac{2^5}{3} - \frac{2^5}{5}\right) = 2^6\left(\frac{1}{3}-\frac{1}{5}\right)\pi = \frac{128\pi}{15}.$$

Example (4): Find the volume of the solid bounded by the surfaces $y = 4 - z^2$, $x = -1$, $x = 2$ and $y = 0$.
Solution:
Let E be the region bounded by the surfaces $y = 4 - z^2$, $x = -1$, $x = 2$ and $y = 0$.









The surface $y = 4-z^2$ intersects the plain $y = 0$ at the two lines passing through $z = \pm 2$.

On $E: \ -1 \leq x \leq 2 \ , \ 0 \leq y \leq 4 - z^2$ and $-2 \leq z \leq 2$.

$$\text{V(E)} = \iiint_E dV = \int_{-1}^{2} \int_{-2}^{2} \int_{0}^{4-z^2} dy \ dz \ dx$$

$$= \int_{-1}^{2} \int_{-2}^{2} [y]_0^{4-z^2} \ dz \ dx = \int_{-1}^{2} \int_{-2}^{2} (4 - z^2) dz \ dx$$

$$= \int_{-1}^{2} \left[4z - \frac{z^3}{3} \right]_{-2}^{2} \ dx = \left[\left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) \right] \ \int_{-1}^{2} dx$$

$$= \left(16 - \frac{16}{3} \right) \ (2 - (-1)) = 3 \left(16 - \frac{16}{3} \right) = 48 - 16 = 32 \ .$$

2.4.3 EXERCISES

1. Evaluate the iterated integral.

   (a). $\displaystyle\int_0^1\int_y^{2y}\int_0^{x+y} 6xyz\ dz\ dx\ dy$ (b). $\displaystyle\int_1^2\int_0^{2z}\int_0^{\ln x} x\ e^{-y}\ dy\ dx\ dz$

2. Evaluate the triple integral.

   (a). $\displaystyle\iiint_E y\ dV$, where

   $E = \{(x,y,z) \mid 0 \le x \le 3\ ,\ 0 \le y \le x\ ,\ x-y \le z \le x+y\}$.

   (b). $\displaystyle\iiint_E \frac{1}{x^3}\ dV$, where

   $E = \big\{(x,y,z) \mid 0 \le y \le 1\ ,\ 0 \le z \le y^2\ ,\ 1 \le x \le z+1\big\}$.

   (c). $\displaystyle\iiint_E 6xy\ dV$, where where E lies under the plain $z = 1+x+y$ and above the region in the xy-plain bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$.

3. Use a triple integral to find the volume of the given solid.

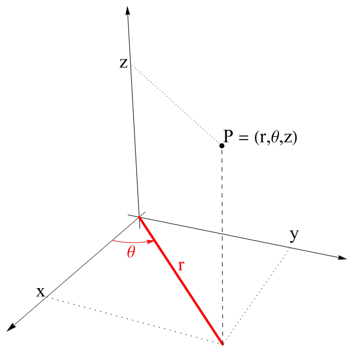
   (a). The tetrahedron enclosed by the coordinate plains and the plain $2x + y + z = 4$.

   (b). The solid enclosed by the paraboloids $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$.

2.5 Triple Integrals in Cylindrical Coordinates

2.5.1 Cylindrical Coordinates

If $P(x, y, z)$ is a point in $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$, then its cylindrical coordinates $P(r, \theta, z)$ are :
$r = \sqrt{x^2 + y^2}$,
$\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$, where $x \neq 0$,
and $z = z$.
Note: $r \in \mathbb{R}$ and $\theta \in [0, 2\pi]$.



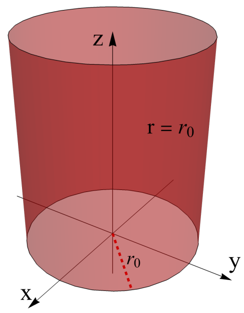
If $P(r, \theta, z)$ is given, the the Cartesian coordinates are :
$x = r\cos\theta$, $y = r\sin\theta$ and $z = z$.

Important equations in Cylindrical coordinates

(1). $r = r_0$, where $r_0 \neq 0$.

$r = r_0 \implies r^2 = r_0^2$
$\implies x^2 + y^2 = r_0^2$.

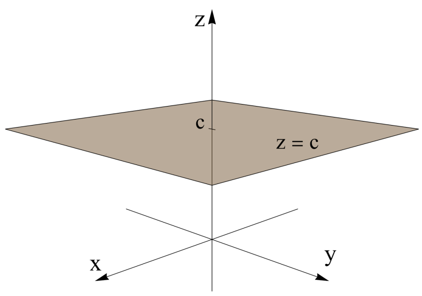
A cylinder,
centered at the origin,
and its radius is r_0.



(2). $z = c$, where $c \in \mathbb{R}$.

All the points (x, y, c) in $\mathbb{R}^3$,
where $x, y \in \mathbb{R}$.

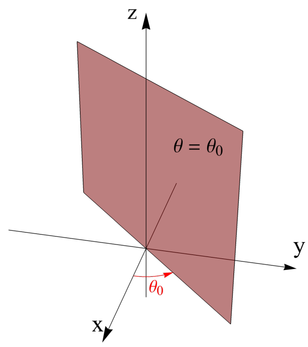
A horizontal plain,
parallel to the xy-plain,
and passes through $(0, 0, c)$.



(3). $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi]$.

$$\begin{aligned} &\theta = \theta_0 \implies \tan(\theta) = \tan(\theta_0), \\ &\implies \frac{y}{x} = \tan(\theta_0) \\ &\implies y = \tan(\theta_0)\, x. \end{aligned}$$

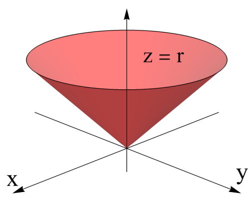
A vertical plain,
passes through the origin.



(4). $z = r$, where $r \neq 0$.

$$z = r \implies z = \sqrt{x^2 + y^2}.$$

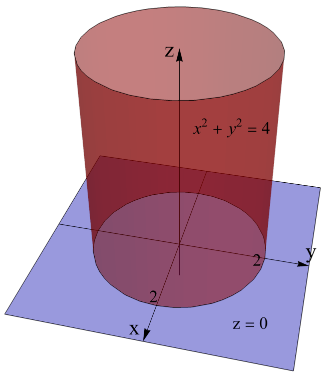
A Cone.

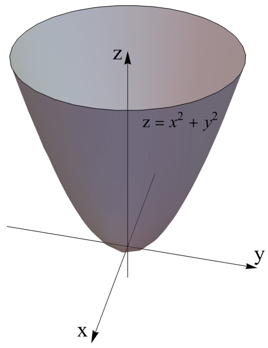


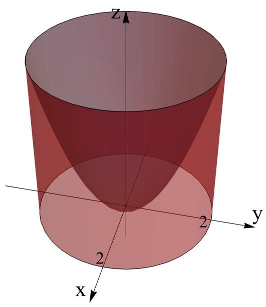
2.5.2 Triple Integrals in Cylindrical Coordinates

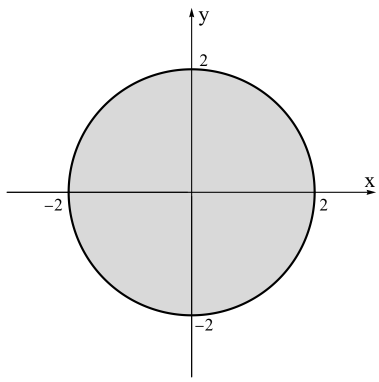
Suppose f is continuous on $E = \{(x, y, z) \mid (x, y) \in D\ ,\ u_1(x, y) \le z \le u_2(x, y)\}$, where D is given in polar coordinates by $D = \{(r, \theta) \mid \theta_1 \le \theta \le \theta_2\ ,\ r_1(\theta) \le r \le r_2(\theta)\}$, then $\iiint_E f(x, y, z)\ dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\ r\ dz\ dr\ d\theta.$

Example (1): Find the volume of the solid within $x^2 + y^2 = 4$, bounded above by $z = x^2 + y^2$ and below by $z = 0$.
Solution:





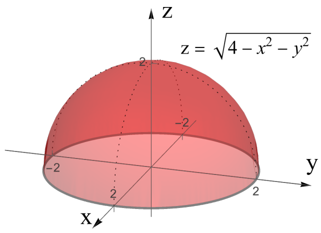


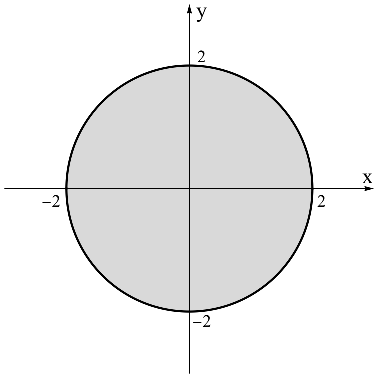


$E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi\ ,\ 0 \le r \le 2\ ,\ 0 \le z \le x^2 + y^2 = r^2\}.$

$$\text{Volume} = \iiint_E dV = \int_0^{2\pi} \int_0^2 \int_0^{x^2+y^2} r\ dz\ dr\ d\theta = \int_0^{2\pi} \int_0^2 \int_0^{r^2} r\ dz\ dr\ d\theta$$
$$= \int_0^{2\pi} \int_0^2 [z]_0^{r^2}\ r\ dr\ d\theta = \int_0^{2\pi} \int_0^2 (r^2 - 0)r\ dr\ d\theta = \int_0^{2\pi} \int_0^2 r^3\ dr\ d\theta$$
$$= \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^2 d\theta = \left[\frac{2^4}{4} - 0\right] \int_0^{2\pi} d\theta = 4(2\pi - 0) = 8\pi.$$

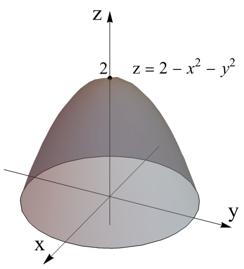
Example (2): Evaluate $\iiint_E z\ dx\ dy\ dz$, where $E = \left\{(x,y,z) \mid 0 \le z \le \sqrt{4-x^2-y^2}\right\}$.
Solution:
$z = \sqrt{4-x^2-y^2}$ is the upper-half of the sphere centered at the origin with radius 2.
$\sqrt{4-x^2-y^2} = 0 \implies x^2+y^2=4$, the upper-half of the sphere intersects the plain $z=0$ at the circle centered at the origin with center 2.
$E = \left\{(r,\theta,z) \mid 0 \le \theta \le 2\pi\ ,\ 0 \le r \le 2\ ,\ 0 \le z \le \sqrt{4-x^2-y^2} = \sqrt{4-r^2}\right\}$.

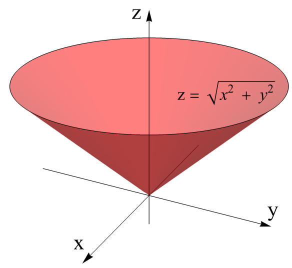


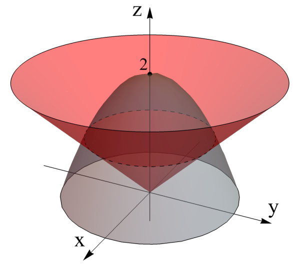


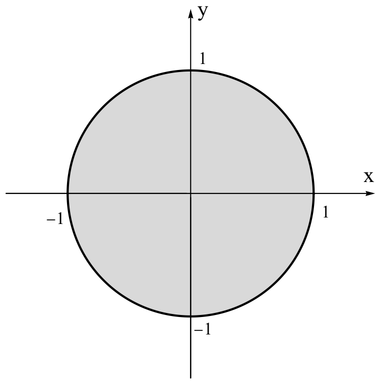
$$\iiint_E z\ dx\ dy\ dz = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{4-r^2}} z\ dz\ r\ drd\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 \left[z^2\right]_0^{\sqrt{4-r^2}} r\ dr\ d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\int_0^2 (4-r^2)r\ dr\ d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 (4r-r^3)\ dr\ d\theta = \frac{1}{2}\int_0^{2\pi}\left[2r^2-\frac{r^4}{4}\right]_0^2 d\theta$$
$$= \frac{1}{2}(8-4)\int_0^{2\pi} d\theta = 2(2\pi-0) = 4\pi\ .$$

Example (3): Find the volume of the solid bounded above by $z = 2-x^2-y^2$ and below by $z = \sqrt{x^2+y^2}$.
Solution:
$z = 2-x^2-y^2 = 2-r^2$ intersects $z = \sqrt{x^2+y^2} = r$ at :
$2-r^2 = r \implies r^2+r-2=0 \implies (r+2)(r-1)=0 \implies r=1$.
(Note that $r=-2$ is excluded because $r \ge 0$).









$E = \left\{(r,\theta,z) \mid 0 \le \theta \le 2\pi\ ,\ 0 \le r \le 1\ ,\ r \le z \le 2-r^2\right\}$.

