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College of Sciences
Department of Mathematics

MATH 201
MULTIVARIABLE CALCULUS

CLASS NOTES
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Chapter 1

Partial Derivatives

1.1 Functions of several variables

1.1.1 Functions of two variables

Definition: A function f of two variables is a map that assigns to each ordered pair of real numbers $(x, y) \in D \subseteq \mathbb{R}^2$ a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is $\{f(x, y) | (x, y) \in D\}$.

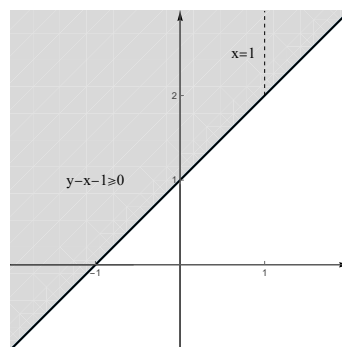
Example (1): If $f(x, y) = \frac{\sqrt{y-x-1}}{x-1}$, evaluate $f(2, 7)$, find the domain of f and sketch it.

Solution : $f(2, 7) = \frac{\sqrt{7-2-1}}{2-1} = \frac{\sqrt{4}}{1} = 2$.

The domain of f is the set $D = \{(x, y) \in \mathbb{R}^2 \mid y - x - 1 \geq 0, x \neq 1\}$.

So, $D = \{(x, y) \in \mathbb{R}^2 \mid y \geq x + 1, x \neq 1\}$.

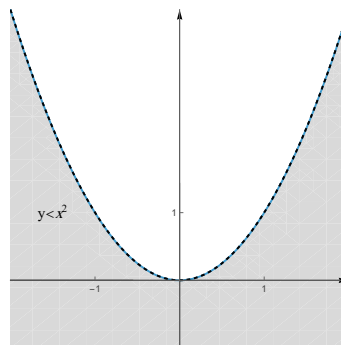
$y - x - 1 \geq 0 \implies y \geq x + 1$
represents the points on and above the line $y = x + 1$.
 $x \neq 1$ means the point on the line $x = 1$ must be excluded from the domain .



Example (2): If $f(x, y) = \ln(x^2 - y)$, find the domain of f and sketch it.
Solution :

The domain of f is the set
 $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y > 0\}$.
 So, $D = \{(x, y) \in \mathbb{R}^2 \mid y < x^2\}$.

$x^2 - y > 0 \implies y < x^2$
 represents the points below the
 parabola $y = x^2$.

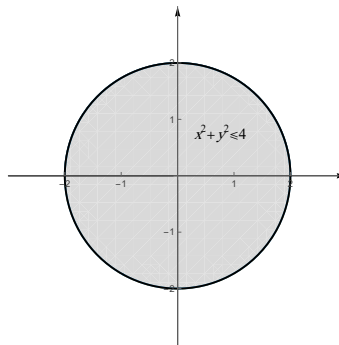


Example (3): If $f(x, y) = \sqrt{4 - x^2 - y^2}$, find the domain and range of f .

Solution :
 The domain of f is the set
 $D = \{(x, y) \in \mathbb{R}^2 \mid 4 - x^2 - y^2 \geq 0\}$.
 So, $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$.

$4 - x^2 - y^2 \geq 0 \implies x^2 + y^2 \leq 4$
 represents the points on and inside the
 disk of center $(0, 0)$ and radius 2.

Note that $0 \leq \sqrt{4 - x^2 - y^2}$
 and $\sqrt{4 - (x^2 + y^2)} \leq \sqrt{4} = 2$
 So, the range of f is
 $\{z \in \mathbb{R} \mid 0 \leq z \leq 2\} = [0, 2]$.



1.1.2 Graphs

Definition: If f is a function of two variables with domain $D \subseteq \mathbb{R}^2$, then the graph of f is the set of all points $(x, y, z) \in \mathbb{R}^3$, such that $z = f(x, y)$ and $(x, y) \in D$.

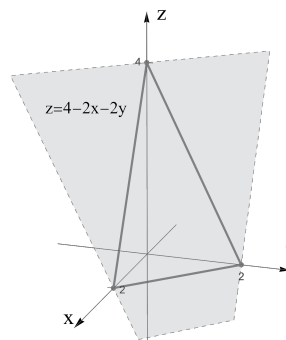
Example (4): Sketch the graph of the function $f(x, y) = 4 - 2x - 2y$.

Solution:
 $z = 4 - 2x - 2y \implies 2x + 2y + z = 4$
 It represents a plain.

To find the x -intercept, put $y = z = 0$
 $2x = 4 \implies x = 2$.
 So the x -intercept is $(2, 0, 0)$.

To find the y -intercept, put $x = z = 0$
 $2y = 4 \implies y = 2$.
 So the y -intercept is $(0, 2, 0)$.

To find the z -intercept, put $x = y = 0$
 $z = 4$, So the z -intercept is $(0, 0, 4)$.



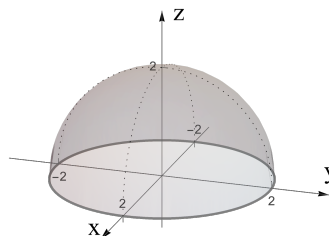
Example (5): Sketch the graph of the function $f(x, y) = \sqrt{4 - x^2 - y^2}$.

Solution:

$$\begin{aligned} z &= \sqrt{4 - x^2 - y^2} \\ \implies z^2 &= 4 - x^2 - y^2 \\ \implies x^2 + y^2 + z^2 &= 2^2 \end{aligned}$$

Note that $z = \sqrt{4 - x^2 - y^2} \geq 0$.

It represents the upper half of the sphere centered at the origin and its radius is 2.



Example (6): Sketch the graph of the function $f(x, y) = x^2 + y^2$.

Solution:

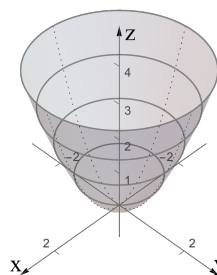
The domain of f is \mathbb{R}^2 .

$$z = x^2 + y^2 \geq 0$$

For each value of $z > 0$,

$x^2 + y^2 = z$ represents a circle centered at the origin and its radius is \sqrt{z} .

$f(x, y) = x^2 + y^2$ represents a paraboloid.



1.1.3 Level curves

Definition: A level curve of a function $f(x, y)$ is the curve $f(x, y) = k$, where k is a constant (in the range of f).

Example (7): Sketch the level curves of the function $f(x, y) = 4 - 2x - 2y$ for the values $k = 0, 4, 8$.

Solution:

$$4 - 2x - 2y = k.$$

$$\implies 2x + 2y = 4 - k$$

$$\text{For } k = 0 : 2y + 2x = 4$$

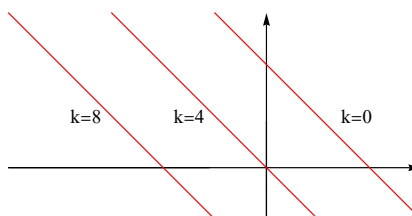
$$\implies y = -x + 2$$

$$\text{For } k = 4 : 2y + 2x = 0$$

$$\implies y = -x$$

$$\text{For } k = 8 : 2y + 2x = -4$$

$$\implies y = -x - 2$$



Example (8): Sketch the level curves of the function $f(x, y) = \sqrt{4 - x^2 - y^2}$ for the values $k = 0, 1, 2$.

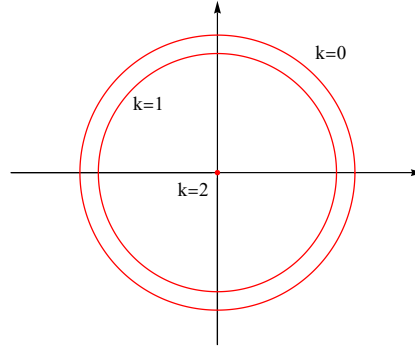
Solution:

$$\begin{aligned}\sqrt{4 - x^2 - y^2} &= k. \\ \implies 4 - x^2 - y^2 &= k^2 \\ \implies x^2 + y^2 &= 4 - k^2\end{aligned}$$

For $k = 0$: $x^2 + y^2 = 4$
Circle: center is $(0, 0)$, radius = 2.

For $k = 1$: $x^2 + y^2 = 3$
Circle: center is $(0, 0)$, radius = $\sqrt{3}$.

For $k = 2$: $x^2 + y^2 = 0$
Just one point, the origin.



Example (9): Sketch the level curves of the function $f(x, y) = 4x^2 + y^2$ for the values $k = 0, 2, 4$.

Solution:

Note that the domain is \mathbb{R}^2 .

$$4x^2 + y^2 = k.$$

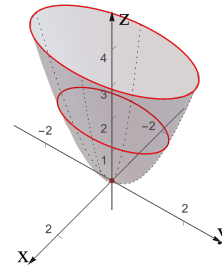
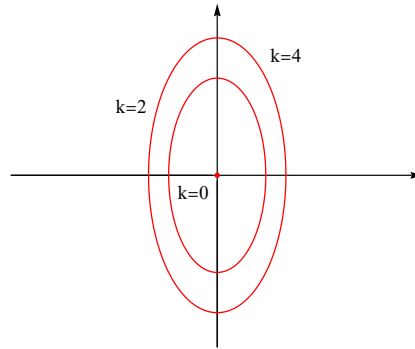
For $k = 0$: $4x^2 + y^2 = 0$
 $\implies x = 0, y = 0$

The level curve is the origin.

For $k > 0$: $4x^2 + y^2 = k$
 $\implies \frac{x^2}{\left(\frac{k}{4}\right)} + \frac{y^2}{k} = 1$

The level curve is an ellipse centered at the origin, the major axis lies on the y -axis and the minor axis lies on the x -axis.

Note that the graph of f is an elliptic paraboloid.



1.1.4 Functions of three variables

Definition: A function f of two variables is a map that assigns to each ordered pair of real numbers $(x, y, z) \in D \subseteq \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. The set D is the domain of f and its range is $\{f(x, y, z) | (x, y, z) \in D\}$.

Example (10): Find the domain of $f(x, y, z) = \ln(z - x) + yz \sin x$.

Solution : $f(x, y, z)$ is defined when $z - x > 0$.

Therefore, $D = \{(x, y, z) \in \mathbb{R}^3 \mid z > x\}$.

Example (11): Find the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$.

Solution : $x^2 + y^2 + z^2 = k$, where $k \geq 0$.

If $k = 0$, then the level surface is just the origin $(0, 0, 0) \in \mathbb{R}^3$.

If $k > 0$, then the level surface is $x^2 + y^2 + z^2 = (\sqrt{k})^2$, which is a sphere centered at the origin, and its radius is \sqrt{k} .

1.1.5 EXERCISES

1. Let $f(x, y) = x^2 \ln(x + y)$
 - (a). Evaluate $f(3, 1)$.
 - (b). Find and sketch the domain of f .
 - (c). Find the range of f .

2. Let $f(x, y, z) = \ln(z - \sqrt{x^2 + y^2})$.
 - (a). Evaluate $f(4, 23, 6)$.
 - (b). Find and describe the domain of f .

3. Find and sketch the domain of the following:
 - (a). $f(x, y) = \sqrt{x - 2} + \sqrt{y - 1}$.
 - (b). $f(x, y) = \ln(9 - x^2 - y^2)$.
 - (c). $f(x, y) = \frac{\ln(2 - x)}{1 - x^2 - y^2}$.

- (*) Find and sketch the domain of the following:
 - (a). $f(x, y) = \sqrt{25 - x^2 - y^2} + \ln(2 + x)$.
 - (b). $f(x, y) = \cos(x + y) + \frac{x^2 - y^2 - 3}{\sqrt{x + y - 4}}$.
 - (c). $f(x, y) = \sqrt{x + y^2} + \sqrt{y - x^2}$.

1.2 Limits and Continuity

1.2.1 Limits of Functions of Two Variables

Definition: Let f be a function of two variables whose domain $D \subseteq \mathbb{R}^2$ includes points arbitrarily close to (a, b) . Then we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$,

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$.

1.2.2 Showing That a Limit Does Not Exist

NOTE: If $f(x, y) \rightarrow L_1$ as $(x, y) \rightarrow (a, b)$ along a path C_1 and $f(x, y) \rightarrow L_2$ as $(x, y) \rightarrow (a, b)$ along a different path C_2 , where $L_1 \neq L_2$, then $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$ does not exist.

Example (1): Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Solution: Note that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Let $y = mx$, where $m \in \mathbb{R}$, note that $x \rightarrow 0$ as $y \rightarrow 0$.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{x^2 - m^2 x^2}{x^2 + m^2 x^2} = \lim_{x \rightarrow 0} \frac{x^2(1 - m^2)}{x^2(1 + m^2)} = \frac{1 - m^2}{1 + m^2}.$$

Note that the limit depends only on m .

Let C_1 be the path $y = 0$ (where $m=0$) then $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{1 - 0}{1 + 0} = 1$.

Let C_2 be the path $y = x$ (where $m=1$) then $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{1 - 1}{1 + 1} = 0$.

Therefore, $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Example (2): Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$ does not exist.

Solution: Note that $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Let $y = mx$, where $m \in \mathbb{R}$, note that $x \rightarrow 0$ as $y \rightarrow 0$.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{mx^2}{x^2 + m^2 x^2} = \lim_{x \rightarrow 0} \frac{mx^2}{x^2(1 + m^2)} = \frac{m}{1 + m^2}.$$

Note that the limit depends only on m .

Let C_1 be the path $y = 0$ (where $m=0$) then $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = \frac{0}{1 + 0} = 0$.

Let C_2 be the path $y = x$ (where $m=1$) then $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = \frac{1}{1 + 1} = \frac{1}{2}$.

Therefore, $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$ does not exist.

Example (3): Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4}$ does not exist.

Solution: Note that $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Let $y = mx$, where $m \in \mathbb{R}$, note that $x \rightarrow 0$ as $y \rightarrow 0$.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{x \rightarrow 0} \frac{x(m^2 x^2)}{x^2 + m^4 x^4} = \lim_{x \rightarrow 0} \frac{mx^3}{x^2(1 + m^4 x^2)} = \lim_{x \rightarrow 0} \frac{mx}{1 + m^4 x^2} = 0$$

Note that the limit here depends on x and m .

Let C_1 be the path $y = 0$ (where $m=0$) then $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4} = 0$.

Let C_2 be the path $x = y^2$ (the parabola with vertex $(0,0)$ and opens to the right) then $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4} = \lim_{y \rightarrow 0} \frac{y^2 y^2}{(y^2)^2 + y^4} = \lim_{y \rightarrow 0} \frac{y^4}{2y^4} = \frac{1}{2}$.

Therefore, $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4}$ does not exist.

Example (4): Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x^2 + y^2}$ does not exist.

Solution: Note that $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x^2 + y^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Let $y = mx$, where $m \in \mathbb{R}$, note that $x \rightarrow 0$ as $y \rightarrow 0$.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{\sin(mx^2)}{x^2 + m^2 x^2} = \lim_{x \rightarrow 0} \frac{\sin(mx^2)}{(1 + m^2)x^2} = \frac{m}{1 + m^2}$$

Note that the limit depends only on m .

Therefore, $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x^2 + y^2}$ does not exist.

1.2.3 Properties of Limits

If $f(x, y)$, $g(x, y)$ are two functions defined on $D \setminus \{(a, b)\}$, where $D \subseteq \mathbb{R}^2$, and $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L_1$, $\lim_{(x,y) \rightarrow (a,b)} g(x, y) = L_2$, where $L_1, L_2 \in \mathbb{R}$ then

- (1). $\lim_{(x,y) \rightarrow (a,b)} [f(x, y) + g(x, y)] = L_1 + L_2$.
- (2). $\lim_{(x,y) \rightarrow (a,b)} [f(x, y) - g(x, y)] = L_1 - L_2$.
- (3). $\lim_{(x,y) \rightarrow (a,b)} k f(x, y) = k L_1$, where $k \in \mathbb{R}$.
- (4). $\lim_{(x,y) \rightarrow (a,b)} [f(x, y).g(x, y)] = L_1 L_2$.
- (5). $\lim_{(x,y) \rightarrow (a,b)} \frac{f(x, y)}{g(x, y)} = \frac{L_1}{L_2}$, where $L_2 \neq 0$.
- (6). If $P(x, y)$ is a polynomial in x, y then $\lim_{(x,y) \rightarrow (a,b)} P(x, y) = P(a, b)$.

Example (5): Evaluate $\lim_{(x,y) \rightarrow (2,1)} (x^2 y^2 - 2xy + x + y - 1)$.

Solution: Note that $P(x, y) = x^2 y^2 - 2xy + x + y - 1$ is a polynomial.

$$\text{So, } \lim_{(x,y) \rightarrow (2,1)} (x^2 y^2 - 2xy + x + y - 1) = P(2, 1)$$

$$= (2)^2(1)^2 - 2(2)(1) + 2 + 1 - 1 = 2.$$

Example (6): Evaluate $\lim_{(x,y) \rightarrow (-1,3)} \frac{2x^2y + 1}{xy^3 - 2x}$.

Solution: Note that $P(x, y) = 2x^2y + 1$ and $Q(x, y) = xy^3 - 2x$ are both polynomials.

$$\lim_{(x,y) \rightarrow (-1,3)} \frac{2x^2y + 1}{xy^3 - 2x} = \frac{P(-1, 3)}{Q(-1, 3)} = \frac{2(-1)^2(3) + 1}{(-1)(3)^3 - 2(-1)} = \frac{7}{-25}.$$

Note that $Q(-1, 3) \neq 0$.

Example (7): Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{2xy^2}{x^2 + y^2} = 0$.

First Solution: $\forall x, y \in \mathbb{R}^*$,

$$y^2 \leq x^2 + y^2 \implies \frac{y^2}{x^2 + y^2} \leq 1.$$

$$0 \leq \left| \frac{2xy^2}{x^2 + y^2} \right| = \frac{|2x| |y^2|}{|x^2 + y^2|} = 2|x| \frac{y^2}{x^2 + y^2} \leq 2|x|.$$

Note that $(x, y) \rightarrow (0, 0) \implies x \rightarrow 0$.

Since $\lim_{(x,y) \rightarrow (0,0)} 0 = 0$ and $\lim_{(x,y) \rightarrow (0,0)} 2|x| = 2(0) = 0$,

By Squeeze Theorem $\lim_{(x,y) \rightarrow (0,0)} \frac{2xy^2}{x^2 + y^2} = 0$.

Second Solution: Using polar coordinates,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2xy^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{2r^3 \cos \theta \sin^2 \theta}{r^2} = \lim_{r \rightarrow 0} 2r \cos \theta \sin^2 \theta = 0.$$

Note that $\lim_{r \rightarrow 0} 2r = 0$ and $\cos \theta \sin^2 \theta$ is bounded.

1.2.4 Continuity

Definition: Let $f(x, y)$ be a function of two variables defined on a set $D \subseteq \mathbb{R}^2$ and $(a, b) \in D$. If $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$, then f is continuous at (a, b) .

If f is continuous at every point in D , then f is continuous on D .

Example (8): Discuss the continuity of $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$.

Solution:

f is not defined at $(0, 0)$, so it is not continuous at $(0, 0)$.

$$\forall (a, b) \neq (0, 0), \quad \lim_{(x,y) \rightarrow (a,b)} f(x, y) = \lim_{(x,y) \rightarrow (a,b)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{a^2 - b^2}{a^2 + b^2} = f(a, b).$$

So, f is continuous on $D = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\} = \mathbb{R}^2 \setminus \{(0, 0)\}$.

Example (9): Discuss the continuity of $f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$.

Solution:

$$\forall (a, b) \neq (0, 0), \quad \lim_{(x,y) \rightarrow (a,b)} f(x, y) = \lim_{(x,y) \rightarrow (a,b)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{a^2 - b^2}{a^2 + b^2} = f(a, b).$$

f is defined at $(0, 0)$ and $f(0, 0) = 0$, but $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist.

So, f is continuous on $D = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\} = \mathbb{R}^2 \setminus \{(0, 0)\}$.

Example (10): Discuss the continuity of $f(x, y) = \begin{cases} \frac{2xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$.

Solution:

$\forall (a, b) \neq (0, 0)$, $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = \lim_{(x,y) \rightarrow (a,b)} \frac{2xy^2}{x^2 + y^2} = \frac{2ab^2}{a^2 + b^2} = f(a, b)$.
 f is defined at $(0, 0)$ and $f(0, 0) = 0$, also $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$.

So, f is continuous on \mathbb{R}^2 .

Example (11): Discuss the continuity of $f(x, y) = e^{-x^2 - y^2}$.

Solution: $f(x, y) = e^{-x^2 - y^2} = e^{-(x^2 + y^2)}$.

f is defined on \mathbb{R}^2 and $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = e^{-(a^2 + b^2)} = f(a, b)$.

Therefore, f is continuous on \mathbb{R}^2 .

Example (12): Discuss the continuity of $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$.

Solution:

f is not defined where $x = 0$.

$\forall (a, b)$ where $a \neq 0$, $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = \tan^{-1}\left(\frac{b}{a}\right) = f(a, b)$.

So, f is continuous on $D = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$ which is \mathbb{R}^2 except the y -axis.

1.2.5 Limit and Continuity of a function of three variables

Definition: Let f be a function of three variables whose domain $D \subseteq \mathbb{R}^3$ includes points arbitrarily close to (a, b, c) . Then we say that the limit of $f(x, y, z)$ as (x, y, z) approaches (a, b, c) is L and we write $\lim_{(x,y,z) \rightarrow (a,b,c)} f(x, y, z) = L$,

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y, z) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta$ then $|f(x, y, z) - L| < \epsilon$.

Definition: Let $f(x, y, z)$ be a function of three variables defined on a set $D \subseteq \mathbb{R}^3$ and $(a, b, c) \in D$. If $\lim_{(x,y,z) \rightarrow (a,b,c)} f(x, y, z) = f(a, b, c)$, then f is continuous at (a, b, c) .

If f is continuous at every point in D , then f is continuous on D .

Example (13): Discuss the continuity of $f(x, y, z) = \frac{1}{1 - x^2 - y^2 - z^2}$.

Solution:

f is not defined where $1 - x^2 - y^2 - z^2 = 0 \implies x^2 + y^2 + z^2 = 1$,

f is not continuous on the unit sphere.

$\forall (a, b, c) \in \mathbb{R}^3$ where $a^2 + b^2 + c^2 \neq 1$,

$\lim_{(x,y,z) \rightarrow (a,b,c)} f(x, y, z) = \frac{1}{1 - a^2 - b^2 - c^2} = f(a, b, c)$.

So, f is continuous on $D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \neq 1\}$ which is \mathbb{R}^3 except the unit sphere.