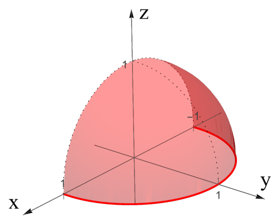
Volume $= \iiint_E dV = \int_0^{2\pi}\int_0^1\int_r^{2-r^2} dz\ r\ dr\ d\theta = \int_0^{2\pi}\int_0^1 [z]_r^{2-r^2}\ r\ dr\ d\theta$

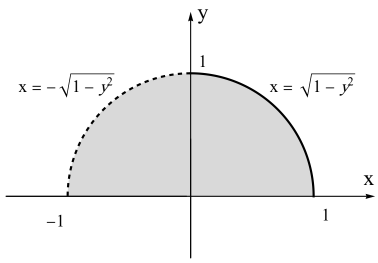
$= \int_0^{2\pi}\int_0^1 (2-r^2-r)r\ dr\ d\theta = \int_0^{2\pi}\int_0^1 (-r^3-r^2+2r)\ dr\ d\theta$

$= \int_0^{2\pi}\left[-\frac{r^4}{4}-\frac{r^3}{3}+r^2\right]_0^1 d\theta = \left(-\frac{1}{4}-\frac{1}{3}+1\right)\int_0^{2\pi} d\theta = \frac{5}{12}(2\pi-0) = \frac{5\pi}{6}$.

Example (4): Evaluate the integral $\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\ dx\ dy$

Solution:
Note that $0 \le y \le 1$, $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$ and $0 \le z \le \sqrt{1-x^2-y^2}$.
$z = \sqrt{1-x^2-y^2} \implies x^2+y^2+z^2 = 1$ represents the upper half of the unit sphere.
$x = \sqrt{1-y^2} \implies x^2+y^2 = 1$ represents the right half of the unit circle.
$x = -\sqrt{1-y^2} \implies x^2+y^2 = 1$ represents the left half of the unit circle.





In cylindrical coordinates: $0 \le \theta \le \pi$, $0 \le r \le 1$ and $0 \le z \le \sqrt{1-r^2}$.

$\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\ dx\ dy = \int_0^{\pi}\int_0^1\int_0^{\sqrt{1-r^2}} dz\ r\ dr\ d\theta$

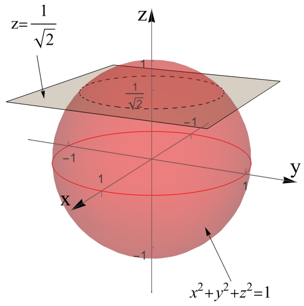
$= \int_0^{\pi}\int_0^1 \sqrt{1-r^2}\ r\ dr\ d\theta = -\frac{1}{2}\int_0^{\pi}\int_0^1 (1-r^2)^{\frac{1}{2}}\ (-2r)\ dr\ d\theta$

$= -\frac{1}{2}\left[\frac{2}{3}(1-r^2)^{\frac{3}{2}}\right]_0^1\int_0^{\pi} d\theta = -\frac{1}{2}\left(0-\frac{2}{3}\right)(\pi-0) = \frac{\pi}{3}$.

Example (5): Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 1$ and above the plain $z = \dfrac{1}{\sqrt{2}}$.

Solution:

Note that $x^2 + y^2 + z^2 = 1$ intersects $z = \dfrac{1}{\sqrt{2}}$ at $x^2 + y^2 = \dfrac{1}{2}$ which is a circle centered at the origin and its radius is $\dfrac{1}{\sqrt{2}}$.



The solid is bounded above by the upper-half of the sphere, and below by the plain.

$x^2 + y^2 + z^2 = 1$

$\Longrightarrow \ z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - r^2}$.

The cylindrical coordinates of the solid : $0 \le r \le \dfrac{1}{\sqrt{2}}$, $0 \le \theta \le 2\pi$ and $\dfrac{1}{\sqrt{2}} \le z \le \sqrt{1 - r^2}$.

$$\mathrm{V} = \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \int_{\frac{1}{\sqrt{2}}}^{\sqrt{1-r^2}} dz \ r \ dr \ d\theta = \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\frac{1}{\sqrt{2}}} \int_{\frac{1}{\sqrt{2}}}^{\sqrt{1-r^2}} dz \ r \ dr \right)$$

$$= [\theta]_0^{2\pi} \ \int_0^{\frac{1}{\sqrt{2}}} [z]_{\frac{1}{\sqrt{2}}}^{\sqrt{1-r^2}} \ r \ dr = (2\pi - 0) \ \int_0^{\frac{1}{\sqrt{2}}} \left(\sqrt{1 - r^2} - \frac{1}{\sqrt{2}} \right) r \ dr$$

$$= 2\pi \int_0^{\frac{1}{\sqrt{2}}} \left(r\sqrt{1 - r^2} - \frac{r}{\sqrt{2}} \right) \ dr = 2\pi \left[-\frac{1}{2} \ \frac{(1 - r^2)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{r^2}{2\sqrt{2}} \right]_0^{\frac{1}{\sqrt{2}}}$$

$$= 2\pi \left[\left(-\frac{1}{2}\frac{2}{3} \left(1 - \frac{1}{2} \right)^{\frac{3}{2}} - \frac{1}{4\sqrt{2}} \right) - \left(-\frac{1}{2}\frac{2}{3} - 0 \right) \right]$$

$$= 2\pi \left(-\frac{1}{3} \frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{2}} + \frac{1}{3} \right) = \frac{2\pi}{3} \left(-\frac{1}{2\sqrt{2}} - \frac{3}{4\sqrt{2}} + 1 \right)$$

$$= \frac{2\pi}{3} \left(1 - \frac{5}{4\sqrt{2}} \right).$$

2.5.3 EXERCISES

1. Evaluate $\iiint_E \sqrt{x^2+y^2}\, dV$, where E is the region that lies inside the cylinder $x^2+y^2=16$ and between the plains $z=-5$ and $z=4$.

2. Evaluate $\iiint_E z\, dV$, where E is enclosed by the paraboloid $z=x^2+y^2$ and the plain $z=4$.

3. Evaluate $\iiint_E (x+y+z)\, dV$, where E is the solid in the first octant that lies under the paraboloid $z=4-x^2-y^2$.

4. Find the volume of the solid that lies within both the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$.

5. Find the volume of the solid that is enclosed by the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=2$.

6. Find the volume of the solid that lies between the paraboloid $z=x^2+y^2$ and the sphere $x^2+y^2+z^2=2$.

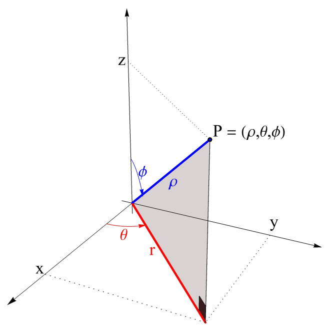
7. Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\; dz\; dx\; dy$.

8. Evaluate $\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}}\int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\; dz\; dx\; dy$.

2.6 Triple Integrals in Spherical Coordinates

2.6.1 Spherical Coordinates

If $P(x, y, z)$ is a point in $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$,
then its spherical coordinates
$P(\rho, \theta, \phi)$ are :
$\rho = \sqrt{x^2 + y^2 + z^2}$,
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$, where $x \neq 0$,
and $\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$, where $\rho \neq 0$.
Note: $\rho \geq 0$, $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$.

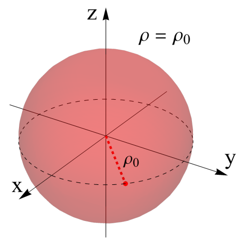


Note that if $P(\rho, \theta, \phi)$ is given then :
$\sin\phi = \frac{r}{\rho} \implies r = \rho\sin\phi$ and $\cos\phi = \frac{z}{\rho} \implies z = \rho\cos\phi$.
So, $x = r\cos\theta = \rho\cos\theta\sin\phi$ and $y = r\sin\theta = \rho\sin\theta\sin\phi$.
Important equations in Spherical coordinates

(1). $\rho = \rho_0$, where $\rho_0 > 0$.

$\rho = \rho_0 \implies \rho^2 = \rho_0^2$
$\implies x^2 + y^2 + z^2 = r_0^2$.

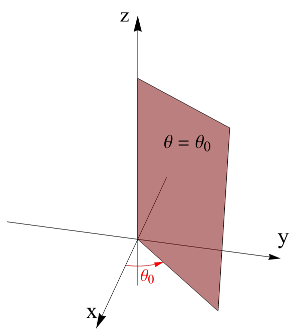
A sphere,
centered at the origin,
and its radius is ρ_0.



(2). $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi]$.

Since $\phi \in [0, \pi]$ then
$\sin\phi \geq 0$.
If $\theta \in [0, \pi]$ then :
$y = \rho\sin\theta\sin\phi \geq 0$
$\theta = \theta_0$ represents a half-plain.

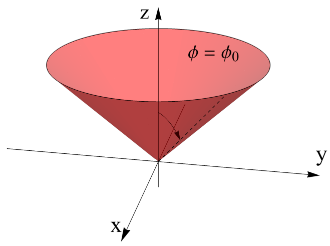
If $\theta \in [\pi, 2\pi]$ then :
$y = \rho\sin\theta\sin\phi \leq 0$
$\theta = \theta_0$ represents the other half-plain.



(3). $\phi = \phi_0$, where $0 < \phi_0 < \dfrac{\pi}{2}$.

$\cot\phi = \dfrac{z}{r} \implies z = r\ \cot\phi$,
$\implies z = \cot\phi\ \sqrt{x^2 + y^2}$

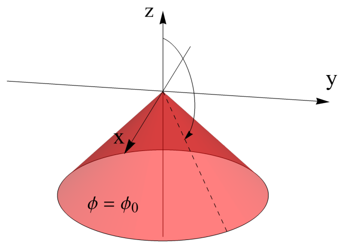
Since $\phi \in \left(0, \dfrac{\pi}{2}\right)$ then
$\cot\phi > 0$.
$\phi = \phi_0$ represents an upper cone.



(4). $\phi = \phi_0$, where $\dfrac{\pi}{2} < \phi_0 < \pi$.

$\cot\phi = \dfrac{z}{r} \implies z = r\ \cot\phi$,
$\implies z = \cot\phi\ \sqrt{x^2 + y^2}$

Since $\phi \in \left(\dfrac{\pi}{2}, \pi\right)$ then
$\cot\phi < 0$.
$\phi = \phi_0$ represents a lower cone.



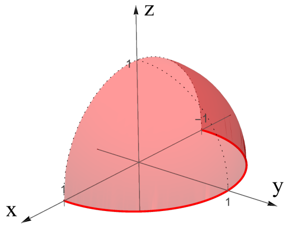
2.6.2 Triple Integrals in Spherical Coordinates

$$\iiint_E f(x,y,z)\ dV = \int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2}\int_a^b f(\rho,\theta,\phi)\ \rho^2\sin\phi\ d\rho\ d\theta\ d\phi\ ,$$
where $E = \{(\rho,\theta,\phi)\mid a\le\rho\le b\ ,\ \theta_1\le\theta\le\theta_2\ ,\ \phi_1\le\phi\le\phi_2\}$.

Example (1): Evaluate the integral $\displaystyle\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\ dx\ dy$

Solution: Referring to Example(4) in section (2.5).

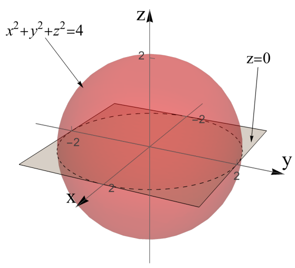
In Spherical coordinates :
$0\le\rho\le 1$, $0\le\theta\le\pi$, and $0\le\phi\le\dfrac{\pi}{2}$.



$$\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\ dx\ dy$$
$$=\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^1 \rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$
$$=\left(\int_0^1\rho^2\ d\rho\right)\left(\int_0^{\frac{\pi}{2}}\sin\phi\ d\phi\right)\left(\int_0^{\pi} d\theta\right)$$
$$=\left[\frac{\rho^3}{3}\right]_0^1\ \left[-\cos\phi\right]_0^{\frac{\pi}{2}}\ \left[\theta\right]_0^{\pi} = \left(\frac{1}{3}-0\right)\ \left(-\cos\left(\frac{\pi}{2}\right)+\cos(0)\right)\ (\pi-0) = \frac{\pi}{3}\ .$$

Example (2): Evaluate the integral $\displaystyle\iiint_E 4z\ dV$, where E is the region bounded above by $x^2+y^2+z^2=4$ and below by $z=0$.
Solution:

Note that $x^2+y^2+z^2=4$ intersects $z=0$ at $x^2+y^2=4$ which is a circle centered at the origin and its radius is 2.