1.2.6 EXERCISES

1. Find the limit of the following:

$$(a). \lim_{(x,y) \rightarrow (3,2)} (x^2y^3 - 4y^2) \quad (b). \lim_{(x,y) \rightarrow (2,-1)} \frac{x^2y + yx^2}{x^2 - y^2}$$

2. Show that the following limits do not exist:

$$(a). \lim_{(x,y) \rightarrow (0,0)} \frac{y^2}{x^2 + y^2} \quad (b). \lim_{(x,y) \rightarrow (0,0)} \frac{(x+y)^2}{x^2 + y^2}$$

$$(c). \lim_{(x,y) \rightarrow (1,1)} \frac{y-x}{1-y+\ln x}$$

3. Discuss the existence of the following limits (and find its value if exists):

$$(a). \lim_{(x,y) \rightarrow (2,3)} \frac{3x-2y}{4x^2-y^2} \quad (b). \lim_{(x,y) \rightarrow (0,0)} \frac{x^2y \cos y}{x^2 + y^4}$$

4. Use the Squeeze Theorem to find the limit of the following :

$$(a). \lim_{(x,y) \rightarrow (0,0)} xy \sin\left(\frac{1}{x^2 + y^2}\right) \quad (b). \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{x^2y^2z^2}{x^2 + y^2 + z^2}$$

5. Determine the set of points of continuity of the following:

$$(a). f(x, y) = \begin{cases} \frac{x^2y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

$$(b). f(x, y) = \begin{cases} \frac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

6. Use polar coordinates to find $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2}$.

(*) Show that the following limits do not exist:

$$(a). \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^2}{x^2 + y^2} \quad (b). \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^4 + y^4}$$

(*) Use Squeeze theorem to find the following limits:

$$(a). \lim_{(x,y) \rightarrow (-1,0)} \frac{y(x+1)^2 + y^2 \sin(\pi x)}{(x+1)^2 + y^2} \quad (b). \lim_{(x,y) \rightarrow (0,0)} \frac{y^2 \sin x + yx^2}{x^2 + y^2}$$

(*) Show that the limit is zero in the following:

$$(a). \lim_{(x,y) \rightarrow (0,0)} \frac{x^5 + y^5}{x^2 + y^2} \quad (b). \lim_{(x,y) \rightarrow (0,0)} \frac{x^6 + y^6}{x^4 + y^4}$$
$$(c). \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + 2xy}{\sqrt{x^2 + y^2}} \quad (d). \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^3 + y^3)}{x^2 + y^2}$$

1.3 Partial Derivatives

1.3.1 Partial Derivatives of Functions of Two Variables

Definition: Let $f(x, y)$ be a function of two variables defined on a set $D \subseteq \mathbb{R}^2$. If $\lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$ and $\lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$ exist, then the partial derivatives of f are denoted by f_x and f_y and are defined as

$$(1). f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

$$(2). f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

Notations for Partial Derivatives: If $z = f(x, y)$ then

$$(1). f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = D_x f .$$

$$(2). f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = D_y f .$$

Rule for Finding Partial Derivatives

- (1). To find f_x : differentiate $f(x, y)$ with respect to x regarding y as a constant.
- (2). To find f_y : differentiate $f(x, y)$ with respect to y regarding x as a constant.

Example (1): If $f(x, y) = x^2 - xy^3 - 3y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$.

Solution:

$$(1). f_x(x, y) = 2x - (1)y^3 - 0 = 2x - y^3 ,$$

$$f_x(1, 1) = 2(1) - (1)^3 = 1.$$

$$(2). f_y(x, y) = 0 - x(3y^2) - 3(2y) = -3xy^2 - 6y ,$$

$$f_y(1, 1) = -3(1)(1)^2 - 6(1) = -9.$$

Example (2): If $f(x, y) = \sin\left(\frac{2x}{1+3y}\right)$, Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution:

$$(1). \frac{\partial f}{\partial x} = \cos\left(\frac{2x}{1+3y}\right) \left(\frac{2}{1+3y}\right) .$$

$$(2). \frac{\partial f}{\partial y} = \cos\left(\frac{2x}{1+3y}\right) ((2x)(-1)(1+3y)^{-2}(3)) = \cos\left(\frac{2x}{1+3y}\right) \left(\frac{-6x}{(1+3y)^2}\right) .$$

1.3.2 Interpretations of Partial Derivatives

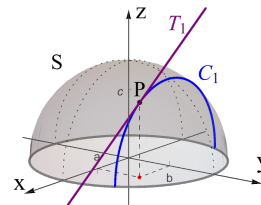
Let S be the surface represented by

$$z = f(x, y) \text{ and } f(a, b) = c.$$

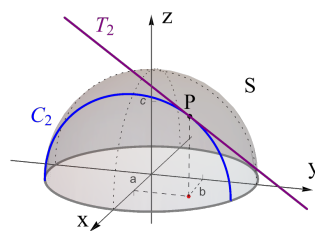
Then $P(a, b, c)$ lies on S .

Let C_1 be the curve where the plane $y = b$ intersects the surface S ,

The slope of the tangent line T_1 to the curve C_1 at $P(a, b, c)$ is $f_x(a, b)$.



Let C_2 be the curve where the plane $x = a$ intersects the surface S ,
 The slope of the tangent line T_2 to the curve C_2 at $P(a, b, c)$ is $f_y(a, b)$.



Example (3): If $f(x, y) = 4 - x^2 - y^2$, find $f_x(1, 1)$, $f_y(1, 1)$ and interpret them as slopes. Solution:

$$z = 4 - x^2 - y^2 .$$

$$f_x(x, y) = -2x .$$

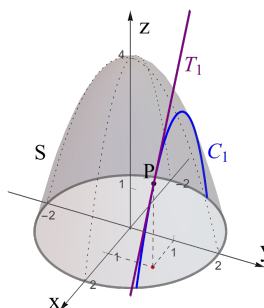
$$f_x(1, 1) = -2(1) = -2 .$$

$$f(1, 1) = 4 - 1 - 1 = 2 .$$

C_1 is the intersection of $z = 4 - x^2 - y^2$ and $y = 1$, and it is the parabola

$$z = 3 - x^2 .$$

The line T_1 is tangent to the curve C_1 at the point $P(1, 1, 2)$ and its slope is $f_x(1, 1) = -2$.



$$z = 4 - x^2 - y^2 .$$

$$f_y(x, y) = -2y .$$

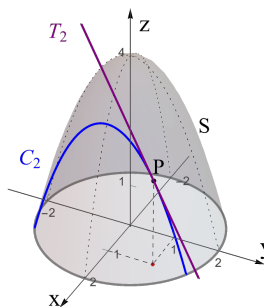
$$f_y(1, 1) = -2(1) = -2 .$$

$$f(1, 1) = 4 - 1 - 1 = 2 .$$

C_2 is the intersection of $z = 4 - x^2 - y^2$ and $x = 1$, and it is the parabola

$$z = 3 - y^2 .$$

The line T_2 is tangent to the curve C_2 at the point $P(1, 1, 2)$ and its slope is $f_y(1, 1) = -2$.



Example (4): If $x^3 + y^3 + z^3 + 6xyz + 4 = 0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution:

Differentiating implicitly with respect to x .

$$3x^2 + 0 + 3z^2 \frac{\partial z}{\partial x} + 6y \left(z + x \frac{\partial z}{\partial x} \right) + 0 = 0$$

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} (3z^2 + 6xy) = -3x^2 - 6yz$$

$$\frac{\partial z}{\partial x} = \frac{-3x^2 - 6yz}{3z^2 + 6xy} = \frac{-x^2 - 2yz}{z^2 + 2xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}.$$

Differentiating implicitly with respect to y .

$$0 + 3y^2 + 3z^2 \frac{\partial z}{\partial y} + 6x \left(z + y \frac{\partial z}{\partial y} \right) + 0 = 0$$

$$3y^2 + 3z^2 \frac{\partial z}{\partial y} + 6xz + 6xy \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} (3z^2 + 6xy) = -3y^2 - 6xz$$

$$\frac{\partial z}{\partial y} = \frac{-3y^2 - 6xz}{3z^2 + 6xy} = \frac{-y^2 - 2xz}{z^2 + 2xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}.$$

1.3.3 Functions of Three Variables

Definition: Let $f(x, y, z)$ be a function of three variables defined on a set $D \subseteq \mathbb{R}^3$. If $\lim_{h \rightarrow 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$, $\lim_{h \rightarrow 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}$ and $\lim_{h \rightarrow 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}$ exist, then the partial derivatives of f are denoted by f_x , f_y and f_z and are defined as

$$(1). f_x(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}.$$

$$(2). f_y(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x, y+h, z) - f(x, y, z)}{h}.$$

$$(3). f_z(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x, y, z+h) - f(x, y, z)}{h}.$$

Example (5): If $f(x, y, z) = e^{xy} \ln(z^2 + 1)$, find f_x , f_y and f_z .

Solution:

$$f_x(x, y, z) = (e^{xy} y) \ln(z^2 + 1) = ye^{xy} \ln(z^2 + 1).$$

$$f_y(x, y, z) = (e^{xy} x) \ln(z^2 + 1) = xe^{xy} \ln(z^2 + 1).$$

$$f_z(x, y, z) = e^{xy} \left(\frac{2z}{z^2 + 1} \right) = \frac{2ze^{xy}}{z^2 + 1}.$$

1.3.4 Higher Derivatives

Definition: If $z = f(x, y)$, then the second partial derivatives of f are

$$(1) f_{xx} = (f_x)_x = \frac{\partial}{\partial x} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}.$$

$$(2) f_{xy} = (f_x)_y = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}.$$

$$(3) f_{yx} = (f_y)_x = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}.$$

$$(y) f_{yy} = (f_y)_y = \frac{\partial}{\partial y} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}.$$

Example (6): Find the second partial derivatives of $f(x, y) = x^3 + x^2y^2 - 2y^3$.

Solution:

$$\begin{aligned}
f_x(x, y) &= 3x^2 + (2x)y^2 - 0 = 3x^2 + 2xy^2 . \\
f_{xx}(x, y) &= 3(2x) + 2y^2(1) = 6x + 2y^2 . \\
f_{xy}(x, y) &= 0 + 2x(2y) = 4xy . \\
f_y(x, y) &= 0 + x^2(2y) - 2(3y^2) = 2x^2y - 6y^2 . \\
f_{yx}(x, y) &= 2y(2x) - 0 = 4xy . \\
f_{yy} &= 2x^2(1) - 6(2y) = 2x^2 - 12y . \\
\text{NOTE : } f_{xy}(x, y) &= f_{yx}(x, y) .
\end{aligned}$$

Clairaut's Theorem: Suppose f is defined on a set $D \subseteq \mathbb{R}^2$ that contains the point (a, b) . If the functions f_{xy} and f_{yx} are both continuous on D , then $f_{xy}(a, b) = f_{yx}(a, b)$.

Example (7): If $f(x, y, z) = \sin(2x - yz)$, find $f_{xyz}(x, y)$.

Solution:

$$\begin{aligned}
f_x(x, y, z) &= \cos(2x - yz)(2) = 2 \cos(2x - yz) . \\
f_{xy}(x, y, z) &= 2 [-\sin(2x - yz) (-z)] = 2z \sin(2x - yz) . \\
f_{xyz}(x, y, z) &= (2) \sin(2x - yz) + 2z [\cos(2x - yz) (-y)] \\
&= 2 \sin(2x - yz) - 2yz \cos(2x - yz) .
\end{aligned}$$

1.3.5 Partial Differential Equations

(1). **Laplace's Equation:** If $u(x, y)$ is a function of two variables and $u_{xx} + u_{yy} = 0$, then u satisfies the Laplace's equation, and u is called a harmonic function.

Example (8): Show that $u(x, y) = e^x \cos y$ is a solution of Laplace's equation.

Solution:

$$\begin{aligned}
u_x(x, y) &= e^x \cos y \text{ and } u_{xx}(x, y) = e^x \cos y . \\
u_y(x, y) &= -e^x \sin y \text{ and } u_{yy}(x, y) = -e^x \cos y . \\
u_{xx} + u_{yy} &= e^x \cos y - e^x \cos y = 0 .
\end{aligned}$$

(2). **Wave Equation:** If $u(x, t)$ is a function of two variables and $u_{tt} = a^2 u_{xx}$, then u satisfies the wave equation.

Example (9): Show that $u(x, t) = \sin(x - at)$ is a solution of the wave equation.

Solution:

$$\begin{aligned}
u_x(x, t) &= \cos(x - at) \text{ and } u_{xx}(x, t) = -\sin(x - at) . \\
u_t(x, t) &= -a \cos(x - at) \text{ and } u_{tt}(x, t) = -a^2 \sin(x - at) . \\
u_{tt} &= a^2 (-\sin(x - at)) = a^2 u_{xx} .
\end{aligned}$$

1.3.6 Differentiability

Definition: If $z = f(x, y)$ and $(a, b) \in D_f$, if x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$, then the increment of z is

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b).$$

Definition: If $z = f(x, y)$, then f is differentiable at (a, b) , if

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \sqrt{(\Delta x)^2 + (\Delta y)^2} \epsilon(\Delta x, \Delta y),$$

where $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \epsilon(\Delta x, \Delta y) = 0$.

NOTE:

If $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{f(a + \Delta x, b + \Delta y) - f(a, b) - f_x(a, b)\Delta x - f_y(a, b)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$, then

f is differentiable at (a, b) .

Theorem: If f is differentiable at (a, b) , then f is continuous at (a, b) .

NOTE: If f is not continuous at (a, b) , then f is not differentiable at (a, b) .

Example (10): Show that $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

is not differentiable at $(0, 0)$.

Solution:

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} f(x, y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r \cos \theta \ r \sin \theta}{r^2} \\ &= \lim_{r \rightarrow 0} \frac{r^2 \cos \theta \sin \theta}{r^2} = \cos \theta \sin \theta. \end{aligned}$$

The limit depends on θ , so the limit does not exist.

f is not continuous at $(0, 0)$, hence, f is not differentiable at $(0, 0)$.

Example (11): Show that $f(x, y) = \begin{cases} \frac{xy^2}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

is differentiable at $(0, 0)$.

Solution:

$$f(0, 0) = 0.$$

$$f(0 + \Delta x, 0 + \Delta y) = f(\Delta x, \Delta y) = \frac{\Delta x (\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}.$$

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0.$$

$$f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, 0 + h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0.$$

$$\begin{aligned} &\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{f(\Delta x, \Delta y) - f(0, 0) - f_x(0, 0)\Delta x - f_y(0, 0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \\ &= \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{\left(\frac{\Delta x (\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{\Delta x (\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} = 0. \end{aligned}$$

$$\text{Note that } 0 \leq \left| \frac{\Delta x (\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \right| = |\Delta x| \left| \frac{(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \right| \leq |\Delta x|$$

By the squeeze theorem $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \left| \frac{\Delta x (\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \right| = 0$,

So, $\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{\Delta x (\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} = 0$.

Therefore, f is differentiable at $(0,0)$.

Example (12): Show that $f(x, y) = \begin{cases} \frac{y^2 \sin x + yx^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

is not differentiable at $(0,0)$.

Solution:

$f(0,0) = 0$.

$$f(0 + \Delta x, 0 + \Delta y) = f(\Delta x, \Delta y) = \frac{(\Delta y)^2 \sin(\Delta x) + \Delta y (\Delta x)^2}{(\Delta x)^2 + (\Delta y)^2}.$$

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0.$$

$$f_y(0,0) = \lim_{h \rightarrow 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0.$$

$$\begin{aligned} & \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{f(\Delta x, \Delta y) - f(0,0) - f_x(0,0)\Delta x - f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \\ &= \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{\left(\frac{(\Delta y)^2 \sin(\Delta x) + \Delta y (\Delta x)^2}{(\Delta x)^2 + (\Delta y)^2} \right)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{(\Delta y)^2 \sin(\Delta x) + \Delta y (\Delta x)^2}{\left[(\Delta x)^2 + (\Delta y)^2 \right]^{\frac{3}{2}}} \end{aligned}$$

Taking the path $\Delta y = \Delta x$:

$$\begin{aligned} & \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{(\Delta y)^2 \sin(\Delta x) + \Delta y (\Delta x)^2}{\left[(\Delta x)^2 + (\Delta y)^2 \right]^{\frac{3}{2}}} = \lim_{\Delta x \rightarrow 0} \frac{(\Delta x)^2 \sin(\Delta x) + \Delta x (\Delta x)^2}{\left[(\Delta x)^2 + (\Delta x)^2 \right]^{\frac{3}{2}}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(\Delta x)^2 [\sin(\Delta x) + \Delta x]}{\left[2(\Delta x)^2 \right]^{\frac{3}{2}}} = \lim_{\Delta x \rightarrow 0} \frac{(\Delta x)^2 [\sin(\Delta x) + \Delta x]}{\sqrt{8} (\Delta x)^3} = \lim_{\Delta x \rightarrow 0} \frac{\sin(\Delta x) + \Delta x}{\sqrt{8} \Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{8}} \left(\frac{\sin(\Delta x)}{\Delta x} + \frac{\Delta x}{\Delta x} \right) = \frac{1}{\sqrt{8}} (1 + 1) = \frac{2}{\sqrt{8}} = \frac{1}{\sqrt{2}} \neq 0. \end{aligned}$$

Therefore, f is not differentiable at $(0,0)$.

1.3.7 EXERCISES

1. Find the first partial derivatives of the function.

$$\begin{aligned} (a). f(x, y) &= x^4 - 5xy^3 & (b). g(x, y) &= x^3 \sin y \\ (c). w(u, v) &= \frac{u}{v^2} & (d). u(r, \theta) &= \sin(r \cos \theta) \\ (e). w(x, y, z) &= y \tan(x + 2z) \end{aligned}$$

2. If $f(x, y) = y \sin^{-1}(xy)$, find $f_y \left(1, \frac{1}{2}\right)$.

3. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$(a). z = f(x) + g(y) \quad (b). z = f(x + y)$$

4. Find all the second partial derivatives of $f(x, y) = x^4y - 2x^3y^2$.

5. Verify that the conclusion of Clairaut's Theorem holds for

$$u(x, y) = x^4y^3 - y^4.$$

6. If $f(x, y) = x^4y^2 - x^3y$, find f_{xxx} and f_{xyx} .

- (*) Discuss the differentiability of the following functions at the given points:

$$(a). f(x, y) = \begin{cases} \frac{x^2(y-2)}{x^2+(y-2)^2} & \text{if } (x, y) \neq (0, 2) \\ 0 & \text{if } (x, y) = (0, 2) \end{cases} \quad \text{at } (0, 2).$$

$$(b). f(x, y) = \begin{cases} \frac{x^2y - xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \quad \text{at } (0, 0).$$

$$(c). f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \quad \text{at } (0, 0).$$

1.4 Chain Rule

1.4.1 The Chain Rule (Case 1)

Theorem: Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t . Then z is a differentiable function of t and $\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$.

Example (1): If $f(x, y) = x^3y + 2xy^3$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{dz}{dt}$ at $t = 0$.

Solution:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2y + 2y^3, \quad \frac{\partial f}{\partial y} = x^3 + 6xy^2. \\ \frac{dx}{dt} &= 2 \cos 2t, \quad \frac{dy}{dt} = -\sin t. \\ \frac{dz}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= (3x^2y + 2y^3)(2 \cos 2t) + (x^3 + 6xy^2)(-\sin t) \\ \left. \frac{dz}{dt} \right|_{t=0} &= (0 + 2)(2) + (0 + 0)(0) = 4. \end{aligned}$$

1.4.2 The Chain Rule (Case 2)

Theorem: Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t .

Then z is a differentiable function of t and $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}$,

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$$

Example (2): If $f(x, y) = e^x \sin y$, where $x = s^2 + t^2$ and $y = s^2 t^2$, find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

Solution:

$$\begin{aligned} \frac{\partial f}{\partial x} &= e^x \sin y, \quad \frac{\partial f}{\partial y} = e^x \cos y. \\ \frac{\partial x}{\partial s} &= 2s, \quad \frac{\partial y}{\partial s} = 2st^2. \\ (1). \quad \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} = (e^x \sin y)(2s) + (e^x \cos y)(2st^2) \\ &= 2se^{s^2+t^2} \sin(s^2 t^2) + 2st^2 e^{s^2+t^2} \cos(s^2 t^2). \\ \text{Also, } \frac{\partial x}{\partial t} &= 2t, \quad \frac{\partial y}{\partial t} = 2ts^2. \\ (2). \quad \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = (e^x \sin y)(2t) + (e^x \cos y)(2ts^2) \\ &= 2te^{s^2+t^2} \sin(s^2 t^2) + 2ts^2 e^{s^2+t^2} \cos(s^2 t^2). \end{aligned}$$

1.4.3 The Chain Rule (The General Case)

Theorem: Suppose that u is a differentiable function of the n variables x_1, x_2, \dots, x_n and each x_j is a differentiable function of the m variables t_1, t_2, \dots, t_m . Then u is a function of t_1, t_2, \dots, t_m and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}, \text{ for each } i = 1, 2, \dots, m.$$

Example (3): If $u(x, y, z) = x^2y^2 + yz^3$, where $x = r^2se^t$, $y = rse^{-t}$ and $z = rs^2 \sin t$, find $\frac{\partial u}{\partial s}$ when $r = 1$, $s = 1$ and $t = 0$.

Solution:

$$\frac{\partial u}{\partial x} = 2xy^2, \quad \frac{\partial u}{\partial y} = 2x^2y + z^3 \text{ and } \frac{\partial u}{\partial z} = 3yz^2.$$

$$\frac{\partial x}{\partial s} = r^2e^t, \quad \frac{\partial y}{\partial s} = re^{-t} \text{ and } \frac{\partial z}{\partial s} = 2rs \sin t.$$

$$\text{So, } \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s}$$

$$\frac{\partial u}{\partial s} = (2xy^2)(r^2e^t) + (2x^2y + z^3)(re^{-t}) + (3yz^2)(2rs \sin t)$$

$$\text{When } r = 1, s = 1, t = 0, \quad \frac{\partial u}{\partial s} = (2)(1) + (2)(1) + (0)(0) = 4.$$

Example (4): If $w = f(x, y)$ is differentiable at (x, y) and $x = s + t$, $y = s - t$.

$$\text{Show that } \frac{\partial w}{\partial s} \frac{\partial w}{\partial t} = \left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{\partial w}{\partial y}\right)^2.$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} = f_x(1) + f_y(1) = f_x + f_y.$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} = f_x(1) + f_y(-1) = f_x - f_y.$$

$$\frac{\partial w}{\partial s} \frac{\partial w}{\partial t} = (f_x + f_y)(f_x - f_y) = (f_x)^2 - (f_y)^2 = \left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{\partial w}{\partial y}\right)^2.$$

Example (5): If $z = f(s^2 - t^2, t^2 - s^2)$ is differentiable at (s, t) ,

$$\text{Show that } t \frac{\partial z}{\partial s} + s \frac{\partial z}{\partial t} = 0.$$

Solution:

Let $z = f(x, y)$, where $x = s^2 - t^2$ and $y = t^2 - s^2$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = f_x(2s) + f_y(-2s) = 2sf_x - 2sf_y.$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = f_x(-2t) + f_y(2t) = -2tf_x + 2tf_y.$$

$$t \frac{\partial z}{\partial s} + s \frac{\partial z}{\partial t} = t(2sf_x - 2sf_y) + s(-2tf_x + 2tf_y) \\ = 2stf_x - 2stf_y - 2stf_x + 2stf_y = 0.$$

Example (6): If $z = f(x, y)$ has continuous second-order partial derivatives

and $x = r^2 + s$, $y = 3rs$. Find $\frac{\partial^2 z}{\partial r^2}$.

Solution:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = f_x(2r) + f_y(3s) = 2rf_x + 3sf_y.$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial r} \right) = \frac{\partial}{\partial r} (2r f_x + 3s f_y) = 2f_x + 2r \left(\frac{\partial f_x}{\partial r} \right) + 3s \left(\frac{\partial f_y}{\partial r} \right) \\
&= 2f_x + 2r \left(\frac{\partial f_x}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f_x}{\partial y} \frac{\partial y}{\partial r} \right) + 3s \left(\frac{\partial f_y}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f_y}{\partial y} \frac{\partial y}{\partial r} \right) \\
&= 2f_x + 2r (f_{xx}(2r) + f_{xy}(3s)) + 3s (f_{yx}(2r) + f_{yy}(3s)) \\
&= 2f_x + 4r^2 f_{xx} + 6rs f_{xy} + 6rs f_{yx} + 9s^2 f_{yy} .
\end{aligned}$$

1.4.4 Implicit Differentiation

Let $F(x, y) = 0$ where $y = f(x)$, using chain rule $\frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0$

$$\implies F_x(1) + F_y \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{F_x}{F_y} .$$

Example (7): If $x^2 + y^2 = 5xy$, Find $\frac{dy}{dx}$.