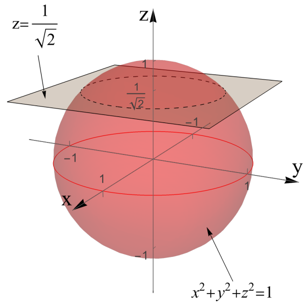


E in spherical coordinates:
$0\le\rho\le 2$, $0\le\theta\le 2\pi$ and $0\le\phi\le\dfrac{\pi}{2}$.
$z=\rho\cos\phi \implies 4z = 4\rho\cos\phi$.

$$\iiint_E 4z\ dV$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^2 (4\rho\cos\phi)\ \rho^2\sin\phi\ d\rho\ d\phi\ d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^2 (4\rho^3)\ (\cos\phi\ \sin\phi)\ d\rho\ d\phi\ d\theta$$
$$=\left(\int_0^2 4\rho^3\ d\rho\right)\left(\int_0^{\frac{\pi}{2}}\cos\phi\ \sin\phi\ d\phi\right)\left(\int_0^{2\pi} d\theta\right) = \left[\rho^4\right]_0^2\ \left[\frac{\sin^2\phi}{2}\right]_0^{\frac{\pi}{2}}\ \left[\theta\right]_0^{2\pi}$$
$$=(16-0)\ \left(\frac{1}{2}-0\right)\ (2\pi-0) = 16\pi\ .$$

Example (3): Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 1$ and above the plain $z = \frac{1}{\sqrt{2}}$.
Solution:



Note that $x^2 + y^2 + z^2 = 1$ intersects $z = \frac{1}{\sqrt{2}}$ at $x^2+y^2 = \frac{1}{2}$ which is a circle centered at the origin and its radius is $\frac{1}{\sqrt{2}}$.

$z = \rho\cos\phi \implies \cos\phi = \frac{1}{\sqrt{2}}$

$\implies \phi = \frac{\pi}{4}$.

$x^2 + y^2 + z^2 = 1 \implies \rho = 1$, $z = \frac{1}{\sqrt{2}} \implies \rho\cos\phi = \frac{1}{\sqrt{2}} \implies \rho = \frac{1}{\sqrt{2}\cos\phi}$.

The spherical coordinates of the solid : $\frac{1}{\sqrt{2}\cos\phi} \le \rho \le 1$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \frac{\pi}{4}$.

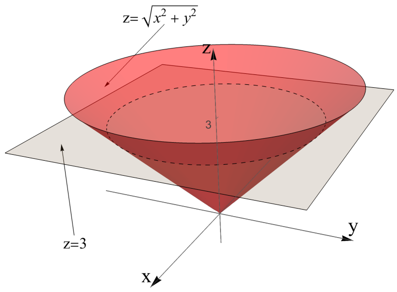
$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\sqrt{2}\cos\phi}}^{1} \rho^2\sin\phi \, d\rho\, d\phi\, d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\sqrt{2}\cos\phi}}^{1} \rho^2\sin\phi\, d\rho\, d\phi\right)$$

$$= [\theta]_0^{2\pi}\int_0^{\frac{\pi}{4}}\left[\frac{\rho^3}{3}\right]_{\frac{1}{\sqrt{2}\cos\phi}}^{1}\sin\phi\, d\phi = \frac{2\pi}{3}\int_0^{\frac{\pi}{4}}\left[1 - \frac{1}{2\sqrt{2}\cos^3\phi}\right]\sin\phi\, d\phi$$

$$= \frac{2\pi}{3}\int_0^{\frac{\pi}{4}}\left[\sin\phi - \frac{(\cos\phi)^{-3}\sin\phi}{2\sqrt{2}}\right] d\phi = \frac{2\pi}{3}\left[-\cos\phi + \frac{1}{2\sqrt{2}}\frac{(\cos\phi)^{-2}}{-2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{2\pi}{3}\left[\left(-\frac{1}{\sqrt{2}} - \frac{2}{4\sqrt{2}}\right) - \left(-1 - \frac{1}{4\sqrt{2}}\right)\right] = \frac{2\pi}{3}\left(1 - \frac{5}{4\sqrt{2}}\right).$$

Example (4): Evaluate $\iiint_E \sqrt{x^2+y^2+z^2}\, dV$, where E is the region bounded above by the palne $z = 3$ and below by the cone $z = \sqrt{x^2+y^2}$.
Solution:



Note that $z = \sqrt{x^2+y^2}$ intersects $z = 3$ at $x^2 + y^2 = 9$ which is a circle centered at the origin and its radius is 3.

$z = \sqrt{x^2+y^2} = r$

$\implies \rho\cos\phi = \rho\sin\phi$

$\implies \cos\phi = \sin\phi$

$\implies \phi = \frac{\pi}{4}$.

$z = 3 \implies \rho\cos\phi = 3 \implies \rho = \dfrac{3}{\cos\phi}$.

The spherical coordinates of E : $0 \le \rho \le \dfrac{3}{\cos\phi}$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \dfrac{\pi}{4}$.

$$\iiint_E \sqrt{x^2+y^2+z^2}\ dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\frac{3}{\cos\phi}} \rho\ \rho^2\ \sin\phi\ d\rho\ d\phi\ d\theta$$

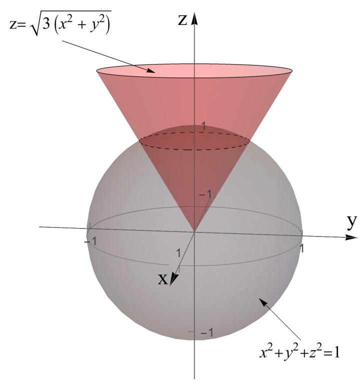
$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{4}}\int_0^{\frac{3}{\cos\phi}} \rho^3\sin\phi\ d\rho\ d\phi\right) = [\theta]_0^{2\pi}\int_0^{\frac{\pi}{4}}\left[\frac{\rho^4}{4}\right]_0^{\frac{3}{\cos\phi}} \sin\phi\ d\phi$$

$$= (2\pi - 0)\ \frac{3^4}{4}\int_0^{\frac{\pi}{4}} (\cos\phi)^{-4}\sin\phi\ d\phi = \frac{81\pi}{2}\left[\frac{-(\cos\phi)^{-3}}{-3}\right]_0^{\frac{\pi}{4}} = \frac{27\pi}{2}\left(2\sqrt{2}-1\right)$$

.

Example (5): Find the volume of the solid inside the sphere $x^2+y^2+z^2=1$ and inside the cone $z=\sqrt{3(x^2+y^2)}$.
Solution:

Note that $x^2+y^2+z^2 = 1$ intersects $z = \sqrt{3(x^2+y^2)}$ at
$3x^2+3y^2 = 1-x^2-y^2$
$\implies 4x^2+4y^2 = 1$
$\implies x^2+y^2 = \dfrac{1}{4}$
which is a circle centered at the origin and its radius is $\dfrac{1}{2}$.



$z = \sqrt{3(x^2+y^2)} = \sqrt{3}r \implies$
$\rho\cos\phi = \sqrt{3}\rho\sin\phi \implies \tan\phi = \dfrac{1}{\sqrt{3}}$
$\implies \phi = \dfrac{\pi}{6}$.

The spherical coordinates of the solid : $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \dfrac{\pi}{6}$.

$$\text{V} = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^1 \rho^2\sin\phi\ d\rho\ d\phi\ d\theta = \left(\int_0^1 \rho^2\ d\rho\right)\left(\int_0^{\frac{\pi}{6}} \sin\phi\ d\phi\right)\left(\int_0^{2\pi} d\theta\right)$$

$$= \left[\frac{\rho^3}{3}\right]_0^1 [-\cos\phi]_0^{\frac{\pi}{6}}\ [\theta]_0^{2\pi} = \left(\frac{1}{3}-0\right)\left(-\frac{\sqrt{3}}{2}-(-1)\right)(2\pi-0)$$

$$= \frac{2\pi}{3}\left(1-\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}\left(2-\sqrt{3}\right)\ .$$

2.6.3 EXERCISES

1. Change from rectangular to spherical coordinates:

   (a). $(3, 3, 0)$. (b). $\left(1, -\sqrt{3}, 2\sqrt{3}\right)$.

2. Identify the surface whose equation is $\rho = \cos\phi$.

3. Sketch the solid described by the given inequalities:

   (a). $\rho \leq 1$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq \dfrac{\pi}{6}$. (b). $\rho \leq 2$, $\rho \leq \csc\phi$.

4. Evaluate $\displaystyle\iiint_E \left(x^2 + y^2 + z^2\right)^2 \, dV$, where E is the ball with center the origin and radius 5.

5. Evaluate $\displaystyle\iiint_E y^2 z^2 \, dV$, where E lies above the cone $\phi = \dfrac{\pi}{3}$ and below the sphere $\rho = 1$.

6. Evaluate $\displaystyle\iiint_E x e^{x^2+y^2+z^2} \, dV$, where E is the portion of the unit ball $x^2 + y^2 + z^2 \leq 1$ that lies in the first octant.

7. Find the volume of the part of the ball $\rho \leq a$ that lies between the cones $\phi = \dfrac{\pi}{6}$ and $\phi = \dfrac{\pi}{3}$.

Chapter 3

Sequences, Series, and Power Series

3.1 Sequences

3.1.1 Infinite Sequences

Definition: An infinite sequence is a function $f : \mathbb{N} \longrightarrow \mathbb{R}$, where $f(n) = a_n$.

Notation: The sequence $\{a_1, a_2, \cdots\}$ is also denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

Example (1): Some sequences can be defined by giving a formula for the n$^{\text{th}}$ term.
(a). $a_n = \dfrac{1}{2^n}$ is the n$^{\text{th}}$ term of the sequence $\left\{\dfrac{1}{2^n}\right\} = \left\{\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \cdots\right\}$.
(b). $\left\{\dfrac{(-1)^n(n+2)}{5^n}\right\} = \left\{-\dfrac{3}{5}, \dfrac{4}{25}, -\dfrac{5}{125}, \cdots\right\}$.

Example (2): The Fibonacci sequences is defined recursively by
$f_1 = 1\ ,\ f_2 = 1\ ,\ f_n = f_{n-1} + f_{n-2}\ ,\ n \geq 3$.
$\{f_n\} = \{1, 1, 2, 3, 5, 8, 13, \cdots\}$.

3.1.2 The Limit of a Sequence

Definition: A sequence $\{a_n\}$ has the limit $L \in \mathbb{R}$, if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that : if $n \geq N$ then $|a_n - L| < \epsilon$, and we write $\lim_{n\to\infty} a_n = L$.

Definition: A sequence $\{a_n\}$ goes to ∞, if for every positive real number M there exists an $N \in \mathbb{N}$ such that : if $n \geq N$ then $a_n \geq M$, and we write $\lim_{n\to\infty} a_n = \infty$.

3.1.3 Properties of Convergent Sequences

Theorem: If $\lim\limits_{x\to\infty} f(x) = L$ and $f(n) = a_n$ where $n \in \mathbb{N}$ then $\lim\limits_{x\to\infty} a_n = L$.

Example (3): Calculate $\lim\limits_{n\to\infty} \dfrac{\ln n}{n}$.
Solution : Let $f(x) = \dfrac{\ln x}{x}$.
$\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{\ln x}{x} = \lim\limits_{x\to\infty} \dfrac{1}{x} = 0$ (By L'Hôpital's Rule).
Therefore, $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$.

Theorem: Suppose $\lim\limits_{n\to\infty} a_n = L_1$, $\lim\limits_{n\to\infty} b_n = L_2$ and $c \in \mathbb{R}$, then:
(1). $\lim\limits_{n\to\infty} c = c$.
(2). $\lim\limits_{n\to\infty} a_n \pm b_n = L_1 \pm L_2$.
(3). $\lim\limits_{n\to\infty} c\ a_n = cL_1$.
(4). $\lim\limits_{n\to\infty} a_n.b_n = L_1.L_2$.
(5). $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L_1}{L_2}$, where $L_2 \neq 0$.

Power Law: $\lim\limits_{n\to\infty} a_n^p = \left[\lim\limits_{n\to\infty} a_n\right]^p$, if $p > 0$ and $a_n > 0$.

Squeeze Theorem for sequences: If $a_n \leq b_n \leq c_n$ for $n \geq N_0$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

Example (4): Calculate $\lim\limits_{n\to\infty} \dfrac{n!}{n^n}$.
Solution:
$0 \leq \dfrac{n!}{n^n} = \dfrac{1.2.3\cdots n}{n.n.n\cdots n} = \dfrac{1}{n}\left(\dfrac{2.3\cdots n}{n.n\cdots n}\right) \leq \dfrac{1}{n}(1) = \dfrac{1}{n}$
Since $\lim\limits_{n\to\infty} 0 = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, then $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$ (By Squeeze Theorem).

Theorem: If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

Example (5): Calculate $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{n}$.
Solution: Let $a_n = \dfrac{(-1)^n}{n}$.
$\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \left|\dfrac{(-1)^n}{n}\right| = \lim\limits_{n\to\infty} \dfrac{|(-1)^n|}{|n|} = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$.
Therefore $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{(-1)^n}{n} = 0$.

Theorem: If $\lim\limits_{n\to\infty} a_n = L$, and the function f is continuous at L , then $\lim\limits_{n\to\infty} f(a_n) = f(L)$.

Example (6): Calculate $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)$.
Solution:
$\lim_{n\to\infty} \frac{\pi}{n} = \lim_{n\to\infty} \pi \ \frac{1}{n} = \pi(0) = 0$.
Since the cosine function is continuous at 0 , then $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$.