Solution:

$$\begin{aligned}
x^2 + y^2 = 5xy &\implies x^2 + y^2 - 5xy = 0, \text{ Let } F(x, y) = x^2 + y^2 - 5xy \text{ then} \\
F(x, y) &= 0. \\
\frac{dy}{dx} &= -\frac{F_x}{F_y} = -\frac{2x - 5y}{2y - 5x}.
\end{aligned}$$

Let $F(x, y, z) = 0$ where $z = f(x, y)$, then $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$.

Example (8): If $x^3 + y^3 + z^2 + 3xyz - 1 = 0$, Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution:

$$\begin{aligned}
\text{Let } F(x, y, z) &= x^3 + y^3 + z^2 + 3xyz - 1 \text{ then } F(x, y, z) = 0. \\
\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = -\frac{3x^2 + 3yz}{2z + 3xy} . \\
\frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = -\frac{3y^2 + 3xz}{2z + 3xy} .
\end{aligned}$$

1.4.5 EXERCISES

1. Find $\frac{dw}{dt}$ of the following:

(a). $w = xy^3 - x^2y$, where $x = t^2 + 1$ and $y = t^2 - 1$.

(b). $w = \sin x \cos y$, where $x = \sqrt{t}$ and $y = \frac{1}{t}$.

(c). $w = xe^{\frac{y}{z}}$, where $x = t^2$, $y = 1 - t$ and $z = 1 + 2t$.

(d). $w = \ln \sqrt{x^2 + y^2 + z^2}$, where $x = \sin t$, $y = \cos t$ and $z = \tan t$.

2. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ of the following:

(a). $z = (x - y)^5$, where $x = s^2t$ and $y = st^2$.

(b). $z = \tan^{-1}(x^2 + y^2)$, where $x = s \ln t$ and $y = t e^s$.

(c). $z = \frac{\sin \theta}{r}$, where $r = st$ and $\theta = s^2 + t^2$.

3. If $z = x^4 + x^2y$, where $x = s + 2t - u$ and $y = stu^2$,

Find $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial u}$ at $s = 4$, $t = 2$, $u = 1$.

4. If $z = f(x, y)$, where $x = r \cos \theta$ and $y = r \sin \theta$, Show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2.$$

5. If $z = f(x + at) + g(x - at)$, Show that $\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$.

(*) If $xe^{yz} - 2ye^{xz} + 3ze^{xy} = 1$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

1.5 Maximum and Minimum Values

1.5.1 Local Maximum and Minimum Values

Definition: If f is a differentiable function of two variables x and y , then the gradient of f is the vector function ∇f and it is defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}.$$

Definition: If f is a differentiable function of three variables x, y and z , then the gradient of f is the vector function ∇f and it is defined by

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}.$$

Definition (Critical point): If $f(x, y)$ is a function of two variables then $(a, b) \in D_f$ is a critical point if both $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or f is not differentiable at (a, b) .

Definition (Local maximum and Local minimum):

- (i) A function f of two variables has a local maximum at $(a, b) \in D_f$ if $f(x, y) \leq f(a, b)$, for all points (x, y) in some disk with center (a, b) .
- (ii) A function f of two variables has a local minimum at $(a, b) \in D_f$ if $f(x, y) \geq f(a, b)$, for all points (x, y) in some disk with center (a, b) .

Theorem: If $f(x, y)$ has a local maximum or minimum at $(a, b) \in D_f$ and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Second Derivatives Test: Suppose the second partial derivatives of $f(x, y)$ are continuous on a disk with center $(a, b) \in D_f$, and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [so (a, b) is a critical point of f].

Let $D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

- (1). If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- (2). If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- (3). If $D < 0$, then (a, b) is a saddle point.

NOTES:

- (1). If $D = 0$, then the test gives no information.

$$(2). D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}f_{yx} = f_{xx}f_{yy} - [f_{xy}]^2.$$

Example (1): Find the local maximum and minimum values and saddle points of $f(x, y) = 6xy - 2x^3 + y^2$.

Solution:

$$f_x(x, y) = 6y - 6x^2 \text{ and } f_y(x, y) = 6x + 2y.$$

$$f_x(x, y) = 0 \implies 6y - 6x^2 = 0 \implies y = x^2.$$

$$f_y(x, y) = 0 \implies 6x + 2y = 0 \implies y = -3x.$$

$$f_x(x, y) = f_y(x, y) \implies x^2 = -3x \implies x^2 + 3x = 0$$

$$\implies x(x + 3) = 0 \implies x = 0, x = -3 \implies y = 0, y = 9.$$

So, the critical points are $(0, 0)$ and $(-3, 9)$.

$$f_{xx}(x, y) = -12x, f_{xy}(x, y) = 6 \text{ and } f_{yy}(x, y) = 2 .$$

First- At the point $(0, 0)$:

$$D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{x,y}(0, 0)]^2 = (0)(2) - (6)^2 = -36 < 0.$$

Therefore, $(0, 0)$ is a saddle point.

Second- At the point $(-3, 9)$:

$$D(-3, 9) = f_{xx}(-3, 9)f_{yy}(-3, 9) - [f_{x,y}(-3, 9)]^2 = (36)(2) - (6)^2 = 36 > 0.$$

Since $f_{xx}(-3, 9) = 36 > 0$, then f attains a local minimum at $(-3, 9)$,

and its value is $f(-3, 9) = 6(-3)(9) - 2(-3)^3 + (-9)^2 = -162 + 54 + 81 = -27$.

Example (2): Find the local maximum and minimum values and saddle points of $f(x, y) = x^3 - y^3 - 3x + 3y + 5$.

Solution:

$$f_x(x, y) = 3x^2 - 3 \text{ and } f_y(x, y) = -3y^2 + 3 .$$

$$f_x(x, y) = 0 \implies 3x^2 - 3 = 0 \implies x^2 - 1 = 0 \implies x = \pm 1 .$$

$$f_y(x, y) = 0 \implies -3y^2 + 3 = 0 \implies y^2 - 1 = 0 \implies y = \pm 1 .$$

So, the critical points are $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$.

$$f_{xx}(x, y) = 6x, f_{xy}(x, y) = 0 \text{ and } f_{yy}(x, y) = -6y .$$

First- At the point $(1, 1)$:

$$D(1, 1) = f_{xx}(1, 1)f_{yy}(1, 1) - [f_{x,y}(1, 1)]^2 = (6)(-6) - (0)^2 = -36 < 0.$$

Therefore, $(1, 1)$ is a saddle point.

Second- At the point $(1, -1)$:

$$D(1, -1) = f_{xx}(1, -1)f_{yy}(1, -1) - [f_{x,y}(1, -1)]^2 = (6)(6) - (0)^2 = 36 > 0.$$

Since $f_{xx}(1, -1) = 6 > 0$, then f attains a local minimum at $(1, -1)$,

and its value is $f(1, -1) = 1 + 1 - 3 - 3 + 5 = 1$.

Third- At the point $(-1, 1)$:

$$D(-1, 1) = f_{xx}(-1, 1)f_{yy}(-1, 1) - [f_{x,y}(-1, 1)]^2 = (-6)(-6) - (0)^2 = 36 > 0.$$

Since $f_{xx}(-1, 1) = -6 < 0$, then f attains a local maximum at $(-1, 1)$,

and its value is $f(-1, 1) = -1 - 1 + 3 + 3 + 5 = 9$.

Fourth- At the point $(-1, -1)$:

$$D(-1, -1) = f_{xx}(-1, -1)f_{yy}(-1, -1) - [f_{x,y}(-1, -1)]^2 \\ = (-6)(6) - (0)^2 = -36 < 0.$$

Therefore, $(-1, -1)$ is a saddle point.

1.5.2 Absolute Maximum and Minimum Values

Definition: Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the

- (1). absolute maximum value of f on D if $f(a, b) \geq f(x, y)$ for all (x, y) in D .
- (2). absolute minimum value of f on D if $f(a, b) \leq f(x, y)$ for all (x, y) in D .

Theorem: If f is continuous on a closed, bounded set D in \mathbb{R}^2 , then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D .

NOTES: To find the absolute maximum and minimum values of a continuous function $f(x, y)$ on a closed, bounded set $D \subseteq \mathbb{R}^2$:

1. Find the values of f at the critical points of f in the interior of D .
2. Find the extreme values of f on the boundary of D .
3. The largest of the values from steps (1) and (2) is the absolute maximum value of f on D , and the smallest of these values is the absolute minimum value of f on D .

Example (3): Find the absolute maximum and minimum values of the function $f(x, y) = xy + 7$ on the plain region bounded by the graphs of the lines $x = 0$, $y = 0$ and $y + x = 2$.

Solution:

$$f_x(x, y) = y \text{ and } f_y(x, y) = x .$$

$$f_x(x, y) = 0 \implies y = 0 .$$

$$f_y(x, y) = 0 \implies x = 0 .$$

The critical point is $(0, 0)$.

Note that the critical point is not inside the given region.

Let L_1 be the line $x = 0$:

$$\text{then } f(x, y) = f(0, y) = 7 .$$

$$f(x, y) = 7 \text{ for all } (x, y) \in L_1 .$$

Let L_2 be the line $y = 0$:

$$\text{then } f(x, y) = f(x, 0) = 7 .$$

$$f(x, y) = 7 \text{ for all } (x, y) \in L_2 .$$

Let L_3 be the line $y + x = 2$:

then $y = -x + 2$ where $0 \leq x \leq 2$.

$$f(x, y) = f(x, -x + 2) \\ = x(-x + 2) + 7 = -x^2 + 2x + 7 .$$

$$f'(x) = -2x + 2 ,$$

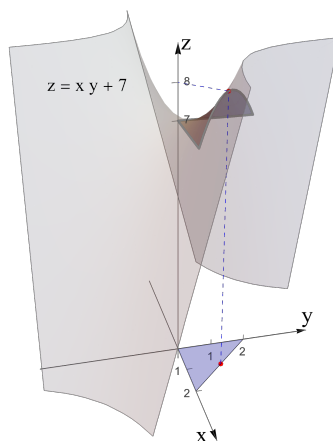
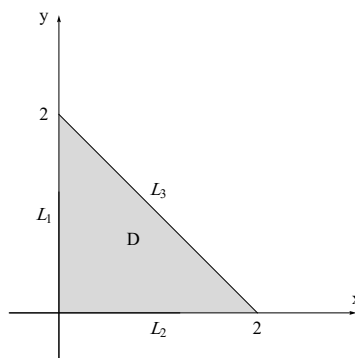
$$f'(x) = 0 \implies -2x + 2 = 0$$

$$\implies x = 1 .$$

So, $y = -2 + 1 = 1$.

$$f(1, 1) = (1)(1) + 7 = 8 .$$

$$\text{Also, } f(0, 2) = 7 \text{ and } f(2, 0) = 7 .$$



The absolute maximum is 8, and f takes it at $(1, 1)$.

The absolute minimum is 7, and f takes it at any point on $L_1 \cup L_2$.

Example (4): Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + 2xy + 3y^2$ on the closed and bounded region

$$D = \{(x, y) \in \mathbb{R}^2 : -2 \leq x \leq 4, -1 \leq y \leq 3\} .$$

Solution:

$$f_x(x, y) = 2x + 2y.$$

$$f_y(x, y) = 2x + 6y.$$

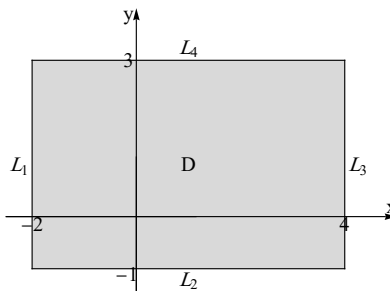
$$f_x(x, y) = 0 \implies x = -y.$$

$$f_y(x, y) = 0 \implies x = -3y.$$

$$-y = -3y \implies y = 0 \implies x = 0.$$

The critical point is $(0, 0)$.

$$f(0, 0) = (0)^2 + 2(0)(0) + 3(0)^3 = 0.$$



Let L_1 be the line between $(-2, -1)$ and $(-2, 3)$.

On L_1 : $x = -2$ and $-1 \leq y \leq 3$.

$$f(x, y) = f(-2, y) = 4 - 4y + 3y^2 \implies f'(y) = -4 + 6y.$$

$$f'(y) = 0 \implies 6y = 4 \implies y = \frac{2}{3}.$$

Note that $\left(-2, \frac{2}{3}\right) \in L_1$.

$$f\left(-2, \frac{2}{3}\right) = (-2)^2 + 2(-2)\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)^2 = 4 - \frac{8}{3} + \frac{4}{3} = \frac{8}{3}.$$

Let L_2 be the line between $(-2, -1)$ and $(4, -1)$.

On L_2 : $y = -1$ and $-2 \leq x \leq 4$.

$$f(x, y) = f(x, -1) = x^2 - 2x + 3 \implies f'(x) = 2x - 2.$$

$$f'(x) = 0 \implies 2x = 2 \implies x = 1.$$

Note that $(1, -1) \in L_2$.

$$f(1, -1) = (1)^2 + 2(1)(-1) + 3(-1)^2 = 1 - 2 + 3 = 2.$$

Let L_3 be the line between $(4, -1)$ and $(4, 3)$.

On L_3 : $x = 4$ and $-1 \leq y \leq 3$.

$$f(x, y) = f(4, y) = 16 + 8y + 3y^2 \implies f'(y) = 8 + 6y.$$

$$f'(y) = 0 \implies 6y = -8 \implies y = -\frac{4}{3}.$$

Note that $\left(4, -\frac{4}{3}\right) \notin L_3$.

Let L_4 be the line between $(-2, 3)$ and $(4, 3)$.

On L_4 : $y = 3$ and $-2 \leq x \leq 4$.

$$f(x, y) = f(x, 3) = x^2 + 6x + 27 \implies f'(x) = 2x + 6.$$

$$f'(x) = 0 \implies 2x = -6 \implies x = -3.$$

Note that $(-3, 3) \notin L_4$.

Evaluating $f(x, y)$ at the four corners of D :

$$f(-2, -1) = (-2)^2 + 2(-2)(-1) + 3(-1)^2 = 4 + 4 + 3 = 11.$$

$$f(-2, 3) = (-2)^2 + 2(-2)(3) + 3(3)^2 = 4 - 12 + 27 = 19.$$

$$f(4, -1) = (4)^2 + 2(4)(-1) + 3(-1)^2 = 16 - 8 + 3 = 11.$$

$$f(4, 3) = (4)^2 + 2(4)(3) + 3(3)^2 = 16 + 24 + 27 = 67.$$

The absolute maximum is 67, and f takes it at $(4, 3)$.

The absolute minimum is 0, and f takes it at $(0, 0)$.

Example (5): Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + y^2 - 2x + 2$ on the closed region with vertices $(0, 0)$, $(2, 1)$ and $(2, -2)$.

Solution:

$$f_x(x, y) = 2x - 2 \text{ and } f_y(x, y) = 2y .$$

$$f_x(x, y) = 0 \implies x = 1 .$$

$$f_y(x, y) = 0 \implies y = 0 .$$

The critical point is $(1, 0)$.

$$f(1, 0) = 1 + 0 - 2 + 2 = 1 .$$

Let L_1 be the line passing through $(0, 0)$ and $(2, 1)$ then $y = \frac{x}{2}$.

$$f\left(x, \frac{x}{2}\right) = x^2 + \frac{x^2}{4} - 2x + 2 .$$

$$f(x) = \frac{5}{4}x^2 - 2x + 2 .$$

$$f'(x) = \frac{5}{2}x - 2 .$$

$$f'(x) = 0 \implies x = \frac{4}{5} \text{ and } y = \frac{2}{5} .$$

$$f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{16}{25} + \frac{4}{25} - \frac{8}{5} + 2$$

$$= \frac{16}{25} + \frac{4}{25} - \frac{40}{25} + \frac{50}{25} = \frac{30}{25} = \frac{6}{5} .$$

Let L_2 be the line passing through $(0, 0)$ and $(2, -2)$ then $y = -x$.

$$f(x, -x) = x^2 + x^2 - 2x + 2$$

$$f(x) = 2x^2 - 2x + 2 .$$

$$f'(x) = 4x - 2 .$$

$$f'(x) = 0 \implies x = \frac{1}{2} \text{ and } y = -\frac{1}{2} .$$

$$f\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} + 1 + 2 = \frac{7}{2} .$$

Let L_3 be the line passing through $(2, -2)$ and $(2, 1)$ then $x = 2$.

$$f(2, y) = 4 + y^2 - 4 + 2 = y^2 + 2, \text{ so } f(y) = y^2 + 2 \implies f'(y) = 2y .$$

$$f'(y) = 0 \implies y = 0 \text{ and } x = 2 .$$

$$f(2, 0) = 4 + 0 - 4 + 2 = 2 .$$

Evaluating $f(x, y)$ at the three corners of D :

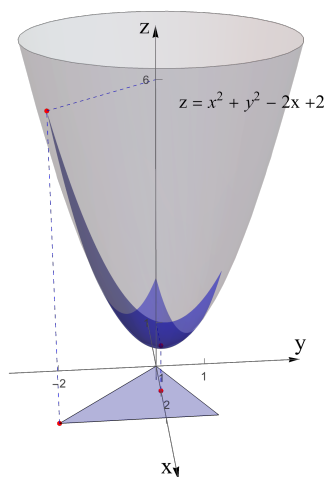
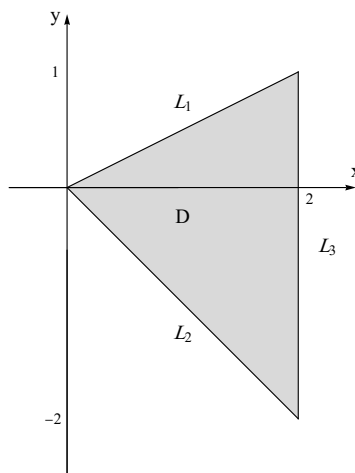
$$f(0, 0) = 0 + 0 - 0 + 2 = 2 .$$

$$f(2, 1) = 4 + 1 - 4 + 2 = 3 .$$

$$f(2, -2) = 4 + 4 - 4 + 2 = 6 .$$

The absolute maximum is 6, and f takes it at $(2, -2)$.

The absolute minimum is 1, and f takes it at $(1, 0)$.



1.5.3 EXERCISES

1. Find the local maximum and minimum values and saddle point(s) of the functions:

(a). $f(x, y) = x^2 + xy + y^2 + y$.

(b). $f(x, y) = x^3 + y^3 + 3xy$.

(c). $f(x, y) = 2 - x^4 + 2x^2 - y^2$.

(d). $f(x, y) = x^4 - 2x^2 + y^3 - 3y$.

2. Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 - 2x$ on the closed region with vertices $(2, 0)$, $(0, 2)$ and $(0, -2)$.

- (*) Find the local maximum and minimum values and saddle point(s) of the functions:

(a). $f(x, y) = 4x^3 - 2x^2y + y^2$.

(b). $f(x, y) = 2x^3 - 3x^2 + 3y^2 - 6xy + 2$.

(c). $f(x, y) = x^4 + y^3 + 32x - 3y$.

1.6 Lagrange Multipliers

1.6.1 Lagrange Multipliers (One Constraint)

To find the extreme values of $f(x, y)$ subject to a constraint $g(x, y) = k$.

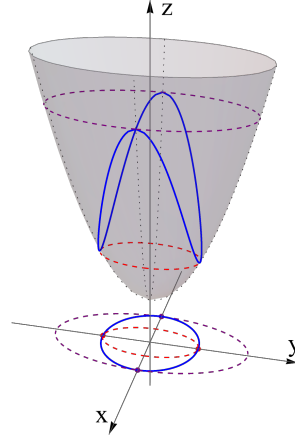
Let $z = f(x, y)$ be the gray surface, and $g(x, y)$ is the blue curve representing the constraint in the xy -plane.

Note that the level curves $f(x, y)$ touches $g(x, y)$ at the points where $f(x, y)$ have minimum and maximum values.

This means that $\nabla f(x, y)$ is parallel to $\nabla g(x, y)$ at these point.

Find these points by solving the equation: $\nabla f(x, y) = \lambda \nabla g(x, y)$, where $\lambda \in \mathbb{R}$ and $\nabla g(x, y) \neq 0$.

Evaluate $f(x, y)$ at these points, The largest is the maximum value of f and the smallest is the minimum value of f .



Example (1): Find the extreme values of the function $f(x, y) = 1 + xy$ on the circle $x^2 + y^2 = 1$.

Solution:

$$f(x, y) = 1 + xy \text{ and } g(x, y) = x^2 + y^2 - 1.$$

$$\nabla f(x, y) = \lambda \nabla g(x, y) \implies \langle y, x \rangle = \lambda \langle 2x, 2y \rangle = \langle 2\lambda x, 2\lambda y \rangle$$

$$\implies \begin{cases} y = 2\lambda x \\ x = 2\lambda y \end{cases} \implies x = 2\lambda(2\lambda x) = 4\lambda^2 x \implies x(1 - 4\lambda^2) = 0$$

$$\implies x = 0, \lambda = \pm \frac{1}{2}.$$

If $x = 0$ then $y = 2\lambda(0) = 0$, but $(0, 0)$ does not lie on the unit circle, so $(0, 0)$ is excluded.

$$\text{If } \lambda = \pm \frac{1}{2} \implies y = 2 \left(\pm \frac{1}{2} \right) x \implies y = \pm x.$$

$$\text{From the constraint : } x^2 + y^2 = 1 \implies 2x^2 = 1 \implies x = \pm \frac{1}{\sqrt{2}}.$$

$$\text{This means } \frac{1}{2} + y^2 = 1 \implies y = \pm \frac{1}{\sqrt{2}}.$$

So, There are 4 points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$.

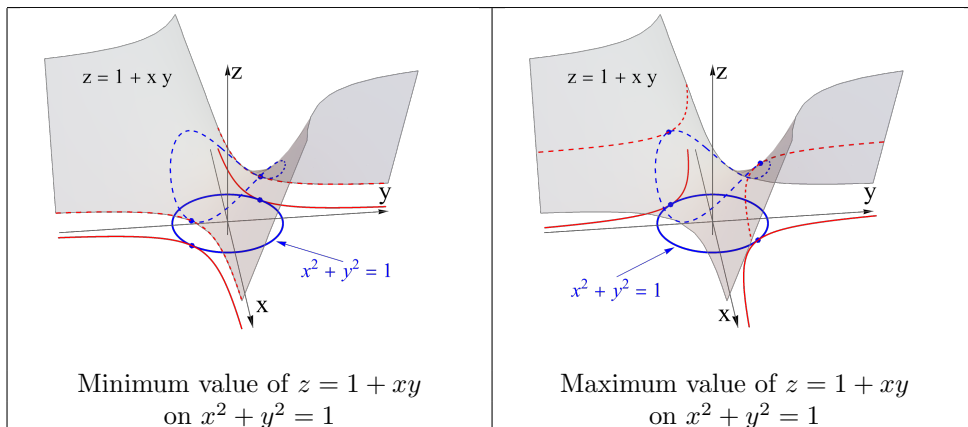
$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = 1 + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = 1 + \frac{1}{2} = \frac{3}{2}.$$

$$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) = 1 + \frac{1}{\sqrt{2}} \frac{-1}{\sqrt{2}} = 1 - \frac{1}{2} = \frac{1}{2}.$$

$$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = 1 + \frac{-1}{\sqrt{2}} \frac{1}{\sqrt{2}} = 1 - \frac{1}{2} = \frac{1}{2}.$$

$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) = 1 + \frac{-1}{\sqrt{2}} \frac{-1}{\sqrt{2}} = 1 + \frac{1}{2} = \frac{3}{2}.$$

The maximum value is $\frac{3}{2}$, and f takes it at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.
 The minimum value is $\frac{1}{2}$, and f takes it at $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.



Note: to find the extreme values of $f(x, y, z)$ subject to a constraint $g(x, y, z) = k$, where $k \in \mathbb{R}$.

- (1). Find the points that satisfy the equation $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$
 $f_x(x, y, z) = \lambda g_x(x, y, z)$, $f_y(x, y, z) = \lambda g_y(x, y, z)$ and $f_z(x, y, z) = \lambda g_z(x, y, z)$.
- (2). Evaluate $f(x, y, z)$ at these points, The largest is the maximum value of f and the smallest is the minimum value of f .

Example (2): Find the points on the sphere $x^2 + y^2 + z^2 = 1$ that are closest to and farthest from the point $(2, 2, 2)$.

Solution:

Let $f(x, y, z)$ be the function of the square of the distance between any point in the sphere and the point $(2, 2, 2)$, then $f(x, y, z) = (x-2)^2 + (y-2)^2 + (z-2)^2$.
 Let $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ be the constraint.

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \implies \langle 2(x-2), 2(y-2), 2(z-2) \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

$$\begin{cases} 2(x-2) = 2\lambda x \\ 2(y-2) = 2\lambda y \\ 2(z-2) = 2\lambda z \end{cases} \implies \begin{cases} x-2 = \lambda x \\ y-2 = \lambda y \\ z-2 = \lambda z \end{cases} \implies \begin{cases} x(1-\lambda) = 2 \\ y(1-\lambda) = 2 \\ z(1-\lambda) = 2 \end{cases}$$

If $\lambda = 1$ then $x-2 = x \implies -2 = 0$, so $\lambda \neq 1$.

Therefore, $x(1-\lambda) = y(1-\lambda) = z(1-\lambda) \implies x = y = z$.

So, $x^2 + y^2 + z^2 = 1 \implies 3x^2 = 1 \implies x = \pm \frac{1}{\sqrt{3}}$

Hence, $x = y = z = \pm \frac{1}{\sqrt{3}}$.

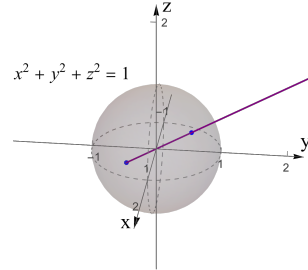
The required points are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$.

$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}} - 2\right)^2 + \left(\frac{1}{\sqrt{3}} - 2\right)^2 + \left(\frac{1}{\sqrt{3}} - 2\right)^2 = 3\left(2 - \frac{1}{\sqrt{3}}\right)^2.$$

$$f\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) = \left(\frac{-1}{\sqrt{3}} - 2\right)^2 + \left(\frac{-1}{\sqrt{3}} - 2\right)^2 + \left(\frac{-1}{\sqrt{3}} - 2\right)^2 = 3\left(2 + \frac{1}{\sqrt{3}}\right)^2.$$

The point $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
is the closest to $(2, 2, 2)$.

The point $\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$
is the farthest to $(2, 2, 2)$.



Example (3): Find the maximum volume of a rectangular box without a lid, where its surface area equals 12 cm^2 .

Solution:

Suppose the sides of the rectangular box are x , y and z .

The volume of the rectangular box is $V(x, y, z) = xyz$, subject to the constraint $2xz + 2yz + xy = 12$.

The constraint is $g(x, y, z) = 2xz + 2yz + xy - 12 = 0$.

$$\nabla V(x, y, z) = \lambda \nabla g(x, y, z) \implies \langle yz, xz, xy \rangle = \lambda \langle 2z + y, 2z + x, 2x + 2z \rangle$$

$$\begin{cases} yz = \lambda(2z + y) \\ xz = \lambda(2z + x) \\ xy = \lambda(2x + 2y) \end{cases} \implies \begin{cases} xyz = \lambda(2xz + xy) & \longrightarrow (1) \\ xyz = \lambda(2yz + xy) & \longrightarrow (2) \\ xyz = \lambda(2xz + 2yz) & \longrightarrow (3) \end{cases}$$

If $\lambda = 0$ then $V(x, y, z) = 0$.

If $x = 0$, $y = 0$ or $z = 0$, then $V(x, y, z) = 0$.

If $\lambda \neq 0$, From equations (1) and (2) :

$$\lambda(2xz + xy) = \lambda(2yz + xy) \implies 2xz + xy = 2yz + xy \implies 2xz = 2yz \implies x = y.$$

From equations (2) and (3), and $x = y$:

$$\begin{aligned} \lambda(2yz + xy) &= \lambda(2xz + 2yz) \implies 2xz + x^2 = 2xz + 2xz \\ \implies x^2 &= 2xz \implies x^2 - 2xz = 0 \implies x(x - 2z) = 0 \implies x = 2z. \end{aligned}$$

From the equation of the constraint and $x = y = 2z$:

$$4z^2 + 4z^2 + 4z^2 = 12 \implies z^2 = 1 \implies z = 1 \text{ and } x = y = 2z = 2.$$

The sides of the rectangular box are 2, 2 and 1, and its maximum volume is 4 cm^3 .

1.6.2 EXERCISES

1. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint:
 - (a). $f(x, y) = x^2 - y^2$, $x^2 + y^2 = 1$.
 - (b). $f(x, y) = xye^{-x^2-y^2}$, $2x - y = 0$.
 - (c). $f(x, y, z) = xy^2z$, $x^2 + y^2 + z^2 = 4$.
 - (d). $f(x, y, z) = x^4 + y^4 + z^4$, $x^2 + y^2 + z^2 = 1$.

2. Find the extreme values of f on the region described by the inequality:
 - (a). $f(x, y) = x^2 + y^2 + 4x - 4y$, $x^2 + y^2 \leq 9$.
 - (b). $f(x, y) = e^{-xy}$, $x^2 + 4y^2 \leq 1$.

3. Show that the problem of finding the minimum value of f subject to the given constraint can be solved using Lagrange multipliers, but f does not have a maximum value with that constraint:
 - (a). $f(x, y) = x^2 + y^2$, $xy = 1$.
 - (b). $f(x, y, z) = x^2 + 2y^2 + 3z^2$, $x + 2y + 3z = 10$.

Chapter 2

Multiple Integrals

2.1 Double Integrals over Rectangles

2.1.1 Iterated Integrals

Suppose that $f(x, y)$ is integrable on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_a^b \left(\int_c^d f(x, y) \, dy \right) dx.$$

Example (1): Evaluate the integral:

(a). $\int_0^2 \int_1^2 x^2 y \, dy \, dx$, (b). $\int_1^2 \int_0^2 x^2 y \, dx \, dy$.

Solution:

(a). $\int_0^2 \int_1^2 x^2 y \, dy \, dx = \int_0^2 \left(\int_1^2 x^2 y \, dy \right) dx = \int_0^2 \left(x^2 \int_1^2 y \, dy \right) dx$
 $= \int_0^2 x^2 \left[\frac{y^2}{2} \right]_1^2 dx = \int_0^2 x^2 \left[\frac{4}{2} - \frac{1}{2} \right] dx$
 $= \frac{3}{2} \int_0^2 x^2 dx = \frac{3}{2} \left[\frac{x^3}{3} \right]_0^2 = \frac{3}{2} \left[\frac{8}{3} - 0 \right] = 4.$

(b). $\int_1^2 \int_0^2 x^2 y \, dx \, dy = \int_1^2 \left(\int_0^2 x^2 y \, dx \right) dy = \int_1^2 \left(y \int_0^2 x^2 dx \right) dy$
 $= \int_1^2 y \left[\frac{x^3}{3} \right]_0^2 dy = \int_1^2 y \left[\frac{8}{3} - 0 \right] dy = \frac{8}{3} \int_1^2 y \, dy$
 $= \frac{8}{3} \left[\frac{y^2}{2} \right]_1^2 = \frac{8}{3} \left[\frac{4}{2} - \frac{1}{2} \right] = \frac{8}{3} \cdot \frac{3}{2} = 4.$

Fubini's Theorem: If f is continuous on the rectangle $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$,

Then $\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$.

Example (2): Evaluate the integral $\iint_R y \cos(xy) \, dA$,

where $R = [0, 1] \times \left[0, \frac{\pi}{2}\right]$.

Solution: Using Fubini's Theorem.

$$\begin{aligned} \iint_R y \cos(xy) \, dA &= \int_0^{\frac{\pi}{2}} \int_0^1 y \cos(xy) \, dx dy = \int_0^{\frac{\pi}{2}} \left(\int_0^1 y \cos(xy) \, dx \right) dy \\ &= \int_0^{\frac{\pi}{2}} [\sin(xy)]_0^1 dy = \int_0^{\frac{\pi}{2}} [\sin y - \sin(0)] dy = \int_0^{\frac{\pi}{2}} \sin y \, dy \\ &= [-\cos y]_0^{\frac{\pi}{2}} = -\cos\left(\frac{\pi}{2}\right) - (-\cos(0)) = 0 + 1 = 1 . \end{aligned}$$

Note : Solving $\iint_R y \cos(x, y) \, dA = \int_0^1 \int_0^{\frac{\pi}{2}} y \cos(xy) \, dy dx$ is hard, it needs integration by parts.

2.1.2 Volume

If $f(x, y) \geq 0$, then the volume V of the solid that lies above the rectangle $R = [a, b] \times [c, d]$ and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy dx = \int_c^d \int_a^b f(x, y) \, dx dy .$$

Example (3): Find the volume of the solid S that is bounded by $z = x^2 + y^2 + 1$, the planes $x = 1$ and $y = 3$, and the three coordinate planes.

Solution:

Note that S is the solid that lies under the surface $z = x^2 + y^2 + 1$ and above the square $R = [0, 1] \times [0, 3]$.

$$\begin{aligned} V &= \iint_R (x^2 + y^2 + 1) \, dA = \int_0^1 \int_0^3 (x^2 + y^2 + 1) \, dy dx \\ &= \int_0^1 \left[x^2 y + \frac{y^3}{3} + y \right]_0^3 dx = \int_0^1 [(3x^2 + 9 + 3) - (0 + 0 + 0)]_0^3 dx \\ &= \int_0^1 (3x^2 + 12) \, dx = [x^3 + 12x]_0^1 = (1 + 12) - (0 + 0) = 13 . \end{aligned}$$

Corollary: If $f(x, y) = g(x)h(y)$ is continuous on the rectangle $R = [a, b] \times [c, d]$,

$$\text{then } \iint_R f(x, y) \, dA = \iint_R g(x)h(y) \, dA = \int_a^b g(x) \, dx \int_c^d h(y) \, dy .$$

Example (4): Evaluate $\iint_R x \cos y \, dA$, where $R = [0, 2] \times \left[0, \frac{\pi}{2}\right]$.

Solution:

$$\begin{aligned} \iint_R x \cos y \, dA &= \int_0^2 \int_0^{\frac{\pi}{2}} x \cos y \, dy dx = \left(\int_0^2 x \, dx \right) \left(\int_0^{\frac{\pi}{2}} \cos y \, dy \right) \\ &= \left[\frac{x^2}{2} \right]_0^2 [\sin y]_0^{\frac{\pi}{2}} = [2 - 0][1 - 0] = 2 . \end{aligned}$$

2.1.3 Average Value

If $f(x, y)$ is defined on a rectangle R then its average value is $f_{avg} = \frac{1}{A(R)} \iint_R f(x, y) \, dA$, where $A(R)$ is the area of the rectangle R .

Example (5): Evaluate f_{avg} of $f(x, y) = x \cos y$ on $R = [0, 2] \times [0, \frac{\pi}{2}]$.

Solution:

$$f_{avg} = \frac{1}{A(R)} \iint_R x \cos y \, dA = \frac{2}{(2-0)(\frac{\pi}{2}-0)} = \frac{2}{\pi}.$$

2.1.4 EXERCISES

1. Calculate the iterated integrals :

$$(a). \int_1^4 \int_0^2 (6x^2y - 2x) dy dx .$$

$$(b). \int_{-3}^1 \int_1^2 (x^2 + y^{-2}) dy dx .$$

$$(c). \int_{-3}^3 \int_0^{\frac{\pi}{2}} (y + y^2 \cos x) dx dy .$$

$$(d). \int_1^3 \int_1^5 \frac{\ln y}{xy} dy dx .$$

$$(e). \int_0^1 \int_0^1 xy\sqrt{x^2 + y^2} dy dx .$$

$$(f). \int_0^1 \int_0^1 v(u + v^2)^4 du dv .$$

2. Calculate the double integrals :

$$(a). \iint_R x \sec^2 y dA, \text{ where } R = \left\{ (x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq \frac{\pi}{4} \right\} .$$

$$(b). \iint_R \frac{xy^2}{x^2 + 1} dA, \text{ where } R = \{ (x, y) \mid 0 \leq x \leq 1, -3 \leq y \leq 3 \} .$$

$$(c). \iint_R \frac{1}{1 + x + y} dA, \text{ where } R = [1, 3] \times [1, 2] .$$

3. Find the volume of the solid that lies under the plane $4x + 6y - 2z + 15 = 0$ and above the rectangle $R = \{ (x, y) \mid -1 \leq x \leq 2, -1 \leq y \leq 1 \}$.

4. Find the volume of the solid that lies under $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

2.2 Double Integrals over General Regions

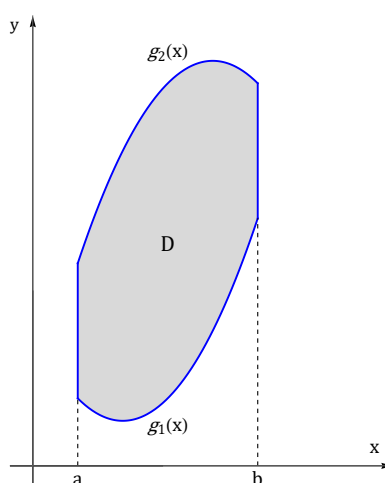
2.2.1 General Regions

First - Regions of Type I:

Let D be the region
 $\{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$

If f is continuous on D , then

$$\begin{aligned} & \iint_D f(x, y) \, dA \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx \\ &= \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \right) \, dx \end{aligned}$$

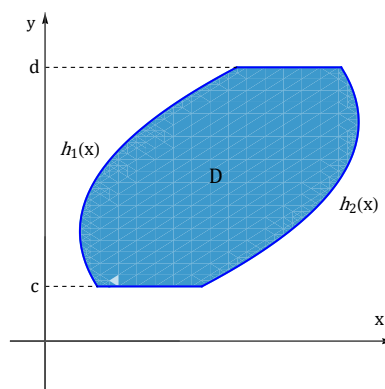


Second - Regions of Type II:

Let D be the region
 $\{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$

If f is continuous on D , then

$$\begin{aligned} & \iint_D f(x, y) \, dA \\ &= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy \\ &= \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \right) \, dy \end{aligned}$$



Example (1): Evaluate $\iint_D 2xy \, dA$, where D is the region bounded by the graphs of $y = 2x^2$ and $y = x^2 + 1$.
 Solution:

Points of intersection:

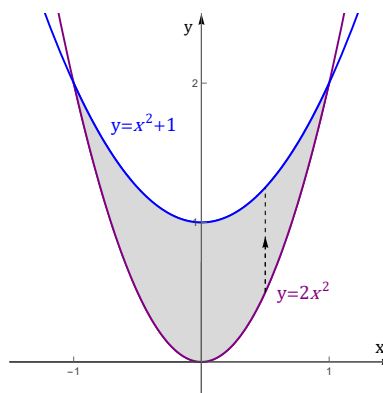
$$2x^2 = x^2 + 1 \implies x^2 = 1$$

$$\implies x = \pm 1$$

So, D is the region where $-1 \leq x \leq 1$

and $2x^2 \leq y \leq x^2 + 1$

$$\begin{aligned} \iint_D 2xy \, dA &= \int_{-1}^1 \int_{2x^2}^{x^2+1} 2xy \, dy \, dx \\ &= \int_{-1}^1 x [y^2]_{2x^2}^{x^2+1} \, dx \\ &= \int_{-1}^1 x [(x^2+1)^2 - (2x^2)^2] \, dx \\ &= \int_{-1}^1 x (x^4 + 2x^2 + 1 - 4x^4) \, dx \\ &= \int_{-1}^1 x (-3x^4 + 2x^2 + 1) \, dx = \int_{-1}^1 (-3x^5 + 2x^3 + x) \, dx \\ &= \left[-\frac{x^6}{2} + \frac{x^4}{2} + \frac{x^2}{2} \right]_{-1}^1 \\ &= \left(-\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) - \left(-\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) = 0. \end{aligned}$$



Example (2): Evaluate $\iint_D (x^2 + y^2) \, dA$, where D is the region bounded by the graphs of $y = x^2$ and $y = x$.

Solution:

Points of intersection:

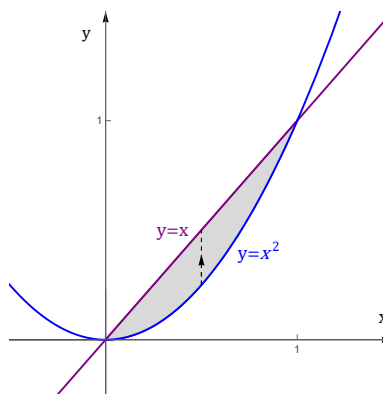
$$x^2 = x \implies x^2 - x = 0$$

$$\implies x(x-1) = 0 \implies x = 0, x = 1$$

So, D is the region where $0 \leq x \leq 1$

and $x^2 \leq y \leq x$

$$\begin{aligned} \iint_D (x^2 + y^2) \, dA &= \int_0^1 \int_{x^2}^x (x^2 + y^2) \, dy \, dx \\ &= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_{x^2}^x \, dx \\ &= \int_0^1 \left[\left(x^3 + \frac{x^3}{3} \right) - \left(x^4 - \frac{x^6}{3} \right) \right] \, dx \\ &= \int_0^1 \left(-\frac{x^6}{3} - x^4 + \frac{4x^3}{3} \right) \, dx = \left[-\frac{x^7}{21} - \frac{x^5}{5} + \frac{x^4}{3} \right]_0^1 \\ &= \left(-\frac{1}{21} - \frac{1}{5} + \frac{1}{3} \right) - (0 + 0 + 0) = \frac{-5 - 21 + 35}{105} = \frac{9}{105} = \frac{3}{35}. \end{aligned}$$



Example (3): Evaluate $\iint_D xy \, dA$, where D is the region bounded by the graphs of $x = y^2$ and $x = y + 2$.

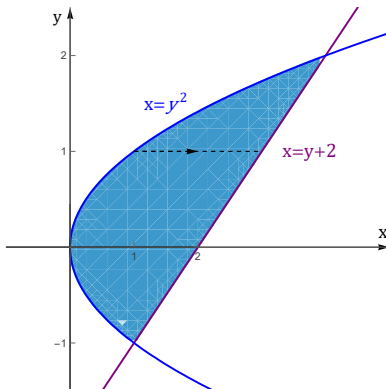
Solution:

Points of intersection:

$$\begin{aligned} y^2 &= y + 2 \implies y^2 - y - 2 = 0 \\ \implies (y - 2)(y + 1) &= 0 \implies y = -1, y = 2 \end{aligned}$$

So, D is the region where $-1 \leq y \leq 2$ and $y^2 \leq x \leq y + 2$

$$\begin{aligned} \iint_D xy \, dA &= \int_{-1}^2 \int_{y^2}^{y+2} xy \, dx \, dy \\ &= \int_{-1}^2 y \left[\frac{x^2}{2} \right]_{y^2}^{y+2} dy \\ &= \frac{1}{2} \int_{-1}^2 y [(y+2)^2 - (y^2)^2] dy \\ &= \frac{1}{2} \int_{-1}^2 y (y^2 + 4y + 4 - y^4) dy \\ &= \frac{1}{2} \int_{-1}^2 (-y^5 + y^3 + 4y^2 + 4y) dy = \frac{1}{2} \left[-\frac{y^6}{6} + \frac{y^4}{4} + \frac{4y^3}{3} + 2y^2 \right]_{-1}^2 \\ &= \frac{1}{2} \left[\left(-\frac{64}{6} + \frac{16}{4} + \frac{32}{3} + 8 \right) - \left(-\frac{1}{6} + \frac{1}{4} - \frac{4}{3} + 2 \right) \right] = \frac{45}{8}. \end{aligned}$$



2.2.2 Changing the Order of Integration

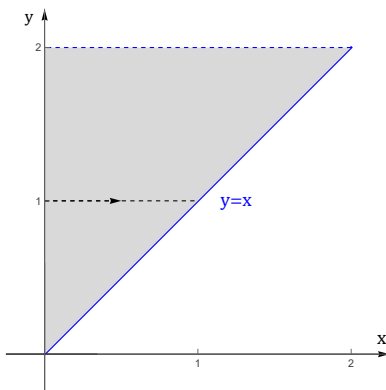
Example (4): Change the order of integration to evaluate $\int_0^2 \int_x^2 e^{5+y^2} dy dx$.

Solution:

D is the region where $0 \leq x \leq 2$ and $x \leq y \leq 2$.

Or, $0 \leq y \leq 2$ and $0 \leq x \leq y$.

$$\begin{aligned} &\int_0^2 \int_x^2 e^{5+y^2} dy dx \\ &= \int_0^2 \int_0^y e^{5+y^2} dx dy \\ &= \int_0^2 e^{5+y^2} [x]_0^y dy = \int_0^2 ye^{5+y^2} dy \\ &= \frac{1}{2} \int_0^2 e^{5+y^2} (2y) dy \\ &= \frac{1}{2} [e^{5+y^2}]_0^2 = \frac{1}{2} (e^9 - e^5). \end{aligned}$$



Example (5): Change the order of integration to evaluate $\int_0^2 \int_{y^2}^4 \cos\left(x^{\frac{3}{2}}\right) dx dy$.

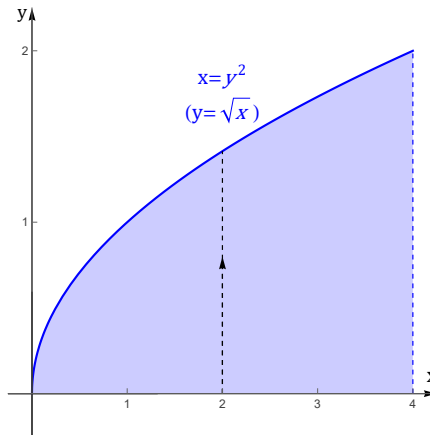
Solution:

D is the region where $0 \leq y \leq 2$ and $y^2 \leq x \leq 4$.