Important Note: The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$, and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad -1 < r < 1 \\ 1 & \text{if} \quad r = 1 \end{cases} .$$

3.1.4 Monotonic and bounded sequences

Definition:
(1). A sequence $\{a_n\}$ is called increasing if $a_{n+1} \geq a_n$ for all $n \geq 1$.
(2). A sequence $\{a_n\}$ is called decreasing if $a_{n+1} \leq a_n$ for all $n \geq 1$.
(3). A sequence is called monotonic if it is either increasing or decreasing.

Definition:
(1). A sequence $\{a_n\}$ is bounded above if there exists a real number M such that $a_n \leq M$ for all $n \geq 1$.
(2). A sequence $\{a_n\}$ is bounded below if there exists a real number m such that $a_n \geq m$ for all $n \geq 1$.
(3). If a sequence is bounded above and below, then it is called a bounded sequence.

Monotonic Sequence Theorem
Every bounded, monotonic sequence is convergent. In particular :
(1). A sequence that is increasing and bounded above converges.
(2). A sequence that is decreasing and bounded below converges.

Important Notes :
(1). If $a_n \leq b_n$ for some $n \geq N_0$, $\{a_n\}$ is an increasing sequence and $\{b_n\}$ is convergent, then $\{a_n\}$ is convergent.
(2). If $a_n \leq b_n$ for some $n \geq N_0$, if $\lim_{n\to\infty} a_n = \infty$, then $\lim_{n\to\infty} b_n = \infty$.

3.2 Series

3.2.1 Infinite Series

Definition:
(1). The sum of the terms of the sequence $\{a_n\}$ is called an infinite series (or a series) and it is denoted by $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.
(2). $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$ is called the n$^{\text{th}}$ partial sum of the series.

Definition:
If the sequence $\{s_n\}$ of the partial sums of the series $\sum_{n=1}^{\infty} a_n$ is convergent, and $\lim_{n\to\infty} s_n = s \in \mathbb{R}$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} a_n = s$.
The number s is called the sum of the series.
If the sequence $\{s_n\}$ is divergent, then the series is divergent.

Example (1): Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.
Solution : Note that $\frac{1}{n} - \frac{1}{n+1} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n(n+1)}$.

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

So, $s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1}$.

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1 .$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

3.2.2 Sum of a Geometric Series

Definition : $\sum_{n=0}^{\infty} ar^n$ (where $a \neq 0$ and $r \in \mathbb{R}$) is called a geometric series.

Note that $s_n = a + ar + ar^2 + \cdots + ar^n$, $rs_n = ra + ar^2 + ar^3 + \cdots + ar^n + ar^{n+1}$.

$$s_n - rs_n = a - ar^{n+1} \implies (1-r)s_n = a\left(1 - a^{n+1}\right) \implies s_n = \frac{a\left(1 - a^{n+1}\right)}{1-r}$$

where $r \neq 1$.

Sum of a geometric series :

(1). If $|r| < 1$, then $\sum_{n=0}^{\infty} ar^n$ is convergent and its sum is $\frac{a}{1-r}$.

(2). If $|r| \geq 1$, then $\sum_{n=0}^{\infty} ar^n$ is divergent.

Example (2): Find the sum of the geometric series $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \cdots$.

Solution : $a = 3$.

$r = \frac{a_2}{a_1} = \frac{-\frac{3}{2}}{3} = -\frac{1}{2}$.

the sum of the geometric series is $\frac{a}{1-r} = \frac{3}{1-\left(-\frac{1}{2}\right)} = \frac{3}{\left(\frac{3}{2}\right)} = 2$.

Example (3): Does the series $\sum_{n=1}^{\infty} 3^{2n} 5^{1-n}$ converge or diverge?.

Solution :

$$\sum_{n=1}^{\infty} 3^{2n} 5^{1-n} = \sum_{n=1}^{\infty} 9^n 5^{1-n} = \sum_{n=1}^{\infty} \frac{9^n}{5^{n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{9\ \left(9^{n-1}\right)}{5^{n-1}} = \sum_{n=1}^{\infty} 9\ \left(\frac{9}{5}\right)^{n-1} = \sum_{n=0}^{\infty} 9\ \left(\frac{9}{5}\right)^{n} .$$

Since $|r| = \left|\frac{9}{5}\right| = \frac{9}{5} > 1$, then the geometric series diverges.

3.2.3 Test for Divergence

Example (4): Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Solution :

$s_2 = s_{2^1} = 1 + \frac{1}{2}$.

$s_4 = s_{2^2} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$.

$s_8 = s_{2^3} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$

$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}$.

So, $s_{2^n} > 1 + \frac{n}{2}$ (By induction).

$\lim_{n\to\infty} s_{2^n} \geq \lim_{n\to\infty} \left(1 + \frac{n}{2}\right) = \infty$.

Therefore, The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem : If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Important Note : The converse of the last theorem is not true.
In other words, if $\lim_{n\to\infty} a_n = 0$ then this does not mean that $\sum_{n=1}^{\infty} a_n$ is convergent.
For example $\lim_{n\to\infty} \frac{1}{n} = 0$, while $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Test of divergence : If $\lim_{n\to\infty} a_n$ does not exist, or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example (5): Show that the series $\sum_{n=1}^{\infty} \frac{n^3}{2n^3+1}$ is divergent.
Solution :
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3}{2n^3+1} = \frac{1}{2} \neq 0$.
Therefore, the series $\sum_{n=1}^{\infty} \frac{n^3}{2n^3+1}$ is divergent.

3.2.4 Properties of Convergent Series

Theorem : If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent series and $c \in \mathbb{R}$ then :
(1). The series $\sum_{n=1}^{\infty} (ca_n)$ is convergent, and $\sum_{n=1}^{\infty} (ca_n) = c\sum_{n=1}^{\infty} a_n$.
(2). The series $\sum_{n=1}^{\infty} (a_n + b_n)$ is convergent, and $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
(3). The series $\sum_{n=1}^{\infty} (a_n - b_n)$ is convergent, and $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

Example (6): Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{4}{n(n+1)} + \frac{2}{3^n} \right)$.
Solution :
From Example (1), $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = \sum_{n=1}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^{n-1} = \sum_{n=0}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^{n} = \frac{2}{3}\ \frac{1}{1-\frac{1}{3}} = \frac{2}{3}\ \frac{3}{2} = 1 \ .$$

$$\sum_{n=1}^{\infty} \left(\frac{4}{n(n+1)} + \frac{2}{3^n} \right) = \sum_{n=1}^{\infty} \frac{4}{n(n+1)} + \sum_{n=1}^{\infty} \frac{2}{3^n}$$

$$= 4 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{2}{3^n} = 4(1) + 1 = 5 \ .$$

3.2.5 EXERCISES

1. Let $a_n = \dfrac{2n}{3n+1}$.

   (a). Determine whether $\{a_n\}$ is convergent.

   (b). Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

2. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

   (1). $\sum_{n=1}^{\infty} \dfrac{5}{\pi^n}$ (2). $\sum_{n=1}^{\infty} \dfrac{(-3)^{n-1}}{4^n}$ (3). $\sum_{n=0}^{\infty} \dfrac{3^{n+1}}{(-2)^n}$

   (4). $\sum_{n=1}^{\infty} \dfrac{e^{2n}}{6^{n-1}}$ (5). $\sum_{n=1}^{\infty} \dfrac{6\ 2^{2n-1}}{3^n}$

3. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

   (1). $\sum_{n=1}^{\infty} \dfrac{2+n}{1-2n}$ (2). $\sum_{n=1}^{\infty} \dfrac{n^2}{n^2-2n+5}$ (3). $\sum_{n=1}^{\infty} \dfrac{1}{4+e^{-n}}$

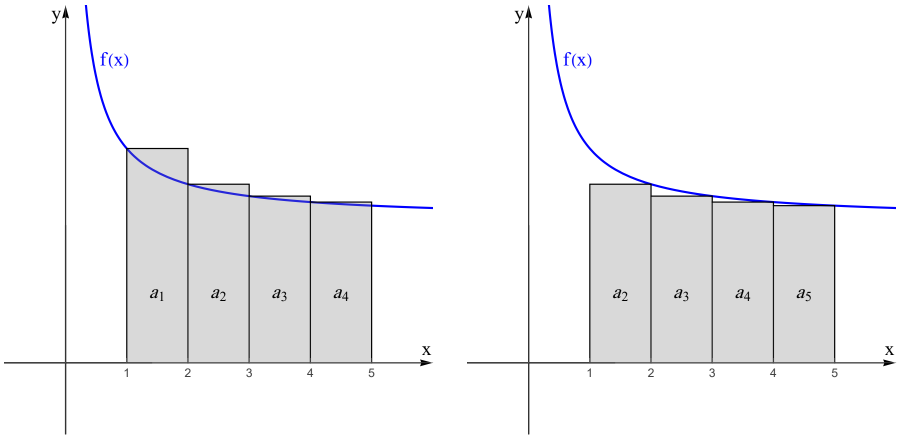
   (4). $\sum_{n=1}^{\infty} \dfrac{2^n+4^n}{e^n}$ (5). $\sum_{n=1}^{\infty} (\sin 100)^n$ (6). $\sum_{n=1}^{\infty} \dfrac{1}{1+\left(\frac{2}{3}\right)^n}$

   (7). $\sum_{n=1}^{\infty} \ln\left(\dfrac{n^2+1}{2n^2+1}\right)$

3.3 The Integral Test

3.3.1 The Integral Test

Theorem (The Integral Test) :
Suppose f is a positive, decreasing continuous function defined on $[1,\infty)$, and let $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:
(1). If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent .
(2). If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent .



Example (1): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Solution :
Let $f(x) = \frac{1}{x^2+1}$, the f is a positive continuous function on $[1,\infty)$.
$f'(x) = \frac{-2x}{(x^2+1)^2} < 0$ on the interval $[1,\infty)$, then f is decreasing on $[1,\infty)$.
$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \left[\tan^{-1} x\right]_1^t$
$= \lim_{t\to\infty} \left[\tan^{-1}(t) - \tan^{-1}(1)\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$
So, $\int_1^{\infty} \frac{1}{x^2+1}\,dx$ is convergent, then $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent.

Example (2): Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n\ \ln n}$.

Solution :
Let $f(x) = \frac{1}{x\ \ln x}$, the f is a positive continuous function on $[2,\infty)$.

$f'(x) = \dfrac{-(1+\ln x)}{(x\ \ln x)^2} < 0$ on $[2,\infty)$, then f is decreasing on $[2,\infty)$.

$\displaystyle\int_2^{\infty} \frac{1}{x\ \ln x}\ dx = \lim_{t\to\infty} \int_2^{t} \frac{\left(\frac{1}{x}\right)}{\ln x}\ dx = \lim_{t\to\infty} [\ln(\ln x)]_2^t$

$\displaystyle = \lim_{t\to\infty} [\ln(\ln t) - \ln(\ln 2)] = \infty$. (Note that $\displaystyle\lim_{t\to\infty} \ln(t) = \infty$).

So, $\displaystyle\int_2^{\infty} \frac{1}{x\ \ln x}\ dx$ is divergent, then $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ \ln n}$ is divergent.

Definition : (The p-series)

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ where $p \in \mathbb{R}$ is called a p-series.

Theorem :

The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$, and divergent if $p \leq 1$.

Proof:

(1). If $p < 0$, then $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} n^{-p} = \infty$. (Note: $-p > 0$).

Since $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent.

(2). If $p = 0$, then $a_n = \dfrac{1}{n^0} = 1$, hence $\displaystyle\lim_{n\to\infty} a_n = 1 \neq 0$.

Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent.

(3). If $0 < p < 1$, then using the integral test:

Note that $f(x) = \dfrac{1}{x^p}$ is a positive, decreasing continuous function on $[1,\infty)$.

$\displaystyle\int_1^{\infty} \frac{1}{x^p}\ dx = \lim_{t\to\infty} \int_1^{t} x^{-p}\ dx = \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^t$

$\displaystyle = \lim_{t\to\infty} \left[\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right] = \infty$. (Note : $1-p > 0$).

Since $\displaystyle\int_1^{\infty} \frac{1}{x^p}\ dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent.

(4). If $p = 1$, then using the integral test:

Note that $f(x) = \dfrac{1}{x}$ is a positive, decreasing continuous function on $[1,\infty)$.

$\displaystyle\int_1^{\infty} \frac{1}{x}\ dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\ dx = \lim_{t\to\infty} [\ln x]_1^t = \lim_{t\to\infty} [\ln(t) - \ln(1)] = \infty$.

Since $\displaystyle\int_1^{\infty} \frac{1}{x}\ dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

(5). If $p > 1$, then using the integral test:

Note that $f(x) = \dfrac{1}{x^p}$ is a positive, decreasing continuous function on $[1,\infty)$.

$\displaystyle\int_1^{\infty} \frac{1}{x^p}\ dx = \lim_{t\to\infty} \int_1^{t} x^{-p}\ dx = \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty} \left[\frac{1}{(1-p)x^{p-1}}\right]_1^t$

$\displaystyle = \lim_{t\to\infty} \left[\frac{1}{(1-p)\ t^{p-1}} - \frac{1}{1-p}\right] = 0 - \frac{1}{1-p} = \frac{1}{p-1}$.

Since $\displaystyle\int_1^\infty \frac{1}{x^p}\ dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

Example (3): Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Solution :
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a p-series, where $p = 4 > 1$, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent.

Example (4): Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

Solution :
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a p-series, where $p = \dfrac{1}{2} < 1$, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

Example (5): Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} n\ e^{-n^2}$.

Solution : Let $f(x) = x\ e^{-x^2}$ then f is a positive continuous function on $[1, \infty)$.
$f'(x) = e^{-x^2} - 2x^2\ e^{-x^2} = \left(1 - 2x^2\right)\ e^{-x^2} < 0$ on $[1, \infty)$.
Therefore, f is decreasing on $[1, \infty)$. Using the integral test :
$$\int_1^\infty x\ e^{-x^2}\ dx = \lim_{t\to\infty}\left(-\frac{1}{2}\int_1^t e^{-x^2}\ (-2x)\ dx\right) = \lim_{t\to\infty}\left(-\frac{1}{2}\left[e^{-x^2}\right]_1^t\right)$$
$$= \lim_{t\to\infty}\left(-\frac{1}{2}\left[e^{-t^2} - e^{-1}\right]\right) = \lim_{t\to\infty}\left(-\frac{1}{2}\left[\frac{1}{e^{t^2}} - \frac{1}{e}\right]\right) = -\frac{1}{2}\left(0 - \frac{1}{e}\right) = \frac{1}{2e}\ .$$
Since $\displaystyle\int_1^\infty x\ e^{-x^2}\ dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} n\ e^{-n^2}$ is convergent.

3.3.2 EXERCISES

1. Use the Integral Test to determine whether the series is convergent or divergent.

(1). $\sum_{n=1}^{\infty} n^{-3}$ (2). $\sum_{n=1}^{\infty} \frac{2}{5n-1}$ (3). $\sum_{n=2}^{\infty} \frac{n^2}{n^3+1}$

(4). $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ (5). $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

2. Determine whether the series is convergent or divergent.

(1). $\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$ (2). $\sum_{n=3}^{\infty} n^{-0.9999}$ (3). $\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots$

(4). $\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2}$ (5). $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ (6). $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots$

(7). $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ (8). $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ (9). $\sum_{n=1}^{\infty} \frac{1}{n^2+n^3}$

3.4 The Comparison Tests

3.4.1 The Direct Comparison Test

Theorem : (The Direct Comparison Test)

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(1). If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

(2). If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

Example (1): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\sin n}{2+3^n}$.

Solution :

For all $n \geq 1$: $\frac{\sin n}{2+3^n} \leq \frac{1}{2+3^n} < \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$.

Since $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is convergent (a geometric series with $r = \frac{1}{3} < 1$), then $\sum_{n=1}^{\infty} \frac{\sin n}{2+3^n}$ is also convergent (by The direct comparison test).

Example (2): Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{2}{\sqrt{n}-1}$.

Solution :

For all $n \geq 2$: $\sqrt{n}-1 < \sqrt{n} \implies \frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$

$\implies \frac{2}{\sqrt{n}-1} > \frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$.

Since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is divergent (a p-series, with $p = \frac{1}{2} \leq 1$), then $\sum_{n=2}^{\infty} \frac{2}{\sqrt{n}-1}$ is also divergent (by The direct comparison test) .

Example (3): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^4+1}$.

Solution :

For all $n \geq 1$: $e^{-n} = \frac{1}{e^n} < 1 \implies \frac{e^{-n}}{n^4+1} < \frac{1}{n^4+1} < \frac{1}{n^4}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent(a p-series, with $p = 4 > 1$), then $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^4+1}$ is also convergent.

Example (4): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Solution :
Since the function $f(x) = \ln(x)$ is increasing on $[1, \infty)$, then for all $n \geq 3$:
$\ln(n) \geq \ln(3) > \ln(e) = 1 \implies \dfrac{\ln n}{n} > \dfrac{1}{n}$.
Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (a p-series, with $p = 1 \leq 1$), then $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is also divergent.

3.4.2 Limit Comparison Test

Theorem: (Limit comparison test)
Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ are series with positive terms.
If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $c \in \mathbb{R}$ and $c > 0$, then either both series converge or both series diverge.

Example (1): Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$.

Solution :
Note that $a_n = \dfrac{1}{2^n - 1}$. Let $b_n = \dfrac{1}{2^n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{2^n-1}\right)}{\left(\frac{1}{2^n}\right)} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = 1 > 0 .$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent (a geometric series with $r = \dfrac{1}{2} < 1$), then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ is convergent (by limit comparison test).

Example (2): Discuss the convergence of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2 - 1}}$.

Solution :
Note that $a_n = \dfrac{1}{\sqrt[3]{n^2 - 1}}$. Let $b_n = \dfrac{1}{\sqrt[3]{n^2}} = \dfrac{1}{n^{\frac{2}{3}}}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt[3]{n^2}}{\sqrt[3]{n^2 - 1}} = \lim_{n\to\infty} \sqrt[3]{\frac{n^2}{n^2 - 1}} = \sqrt[3]{1} = 1 > 0 .$$

Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{2}{3}}}$ is divergent (a p-series, with $p = \dfrac{2}{3} \leq 1$), then $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2 - 1}}$ is divergent (by limit comparison test).

Example (3): Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2 + 3n}{3^n(n^2 + 2)}$.

Solution :

Note that $a_n = \dfrac{5n^2+3n}{3^n(n^2+2)}$. Let $b_n = \dfrac{5n^2}{3^n\ n^2} = \dfrac{5}{3^n}$.
$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \left(\dfrac{5n^2+3n}{3^n(n^2+2)} . \dfrac{3^n}{5} \right) = \lim\limits_{n\to\infty} \dfrac{5n^2+3n}{5n^2+10} = 1 > 0$.
Since $\sum\limits_{n=1}^{\infty} \dfrac{5}{3^n} = \sum\limits_{n=1}^{\infty} 5 \left(\dfrac{1}{3}\right)^n$ is convergent (a geometric series with $r = \dfrac{1}{3} < 1$), then $\sum\limits_{n=1}^{\infty} \dfrac{5n^2+3n}{3^n(n^2+2)}$ is convergent.

Example (4): Discuss the convergence of the series $\sum\limits_{n=1}^{\infty} \dfrac{n+3n^2}{\sqrt{1+n^5}}$.
Solution :
Note that $a_n = \dfrac{3n^2+n}{\sqrt{n^5+1}}$. Let $b_n = \dfrac{3n^2}{\sqrt{n^5}} = \dfrac{3n^2}{n^{\frac{5}{2}}} = \dfrac{3}{n^{\frac{1}{2}}}$.
$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \left(\dfrac{3n^2+n}{\sqrt{n^5+1}} . \dfrac{n^{\frac{1}{2}}}{3} \right) = \lim\limits_{n\to\infty} \dfrac{3n^{\frac{5}{2}}+n^{\frac{3}{2}}}{3\sqrt{n^5+1}} = 1 > 0$.
Since $\sum\limits_{n=1}^{\infty} \dfrac{3}{n^{\frac{1}{2}}}$ is divergent (a p-series, with $p = \dfrac{1}{2} < 1$), then $\sum\limits_{n=1}^{\infty} \dfrac{n+3n^2}{\sqrt{1+n^5}}$ is divergent.

3.4.3 EXERCISES

1. (a). Use the Direct Comparison Test to show that the first series converges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n}{n^3+5}\,, \qquad \sum_{n=2}^{\infty} \frac{1}{n^2}\,.$$

(b). Use the Limit Comparison Test to show that that the first series converges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n}{n^3-5}\,, \qquad \sum_{n=2}^{\infty} \frac{1}{n^2}\,.$$

2. (a). Use the Direct Comparison Test to show that the first series diverges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n^2+n}{n^3-2}\,, \qquad \sum_{n=2}^{\infty} \frac{1}{n}\,.$$

(b). Use the Limit Comparison Test to show that that the first series diverges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n^2-n}{n^3+2}\,, \qquad \sum_{n=2}^{\infty} \frac{1}{n}\,.$$

3. Which of the following inequalities can be used to show that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ converges?

(a). $\dfrac{n}{n^3+1} \geq \dfrac{1}{n^3+1}$ (b). $\dfrac{n}{n^3+1} \leq \dfrac{1}{n}$ (c). $\dfrac{n}{n^3+1} \leq \dfrac{1}{n^2}$.

4. Determine whether the series converges or diverges.

(1). $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+8}$ (2). $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ (3). $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$

(4). $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$ (5). $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$ (6). $\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$

(7). $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ (8). $\displaystyle\sum_{n=1}^{\infty} \frac{1+\cos n}{e^n}$ (9). $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

(10). $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3+n}$ (11). $\displaystyle\sum_{n=1}^{\infty} \frac{2+\sin n}{n^2}$ (12). $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

3.5 Alternating Series and Absolute Convergence

3.5.1 Alternating Series

Definition : If $a_n \geq 0$, the the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is called an alternating series.

Theorem : (Alternating Series Test)
If $a_n \geq 0$ and the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ satisfies :
(1). The sequence $\{a_n\}$ is decreasing, i.e. $a_{n+1} \leq a_n$ for all $n \geq 1$.
(2). $\lim_{n\to\infty} a_n = 0$.
Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.

Example (1): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.
Solution :
(1). For all $n \geq 1$: $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$.
The sequence $\{a_n\} = \left\{\frac{1}{n}\right\}$ is decreasing.
(2). $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$.
By the alternating series test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent .

Example (2): Discuss the convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{4n^2-1}$.
Solution :
(1). Let $f(x) = \frac{2x}{4x^2-1}$, then for $x \geq 1$:
$f'(x) = \frac{2(4x^2-1) - 2x(8x)}{(4x^2-1)^2} = \frac{8x^2 - 2 - 16x^2}{(4x^2-1)^2} = \frac{-2-8x^2}{(4x^2-1)^2} < 0$.
The sequence $\{a_n\} = \left\{\frac{2n}{4n^2-1}\right\}$ is decreasing.
(2). $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{4n^2-1} = 0$.
By the alternating series test, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{4n^2-1}$ is convergent .

Example (3): Discuss the convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n+1}$.
Solution :
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$.

So, the alternating series test does not apply.
Note that if $b_n = (-1)^{n+1}\dfrac{n}{2n+1}$, then $\lim\limits_{n\to\infty} b_{2n} = \dfrac{1}{2}$ and $\lim\limits_{n\to\infty} b_{2n+1} = -\dfrac{1}{2}$.
So, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} (-1)^{n+1}\dfrac{n}{2n+1}$ does not exist.
Therefore, $\sum\limits_{n=1}^{\infty} (-1)^{n+1}\dfrac{n}{2n+1}$ is divergent (by test of divergence).

3.5.2 Absolute Convergence and Conditional Convergence

Definition : (Absolute Convergence)
A series $\sum\limits_{n=1}^{\infty} a_n$ is called absolutely convergent if the series of absolute values $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent .

Example (4): The series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^p}$, where $p > 1$ is absolutely convergent.
Solution :
$\sum\limits_{n=1}^{\infty} \left|\dfrac{(-1)^{n+1}}{n^p}\right| = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ is convergent (a p-series, with $p > 1$).
Therefore, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^p}$ is absolutely convergent.

Definition : (Conditional Convergence)
A series $\sum\limits_{n=1}^{\infty} a_n$ is called conditionally convergent if it is convergent but not absolutely convergent. That is, if $\sum\limits_{n=1}^{\infty} a_n$ is convergent but $\sum\limits_{n=1}^{\infty} |a_n|$ is divergent.

Example (5): The series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}$ is conditionally convergent.
Solution :
Note that $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}$ is convergent (see Example (1)).
But $\sum\limits_{n=1}^{\infty} \left|\dfrac{(-1)^{n+1}}{n}\right| = \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ is divergent.
Therefore, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}$ is conditionally convergent.

Theorem :
If a series is absolutely convergent then it is convergent.

Example (6): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$.

Solution :

$\left|\frac{\cos n}{n^2}\right| = \frac{|\cos n|}{|n^2|} \leq \frac{1}{n^2}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent then by direct comparison test $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ is convergent.

Therefore, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent and hence convergent.

Example (7): Discuss the convergence of the following series :

(a). $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1}$ (b). $\sum_{n=1}^{\infty} (-1)^n \frac{1}{1+n\sqrt{n}}$ (c). $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{(2n-3)^2}$.

Solution :

(a). $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{n+1}{n^2+1}\right| = \sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$

Let $a_n = \frac{n+1}{n^2+1}$, put $b_n = \frac{1}{n}$, then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n+1}{n^2+1} \frac{n}{1}\right) = \lim_{n\to\infty} \frac{n^2+n}{n^2+1} = 1 > 0$.

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, then $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{n+1}{n^2+1}\right|$ is divergent.

(1). Let $f(x) = \frac{x+1}{x^2+1}$, then for $x \geq 1$:

$f'(x) = \frac{(1)(x^2+1) - (x+1)(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2-2x}{(x^2+1)^2}$

$= \frac{-x^2-2x+1}{(x^2+1)^2} = \frac{1-(x^2+2x)}{(x^2+1)^2} < 0.$

(Note that $1-(x^2+2x) < 0$ when $x \geq 1$).