Or, $0 \leq x \leq 4$ and $0 \leq y \leq \sqrt{x}$.

$$\begin{aligned} & \int_0^2 \int_{y^2}^4 \cos\left(x^{\frac{3}{2}}\right) dx dy \\ &= \int_0^4 \int_0^{\sqrt{x}} \cos\left(x^{\frac{3}{2}}\right) dy dx \\ &= \int_0^4 \cos\left(x^{\frac{3}{2}}\right) [y]_0^{\sqrt{x}} dx \\ &= \frac{2}{3} \int_0^4 \cos\left(x^{\frac{3}{2}}\right) \left(\frac{3}{2}x^{\frac{1}{2}}\right) dx \end{aligned}$$

$$= \frac{2}{3} \left[\sin\left(x^{\frac{3}{2}}\right) \right]_0^4 = \frac{2}{3} [\sin(8) - \sin(0)] = \frac{2}{3} \sin(8).$$



Example (6): Change the order of integration to evaluate $\int_0^1 \int_{\sqrt{y}}^1 \sin(x^3) dx dy$.

Solution:

D is the region where $0 \leq y \leq 1$ and

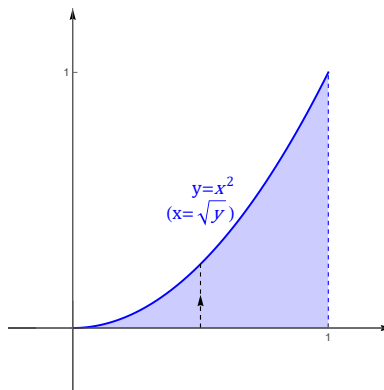
$\sqrt{y} \leq x \leq 1$.

Or, $0 \leq x \leq 1$ and $0 \leq y \leq x^2$.

$$\begin{aligned} & \int_0^1 \int_{\sqrt{y}}^1 \sin(x^3) dx dy \\ &= \int_0^1 \int_0^{x^2} \sin(x^3) dy dx \\ &= \int_0^1 \sin(x^3) [y]_0^{x^2} dx \\ &= \int_0^1 \sin(x^3) x^2 dx \end{aligned}$$

$$= \frac{1}{3} \int_0^1 \sin(x^3) (3x^2) dx = \frac{1}{3} [-\cos(x^3)]_0^1.$$

$$= \frac{1}{3} [-\cos(1) - (-\cos(0))] = \frac{1 - \cos(1)}{3}.$$



2.2.3 Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are both integrable on $D \subseteq \mathbb{R}^2$, then

- (1). $\iint_D [f(x, y) + g(x, y)] dA = \iint_D f(x, y) dA + \iint_D g(x, y) dA$.
- (2). $\iint_D k f(x, y) dA = k \iint_D f(x, y) dA$, where $k \in \mathbb{R}$.
- (3). If $f(x, y) \geq g(x, y)$ on D , then $\iint_D f(x, y) dA \geq \iint_D g(x, y) dA$.

(4). If $D = D_1 \cup D_2$, where $D_1 \cap D_2 = \phi$, then

$$\iint_D f(x, y) \, dA = \iint_{D_1} f(x, y) \, dA + \iint_{D_2} f(x, y) \, dA.$$

(5). $\iint_D 1 \, dA = A(D)$, where $A(D)$ is the area of the region D .

(6). If $m \leq f(x, y) \leq M$ on D , then $m A(D) \leq \iint_D f(x, y) \, dA \leq M A(D)$.

2.2.4 EXERCISES

1. Evaluate the iterated integrals :

$$(a). \int_1^5 \int_0^x (8x - 2y) dy dx \quad (b). \int_0^2 \int_0^{y^2} x^2 y dx dy$$

$$(c). \int_0^1 \int_0^{e^x} \sqrt{1 + e^x} dy dx$$

2. Evaluate $\iint_D 2y dA$, where D is the region bounded by the graphs of $y = 3x - x^2$ and $y = x$.

3. Evaluate the double integrals:

$$(a). \iint_D \frac{y}{x^2 + 1} dA, D = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq \sqrt{x}\}.$$

$$(b). \iint_D (2x + y) dA, D = \{(x, y) \mid 1 \leq y \leq 2, y - 1 \leq x \leq 1\}.$$

$$(c). \iint_D x dA, D \text{ is enclosed by the lines } y = x, y = 0 \text{ and } x = 1.$$

$$(d). \iint_D xy dA, D \text{ is enclosed by the curves } y = x^2 \text{ and } y = 3x.$$

$$(e). \iint_D x \cos y dA, D \text{ is bounded by the } y = 0, y = x^2 \text{ and } x = 1.$$

$$(f). \iint_D y^2 dA, D \text{ is the triangular region with vertices } (0, 1), (1, 2) \text{ and } (4, 1).$$

4. Evaluate the integral by reversing the order of integration:

$$(a). \int_0^1 \int_{3y}^3 e^{x^2} dx dy \quad (b). \int_0^1 \int_{x^2}^1 \sqrt{y} \sin y dy dx$$

$$(c). \int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1} dy dx \quad (d). \int_0^2 \int_{\frac{y}{2}}^1 y \cos(x^3 - 1) dx dy$$

2.3 Double Integrals in Polar Coordinates

2.3.1 Double Integrals in Polar Coordinates

If f is continuous on the polar region R given by $a \leq r \leq b$ and $\theta_1 \leq \theta \leq \theta_2$, where $0 \leq \theta_2 - \theta_1 \leq 2\pi$, then $\iint_R f(x, y) dA = \int_{\theta_1}^{\theta_2} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$.

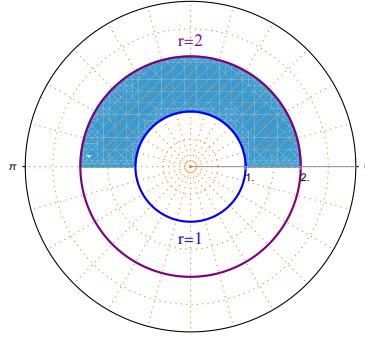
Example (1): Evaluate $\iint_R (4x^2 + 3y) dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution:

R is the region where $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi$.

$$\begin{aligned} & \iint_R (4x^2 + 3y) dA \\ &= \int_0^\pi \int_1^2 (4r^2 \cos^2 \theta + 3r \sin \theta) r dr d\theta \\ &= \int_0^\pi \int_1^2 (4r^3 \cos^2 \theta + 3r^2 \sin \theta) dr d\theta \\ &= \int_0^\pi [r^4 \cos^2 \theta + r^3 \sin \theta]_1^2 d\theta \end{aligned}$$

$$\begin{aligned} &= \int_0^\pi [(16 \cos^2 \theta + 8 \sin \theta) - (\cos^2 \theta + \sin \theta)] d\theta = \int_0^\pi (15 \cos^2 \theta + 7 \sin \theta) d\theta \\ &= \int_0^\pi \left[15 \left(\frac{1 + \cos 2\theta}{2} \right) + 7 \sin \theta \right] d\theta = \frac{15}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^\pi + 7 [-\cos \theta]_0^\pi \\ &= \frac{15}{2} [(\pi + 0) - (0 + 0)] + 7[-(-1) - (-1)] = \frac{15\pi}{2} + 14. \end{aligned}$$



Example (2): Evaluate $\int_{-1}^1 \int_0^{\sqrt{1-y^2}} \frac{8(x^2 + y^2)}{9 + (x^2 + y^2)^2} dx dy$

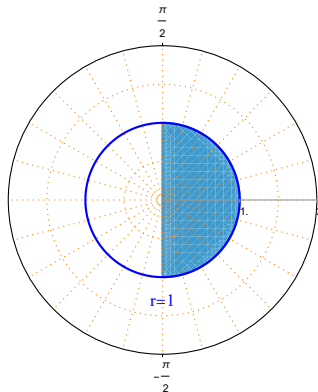
Solution:

R is the region where $-1 \leq y \leq 1$ and $0 \leq x \leq \sqrt{1 - y^2}$

In polar coordinates. $0 \leq r \leq 1$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

$$\begin{aligned} & \int_{-1}^1 \int_0^{\sqrt{1-y^2}} \frac{8(x^2 + y^2)}{9 + (x^2 + y^2)^2} dx dy \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \frac{8r^2}{9 + r^4} r dr d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 2 \left(\frac{4r^3}{9 + r^4} \right) dr d\theta \end{aligned}$$

$$= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\ln(9 + r^4)]_0^1 d\theta = 2(\ln 10 - \ln 9) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta$$



$$= 2 \ln \left(\frac{10}{9} \right) [\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2 \ln \left(\frac{10}{9} \right) \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = 2\pi \ln \left(\frac{10}{9} \right).$$

Example (3): Find the volume of the solid bounded by $z = 4 - x^2 - y^2$ and $z = 0$

Solution:

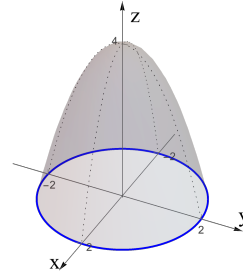
The surface of intersection is :

$$4 - x^2 - y^2 = 0 \implies x^2 + y^2 = 4.$$

R is the region inside the circle centered at the origin with radius 2.

In polar coordinates, $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$.

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^2 (4 - r^2) r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta \\ &= \int_0^{2\pi} \left[2r^2 - \frac{r^4}{4} \right]_0^2 \, d\theta = \int_0^{2\pi} [(8 - 4) - (0 - 0)] \, d\theta \\ &= 4 \int_0^{2\pi} d\theta = 4 [\theta]_0^{2\pi} = 4(2\pi - 0) = 8\pi. \end{aligned}$$



NOTE: If f is continuous on the polar region $R = \{(r, \theta) | \theta_1 \leq \theta \leq \theta_2, g_1(\theta) \leq r \leq g_2(\theta)\}$,

$$\text{then } \iint_R f(x, y) \, dA = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta.$$

Example (4): Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2 + y^2)^{\frac{1}{2}} \, dy \, dx$.

Solution:

R is the region where $0 \leq x \leq 2$ and

$$0 \leq y \leq \sqrt{2x - x^2}.$$

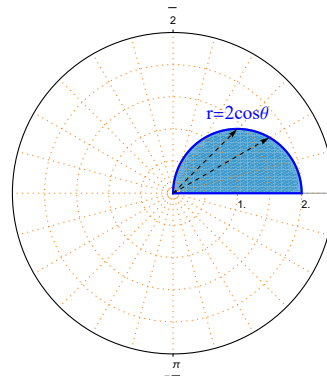
$$y = \sqrt{2x - x^2} \implies y^2 = 2x - x^2$$

$$\implies x^2 - 2x + y^2 = 0$$

$$\implies (x^2 - 2x + 1) + y^2 = 1$$

$$\implies (x - 1)^2 + y^2 = 1.$$

$y = \sqrt{2x - x^2}$ is the upper-half of the circle centered at $(1, 0)$ with radius 1.



$$\begin{aligned} \text{Note that : } y &= \sqrt{2x - x^2} \implies y^2 = 2x - x^2 \implies x^2 + y^2 = 2x \\ \implies r^2 &= 2r \cos \theta \implies r = 2 \cos \theta. \end{aligned}$$

In polar coordinates, $0 \leq r \leq 2 \cos \theta$ and $0 \leq \theta \leq \frac{\pi}{2}$.

$$\begin{aligned} \int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2 + y^2)^{\frac{1}{2}} dy dx &= \int_0^{\frac{\pi}{2}} \int_0^{2 \cos \theta} r r dr d\theta = \int_0^{\frac{\pi}{2}} \int_0^{2 \cos \theta} r^2 dr d\theta \\ &= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3} \right]_0^{2 \cos \theta} d\theta = \frac{8}{3} \int_0^{\frac{\pi}{2}} \cos^3 \theta d\theta = \frac{8}{3} \int_0^{\frac{\pi}{2}} \cos^2 \theta \cos \theta d\theta \\ &= \frac{8}{3} \int_0^{\frac{\pi}{2}} (1 - \sin^2 \theta) \cos \theta d\theta = \frac{8}{3} \int_0^{\frac{\pi}{2}} (\cos \theta - \sin^2 \theta \cos \theta) d\theta \\ &= \frac{8}{3} \left[\sin \theta - \frac{\sin^3 \theta}{3} \right]_0^{\frac{\pi}{2}} = \frac{8}{3} \left[\left(1 - \frac{1}{3} \right) - (0 - 0) \right] = \frac{16}{9}. \end{aligned}$$

2.3.2 EXERCISES

1. Evaluate the given integral by changing to polar coordinates.

(a). $\iint_R (2x - y) \, dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.

(b). $\iint_R e^{-x^2-y^2} \, dA$, where R is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y -axis.

(c). $\iint_R \cos \sqrt{x^2 + y^2} \, dA$, where R is the disk with center the origin and radius 2.

2. Use polar coordinates to find the volume of the given solid.

(a). Under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leq 25$.

(b). Below the cone $z = \sqrt{x^2 + y^2}$ and above the ring $1 \leq x^2 + y^2 \leq 4$.

(c). Below the plane $2x + y + z = 4$ and above the disk $x^2 + y^2 \leq 1$.

(d). Inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.

(e). Above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

(f). Bounded by the paraboloids $z = 6 - x^2 - y^2$ and $z = 2x^2 + 2y^2$.

3. Evaluate the iterated integral by converting to polar coordinates.

(a). $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2} \, dy \, dx$.

(b). $\int_0^{\frac{1}{2}} \int_{\sqrt{3y}}^{\sqrt{1-y^2}} xy^2 \, dx \, dy$.

2.4 Triple Integrals

2.4.1 Triple Integrals over Rectangular Boxes

Fubini's Theorem:

If f is continuous on the rectangular box $E = [a, b] \times [c, d] \times [r, s]$, then

$$\begin{aligned} \iiint_E f(x, y, z) \, dV &= \int_r^s \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz = \int_r^s \int_a^b \int_c^d f(x, y, z) \, dy \, dx \, dz \\ &= \int_c^d \int_r^s \int_a^b f(x, y, z) \, dx \, dz \, dy = \int_c^d \int_a^b \int_r^s f(x, y, z) \, dz \, dx \, dy \\ &= \int_a^b \int_r^s \int_c^d f(x, y, z) \, dy \, dz \, dx = \int_a^b \int_c^d \int_r^s f(x, y, z) \, dz \, dy \, dx . \end{aligned}$$

Example (1): Evaluate $\iiint_E xyz^2 \, dV$, where $E = [0, 1] \times [0, 2] \times [0, 3]$.

Solution:

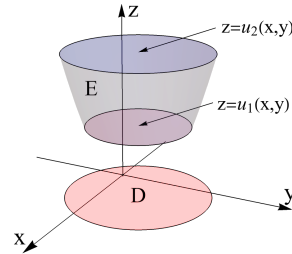
$$\begin{aligned} \iiint_E xyz^2 \, dV &= \int_0^1 \int_0^2 \int_0^3 xyz^2 \, dz \, dy \, dx = \int_0^1 \int_0^2 xy \left[\frac{z^3}{3} \right]_0^3 \, dy \, dx \\ &= \left[\frac{27}{3} - \frac{0}{3} \right] \int_0^1 \int_0^2 xy \, dy \, dx = 9 \int_0^1 x \left[\frac{y^2}{2} \right]_0^2 \, dx = 9 \int_0^1 x \left[\frac{2^2}{2} - \frac{0}{2} \right] \, dx \\ &= 9 \int_0^1 2x \, dx = 9 [x^2]_0^1 = 9[1 - 0] = 9 . \end{aligned}$$

2.4.2 Triple Integrals over General Regions

First - Regions of Type I:

Case (1) :

Let E be the region where $(x, y) \in D$ and $u_1(x, y) \leq z \leq u_2(x, y)$.

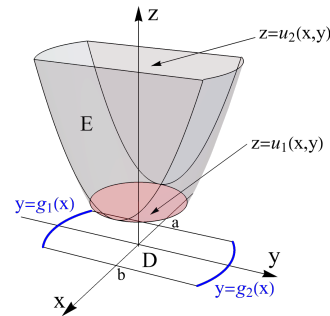


If f is continuous on E , then

$$\begin{aligned} \iiint_E f(x, y, z) \, dV \\ &= \iint_D \left(\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \right) \, dA. \end{aligned}$$

Case (2) :

Let E be the region where $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$ and $u_1(x, y) \leq z \leq u_2(x, y)$.



If f is continuous on E , then

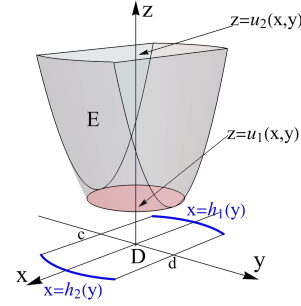
$$\begin{aligned} \iiint_E f(x, y, z) \, dV \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \, dy \, dx. \end{aligned}$$

Case (3) :

Let E be the region where $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$ and $u_1(x, y) \leq z \leq u_2(x, y)$.

If f is continuous on E , then

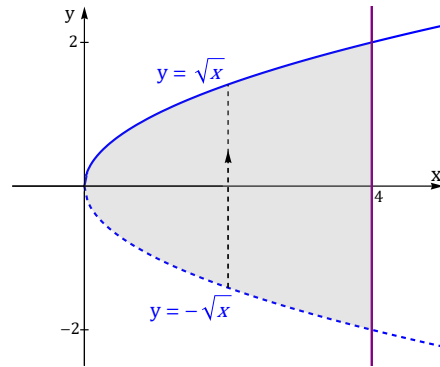
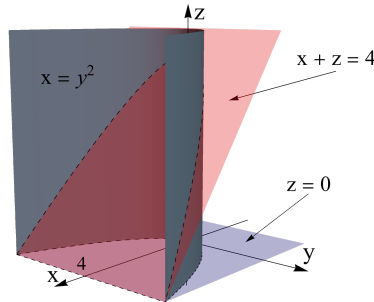
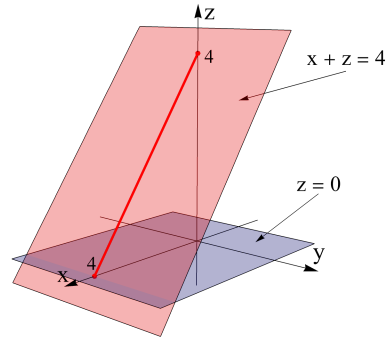
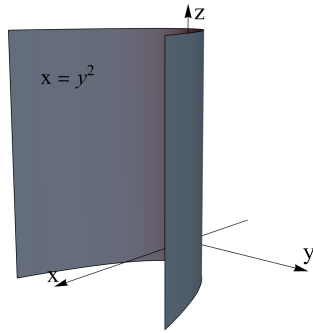
$$\begin{aligned} & \iiint_E f(x, y, z) \, dV \\ &= \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \, dx \, dy. \end{aligned}$$



NOTE: $\iiint_E 1 \, dV = V(E)$, where $V(E)$ is the volume of the region E .

Example (2): Find the volume of the solid bounded by the cylinder $x = y^2$ and the plains $z = 0$ and $x + z = 4$.

Solution:



Let E be the region bounded by the cylinder $x = y^2$ and the plains $z = 0$ and $x + z = 4$.

Note that $z = 0$ intersects $x + z = 4$ at the line where $x = 4$ and $z = 4$.

On E : $0 \leq x \leq 4$, $-\sqrt{x} \leq y \leq \sqrt{x}$, $0 \leq z \leq 4 - x$.

$$V(E) = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{4-x} 1 \, dz \, dy \, dx = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} [z]_0^{4-x} \, dy \, dx = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} (4 - x) \, dy \, dx$$

$$\begin{aligned}
&= \int_0^4 (4-x) [y]_{-\sqrt{x}}^{\sqrt{x}} dx = \int_0^4 2\sqrt{x} (4-x) dx = 2 \int_0^4 (4x^{\frac{1}{2}} - x^{\frac{3}{2}}) dx \\
&= 2 \left[\frac{8}{3} x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}} \right]_0^4 = 2 \left[\frac{8}{3} (4)^{\frac{3}{2}} - \frac{2}{5} (4)^{\frac{5}{2}} \right] = 2 \left[\frac{8}{3} (2^3) - \frac{2}{5} (2^5) \right] \\
&= 2 \left[\frac{2^6}{3} - \frac{2^6}{5} \right] = 2^7 \left(\frac{1}{3} - \frac{1}{5} \right) = 2^7 \left(\frac{2}{15} \right) = \frac{2^8}{15} = \frac{256}{15}.
\end{aligned}$$

Another Solution :

On E : $-2 \leq y \leq 2$, $y^2 \leq x \leq 4$, $0 \leq z \leq 4-x$.

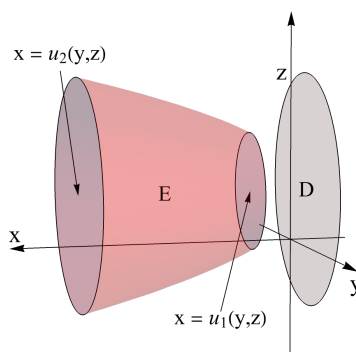
$$V(E) = \int_{-2}^2 \int_{y^2}^4 \int_0^{4-x} 1 dz dx dy .$$

Second - Regions of Type II:

Let E be the region where
 $(y, z) \in D$ and
 $u_1(y, z) \leq x \leq u_2(y, z)$.

If f is continuous on E , then

$$\begin{aligned}
&\iiint_E f(x, y, z) dV \\
&= \iint_D \left(\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z) dx \right) dA.
\end{aligned}$$

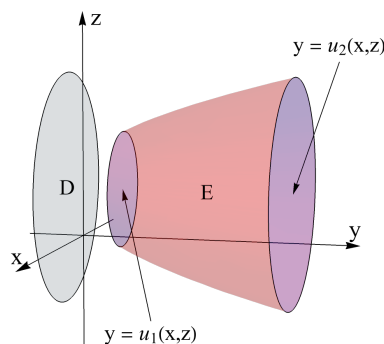


Third - Regions of Type III:

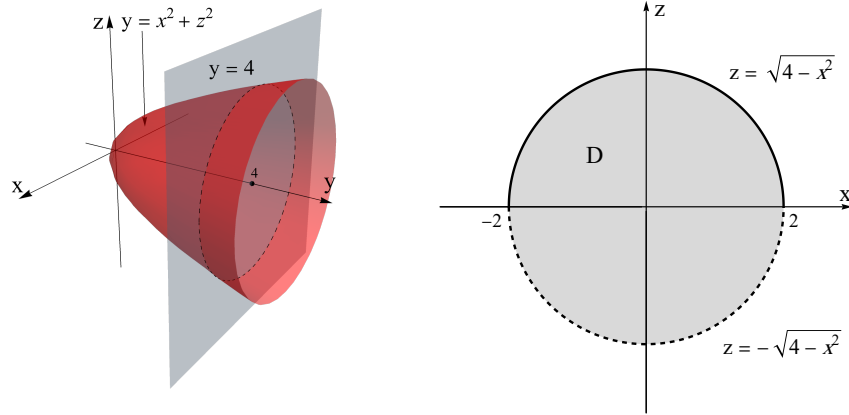
Let E be the region where
 $(x, z) \in D$ and
 $u_1(x, z) \leq y \leq u_2(x, z)$.

If f is continuous on E , then

$$\begin{aligned}
&\iiint_E f(x, y, z) dV \\
&= \iint_D \left(\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z) dy \right) dA.
\end{aligned}$$



Example (3): Evaluate $\iiint_E \sqrt{x^2 + z^2} dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plain $y = 4$.
 Solution:



Note that $y = x^2 + z^2$ intersects $y = 4$ at $x^2 + z^2 = 4$.