Therefore, the sequence $\left\{\frac{n+1}{n^2+1}\right\}$ is decreasing.

(2). $\lim_{n\to\infty} \frac{n+1}{n^2+1} = 0.$

From (1) and (2), the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1}$ is convergent.

Hence, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1}$ is conditionally convergent.

(b). $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{1+n\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{1+n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{1+n^{\frac{3}{2}}}$.

Let $a_n = \frac{1}{1+n^{\frac{3}{2}}}$, put $b_n = \frac{1}{n^{\frac{3}{2}}}$, then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{\frac{3}{2}}}{n^{\frac{3}{2}}+1} = 1 > 0.$

Since $\sum_{n=1}^{\infty} \frac{1}{1+n^{\frac{3}{2}}}$ is convergent, then $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{1+n\sqrt{n}}\right|$ is convergent.
Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{1+n\sqrt{n}}$ is absolutely convergent.

(c). Let $a_n = (-1)^n \frac{n^2+1}{(2n-3)^2} = (-1)^n \frac{n^2+1}{4n^2-12n+9}$
$\lim_{n\to\infty} a_{2n} = \frac{1}{4}$ and $\lim_{n\to\infty} a_{2n+1} = -\frac{1}{4}$.
Hence $\lim_{n\to\infty} a_n$ does not exist.
Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{(2n-3)^2}$ is divergent.

3.5.3 EXERCISES

1. Test the series for convergence or divergence.

(1). $\frac{2}{3}-\frac{2}{5}+\frac{2}{7}-\frac{2}{9}+\frac{2}{11}-\cdots$ (2). $\frac{1}{\ln 3}-\frac{1}{\ln 4}+\frac{1}{\ln 5}-\frac{1}{\ln 6}+\frac{1}{\ln 7}-\cdots$

(3). $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3+5n}$ (4). $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n+1}}$ (5). $\sum_{n=1}^{\infty}(-1)^n\ \frac{3n-1}{2n+1}$

(6). $\sum_{n=1}^{\infty}(-1)^n e^{-n}$ (7). $\sum_{n=1}^{\infty}(-1)^{n+1}\ \frac{n^2}{n^3+4}$ (8). $\sum_{n=1}^{\infty}(-1)^n\ \frac{n}{2^n}$

(9). $\sum_{n=1}^{\infty}\frac{n\cos(n\pi)}{2^n}$ (10). $\sum_{n=1}^{\infty}(-1)^n\ \sin\left(\frac{\pi}{n}\right)$ (11). $\sum_{n=1}^{\infty}(-1)^n\ \cos\left(\frac{\pi}{n}\right)$

2. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

(1). $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$ (2). $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt[3]{n^2}}$ (3). $\sum_{n=0}^{\infty}(-1)^{n+1}\ \frac{n^2}{n^2+1}$

(4). $\sum_{n=1}^{\infty}\frac{(-1)^n}{5n+1}$ (5). $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$ (6). $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$

(7). $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$

3.6 The Ratio and Root Tests

3.6.1 The Ratio Test

Theorem : (the Ratio Test)

(1). If $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(2). If $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

(3). If $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$, the ratio test is inconclusive.

Example (1): Discuss the convergence of the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{n^4}{2^n}$.

Solution :

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^4}{2^{n+1}}\ \frac{2^n}{n^4}\right| = \lim_{n\to\infty}\frac{1}{2}\left(\frac{n+1}{n}\right)^4 = \frac{1}{2}(1)^4 = \frac{1}{2} < 1 \ .$$

Therefore, $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{n^4}{2^n}$ is absolutely convergent.

Example (2): Discuss the convergence of the series $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{5^n}{n!}$.

Solution :

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5^{n+1}}{(n+1)!}\ \frac{n!}{5^n}\right| = \lim_{n\to\infty}\left|\frac{5^{n+1}}{n!\ (n+1)}\ \frac{n!}{5^n}\right|$$

$$= \lim_{n\to\infty}\frac{5}{n+1} = 0 < 1 \ .$$

Therefore, $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{5^n}{n!}$ is absolutely convergent.

Example (3): Discuss the convergence of the series $\sum\limits_{n=1}^{\infty} \dfrac{n^n}{n!}$.

Solution :

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{(n+1)!}\ \frac{n!}{n^n}\right] = \lim_{n\to\infty}\left[\frac{(n+1)^n\ (n+1)}{n!\ (n+1)}\ \frac{n!}{n^n}\right]$$

$$= \lim_{n\to\infty}\frac{(n+1)^n}{n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1 \ .$$

Therefore, $\sum\limits_{n=1}^{\infty} \dfrac{n^n}{n!}$ is divergent.

Note that $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$, using L'Hôpital's rule.

Example (4): Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{3^n}$.

Solution : For $n \geq 2$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\ln(n+1)}{3^{n+1}}\ \frac{3^n}{\ln(n)}\right| = \lim_{n\to\infty}\frac{1}{3}\ \frac{\ln(n+1)}{\ln(n)}$$

$$= \lim_{n\to\infty}\frac{1}{3}\left(\frac{\frac{1}{n+1}}{\frac{1}{n}}\right) = \lim_{n\to\infty}\frac{1}{3}\ \frac{n}{n+1} = \frac{1}{3}(1) = \frac{1}{3} < 1 \ .$$

Therefore, $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{3^n}$ is convergent.

Notes :

(1). The harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{\frac{1}{n+1}}{\frac{1}{n}}\right) = \lim_{n\to\infty}\frac{n}{n+1} = 1 \ .$$

(2). The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right) = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = (1)^2 = 1 \ .$$

3.6.2 The Root Test

Theorem : (the Root Test)

(1). If $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then the series $\displaystyle\sum_{n=1}^{\infty}a_n$ is absolutely convergent (and therefore convergent).

(2). If $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ or $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, then the series $\displaystyle\sum_{n=1}^{\infty}a_n$ is divergent.

(3). If $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, the root test is inconclusive.

Example (5): Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+3}{3n+2}\right)^n$.

Solution :

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n+3}{3n+2}\right)^n\right|} = \lim_{n\to\infty}\left[\left(\frac{n+3}{3n+2}\right)^n\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{n+3}{3n+2} = \frac{1}{3} < 1 \ .$$

Therefore, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+3}{3n+2}\right)^n$ is absolutely convergent.

Example (6): Discuss the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{2n+1}{n+3}\right)^n$.

Solution :

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{2n+1}{n+3}\right)^n\right|}=\lim_{n\to\infty}\left[\left(\frac{2n+1}{n+3}\right)^n\right]^{\frac{1}{n}}$$
$$=\lim_{n\to\infty}\frac{2n+1}{n+3}=2>1\ .$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{2n+1}{n+3}\right)^n$ is divergent.

Example (7): Discuss the convergence of the series $\sum_{n=1}^{\infty}\frac{2^{3n+1}}{n^n}$.

Solution :

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\frac{2^{3n+1}}{n^n}\right|}=\lim_{n\to\infty}\left(\frac{2^{3n+1}}{n^n}\right)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{2^{3+\frac{1}{n}}}{n}=0<1\ .$$

Therefore, $\sum_{n=1}^{\infty}\frac{2^{3n+1}}{n^n}$ is absolutely convergent.

Another solution :

$$\sum_{n=1}^{\infty}\frac{2^{3n+1}}{n^n}=\sum_{n=1}^{\infty}\frac{(2)\ 2^{3n}}{n^n}=\sum_{n=1}^{\infty}\frac{(2)\ \left(2^3\right)^n}{n^n}=2\sum_{n=1}^{\infty}\left(\frac{2^3}{n}\right)^n=2\sum_{n=1}^{\infty}\left(\frac{8}{n}\right)^n$$
$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{8}{n}\right)^n\right|}=\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{8}{n}\right)\right|^n}=\lim_{n\to\infty}\frac{8}{n}=0<1\ .$$

Therefore, $\sum_{n=1}^{\infty}\frac{2^{3n+1}}{n^n}$ is absolutely convergent.

Example (8): Discuss the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^n$.

Solution :

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n+1}{n}\right)^n\right|}=\lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^n\right]^{\frac{1}{n}}=\lim_{n\to\infty}\frac{n+1}{n}=1.$$

So, The root test is inconclusive.

But $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\neq 0$.

Therefore, $\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^n$ is divergent (by test of divergence).

3.6.3 EXERCISES

1. What can you say about the series $\sum_{n=1}^{\infty} a_n$ an in each of the following cases?

   (a). $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 8$ (b). $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0.8$ (c).$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

2. Use the Ratio Test to determine whether the series is convergent or divergent.

   (1). $\sum_{n=1}^{\infty} \frac{n}{5^n}$ (2). $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$ (3). $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{3^n}{2^n n^3}$

   (4). $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$ (5). $\sum_{n=1}^{\infty} \frac{1}{n!}$ (6). $\sum_{n=1}^{\infty} \frac{n}{100^n}$

   (7). $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

3. Use the Root Test to determine whether the series is convergent or divergent.

   (1). $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$ (2). $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$ (3). $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

3.7 Strategy for Testing Series

(1). Divergence test :

If $\lim\limits_{n\to\infty} a_n \neq 0$, or $\lim\limits_{n\to\infty} a_n$ does not exist, then $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

$\sum\limits_{n=1}^{\infty} \dfrac{n^2+2n}{n^2+4}$ is divergent, because $\lim\limits_{n\to\infty} \dfrac{n^2+2n}{n^2+4} = 1 \neq 0$.

$\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{n-1}{n+2}$ is divergent, because $\lim\limits_{n\to\infty} (-1)^n \dfrac{n-1}{4n+2}$ does not exist.

(2). p-series : It has the form $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$:

If $p > 1$ then $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ is convergent, like $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{\frac{3}{2}}}$.

If $p \leq 1$ then $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ is divergent, like $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$.

(3). Geometric series : It has the form $\sum\limits_{n=0}^{\infty} a\ r^n$ where $a \neq 0$ and $r \in \mathbb{R}$:

If $|r| < 1$ then $\sum\limits_{n=0}^{\infty} a\ r^n$ is convergent, like $\sum\limits_{n=1}^{\infty} \dfrac{1}{e^n} = \sum\limits_{n=1}^{\infty} \left(\dfrac{1}{e}\right)^n$.

If $|r| \geq 1$ then $\sum\limits_{n=0}^{\infty} a\ r^n$ is divergent, like $\sum\limits_{n=1}^{\infty} \dfrac{3^n}{2}$.

(4). Comparison test :
It is used when the series has a form that is similar to a p-series or a geometric series, it can be used also when a_n is a rational function or an algebraic function of n.

It is used on series of positive terms, it can be used on $\sum\limits_{n=1}^{\infty} |a_n|$ to test for absolute convergence.

$\sum\limits_{n=1}^{\infty} \dfrac{3}{n+2}$ is divergent, comparing it with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$.

$\sum\limits_{n=1}^{\infty} \dfrac{2}{7^n+1}$ is convergent, comparing it with $\sum\limits_{n=1}^{\infty} \dfrac{1}{7^n}$.

$\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n}}{n^2+5}$ is convergent, comparing it with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{\frac{3}{2}}}$.

(5). Alternating series test :

It is used on the series $\sum\limits_{n=1}^{\infty} (-1)^n a_n$, where the sequence $\{a_n\}$ is decreasing and

$\lim_{n\to\infty} a_n = 0$. Note that if $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} (-1)^n a_n$ is absolutely convergent.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+2}$ satisfies all the conditions of the alternating series test, hence it is absolutely convergent. Also, $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^3+2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^3+2}$ is convergent (using comparison test), hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+2}$ is absolutely convergent.

(6). The Ratio test :
it is used on series involving factorial or other products like a constant raised to a power n.
It is not used on p-series or rational functions or algebraic functions of n.
$\sum_{n=1}^{\infty} \frac{2^n}{n!}$ is convergent since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$.

(7). The Root test :
It is used on series of the form $\sum_{n=1}^{\infty} (a_n)^n$.
$\sum_{n=1}^{\infty} \left(\frac{3n^2+1}{4n^2+5}\right)^n$ is convergent, since $\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n^2+1}{4n^2+5}\right)^n} = \frac{3}{4} < 1$.

(8). The Integral test :
It is used when $a_n = f(n)$, where f is a decreasing function on $[1,\infty)$ and $\int_1^{\infty} f(x)\, dx$ is easy to calculate.
$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent, since $f(x) = \frac{1}{x(\ln x)^2}$ is decreasing on $[2,\infty)$ and

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} = \lim_{t\to\infty} \int_2^t (\ln x)^{-2} \frac{1}{x}\, dx = \lim_{t\to\infty} \left[\frac{(\ln x)^{-1}}{-1}\right]_2^t = \lim_{t\to\infty} \left[\frac{-1}{\ln x}\right]_2^t$$
$$= \lim_{t\to\infty} \left[\frac{-1}{\ln t} - \left(\frac{-1}{\ln 2}\right)\right] = 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}\ .$$

3.8 Power Series

3.8.1 Power Series

Definition : A power series is a series of the form $\sum_{n=1}^{\infty} c_n x^n$, where x is a variable and c_n's are real constants called the coefficients.