So, E is the region where $x^2 + z^2 \leq y \leq 4$, $-\sqrt{4-x^2} \leq z \leq \sqrt{4-x^2}$ and $-2 \leq x \leq 2$.

$$\begin{aligned} \iiint_E \sqrt{x^2 + z^2} \, dV &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+z^2}^4 \sqrt{x^2 + z^2} \, dy \, dz \, dx \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [y]_{x^2+z^2}^4 \sqrt{x^2 + z^2} \, dz \, dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4 - (x^2 + z^2)] \sqrt{x^2 + z^2} \, dz \, dx \end{aligned}$$

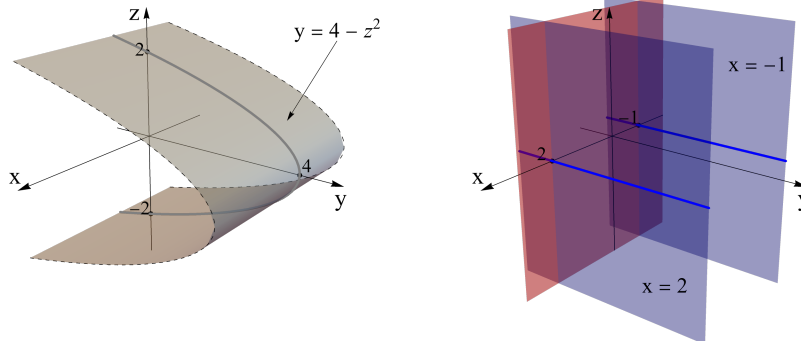
Using the polar coordinates $x = r \cos \theta$ and $z = r \sin \theta$,

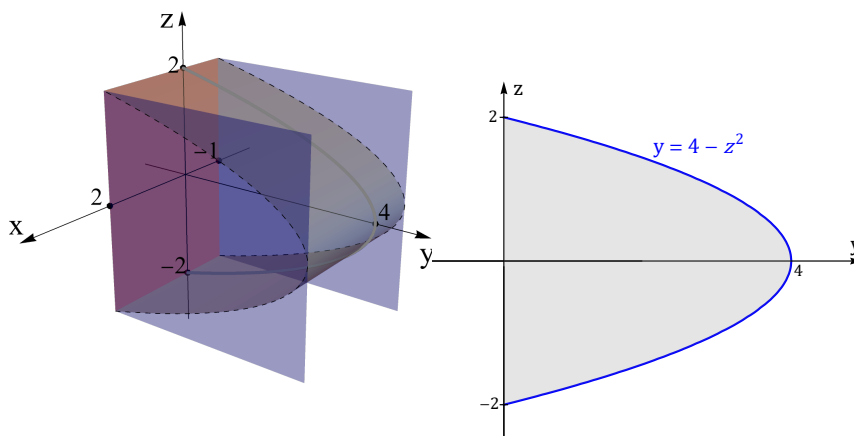
$$\begin{aligned} \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4 - (x^2 + z^2)] \sqrt{x^2 + z^2} \, dz \, dx &= \int_0^{2\pi} \int_0^2 (4 - r^2) r \, r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 (4 - r^2) r^2 \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (4r^2 - r^4) \, dr \, d\theta = \int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5} \right]_0^2 \, d\theta \\ &= \left(4 \frac{2^3}{3} - \frac{2^5}{5} \right) (2\pi - 0) = 2\pi \left(\frac{2^5}{3} - \frac{2^5}{5} \right) = 2^6 \left(\frac{1}{3} - \frac{1}{5} \right) \pi = \frac{128\pi}{15}. \end{aligned}$$

Example (4): Find the volume of the solid bounded by the surfaces $y = 4 - z^2$, $x = -1$, $x = 2$ and $y = 0$.

Solution:

Let E be the region bounded by the surfaces $y = 4 - z^2$, $x = -1$, $x = 2$ and $y = 0$.





The surface $y = 4 - z^2$ intersects the plane $y = 0$ at the two lines passing through $z = \pm 2$.

On E : $-1 \leq x \leq 2$, $0 \leq y \leq 4 - z^2$ and $-2 \leq z \leq 2$.

$$\begin{aligned}
 V(E) &= \iiint_E dV = \int_{-1}^2 \int_{-2}^2 \int_0^{4-z^2} dy \, dz \, dx \\
 &= \int_{-1}^2 \int_{-2}^2 [y]_0^{4-z^2} dz \, dx = \int_{-1}^2 \int_{-2}^2 (4 - z^2) dz \, dx \\
 &= \int_{-1}^2 \left[4z - \frac{z^3}{3} \right]_{-2}^2 dx = \left[\left(8 - \frac{8}{3} \right) - \left(-8 + \frac{8}{3} \right) \right] \int_{-1}^2 dx \\
 &= \left(16 - \frac{16}{3} \right) (2 - (-1)) = 3 \left(16 - \frac{16}{3} \right) = 48 - 16 = 32 .
 \end{aligned}$$

2.4.3 EXERCISES

1. Evaluate the iterated integral.

$$(a). \int_0^1 \int_y^{2y} \int_0^{x+y} 6xyz \, dz \, dx \, dy \quad (b). \int_1^2 \int_0^{2z} \int_0^{\ln x} x e^{-y} \, dy \, dx \, dz$$

2. Evaluate the triple integral.

$$(a). \iiint_E y \, dV, \text{ where}$$

$$E = \{(x, y, z) \mid 0 \leq x \leq 3, 0 \leq y \leq x, x - y \leq z \leq x + y\}.$$

$$(b). \iiint_E \frac{1}{x^3} \, dV, \text{ where}$$

$$E = \{(x, y, z) \mid 0 \leq y \leq 1, 0 \leq z \leq y^2, 1 \leq x \leq z + 1\}.$$

$$(c). \iiint_E 6xy \, dV, \text{ where } E \text{ lies under the plane } z = 1 + x + y \text{ and above the region in the } xy\text{-plane bounded by the curves } y = \sqrt{x}, y = 0, \text{ and } x = 1.$$

3. Use a triple integral to find the volume of the given solid.

$$(a). \text{ The tetrahedron enclosed by the coordinate planes and the plane } 2x + y + z = 4.$$

$$(b). \text{ The solid enclosed by the paraboloids } y = x^2 + z^2 \text{ and } y = 8 - x^2 - z^2.$$

2.5 Triple Integrals in Cylindrical Coordinates

2.5.1 Cylindrical Coordinates

If $P(x, y, z)$ is a point in $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$,
then its cylindrical coordinates

$P(r, \theta, z)$ are :

$$r = \sqrt{x^2 + y^2},$$

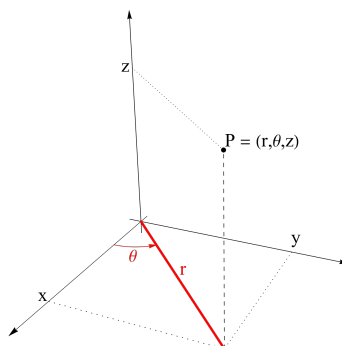
$$\theta = \tan^{-1} \left(\frac{y}{x} \right), \text{ where } x \neq 0,$$

and $z = z$.

Note: $r \in \mathbb{R}$ and $\theta \in [0, 2\pi]$.

If $P(r, \theta, z)$ is given, the the Cartesian
coordinates are :

$$x = r \cos \theta, \quad y = r \sin \theta \text{ and } z = z.$$



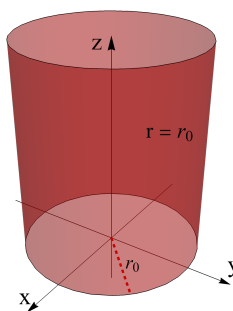
Important equations in Cylindrical coordinates

(1). $r = r_0$, where $r_0 \neq 0$.

$$r = r_0 \implies r^2 = r_0^2$$

$$\implies x^2 + y^2 = r_0^2.$$

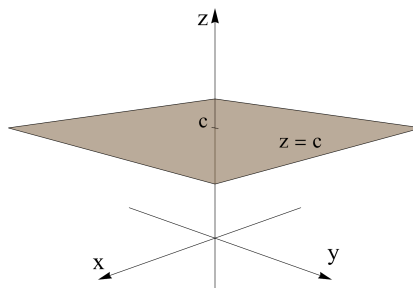
A cylinder,
centered at the origin,
and its radius is r_0 .



(2). $z = c$, where $c \in \mathbb{R}$.

All the points (x, y, c) in \mathbb{R}^3 ,
where $x, y \in \mathbb{R}$.

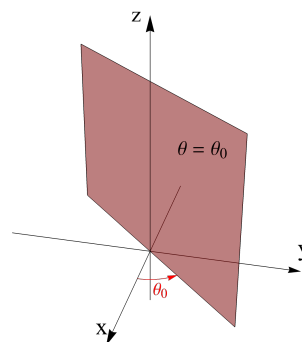
A horizontal plain,
parallel to the xy -plain,
and passes through $(0, 0, c)$.



(3). $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi]$.

$$\begin{aligned}\theta = \theta_0 &\implies \tan(\theta) = \tan(\theta_0), \\ \implies \frac{y}{x} &= \tan(\theta_0) \\ \implies y &= \tan(\theta_0)x.\end{aligned}$$

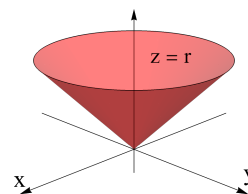
A vertical plain,
passes through the origin.



(4). $z = r$, where $r \neq 0$.

$$z = r \implies z = \sqrt{x^2 + y^2}.$$

A Cone.



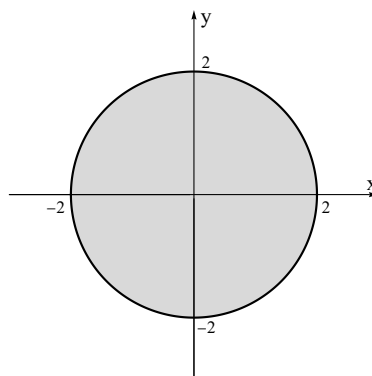
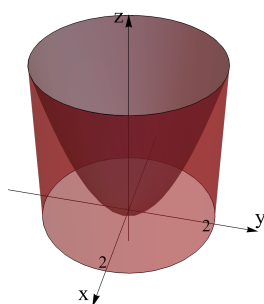
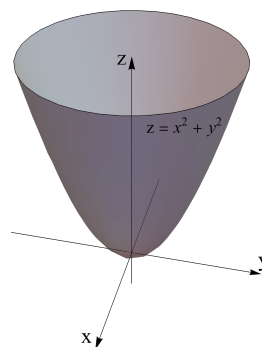
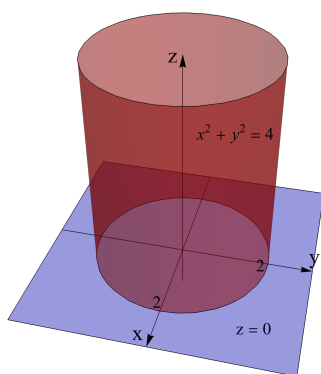
2.5.2 Triple Integrals in Cylindrical Coordinates

Suppose f is continuous on $E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$, where D is given in polar coordinates by $D = \{(r, \theta) \mid \theta_1 \leq \theta \leq \theta_2, r_1(\theta) \leq r \leq r_2(\theta)\}$,

$$\text{then } \iiint_E f(x, y, z) \, dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{u_1(r, \theta)}^{u_2(r, \theta)} f(r \cos \theta, r \sin \theta, z) \, r \, dz \, dr \, d\theta.$$

Example (1): Find the volume of the solid within $x^2 + y^2 = 4$, bounded above by $z = x^2 + y^2$ and below by $z = 0$.

Solution:



$$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2, 0 \leq z \leq x^2 + y^2 = r^2\}.$$

$$\begin{aligned} \text{Volume} &= \iiint_E dV = \int_0^{2\pi} \int_0^2 \int_0^{x^2+y^2} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^2 \int_0^{r^2} r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 [z]_0^{r^2} r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (r^2 - 0)r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 r^3 \, dr \, d\theta \\ &= \int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^2 d\theta = \left[\frac{2^4}{4} - 0 \right] \int_0^{2\pi} d\theta = 4(2\pi - 0) = 8\pi. \end{aligned}$$

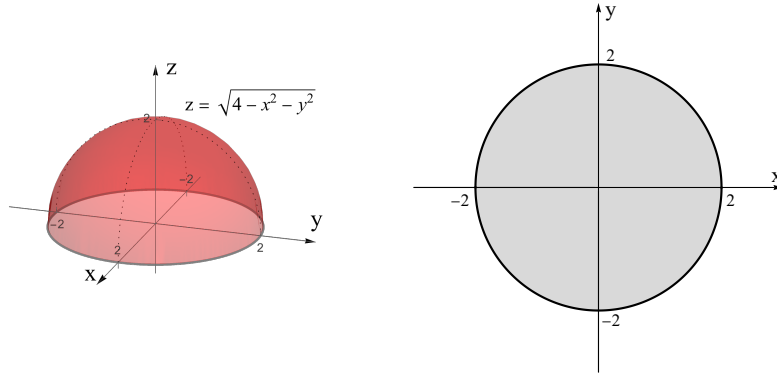
Example (2): Evaluate $\iiint_E z \, dx \, dy \, dz$, where $E = \{(x, y, z) \mid 0 \leq z \leq \sqrt{4 - x^2 - y^2}\}$.

Solution:

$z = \sqrt{4 - x^2 - y^2}$ is the upper-half of the sphere centered at the origin with radius 2.

$\sqrt{4 - x^2 - y^2} = 0 \implies x^2 + y^2 = 4$, the upper-half of the sphere intersects the plain $z = 0$ at the circle centered at the origin with center 2.

$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2, 0 \leq z \leq \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}\}$.



$$\begin{aligned} \iiint_E z \, dx \, dy \, dz &= \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{4-r^2}} z \, dz \, r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^2 [z^2]_0^{\sqrt{4-r^2}} r \, dr \, d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \int_0^2 (4 - r^2)r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^2 (4r - r^3) \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} \left[2r^2 - \frac{r^4}{4} \right]_0^2 d\theta \\ &= \frac{1}{2}(8 - 4) \int_0^{2\pi} d\theta = 2(2\pi - 0) = 4\pi. \end{aligned}$$

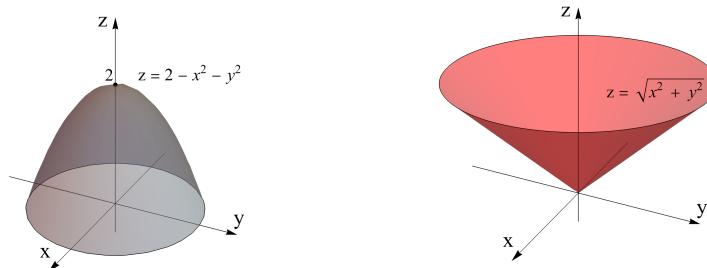
Example (3): Find the volume of the solid bounded above by $z = 2 - x^2 - y^2$ and below by $z = \sqrt{x^2 + y^2}$.

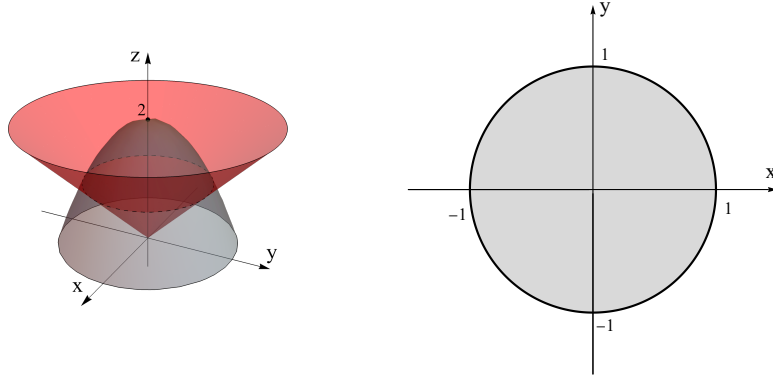
Solution:

$z = 2 - x^2 - y^2 = 2 - r^2$ intersects $z = \sqrt{x^2 + y^2} = r$ at :

$2 - r^2 = r \implies r^2 + r - 2 = 0 \implies (r + 2)(r - 1) = 0 \implies r = 1$.

(Note that $r = -2$ is excluded because $r \geq 0$).





$$E = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, r \leq z \leq 2 - r^2\}.$$

$$\begin{aligned} \text{Volume} &= \iiint_E dV = \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} dz r dr d\theta = \int_0^{2\pi} \int_0^1 [z]_r^{2-r^2} r dr d\theta \\ &= \int_0^{2\pi} \int_0^1 (2 - r^2 - r)r dr d\theta = \int_0^{2\pi} \int_0^1 (-r^3 - r^2 + 2r) dr d\theta \\ &= \int_0^{2\pi} \left[-\frac{r^4}{4} - \frac{r^3}{3} + r^2 \right]_0^1 d\theta = \left(-\frac{1}{4} - \frac{1}{3} + 1 \right) \int_0^{2\pi} d\theta = \frac{5}{12}(2\pi - 0) = \frac{5\pi}{6}. \end{aligned}$$

Example (4): Evaluate the integral $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz dx dy$

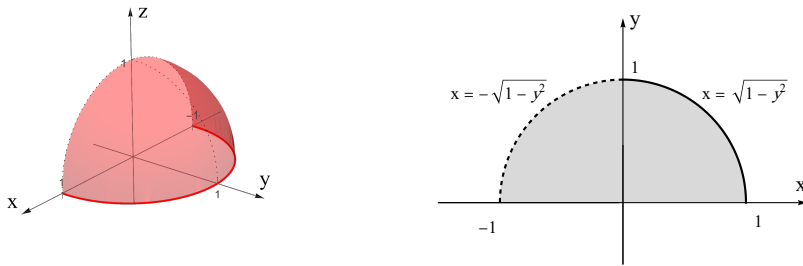
Solution:

Note that $0 \leq y \leq 1$, $-\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}$ and $0 \leq z \leq \sqrt{1-x^2-y^2}$.

$z = \sqrt{1-x^2-y^2} \implies x^2 + y^2 + z^2 = 1$ represents the upper half of the unit sphere.

$x = \sqrt{1-y^2} \implies x^2 + y^2 = 1$ represents the right half of the unit circle.

$x = -\sqrt{1-y^2} \implies x^2 + y^2 = 1$ represents the left half of the unit circle.



In cylindrical coordinates: $0 \leq \theta \leq \pi$, $0 \leq r \leq 1$ and $0 \leq z \leq \sqrt{1-r^2}$.

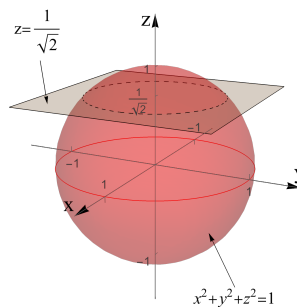
$$\begin{aligned} \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz dx dy &= \int_0^\pi \int_0^1 \int_0^{\sqrt{1-r^2}} dz r dr d\theta \\ &= \int_0^\pi \int_0^1 \sqrt{1-r^2} r dr d\theta = -\frac{1}{2} \int_0^\pi \int_0^1 (1-r^2)^{\frac{1}{2}} (-2r) dr d\theta \\ &= -\frac{1}{2} \left[\frac{2}{3} (1-r^2)^{\frac{3}{2}} \right]_0^1 \int_0^\pi d\theta = -\frac{1}{2} \left(0 - \frac{2}{3} \right) (\pi - 0) = \frac{\pi}{3}. \end{aligned}$$

Example (5): Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 1$ and above the plain $z = \frac{1}{\sqrt{2}}$.

Solution:

Note that $x^2 + y^2 + z^2 = 1$ intersects $z = \frac{1}{\sqrt{2}}$ at $x^2 + y^2 = \frac{1}{2}$ which is a circle centered at the origin and its radius is $\frac{1}{\sqrt{2}}$.

The solid is bounded above by the upper-half of the sphere, and below by the plain.



$$x^2 + y^2 + z^2 = 1$$

$$\implies z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - r^2}.$$

The cylindrical coordinates of the solid : $0 \leq r \leq \frac{1}{\sqrt{2}}$, $0 \leq \theta \leq 2\pi$

and $\frac{1}{\sqrt{2}} \leq z \leq \sqrt{1 - r^2}$.

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \int_{\frac{1}{\sqrt{2}}}^{\sqrt{1-r^2}} dz r dr d\theta = \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\frac{1}{\sqrt{2}}} \int_{\frac{1}{\sqrt{2}}}^{\sqrt{1-r^2}} dz r dr \right) \\ &= [\theta]_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} [z]_{\frac{1}{\sqrt{2}}}^{\sqrt{1-r^2}} r dr = (2\pi - 0) \int_0^{\frac{1}{\sqrt{2}}} \left(\sqrt{1-r^2} - \frac{1}{\sqrt{2}} \right) r dr \\ &= 2\pi \int_0^{\frac{1}{\sqrt{2}}} \left(r\sqrt{1-r^2} - \frac{r}{\sqrt{2}} \right) dr = 2\pi \left[-\frac{1}{2} \frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{r^2}{2\sqrt{2}} \right]_0^{\frac{1}{\sqrt{2}}} \\ &= 2\pi \left[\left(-\frac{1}{2} \frac{2}{3} \left(1 - \frac{1}{2} \right)^{\frac{3}{2}} - \frac{1}{4\sqrt{2}} \right) - \left(-\frac{1}{2} \frac{2}{3} - 0 \right) \right] \\ &= 2\pi \left(-\frac{1}{3} \frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{2}} + \frac{1}{3} \right) = \frac{2\pi}{3} \left(-\frac{1}{2\sqrt{2}} - \frac{3}{4\sqrt{2}} + 1 \right) \\ &= \frac{2\pi}{3} \left(1 - \frac{5}{4\sqrt{2}} \right). \end{aligned}$$

2.5.3 EXERCISES

1. Evaluate $\iiint_E \sqrt{x^2 + y^2} \, dV$, where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.
2. Evaluate $\iiint_E z \, dV$, where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.
3. Evaluate $\iiint_E (x + y + z) \, dV$, where E is the solid in the first octant that lies under the paraboloid $z = 4 - x^2 - y^2$.
4. Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.
5. Find the volume of the solid that is enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$.
6. Find the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$.
7. Evaluate $\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^2 xz \, dz \, dx \, dy$.
8. Evaluate $\int_{-3}^3 \int_0^{\sqrt{9-y^2}} \int_0^{9-x^2-y^2} \sqrt{x^2 + y^2} \, dz \, dx \, dy$.

2.6 Triple Integrals in Spherical Coordinates

2.6.1 Spherical Coordinates

If $P(x, y, z)$ is a point in $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$, then its spherical coordinates

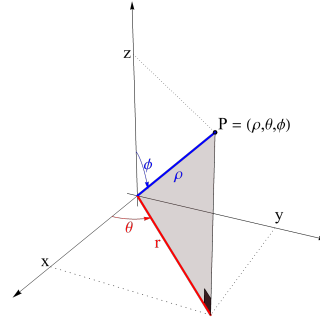
$P(\rho, \theta, \phi)$ are :

$$\rho = \sqrt{x^2 + y^2 + z^2},$$

$$\theta = \tan^{-1} \left(\frac{y}{x} \right), \text{ where } x \neq 0,$$

$$\text{and } \phi = \cos^{-1} \left(\frac{z}{\rho} \right), \text{ where } \rho \neq 0.$$

Note: $\rho \geq 0$, $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$.