Notes :

(1). If $c_n = 1$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} c_n x^n = \sum_{n=1}^{\infty} x^n$ is the geometric series and it converges when $|x| < 1$ and diverges when $|x| > 1$.

If $x = \frac{1}{2}$, then $\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges to 2.

If $x = 2$, then $\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} 2^n$ diverges.

(2). $\sum_{n=1}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-c)^2 + \cdots$ is called a power series in $(x-a)$ or a power series centered at a or a power series about a.

Example (1): For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$ converge?

Solution :

Let $a_n = \frac{(x-2)^n}{n}$, using the ratio test :

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{n+1} \; \frac{n}{(x-2)^n} \right| = \lim_{n\to\infty} \frac{n}{n+1} |x-2| = |x-2| \ .$$

The series converges when $|x-2| < 1 \implies -1 < x-2 < 1 \implies 1 < x < 3$.

The series diverges when $|x-2| > 1 \implies x-2 < -1$ or $x-2 > 1$

$\implies x < 1$ or $x > 3$.

If $x = 1$ then $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is the convergent alternating harmonic series.

If $x = 3$ then $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Therefore, $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$ converges when $x \in [1, 3)$ or $1 \leq x < 3$.

Example (2): For what values of x does the series $\sum_{n=1}^{\infty} n! \; x^n$ converge?

Solution :

Let $a_n = n! \; x^n$, using the ratio test :

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)! \; x^{n+1}}{n! \; x^n} \right| = \lim_{n\to\infty} (n+1) \; |x|$$

If $x \neq 0$, then $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$. Therefore the series diverges when $x \neq 0$.

If $x = 0$, then $\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 0$. Therefore the series converges when $x = 0$.

Example (3): For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converge?
Solution :
Let $a_n = \dfrac{x^n}{n!}$, using the ratio test :
$\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \left| \dfrac{x^{n+1}}{(n+1)!} \; \dfrac{n!}{x^n} \right| = \lim\limits_{n\to\infty} \dfrac{|x|}{n+1} = 0 < 1.$
Therefore, the series converges for all $x \in \mathbb{R}$.

3.8.2 Interval of Convergence

Theorem : For a power series $\displaystyle\sum_{n=1}^{\infty} c_n (x-a)^n$, there are only three possibilities:
(1). The series converges only when $x = a$.
(2). The series converges for all $x \in \mathbb{R}$.
(3). There is a positive real number R such that the series converges when $|x - a| < R$ and diverges when $|x - a| > R$.

Notes :
(1). The positive real number R is called the radius of convergence. If the series converges only when $x = a$ then $R = 0$, and if the series converges for all $x \in \mathbb{R}$ then $R = \infty$.
(2). The interval of convergence of a power series is the interval that consists of all values of x for which the series converges.
If the series converges only when $x = a$ then the interval of convergence consists of only one point a.
If the series converges for all $x \in \mathbb{R}$ then the interval of convergence is $(-\infty, \infty)$.
If the radius of convergence is R, then interval of convergence is $(a - R, a + R)$.

Example (4): Find the radius of convergence and interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$.
Solution :
Let $a_n = \dfrac{(x-2)^n}{n^2+1}$, using the ratio test :
$\lim\limits_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n\to\infty} \left| \dfrac{(x-2)^{n+1}}{(n+1)^2+1} \; \dfrac{n^2+1}{(x-2)^n} \right| = \lim\limits_{n\to\infty} \dfrac{n^2+1}{(n+1)^2+1} \; |x-2|$
$= \lim\limits_{n\to\infty} \dfrac{n^2+1}{n^2+2n+2} \; |x-2| = (1) \; |x-2| = |x-2|.$
The series converges when $|x - 2| < 1$, so the radius of convergence is $R = 1$.
Therefore, the series converges on $(1, 3)$.
If $x = 1$, then $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ which is convergent.

If $x = 3$, then $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1} = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ which is convergent.
Therefore, the interval of convergence is $[1, 3]$.

Example (5): Find the radius of convergence and interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n\ 5^n}$.
Solution :
Let $a_n = \dfrac{(2x-6)^n}{n\ 5^n}$, using the ratio test :

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2x-6)^{n+1}}{(n+1)\ 5^{n+1}} \ \frac{n\ 5^n}{(2x-6)^n}\right| = \lim_{n\to\infty} \frac{n}{5(n+1)}\ |2x-6|$$

$$= \lim_{n\to\infty} \frac{n}{5n+5}\ |2x-6| = \frac{1}{5}\ |2x-6| = \frac{|2x-6|}{5}\ .$$

The series converges when $\dfrac{|2x-6|}{5} < 1 \implies 2|x-3| < 5 \implies |x-3| < \dfrac{5}{2}$, so the radius of convergence is $R = \dfrac{5}{2}$.
Therefore, the series converges on $\left(3 - \dfrac{5}{2}, 3 + \dfrac{5}{2}\right) = \left(\dfrac{1}{2}, \dfrac{11}{2}\right)$.
If $x = \dfrac{1}{2}$, then $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n\ 5^n} = \sum_{n=1}^{\infty} \frac{(-5)^n}{n\ 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is convergent.
If $x = \dfrac{11}{2}$, then $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n\ 5^n} = \sum_{n=1}^{\infty} \frac{5^n}{n\ 5^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent.
Therefore, the interval of convergence is $\left[\dfrac{1}{2}, \dfrac{11}{2}\right)$.

Example (6): Find the radius of convergence and interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{7^n}(x+3)^n$.
Solution :
Let $a_n = \dfrac{n}{7^n}(x+3)^n$, using the ratio test :

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(x+3)^{n+1}}{7^{n+1}} \ \frac{7^n}{n(x+3)^n}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{7n}|x+3| = \frac{1}{7}|x+3| = \frac{|x+3|}{7}\ .$$

The series converges when $\dfrac{|x+3|}{7} < 1 \implies |x+3| < 7$, so the radius of convergence is $R = 7$.
Therefore, the series converges on $(-3-7, -3+7) = (-10, 4)$.
If $x = -10$, then $\displaystyle\sum_{n=1}^{\infty} \frac{n}{7^n}(x+3)^n = \sum_{n=1}^{\infty} (-1)^n n$ which is divergent.
If $x = 4$, then $\displaystyle\sum_{n=1}^{\infty} \frac{n}{7^n}(x+3)^n = \sum_{n=1}^{\infty} n$ which is divergent.
Therefore, the interval of convergence is $(-10, 4)$.

3.8.3 EXERCISES

1. Find the radius of convergence and interval of convergence of the power series.

(1). $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n}$ (2). $\displaystyle\sum_{n=1}^{\infty}(-1)^n n x^n$ (3). $\displaystyle\sum_{n=1}^{\infty}\sqrt{n}x^n$

(4). $\displaystyle\sum_{n=1}^{\infty}\frac{n}{5^n}x^n$ (5). $\displaystyle\sum_{n=1}^{\infty}\frac{5^n}{n}x^n$ (6). $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n3^n}$

(7). $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n^2}$ (8). $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ (9). $\displaystyle\sum_{n=1}^{\infty}n^n x^n$

(10). $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n5^n}x^n$ (11). $\displaystyle\sum_{n=1}^{\infty}\frac{x^{2n}}{n!}$ (12). $\displaystyle\sum_{n=0}^{\infty}\frac{(x-2)^n}{n^2+1}$

(13). $\displaystyle\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^n}$ (14). $\displaystyle\sum_{n=1}^{\infty}\frac{(2x-1)^n}{5^n\sqrt{n}}$

3.9 Representations of Functions as Power Series

3.9.1 Representations of Functions using Geometric Series

$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$, when $|x| < 1$.

Example (1): Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.
Solution :
$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, where $\left|-x^2\right| < 1$.
$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - \cdots$, where $\left|-x^2\right| < 1$.
$\left|-x^2\right| < 1 \implies \left|x^2\right| < 1 \implies |x| < 1$.
If $x = \pm 1$, the series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ diverge.
Therefore, The interval of convergence is $(-1, 1)$.

Example (2): Express $\frac{x^2}{1-x^6}$ as the sum of a power series and find the interval of convergence.
Solution :
$\frac{x^2}{1-x^6} = x^2 \frac{1}{1-x^6} = x^2 \sum_{n=0}^{\infty} \left(x^6\right)^n = x^2 \sum_{n=0}^{\infty} x^{6n} = \sum_{n=0}^{\infty} x^{6n+2}$, where $\left|x^6\right| < 1$.
$\left|x^6\right| < 1 \implies |x| < 1$.
If $x = \pm 1$, the series $\sum_{n=0}^{\infty} x^{6n+2}$ diverge.
Therefore, The interval of convergence is $(-1, 1)$.

Example (3): Express $\frac{x}{5-x}$ as the sum of a power series and find the interval of convergence.
Solution :
$\frac{x}{5-x} = \frac{x}{5\left(1-\frac{x}{5}\right)} = \frac{x}{5} \frac{1}{1-\frac{x}{5}} = \frac{x}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n = \frac{x}{5} \sum_{n=0}^{\infty} \frac{x^n}{5^n} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}}$,
where $\left|\frac{x}{5}\right| < 1$.
$\left|\frac{x}{5}\right| < 1 \implies \frac{|x|}{5} < 1 \implies |x| < 5$.
If $x = \pm 5$, the series $\sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}}$ diverge.
Therefore, The interval of convergence is $(-5, 5)$.

Example (4): Express $\dfrac{1}{2-x}$ as power series in $x-1$, and find the interval of convergence.
Solution :
$\dfrac{1}{2-x} = \dfrac{1}{1-x+1} = \dfrac{1}{1-(x-1)} = \sum_{n=0}^{\infty}(x-1)^n$, where $|x-1|<1$.
$|x-1|<1 \implies -1<x-1<1 \implies 0<x<2$.
If $x=0$ or $x=2$, the series $\sum_{n=0}^{\infty}(x-1)^n$ diverge.
Therefore, The interval of convergence is $(0,2)$.

3.9.2 Differentiation and Integration of Power Series

Theorem : If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R>0$, then the function f defined by
$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$
is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and
(1). $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.
(2). $\int f(x)\ dx = C + c_0(x-a) + c_1\dfrac{(x-a)^2}{2} + c_2\dfrac{(x-a)^3}{3} + \cdots$
$= C + \sum_{n=0}^{\infty} c_n \dfrac{(x-a)^{n+1}}{n+1}$, where C is a constant.
The radii of convergence of the power series in (1) and (2) are both R.

Notes : Equations (1) and (2) in the last theorem can be rewritten as :
(1). $\dfrac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty}\left[\dfrac{d}{dx}c_n(x-a)^n\right]$.
(2). $\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]\ dx = \sum_{n=0}^{\infty}\left[\int c_n(x-a)^n\ dx\right]$.

Example (5): Express $\dfrac{1}{(1-x)^2}$ as a power series, and find its radius of convergence.
Solution :
$\dfrac{d}{dx}\left[\dfrac{1}{1-x}\right] = \dfrac{d}{dx}\left[(1-x)^{-1}\right] = (-1)(1-x)^{-2}(-1) = \dfrac{1}{(1-x)^2}$.
Note that $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$, where $|x|<1$.
$\dfrac{1}{(1-x)^2} = \dfrac{d}{dx}\left[\dfrac{1}{1-x}\right] = \dfrac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\dfrac{d}{dx}[x^n] = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}$.

$\dfrac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$.

Since the radius of convergence of $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ is $R = 1$ then the radius of convergence of $\dfrac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ is also $R = 1$.

Example (6): Express $\ln(1+x)$ as a power series, and find its radius of convergence.

Solution :

$\int \dfrac{1}{1+x}\, dx = \ln(1+x) + c$

Note that $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$, where $|x| < 1$.

$\ln(1+x) = \int \dfrac{1}{1+x}\, dx = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n\right] dx = \sum_{n=0}^{\infty} \left[\int (-1)^n x^n\, dx\right]$

$= \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{n+1}}{n+1} + C = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} + C$, where $|x| < 1$.

Put $x = 0$ in the last equation : $\ln(1+0) = 0 + C \implies C = 0$.

Therefore, $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots$.

The radius of convergence of $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n}$ is $R = 1$.

Example (7): Express $\tan^{-1} x$ as a power series, and find its radius of convergence.

Solution :

$\int \dfrac{1}{1+x^2}\, dx = \tan^{-1} x + c$.

From Example (1), $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, where $|x| < 1$.

$\tan^{-1} x = \int \dfrac{1}{1+x^2}\, dx = \int \left[\sum_{n=0}^{\infty} (-1)^n x^{2n}\right] dx = \sum_{n=0}^{\infty} \int \left[(-1)^n x^{2n}\right] dx$

$= \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} + C$, where $|x| < 1$.

Put $x = 0$ in the last equation : $\tan^{-1}(0) = 0 + C \implies C = 0$.

Therefore, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots$.

The radius of convergence of $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ is $R = 1$.