Note that if $P(\rho, \theta, \phi)$ is given then :

$$\sin \phi = \frac{r}{\rho} \implies r = \rho \sin \phi \text{ and } \cos \phi = \frac{z}{\rho} \implies z = \rho \cos \phi .$$

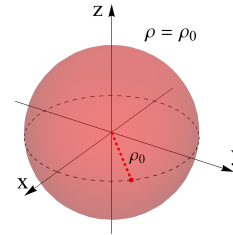
$$\text{So, } x = r \cos \theta = \rho \cos \theta \sin \phi \text{ and } y = r \sin \theta = \rho \sin \theta \sin \phi .$$

Important equations in Spherical coordinates

(1). $\rho = \rho_0$, where $\rho_0 > 0$.

$$\rho = \rho_0 \implies \rho^2 = \rho_0^2 \\ \implies x^2 + y^2 + z^2 = \rho_0^2 .$$

A sphere,
centered at the origin,
and its radius is ρ_0 .



(2). $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi]$.

Since $\phi \in [0, \pi]$ then

$$\sin \phi \geq 0 .$$

If $\theta \in [0, \pi]$ then :

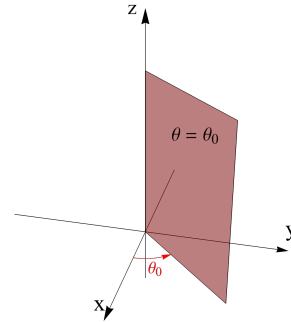
$$y = \rho \sin \theta \sin \phi \geq 0$$

$\theta = \theta_0$ represents a half-plane.

If $\theta \in [\pi, 2\pi]$ then :

$$y = \rho \sin \theta \sin \phi \leq 0$$

$\theta = \theta_0$ represents the other half-plane.



(3). $\phi = \phi_0$, where $0 < \phi_0 < \frac{\pi}{2}$.

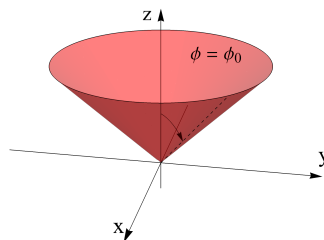
$$\cot \phi = \frac{z}{r} \implies z = r \cot \phi,$$

$$\implies z = \cot \phi \sqrt{x^2 + y^2}$$

Since $\phi \in \left(0, \frac{\pi}{2}\right)$ then

$\cot \phi > 0$.

$\phi = \phi_0$ represents an upper cone.



(4). $\phi = \phi_0$, where $\frac{\pi}{2} < \phi_0 < \pi$.

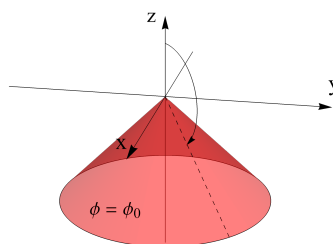
$$\cot \phi = \frac{z}{r} \implies z = r \cot \phi,$$

$$\implies z = \cot \phi \sqrt{x^2 + y^2}$$

Since $\phi \in \left(\frac{\pi}{2}, \pi\right)$ then

$\cot \phi < 0$.

$\phi = \phi_0$ represents a lower cone.



2.6.2 Triple Integrals in Spherical Coordinates

$$\iiint_E f(x, y, z) dV = \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \int_a^b f(\rho, \theta, \phi) \rho^2 \sin \phi d\rho d\theta d\phi,$$

where $E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \theta_1 \leq \theta \leq \theta_2, \phi_1 \leq \phi \leq \phi_2\}$.

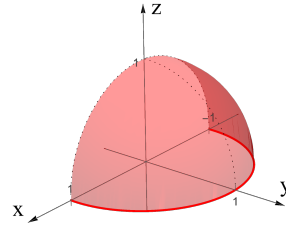
Example (1): Evaluate the integral $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz dx dy$

Solution: Referring to Example(4) in section (2.5).

In Spherical coordinates :

$$0 \leq \rho \leq 1, 0 \leq \theta \leq \pi, \text{ and } 0 \leq \phi \leq \frac{\pi}{2}.$$

$$\begin{aligned} & \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz dx dy \\ &= \int_0^{\frac{\pi}{2}} \int_0^{\pi} \int_0^1 \rho^2 \sin \phi d\rho d\theta d\phi \\ &= \left(\int_0^1 \rho^2 d\rho \right) \left(\int_0^{\frac{\pi}{2}} \sin \phi d\phi \right) \left(\int_0^{\pi} d\theta \right) \end{aligned}$$



$$= \left[\frac{\rho^3}{3} \right]_0^1 [-\cos \phi]_0^{\frac{\pi}{2}} [\theta]_0^{\pi} = \left(\frac{1}{3} - 0 \right) \left(-\cos \left(\frac{\pi}{2} \right) + \cos(0) \right) (\pi - 0) = \frac{\pi}{3}.$$

Example (2): Evaluate the integral $\iiint_E 4z dV$, where E is the region bounded above by $x^2 + y^2 + z^2 = 4$ and below by $z = 0$.

Solution:

Note that $x^2 + y^2 + z^2 = 4$ intersects $z = 0$ at $x^2 + y^2 = 4$ which is a circle centered at the origin and its radius is 2.

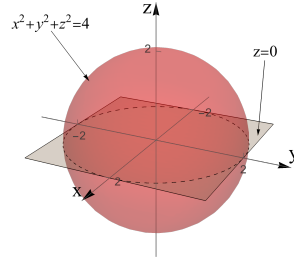
E in spherical coordinates:

$$0 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi \text{ and } 0 \leq \phi \leq \frac{\pi}{2}.$$

$$z = \rho \cos \phi \implies 4z = 4\rho \cos \phi.$$

$$\iiint_E 4z dV$$

$$\begin{aligned} &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 (4\rho \cos \phi) \rho^2 \sin \phi d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^2 (4\rho^3) (\cos \phi \sin \phi) d\rho d\phi d\theta \\ &= \left(\int_0^2 4\rho^3 d\rho \right) \left(\int_0^{\frac{\pi}{2}} \cos \phi \sin \phi d\phi \right) \left(\int_0^{2\pi} d\theta \right) = [\rho^4]_0^2 \left[\frac{\sin^2 \phi}{2} \right]_0^{\frac{\pi}{2}} [\theta]_0^{2\pi} \\ &= (16 - 0) \left(\frac{1}{2} - 0 \right) (2\pi - 0) = 16\pi. \end{aligned}$$

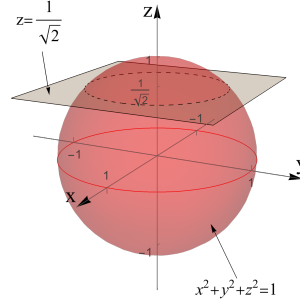


Example (3): Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 1$ and above the plane $z = \frac{1}{\sqrt{2}}$.

Solution:

Note that $x^2 + y^2 + z^2 = 1$ intersects $z = \frac{1}{\sqrt{2}}$ at $x^2 + y^2 = \frac{1}{2}$ which is a circle centered at the origin and its radius is $\frac{1}{\sqrt{2}}$.

$$\begin{aligned} z &= \rho \cos \phi \implies \cos \phi = \frac{1}{\sqrt{2}} \\ \implies \phi &= \frac{\pi}{4}. \end{aligned}$$



$$x^2 + y^2 + z^2 = 1 \implies \rho = 1, z = \frac{1}{\sqrt{2}} \implies \rho \cos \phi = \frac{1}{\sqrt{2}} \implies \rho = \frac{1}{\sqrt{2} \cos \phi}.$$

The spherical coordinates of the solid : $\frac{1}{\sqrt{2} \cos \phi} \leq \rho \leq 1, 0 \leq \theta \leq 2\pi$

and $0 \leq \phi \leq \frac{\pi}{4}$.

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_{\frac{1}{\sqrt{2} \cos \phi}}^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\frac{\pi}{4}} \int_{\frac{1}{\sqrt{2} \cos \phi}}^1 \rho^2 \sin \phi \, d\rho \, d\phi \right) \\ &= [\theta]_0^{2\pi} \int_0^{\frac{\pi}{4}} \left[\frac{\rho^3}{3} \right]_{\frac{1}{\sqrt{2} \cos \phi}}^1 \sin \phi \, d\phi = \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \left[1 - \frac{1}{2\sqrt{2} \cos^3 \phi} \right] \sin \phi \, d\phi \\ &= \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \left[\sin \phi - \frac{(\cos \phi)^{-3} \sin \phi}{2\sqrt{2}} \right] d\phi = \frac{2\pi}{3} \left[-\cos \phi + \frac{1}{2\sqrt{2}} \frac{(\cos \phi)^{-2}}{-2} \right]_0^{\frac{\pi}{4}} \\ &= \frac{2\pi}{3} \left[\left(-\frac{1}{\sqrt{2}} - \frac{2}{4\sqrt{2}} \right) - \left(-1 - \frac{1}{4\sqrt{2}} \right) \right] = \frac{2\pi}{3} \left(1 - \frac{5}{4\sqrt{2}} \right). \end{aligned}$$

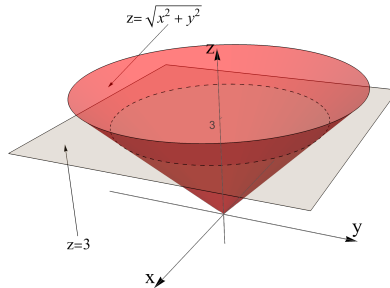
Example (4): Evaluate $\iiint_E \sqrt{x^2 + y^2 + z^2} \, dV$, where E is the region bounded

above by the plane $z = 3$ and below by the cone $z = \sqrt{x^2 + y^2}$.

Solution:

Note that $z = \sqrt{x^2 + y^2}$ intersects $z = 3$ at $x^2 + y^2 = 9$ which is a circle centered at the origin and its radius is 3.

$$\begin{aligned} z &= \sqrt{x^2 + y^2} = r \\ \implies \rho \cos \phi &= \rho \sin \phi \\ \implies \cos \phi &= \sin \phi \\ \implies \phi &= \frac{\pi}{4}. \end{aligned}$$



$$z = 3 \implies \rho \cos \phi = 3 \implies \rho = \frac{3}{\cos \phi}.$$

The spherical coordinates of $E : 0 \leq \rho \leq \frac{3}{\cos \phi}$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{\pi}{4}$.

$$\begin{aligned} \iiint_E \sqrt{x^2 + y^2 + z^2} dV &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\frac{3}{\cos \phi}} \rho \rho^2 \sin \phi d\rho d\phi d\theta \\ &= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\frac{\pi}{4}} \int_0^{\frac{3}{\cos \phi}} \rho^3 \sin \phi d\rho d\phi \right) = [\theta]_0^{2\pi} \int_0^{\frac{\pi}{4}} \left[\frac{\rho^4}{4} \right]_0^{\frac{3}{\cos \phi}} \sin \phi d\phi \\ &= (2\pi - 0) \frac{3^4}{4} \int_0^{\frac{\pi}{4}} (\cos \phi)^{-4} \sin \phi d\phi = \frac{81\pi}{2} \left[\frac{-(\cos \phi)^{-3}}{-3} \right]_0^{\frac{\pi}{4}} = \frac{27\pi}{2} (2\sqrt{2} - 1) \end{aligned}$$

Example (5): Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 1$ and inside the cone $z = \sqrt{3(x^2 + y^2)}$.

Solution:

Note that $x^2 + y^2 + z^2 = 1$ intersects

$$z = \sqrt{3(x^2 + y^2)} \text{ at}$$

$$3x^2 + 3y^2 = 1 - x^2 - y^2$$

$$\implies 4x^2 + 4y^2 = 1$$

$$\implies x^2 + y^2 = \frac{1}{4}$$

which is a circle centered at the origin

and its radius is $\frac{1}{2}$.

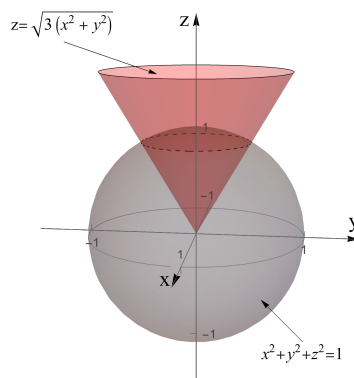
$$z = \sqrt{3(x^2 + y^2)} = \sqrt{3}r \implies$$

$$\rho \cos \phi = \sqrt{3}\rho \sin \phi \implies \tan \phi = \frac{1}{\sqrt{3}}$$

$$\implies \phi = \frac{\pi}{6}.$$

The spherical coordinates of the solid : $0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{\pi}{6}$.

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^1 \rho^2 \sin \phi d\rho d\phi d\theta = \left(\int_0^1 \rho^2 d\rho \right) \left(\int_0^{\frac{\pi}{6}} \sin \phi d\phi \right) \left(\int_0^{2\pi} d\theta \right) \\ &= \left[\frac{\rho^3}{3} \right]_0^1 [-\cos \phi]_0^{\frac{\pi}{6}} [\theta]_0^{2\pi} = \left(\frac{1}{3} - 0 \right) \left(-\frac{\sqrt{3}}{2} - (-1) \right) (2\pi - 0) \\ &= \frac{2\pi}{3} \left(1 - \frac{\sqrt{3}}{2} \right) = \frac{\pi}{3} (2 - \sqrt{3}). \end{aligned}$$



2.6.3 EXERCISES

1. Change from rectangular to spherical coordinates:

(a). $(3, 3, 0)$. (b). $(1, -\sqrt{3}, 2\sqrt{3})$.

2. Identify the surface whose equation is $\rho = \cos \phi$.

3. Sketch the solid described by the given inequalities:

(a). $\rho \leq 1$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq \frac{\pi}{6}$. (b). $\rho \leq 2$, $\rho \leq \csc \phi$.

4. Evaluate $\iiint_E (x^2 + y^2 + z^2)^2 dV$, where E is the ball with center the origin and radius 5.

5. Evaluate $\iiint_E y^2 z^2 dV$, where E lies above the cone $\phi = \frac{\pi}{3}$ and below the sphere $\rho = 1$.

6. Evaluate $\iiint_E x e^{x^2+y^2+z^2} dV$, where E is the portion of the unit ball $x^2 + y^2 + z^2 \leq 1$ that lies in the first octant.

7. Find the volume of the part of the ball $\rho \leq a$ that lies between the cones $\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$.

Chapter 3

Sequences, Series, and Power Series

3.1 Sequences

3.1.1 Infinite Sequences

Definition: An infinite sequence is a function $f : \mathbb{N} \rightarrow \mathbb{R}$, where $f(n) = a_n$.

Notation: The sequence $\{a_1, a_2, \dots\}$ is also denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

Example (1): Some sequences can be defined by giving a formula for the n^{th} term.

(a). $a_n = \frac{1}{2^n}$ is the n^{th} term of the sequence $\left\{ \frac{1}{2^n} \right\} = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \right\}$.

(b). $\left\{ \frac{(-1)^n(n+2)}{5^n} \right\} = \left\{ -\frac{3}{5}, \frac{4}{25}, -\frac{5}{125}, \dots \right\}$.

Example (2): The Fibonacci sequences is defined recursively by

$f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}, n \geq 3$.

$\{f_n\} = \{1, 1, 2, 3, 5, 8, 13, \dots\}$.

3.1.2 The Limit of a Sequence

Definition: A sequence $\{a_n\}$ has the limit $L \in \mathbb{R}$, if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that : if $n \geq N$ then $|a_n - L| < \epsilon$, and we write $\lim_{n \rightarrow \infty} a_n = L$.

Definition: A sequence $\{a_n\}$ goes to ∞ , if for every positive real number M there exists an $N \in \mathbb{N}$ such that : if $n \geq N$ then $a_n \geq M$, and we write

$\lim_{n \rightarrow \infty} a_n = \infty$.

3.1.3 Properties of Convergent Sequences

Theorem: If $\lim_{x \rightarrow \infty} f(x) = L$ and $f(n) = a_n$ where $n \in \mathbb{N}$ then $\lim_{x \rightarrow \infty} a_n = L$.

Example (3): Calculate $\lim_{n \rightarrow \infty} \frac{\ln n}{n}$.

Solution : Let $f(x) = \frac{\ln x}{x}$.

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0 \text{ (By L'Hôpital's Rule).}$$

Therefore, $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$.

Theorem: Suppose $\lim_{n \rightarrow \infty} a_n = L_1$, $\lim_{n \rightarrow \infty} b_n = L_2$ and $c \in \mathbb{R}$, then:

- (1). $\lim_{n \rightarrow \infty} c = c$.
- (2). $\lim_{n \rightarrow \infty} a_n \pm b_n = L_1 \pm L_2$.
- (3). $\lim_{n \rightarrow \infty} c a_n = c L_1$.
- (4). $\lim_{n \rightarrow \infty} a_n \cdot b_n = L_1 \cdot L_2$.
- (5). $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L_1}{L_2}$, where $L_2 \neq 0$.

Power Law: $\lim_{n \rightarrow \infty} a_n^p = \left[\lim_{n \rightarrow \infty} a_n \right]^p$, if $p > 0$ and $a_n > 0$.

Squeeze Theorem for sequences: If $a_n \leq b_n \leq c_n$ for $n \geq N_0$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$.

Example (4): Calculate $\lim_{n \rightarrow \infty} \frac{n!}{n^n}$.

Solution:

$$0 \leq \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \frac{1}{n} \left(\frac{2 \cdot 3 \cdots n}{n \cdot n \cdots n} \right) \leq \frac{1}{n} (1) = \frac{1}{n}$$

Since $\lim_{n \rightarrow \infty} 0 = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$, then $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$ (By Squeeze Theorem).

Theorem: If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Example (5): Calculate $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n}$.

Solution: Let $a_n = \frac{(-1)^n}{n}$.

$$\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \rightarrow \infty} \frac{|(-1)^n|}{|n|} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Therefore $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$.

Theorem: If $\lim_{n \rightarrow \infty} a_n = L$, and the function f is continuous at L , then $\lim_{n \rightarrow \infty} f(a_n) = f(L)$.

Example (6): Calculate $\lim_{n \rightarrow \infty} \cos\left(\frac{\pi}{n}\right)$.

Solution:

$$\lim_{n \rightarrow \infty} \frac{\pi}{n} = \lim_{n \rightarrow \infty} \pi \frac{1}{n} = \pi(0) = 0 .$$

Since the cosine function is continuous at 0 , then $\lim_{n \rightarrow \infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$.

Important Note: The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$, and divergent for all other values of r .

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases} .$$

3.1.4 Monotonic and bounded sequences

Definition:

- (1). A sequence $\{a_n\}$ is called increasing if $a_{n+1} \geq a_n$ for all $n \geq 1$.
- (2). A sequence $\{a_n\}$ is called decreasing if $a_{n+1} \leq a_n$ for all $n \geq 1$.
- (3). A sequence is called monotonic if it is either increasing or decreasing.

Definition:

- (1). A sequence $\{a_n\}$ is bounded above if there exists a real number M such that $a_n \leq M$ for all $n \geq 1$.
- (2). A sequence $\{a_n\}$ is bounded below if there exists a real number m such that $a_n \geq m$ for all $n \geq 1$.
- (3). If a sequence is bounded above and below, then it is called a bounded sequence.

Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent. In particular :

- (1). A sequence that is increasing and bounded above converges.
- (2). A sequence that is decreasing and bounded below converges.

Important Notes :

- (1). If $a_n \leq b_n$ for some $n \geq N_0$, $\{a_n\}$ is an increasing sequence and $\{b_n\}$ is convergent, then $\{a_n\}$ is convergent.
- (2). If $a_n \leq b_n$ for some $n \geq N_0$, if $\lim_{n \rightarrow \infty} a_n = \infty$, then $\lim_{n \rightarrow \infty} b_n = \infty$.

3.2 Series

3.2.1 Infinite Series

Definition:

(1). The sum of the terms of the sequence $\{a_n\}$ is called an infinite series (or a series) and it is denoted by $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

(2). $s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$ is called the n^{th} partial sum of the series.

Definition:

If the sequence $\{s_n\}$ of the partial sums of the series $\sum_{n=1}^{\infty} a_n$ is convergent, and

$\lim_{n \rightarrow \infty} s_n = s \in \mathbb{R}$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} a_n = s$.

The number s is called the sum of the series.

If the sequence $\{s_n\}$ is divergent, then the series is divergent.

Example (1): Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Solution : Note that $\frac{1}{n} - \frac{1}{n+1} = \frac{(n+1) - n}{n(n+1)} = \frac{1}{n(n+1)}$.

$$\begin{aligned} s_n &= \sum_{i=1}^n \frac{1}{i(i+1)} = \sum_{i=1}^n \left(\frac{1}{i} - \frac{1}{i+1} \right) \\ &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \end{aligned}$$

$$\text{So, } s_n = \sum_{i=1}^n \frac{1}{i(i+1)} = 1 - \frac{1}{n+1}.$$

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) = 1 - 0 = 1.$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

3.2.2 Sum of a Geometric Series

Definition : $\sum_{n=0}^{\infty} ar^n$ (where $a \neq 0$ and $r \in \mathbb{R}$) is called a geometric series.

Note that $s_n = a + ar + ar^2 + \cdots + ar^n$, $rs_n = ra + ar^2 + ar^3 + \cdots + ar^n + ar^{n+1}$.

$$s_n - rs_n = a - ar^{n+1} \implies (1-r)s_n = a(1 - a^{n+1}) \implies s_n = \frac{a(1 - a^{n+1})}{1-r}$$

where $r \neq 1$.

Sum of a geometric series :

- (1). If $|r| < 1$, then $\sum_{n=0}^{\infty} ar^n$ is convergent and its sum is $\frac{a}{1-r}$.
- (2). If $|r| \geq 1$, then $\sum_{n=0}^{\infty} ar^n$ is divergent.

Example (2): Find the sum of the geometric series $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \dots$.

Solution : $a = 3$.

$$r = \frac{a_2}{a_1} = \frac{-\frac{3}{2}}{3} = -\frac{1}{2}.$$

the sum of the geometric series is $\frac{a}{1-r} = \frac{3}{1 - (-\frac{1}{2})} = \frac{3}{(\frac{3}{2})} = 2$.

Example (3): Does the series $\sum_{n=1}^{\infty} 3^{2n}5^{1-n}$ converge or diverge?.

Solution :

$$\begin{aligned} \sum_{n=1}^{\infty} 3^{2n}5^{1-n} &= \sum_{n=1}^{\infty} 9^n 5^{1-n} = \sum_{n=1}^{\infty} \frac{9^n}{5^{n-1}} \\ &= \sum_{n=1}^{\infty} \frac{9(9^{n-1})}{5^{n-1}} = \sum_{n=1}^{\infty} 9 \left(\frac{9}{5}\right)^{n-1} = \sum_{n=0}^{\infty} 9 \left(\frac{9}{5}\right)^n. \end{aligned}$$

Since $|r| = \left|\frac{9}{5}\right| = \frac{9}{5} > 1$, then the geometric series diverges.

3.2.3 Test for Divergence

Example (4): Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Solution :

$$s_2 = s_{2^1} = 1 + \frac{1}{2}.$$

$$s_4 = s_{2^2} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{3}{4}.$$

$$\begin{aligned} s_8 = s_{2^3} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{5}{8}. \end{aligned}$$

So, $s_{2^n} > 1 + \frac{n}{2}$ (By induction).

$$\lim_{n \rightarrow \infty} s_{2^n} \geq \lim_{n \rightarrow \infty} \left(1 + \frac{n}{2}\right) = \infty.$$

Therefore, The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Theorem : If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$.

Important Note : The converse of the last theorem is not true.

In other words, if $\lim_{n \rightarrow \infty} a_n = 0$ then this does not mean that $\sum_{n=1}^{\infty} a_n$ is convergent.

For example $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, while $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Test of divergence : If $\lim_{n \rightarrow \infty} a_n$ does not exist, or $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series

$\sum_{n=1}^{\infty} a_n$ is divergent.