3.9.3 EXERCISES

1. Find a power series representation for the function and determine the interval of convergence.

   (1). $f(x) = \dfrac{x}{x+1}$ (2). $f(x) = \dfrac{1}{1-x^2}$ (3). $f(x) = \dfrac{5}{1-4x^2}$

   (4). $f(x) = \dfrac{2}{3-x}$ (5). $f(x) = \dfrac{4}{2x+3}$ (6). $f(x) = \dfrac{x^2}{x^4+16}$

   (7). $f(x) = \dfrac{x}{2x^2+1}$ (8). $f(x) = \dfrac{x-1}{x+2}$

2. (a). Use differentiation to find a power series representation for

   $f(x) = \dfrac{1}{(x+1)^2}$, What is the radius of convergence?.

   (b). Use part (a) to find a power series for $f(x) = \dfrac{1}{(x+1)^3}$.

   (c). Use part (b) to find a power series for $f(x) = \dfrac{x^2}{(x+1)^3}$.

3. Find a power series representation for the function and determine the radius of convergence.

   (1). $f(x) = \ln(5-x)$ (2). $f(x) = x^2 \tan^{-1}(x^3)$.

3.10 Taylor and Maclaurin Series

3.10.1 Definitions of Taylor Series and Maclaurin Series

Theorem : If f has a power series representation at a, that is $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$,
where $|x-a| < R$, then the coefficient c_n is given by $c_n = \dfrac{f^{(n)}(a)}{n!}$.
Therefore, $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$,
or $f(x) = f(a) + \dfrac{f'(a)}{1!}(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \dfrac{f'''(a)}{3!}(x-a)^3 + \cdots$.

Definition : $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$ is called the Taylor series of the function f at a (or about a, or centered at a).

Definition : If $a = 0$ in the Taylor series of f, then $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!}x^n$ is called the Maclaurin series of f. In this case :
$f(x) = f(0) + \dfrac{f'(0)}{1!}x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 + \cdots$.

Example (1): Find the Maclaurin Series and its radius of convergence for the function $f(x) = \dfrac{1}{1-x}$.
Solution :
$f(x) = \dfrac{1}{1-x} \implies f(0) = 1 = 0!$.
$f'(x) = \dfrac{1}{(1-x)^2} \implies f'(0) = 1 = 1!$.
$f''(x) = \dfrac{2}{(1-x)^3} \implies f''(0) = 2 = 2!$.
$f'''(x) = \dfrac{6}{(1-x)^4} \implies f'''(0) = 6 = 3!$.
$f^{(n)}(x) = \dfrac{1.2.3\cdots n}{(1-x)^{n+1}} \implies f^{(n)}(0) = n!$.
The Maclaurin series for $f(x) = \dfrac{1}{1-x}$ is $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} \dfrac{n!}{n!}x^n = \sum_{n=0}^{\infty} x^n$.
Let $a_n = x^n$, then $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{x^{n+1}}{x^n}\right| = |x|$.
Therefore, the Maclaurin series converges when $|x| < 1$. Hence $R = 1$.

Example (2): Find the Maclaurin Series and its radius of convergence for the function $f(x) = e^x$.
Solution :
Note that $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$.

The Maclaurin series for $f(x) = e^x$ is $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Let $a_n = \frac{x^n}{n!}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$.
Therefore, the Maclaurin series converges for all $x \in \mathbb{R}$. Hence $R = \infty$.

3.10.2 Remainder of a Taylor series

Definition :
Suppose $f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$ then $f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i + \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$,
$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$ is called the nth-degree polynomial of f at a, and
$R_n(x) = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$ is called the remainder of the Taylor series.

Theorem : If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the nth-degree polynomial of f at a , and If $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Theorem : (Taylor's Inequality)
Suppose $f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i + \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i = T_n(x) + R_n(x)$.
If $\left|f^{(i)}(x)\right| \le M$ for $|x-a| \le d$ and for all $i \ge n+1$, where M and d are positive real numbers, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality : $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$, for $|x-a| \le d$.
In this case, $\lim_{n\to\infty} |R_n(x)| = 0$, hence $\lim_{n\to\infty} R_n(x) = 0$ for all values of x.

Example (3): Show that e^x is equal to the sum of its Maclaurin series.
Solution :
$f(x) = e^x \implies f^{(i)}(x) = e^x$ for all $i \ge 1$.
If $|x| \le d$ where $d > 0$, then $\left|f^{(i)}(x)\right| = e^x \le e^d$, for all $i \ge n+1$.
So Taylor's inequality, with $a = 0$ and $M = e^d$ is
$|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1}$ for $|x| \le d$.
Therefore, $\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = 0$.
So, $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \cdots$, for all $x \in \mathbb{R}$.
Note : If $x = 1$ then $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$.

Example (4): Find the Taylor series of $f(x) = e^x$ at $a = 2$.
Solution :
$f(x) = e^x \implies f^{(n)}(x) = e^x \implies f^{(n)}(2) = e^2$, for all n.
So, $e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2(x-2)^n}{n!}$ for all for all $x \in \mathbb{R}$.
Another Solution : $e^{x-2} = \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} \implies \frac{e^x}{e^2} = \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$
$\implies e^x = e^2 \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^2(x-2)^n}{n!}$.

Example (5): Write the function $f(x) = e^{x^2-1}$ as a Taylor series of x.
Solution :
$f(x) = e^{x^2-1} = e^{-1}e^{x^2} = e^{-1} \sum_{n=0}^{\infty} \frac{\left(x^2\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{e} \frac{x^{2n}}{n!}$ for all x .

3.10.3 Taylor Series of Important Functions

Example (6): Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.
Solution :
$f(x) = \sin x \implies f(0) = \sin(0) = 0$.
$f'(x) = \cos x \implies f'(0) = \cos(0) = 1$.
$f''(x) = -\sin x \implies f''(0) = -\sin(0) = 0$.
$f'''(x) = -\cos x \implies f'(0) = -\cos(0) = -1$.
$f^{(4)}(x) = \sin x \implies f^{(4)}(0) = \sin(0) = 0$.
Therefore, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.
$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!} \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$
$= \lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0 < 1.$
Therefore, the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.
Since $f^{(i)}(x) = \pm \sin x$ or $\pm \cos x$ for all $i \geq 1$, then $\left| f^{(i)}(x) \right| \leq 1$.
Taylor's inequality, with $a = 0$ and $M = 1$ is
$|R_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1}$ for $|x| \leq d$, where $d > 0$.
Therefore, $\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$.
So, $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for all x.

Example (7): Find the Maclaurin series for $\cos x$ and prove that it represents $\cos x$ for all x.

Solution :
$f(x) = \cos x \implies f(0) = \cos(0) = 1$.
$f'(x) = -\sin x \implies f'(0) = -\sin(0) = 0$.
$f''(x) = -\cos x \implies f''(0) = -\cos(0) = -1$.
$f'''(x) = \sin x \implies f'(0) = \sin(0) = 0$.
$f^{(4)}(x) = \cos x \implies f^{(4)}(0) = \cos(0) = 1$.
Therefore, $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.
$\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!} \frac{(2n)!}{(-1)^n x^{2n}}\right|$
$\displaystyle= \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+1)} = 0 < 1.$
Therefore, the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.
Since $f^{(i)}(x) = \pm\sin x$ or $\pm\cos x$ for all $i \geq 1$, then $\left|f^{(i)}(x)\right| \leq 1$.
Taylor's inequality, with $a = 0$ and $M = 1$ is
$|R_n(x)| \leq \dfrac{1}{(n+1)!}|x|^{n+1}$ for $|x| \leq d$, where $d > 0$.
Therefore, $\displaystyle\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$.
So, $\cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for all x .

Example (8): Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.
Solution :
$f(x) = (1+x)^k \implies f(0) = 1$.
$f'(x) = k(1+x)^{k-1} \implies f(0) = k$.
$f''(x) = k(k-1)(1+x)^{k-2} \implies f(0) = k(k-1)$.
$f'''(x) = k(k-1)(k-2)(1+x)^{k-3} \implies f(0) = k(k-1)(k-2)$.
$f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n} \implies f(0) = k(k-1)\cdots(k-n+1)$.
So, $(1+x)^k = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n$.
Let $a_n = \dfrac{k(k-1)\cdots(k-n+1)}{n!}x^n$ for all $n \geq 1$.
$\displaystyle\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right|$
$\displaystyle= \left|\frac{k-n}{n+1}\, x\right| = \frac{\left|1-\frac{k}{n}\right|}{1+\frac{1}{n}}|x|.$
$\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\left|1-\frac{k}{n}\right|}{1+\frac{1}{n}}|x| = |x|$.
So, $(1+x)^k = \displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n$ converges when $|x| < 1$. Therefore, $R = 1$.
Note : $(1+x)^k = \displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n$ is called the binomial series.

Example (9): For $f(x) = \dfrac{1}{\sqrt{4-x}}$, find the Maclaurin series and its radius of convergence.
Solution :
$f(x) = \dfrac{1}{\sqrt{4-x}} = \dfrac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} = \dfrac{1}{2\sqrt{1-\frac{x}{4}}} = \dfrac{1}{2}\left(1+\left(-\dfrac{x}{4}\right)\right)^{-\frac{1}{2}}$.
$\dfrac{1}{\sqrt{4-x}} = \dfrac{1}{2}\left(1+\left(-\dfrac{x}{4}\right)\right)^{-\frac{1}{2}} = \dfrac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\dfrac{x}{4}\right)^{n}$, where $\left|-\dfrac{x}{4}\right| < 1$.
$\left|-\dfrac{x}{4}\right| < 1 \implies \dfrac{|x|}{4} < 1 \implies |x| < 4$.
The radius of convergence of the Maclaurin series is $R = 4$.

Table of Important Series :

No.	Series	R
1	$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$	1
2	$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1}\dfrac{x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots$	1
3	$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots$	1
4	$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \dfrac{k(k-1)}{2!}x^2 + \cdots$	1
5	$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	∞
6	$\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$	∞
7	$\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots$	∞

3.10.4 New Taylor Series from Old

Example (10): Find the Maclaurin series for $f(x) = x^2 \cos x$.
Solution :
$f(x) = x^2 \cos x = x^2 \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+2}}{(2n)!}$ for all x.

Example (11): Find the Maclaurin series for $f(x) = \ln(1+4x^2)$.
Solution :
$f(x) = \ln(1+4x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{(4x^2)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{4^n x^{2n}}{n}$,
Where $\left|4x^2\right| < 1 \implies |x|^2 < \dfrac{1}{4} \implies |x| < \dfrac{1}{2}$.

Example (12): Find the function represented by the power series $\sum_{n=0}^{\infty}(-1)^n\frac{3^n x^n}{n!}$.

Solution :

$$\sum_{n=0}^{\infty}(-1)^n\frac{3^n x^n}{n!}=\sum_{n=0}^{\infty}\frac{(-3x)^n}{n!}=e^{-3x}\ .$$

Example (13): Find the sum of the series $\frac{1}{1\ (2)}-\frac{1}{2\ (2^2)}+\frac{1}{3\ (2^3)}-\cdots$.

Solution :

$$\frac{1}{1\ (2)}-\frac{1}{2\ (2^2)}+\frac{1}{3\ (2^3)}-\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n\ (2^n)}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\left(\frac{1}{2}\right)^n}{n}$$

$$=\ln\left(1+\frac{1}{2}\right)=\ln\left(\frac{3}{2}\right)\ .$$

Example (14): Use series to evaluate $\lim_{x\to 0}\frac{e^x-1-x}{x^2}$.

Solution : For all $x\neq 0$,

$$\frac{e^x-1-x}{x^2}=\frac{1}{x^2}\left[\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)-1-x\right]$$

$$=\frac{1}{x^2}\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)=\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots$$

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2}=\lim_{x\to 0}\left(\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots\right)=\frac{1}{2}.$$

3.10.5 EXERCISES

1. Use the definition of a Taylor series to find the first four nonzero terms of the series for $f(x)$ centered at the given value of a.

   (1). $f(x) = xe^x \quad , \quad a = 0$ (2). $f(x) = \dfrac{1}{1+x} \quad , \quad a = 2$

   (3). $f(x) = \ln x \quad , \quad a = 1$ (4). $f(x) = \cos^2 x \quad , \quad a = 0$

2. Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. Also find the associated radius of convergence.

   (1). $f(x) = (1-x)^{-2}$ (2). $f(x) = \ln(1+x)$ (3). $f(x) = \cos x$

   (4). $f(x) = 2x^4 - 3x^2 + 3$ (5). $f(x) = x\cos x$ (6). $f(x) = \sinh x$

3. Find the Taylor series for $f(x)$ centered at the given value of a. Also find the associated radius of convergence.

   (1). $f(x) = x^5 + 2x^3 + x \quad , \quad a = 2$ (2). $f(x) = \ln x \quad , \quad a = 2$

   (3). $f(x) = \sin x \quad , \quad a = \pi$ (4). $f(x) = \cos x \quad , \quad a = \dfrac{\pi}{2}$

4. Show that : (a). $\sinh x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ (b). $\cosh x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

   Hint : use $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$.