Example (5): Show that the series $\sum_{n=1}^{\infty} \frac{n^3}{2n^3 + 1}$ is divergent.

Solution :

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^3}{2n^3 + 1} = \frac{1}{2} \neq 0 .$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{n^3}{2n^3 + 1}$ is divergent.

3.2.4 Properties of Convergent Series

Theorem : If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent series and $c \in \mathbb{R}$ then :

- (1). The series $\sum_{n=1}^{\infty} (ca_n)$ is convergent, and $\sum_{n=1}^{\infty} (ca_n) = c \sum_{n=1}^{\infty} a_n$.
- (2). The series $\sum_{n=1}^{\infty} (a_n + b_n)$ is convergent, and $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
- (3). The series $\sum_{n=1}^{\infty} (a_n - b_n)$ is convergent, and $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

Example (6): Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{4}{n(n+1)} + \frac{2}{3^n} \right)$.

Solution :

$$\text{From Example (1), } \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 .$$

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = \sum_{n=1}^{\infty} \frac{2}{3} \left(\frac{1}{3} \right)^{n-1} = \sum_{n=0}^{\infty} \frac{2}{3} \left(\frac{1}{3} \right)^n = \frac{2}{3} \frac{1}{1 - \frac{1}{3}} = \frac{2}{3} \frac{3}{2} = 1 .$$

$$\sum_{n=1}^{\infty} \left(\frac{4}{n(n+1)} + \frac{2}{3^n} \right) = \sum_{n=1}^{\infty} \frac{4}{n(n+1)} + \sum_{n=1}^{\infty} \frac{2}{3^n}$$

$$= 4 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{2}{3^n} = 4(1) + 1 = 5 .$$

3.2.5 EXERCISES

1. Let $a_n = \frac{2n}{3n+1}$.

(a). Determine whether $\{a_n\}$ is convergent.(b). Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

2. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

(1). $\sum_{n=1}^{\infty} \frac{5}{\pi^n}$ (2). $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$ (3). $\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^n}$

(4). $\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$ (5). $\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n}$

3. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

(1). $\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$ (2). $\sum_{n=1}^{\infty} \frac{n^2}{n^2-2n+5}$ (3). $\sum_{n=1}^{\infty} \frac{1}{4+e^{-n}}$

(4). $\sum_{n=1}^{\infty} \frac{2^n+4^n}{e^n}$ (5). $\sum_{n=1}^{\infty} (\sin 100)^n$ (6). $\sum_{n=1}^{\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n}$

(7). $\sum_{n=1}^{\infty} \ln \left(\frac{n^2+1}{2n^2+1} \right)$

3.3 The Integral Test

3.3.1 The Integral Test

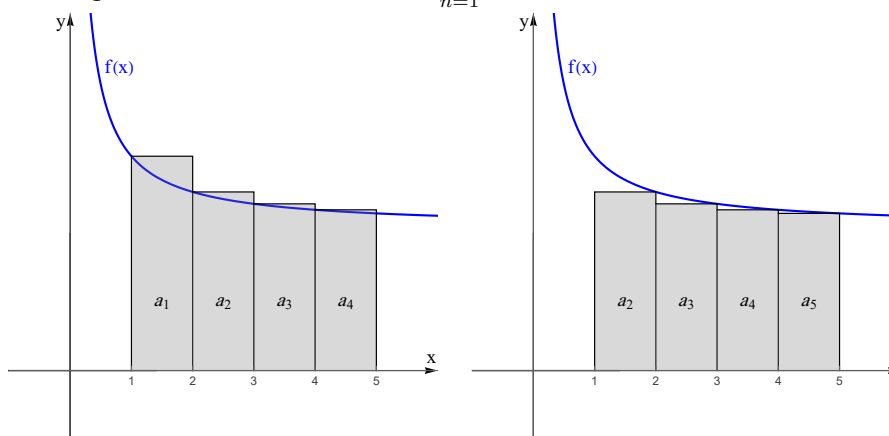
Theorem (The Integral Test) :

Suppose f is a positive, decreasing continuous function defined on $[1, \infty)$, and let $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper

integral $\int_1^{\infty} f(x) dx$ is convergent. In other words:

(1). If $\int_1^{\infty} f(x) dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent .

(2). If $\int_1^{\infty} f(x) dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent .



Example (1): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$.

Solution :

Let $f(x) = \frac{1}{x^2 + 1}$, the f is a positive continuous function on $[1, \infty)$.

$f'(x) = \frac{-2x}{(x^2 + 1)^2} < 0$ on the interval $[1, \infty)$, then f is decreasing on $[1, \infty)$.

$$\int_1^{\infty} \frac{1}{x^2 + 1} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2 + 1} dx = \lim_{t \rightarrow \infty} [\tan^{-1} x]_1^t$$

$$= \lim_{t \rightarrow \infty} [\tan^{-1}(t) - \tan^{-1}(1)] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} .$$

So, $\int_1^{\infty} \frac{1}{x^2 + 1} dx$ is convergent, then $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ is convergent.

Example (2): Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

Solution :

Let $f(x) = \frac{1}{x \ln x}$, the f is a positive continuous function on $[2, \infty)$.

$f'(x) = \frac{-(1 + \ln x)}{(x \ln x)^2} < 0$ on $[2, \infty)$, then f is decreasing on $[2, \infty)$.

$$\int_2^\infty \frac{1}{x \ln x} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{\left(\frac{1}{x}\right)}{\ln x} dx = \lim_{t \rightarrow \infty} [\ln(\ln x)]_2^t \\ = \lim_{t \rightarrow \infty} [\ln(\ln t) - \ln(\ln 2)] = \infty. \text{ (Note that } \lim_{t \rightarrow \infty} \ln(t) = \infty \text{).}$$

So, $\int_2^\infty \frac{1}{x \ln x} dx$ is divergent, then $\sum_{n=2}^\infty \frac{1}{n \ln n}$ is divergent.

Definition : (The p -series)

The series $\sum_{n=1}^\infty \frac{1}{n^p}$ where $p \in \mathbb{R}$ is called a p -series.

Theorem :

The p -series $\sum_{n=1}^\infty \frac{1}{n^p}$ is convergent if $p > 1$, and divergent if $p \leq 1$.

Proof:

(1). If $p < 0$, then $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n^p} = \lim_{n \rightarrow \infty} n^{-p} = \infty$. (Note: $-p > 0$).

Since $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=1}^\infty \frac{1}{n^p}$ is divergent.

(2). If $p = 0$, then $a_n = \frac{1}{n^0} = 1$, hence $\lim_{n \rightarrow \infty} a_n = 1 \neq 0$.

Therefore, $\sum_{n=1}^\infty \frac{1}{n^p}$ is divergent.

(3). If $0 < p < 1$, then using the integral test:

Note that $f(x) = \frac{1}{x^p}$ is a positive, decreasing continuous function on $[1, \infty)$.

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \int_1^t x^{-p} dx = \lim_{t \rightarrow \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^t = \lim_{t \rightarrow \infty} \left[\frac{x^{1-p}}{1-p} \right]_1^t \\ = \lim_{t \rightarrow \infty} \left[\frac{t^{1-p}}{1-p} - \frac{1}{1-p} \right] = \infty. \text{ (Note : } 1-p > 0 \text{).}$$

Since $\int_1^\infty \frac{1}{x^p} dx$ is divergent, then $\sum_{n=1}^\infty \frac{1}{n^p}$ is divergent.

(4). If $p = 1$, then using the integral test:

Note that $f(x) = \frac{1}{x}$ is a positive, decreasing continuous function on $[1, \infty)$.

$$\int_1^\infty \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx = \lim_{t \rightarrow \infty} [\ln x]_1^t = \lim_{t \rightarrow \infty} [\ln(t) - \ln(1)] = \infty.$$

Since $\int_1^\infty \frac{1}{x} dx$ is divergent, then $\sum_{n=1}^\infty \frac{1}{n}$ is divergent.

(5). If $p > 1$, then using the integral test:

Note that $f(x) = \frac{1}{x^p}$ is a positive, decreasing continuous function on $[1, \infty)$.

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \int_1^t x^{-p} dx = \lim_{t \rightarrow \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^t = \lim_{t \rightarrow \infty} \left[\frac{1}{(1-p)x^{p-1}} \right]_1^t \\ = \lim_{t \rightarrow \infty} \left[\frac{1}{(1-p)t^{p-1}} - \frac{1}{1-p} \right] = 0 - \frac{1}{1-p} = \frac{1}{p-1}.$$

Since $\int_1^{\infty} \frac{1}{x^p} dx$ is convergent, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

Example (3): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Solution :

$\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a p -series, where $p = 4 > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent.

Example (4): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

Solution :

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a p -series, where $p = \frac{1}{2} < 1$, then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

Example (5): Discuss the convergence of the series $\sum_{n=1}^{\infty} n e^{-n^2}$.

Solution : Let $f(x) = x e^{-x^2}$ then f is a positive continuous function on $[1, \infty)$.

$f'(x) = e^{-x^2} - 2x^2 e^{-x^2} = (1 - 2x^2) e^{-x^2} < 0$ on $[1, \infty)$.

Therefore, f is decreasing on $[1, \infty)$. Using the integral test :

$$\begin{aligned} \int_1^{\infty} x e^{-x^2} dx &= \lim_{t \rightarrow \infty} \left(-\frac{1}{2} \int_1^t e^{-x^2} (-2x) dx \right) = \lim_{t \rightarrow \infty} \left(-\frac{1}{2} [e^{-x^2}]_1^t \right) \\ &= \lim_{t \rightarrow \infty} \left(-\frac{1}{2} [e^{-t^2} - e^{-1}] \right) = \lim_{t \rightarrow \infty} \left(-\frac{1}{2} \left[\frac{1}{e^{t^2}} - \frac{1}{e} \right] \right) = -\frac{1}{2} \left(0 - \frac{1}{e} \right) = \frac{1}{2e}. \end{aligned}$$

Since $\int_1^{\infty} x e^{-x^2} dx$ is convergent, then $\sum_{n=1}^{\infty} n e^{-n^2}$ is convergent.

3.3.2 EXERCISES

1. Use the Integral Test to determine whether the series is convergent or divergent.

$$(1). \sum_{n=1}^{\infty} n^{-3} \quad (2). \sum_{n=1}^{\infty} \frac{2}{5n-1} \quad (3). \sum_{n=2}^{\infty} \frac{n^2}{n^3+1}$$

$$(4). \sum_{n=1}^{\infty} n^2 e^{-n^3} \quad (5). \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$$

2. Determine whether the series is convergent or divergent.

$$(1). \sum_{n=1}^{\infty} \frac{1}{n\sqrt{2}} \quad (2). \sum_{n=3}^{\infty} n^{-0.9999} \quad (3). \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \dots$$

$$(4). \sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2} \quad (5). \sum_{n=1}^{\infty} \frac{1}{n^2+4} \quad (6). 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \dots$$

$$(7). \sum_{n=2}^{\infty} \frac{1}{n \ln n} \quad (8). \sum_{n=2}^{\infty} \frac{\ln n}{n^2} \quad (9). \sum_{n=1}^{\infty} n e^{-n^2}$$

$$(10). \sum_{n=1}^{\infty} \frac{1}{n^2+n^3}$$

3.4 The Comparison Tests

3.4.1 The Direct Comparison Test

Theorem : (The Direct Comparison Test)

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

- (1). If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all n , then $\sum_{n=1}^{\infty} a_n$ is also convergent.
- (2). If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all n , then $\sum_{n=1}^{\infty} a_n$ is also divergent.

Example (1): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\sin n}{2+3^n}$.

Solution :

$$\text{For all } n \geq 1 : \frac{\sin n}{2+3^n} \leq \frac{1}{2+3^n} < \frac{1}{3^n} = \left(\frac{1}{3}\right)^n .$$

Since $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is convergent (a geometric series with $r = \frac{1}{3} < 1$), then

$\sum_{n=1}^{\infty} \frac{\sin n}{2+3^n}$ is also convergent (by The direct comparison test).

Example (2): Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{2}{\sqrt{n-1}}$.

Solution :

$$\text{For all } n \geq 2 : \sqrt{n-1} < \sqrt{n} \implies \frac{1}{\sqrt{n-1}} > \frac{1}{\sqrt{n}}$$

$$\implies \frac{2}{\sqrt{n-1}} > \frac{1}{\sqrt{n-1}} > \frac{1}{\sqrt{n}} .$$

Since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is divergent (a p -series, with $p = \frac{1}{2} \leq 1$), then

$\sum_{n=2}^{\infty} \frac{2}{\sqrt{n-1}}$ is also divergent (by The direct comparison test) .

Example (3): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^4+1}$.

Solution :

$$\text{For all } n \geq 1 : e^{-n} = \frac{1}{e^n} < 1 \implies \frac{e^{-n}}{n^4+1} < \frac{1}{n^4+1} < \frac{1}{n^4} .$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent(a p -series, with $p = 4 > 1$), then $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^4+1}$ is also convergent.

Example (4): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Solution :

Since the function $f(x) = \ln(x)$ is increasing on $[1, \infty)$, then for all $n \geq 3$:

$$\ln(n) \geq \ln(3) > \ln(e) = 1 \implies \frac{\ln n}{n} > \frac{1}{n} .$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (a p -series, with $p = 1 \leq 1$), then $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is also divergent.

3.4.2 Limit Comparison Test

Theorem: (Limit comparison test)

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$, where $c \in \mathbb{R}$ and $c > 0$, then either both series converge or both series diverge.

Example (1): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$.

Solution :

Note that $a_n = \frac{1}{2^n - 1}$. Let $b_n = \frac{1}{2^n}$.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{2^n - 1}\right)}{\left(\frac{1}{2^n}\right)} = \lim_{n \rightarrow \infty} \frac{2^n}{2^n - 1} = 1 > 0 .$$

Since $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent (a geometric series with $r = \frac{1}{2} < 1$),

then $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ is convergent (by limit comparison test).

Example (2): Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2 - 1}}$.

Solution :

Note that $a_n = \frac{1}{\sqrt[3]{n^2 - 1}}$. Let $b_n = \frac{1}{\sqrt[3]{n^2}} = \frac{1}{n^{\frac{2}{3}}}$.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sqrt[3]{n^2}}{\sqrt[3]{n^2 - 1}} = \lim_{n \rightarrow \infty} \sqrt[3]{\frac{n^2}{n^2 - 1}} = \sqrt[3]{1} = 1 > 0 .$$

Since $\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{2}{3}}}$ is divergent (a p -series, with $p = \frac{2}{3} \leq 1$), then

$\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2 - 1}}$ is divergent (by limit comparison test).

Example (3): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{5n^2 + 3n}{3^n(n^2 + 2)}$.

Solution :

Note that $a_n = \frac{5n^2 + 3n}{3^n(n^2 + 2)}$. Let $b_n = \frac{5n^2}{3^n n^2} = \frac{5}{3^n}$.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left(\frac{5n^2 + 3n}{3^n(n^2 + 2)} \cdot \frac{3^n}{5} \right) = \lim_{n \rightarrow \infty} \frac{5n^2 + 3n}{5n^2 + 10} = 1 > 0.$$

Since $\sum_{n=1}^{\infty} \frac{5}{3^n} = \sum_{n=1}^{\infty} 5 \left(\frac{1}{3}\right)^n$ is convergent (a geometric series with $r = \frac{1}{3} < 1$),

then $\sum_{n=1}^{\infty} \frac{5n^2 + 3n}{3^n(n^2 + 2)}$ is convergent.

Example (4): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{n + 3n^2}{\sqrt{1 + n^5}}$.

Solution :

Note that $a_n = \frac{3n^2 + n}{\sqrt{n^5 + 1}}$. Let $b_n = \frac{3n^2}{\sqrt{n^5}} = \frac{3n^2}{n^{\frac{5}{2}}} = \frac{3}{n^{\frac{1}{2}}}$.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left(\frac{3n^2 + n}{\sqrt{n^5 + 1}} \cdot \frac{n^{\frac{1}{2}}}{3} \right) = \lim_{n \rightarrow \infty} \frac{3n^{\frac{5}{2}} + n^{\frac{3}{2}}}{3\sqrt{n^5 + 1}} = 1 > 0.$$

Since $\sum_{n=1}^{\infty} \frac{3}{n^{\frac{1}{2}}}$ is divergent (a p -series, with $p = \frac{1}{2} < 1$), then $\sum_{n=1}^{\infty} \frac{n + 3n^2}{\sqrt{1 + n^5}}$ is divergent.

3.4.3 EXERCISES

1. (a). Use the Direct Comparison Test to show that the first series converges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n}{n^3 + 5}, \quad \sum_{n=2}^{\infty} \frac{1}{n^2}.$$

- (b). Use the Limit Comparison Test to show that that the first series converges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n}{n^3 - 5}, \quad \sum_{n=2}^{\infty} \frac{1}{n^2}.$$

2. (a). Use the Direct Comparison Test to show that the first series diverges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n^2 + n}{n^3 - 2}, \quad \sum_{n=2}^{\infty} \frac{1}{n}.$$

- (b). Use the Limit Comparison Test to show that that the first series diverges by comparing it to the second series.

$$\sum_{n=2}^{\infty} \frac{n^2 - n}{n^3 + 2}, \quad \sum_{n=2}^{\infty} \frac{1}{n}.$$

3. Which of the following inequalities can be used to show that $\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$ converges?

(a). $\frac{n}{n^3 + 1} \geq \frac{1}{n^3 + 1}$ (b). $\frac{n}{n^3 + 1} \leq \frac{1}{n}$ (c). $\frac{n}{n^3 + 1} \leq \frac{1}{n^2}$.

4. Determine whether the series converges or diverges.

(1). $\sum_{n=1}^{\infty} \frac{1}{n^3 + 8}$ (2). $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$ (3). $\sum_{n=1}^{\infty} \frac{n + 1}{n\sqrt{n}}$

(4). $\sum_{n=1}^{\infty} \frac{n - 1}{n^3 + 1}$ (5). $\sum_{n=1}^{\infty} \frac{9^n}{3 + 10^n}$ (6). $\sum_{n=1}^{\infty} \frac{6^n}{5^n - 1}$

(7). $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ (8). $\sum_{n=1}^{\infty} \frac{1 + \cos n}{e^n}$ (9). $\sum_{n=1}^{\infty} \frac{1}{n^n}$

(10). $\sum_{n=1}^{\infty} \frac{n + 1}{n^3 + n}$ (11). $\sum_{n=1}^{\infty} \frac{2 + \sin n}{n^2}$ (12). $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

3.5 Alternating Series and Absolute Convergence

3.5.1 Alternating Series

Definition : If $a_n \geq 0$, the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is called an alternating series.

Theorem : (Alternating Series Test)

If $a_n \geq 0$ and the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ satisfies :

- (1). The sequence $\{a_n\}$ is decreasing, i.e. $a_{n+1} \leq a_n$ for all $n \geq 1$.
- (2). $\lim_{n \rightarrow \infty} a_n = 0$.

Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.

Example (1): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Solution :

- (1). For all $n \geq 1$: $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$.

The sequence $\{a_n\} = \left\{ \frac{1}{n} \right\}$ is decreasing.

- (2). $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

By the alternating series test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent.

Example (2): Discuss the convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{4n^2 - 1}$.

Solution :

- (1). Let $f(x) = \frac{2x}{4x^2 - 1}$, then for $x \geq 1$:

$$f'(x) = \frac{2(4x^2 - 1) - 2x(8x)}{(4x^2 - 1)^2} = \frac{8x^2 - 2 - 16x^2}{(4x^2 - 1)^2} = \frac{-2 - 8x^2}{(4x^2 - 1)^2} < 0.$$

The sequence $\{a_n\} = \left\{ \frac{2n}{4n^2 - 1} \right\}$ is decreasing.

- (2). $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2n}{4n^2 - 1} = 0$.

By the alternating series test, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{4n^2 - 1}$ is convergent.

Example (3): Discuss the convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n+1}$.

Solution :

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0.$$

So, the alternating series test does not apply.

Note that if $b_n = (-1)^{n+1} \frac{n}{2n+1}$, then $\lim_{n \rightarrow \infty} b_{2n} = \frac{1}{2}$ and $\lim_{n \rightarrow \infty} b_{2n+1} = -\frac{1}{2}$.

So, $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} (-1)^{n+1} \frac{n}{2n+1}$ does not exist.

Therefore, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n+1}$ is divergent (by test of divergence).

3.5.2 Absolute Convergence and Conditional Convergence

Definition : (Absolute Convergence)

A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if the series of absolute values

$\sum_{n=1}^{\infty} |a_n|$ is convergent .

Example (4): The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$, where $p > 1$ is absolutely convergent.

Solution :

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^p} \right| = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent (a p -series, with $p > 1$).

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ is absolutely convergent.

Definition : (Conditional Convergence)

A series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if it is convergent but not

absolutely convergent. That is, if $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent.

Example (5): The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Solution :

Note that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent (see Example (1)).

But $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Theorem :

If a series is absolutely convergent then it is convergent.

Example (6): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$.

Solution :

$$\left| \frac{\cos n}{n^2} \right| = \frac{|\cos n|}{|n^2|} \leq \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent then by direct comparison test $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$ is convergent.

Therefore, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent and hence convergent.

Example (7): Discuss the convergence of the following series :

$$(a). \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1} \quad (b). \sum_{n=1}^{\infty} (-1)^n \frac{1}{1+n\sqrt{n}} \quad (c). \sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{(2n-3)^2}.$$

Solution :

$$(a). \sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{n+1}{n^2+1} \right| = \sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$$

Let $a_n = \frac{n+1}{n^2+1}$, put $b_n = \frac{1}{n}$, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n^2+1} \cdot n \right) = \lim_{n \rightarrow \infty} \frac{n^2+n}{n^2+1} = 1 > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, then $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{n+1}{n^2+1} \right|$ is divergent.

(1). Let $f(x) = \frac{x+1}{x^2+1}$, then for $x \geq 1$:

$$f'(x) = \frac{(1)(x^2+1) - (x+1)(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2-2x}{(x^2+1)^2} = \frac{-x^2-2x+1}{(x^2+1)^2} < 0.$$

Therefore, the sequence $\left\{ \frac{n+1}{n^2+1} \right\}$ is decreasing.

$$(2). \lim_{n \rightarrow \infty} \frac{n+1}{n^2+1} = 0.$$

From (1) and (2), the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1}$ is convergent.

Hence, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1}$ is conditionally convergent.

$$(b). \sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{1+n\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{1+n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{1+n^{\frac{3}{2}}}.$$

Let $a_n = \frac{1}{1+n^{\frac{3}{2}}}$, put $b_n = \frac{1}{n^{\frac{3}{2}}}$, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^{\frac{3}{2}}}{n^{\frac{3}{2}}+1} = 1 > 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{1+n^{\frac{3}{2}}}$ is convergent, then $\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{1+n\sqrt{n}} \right|$ is convergent.

Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{1+n\sqrt{n}}$ is absolutely convergent.

(c). Let $a_n = (-1)^n \frac{n^2+1}{(2n-3)^2} = (-1)^n \frac{n^2+1}{4n^2-12n+9}$

$\lim_{n \rightarrow \infty} a_{2n} = \frac{1}{4}$ and $\lim_{n \rightarrow \infty} a_{2n+1} = -\frac{1}{4}$.
Hence $\lim_{n \rightarrow \infty} a_n$ does not exist.

Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{(2n-3)^2}$ is divergent.

3.5.3 EXERCISES

1. Test the series for convergence or divergence.

$$(1). \frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \cdots \quad (2). \frac{1}{\ln 3} - \frac{1}{\ln 4} + \frac{1}{\ln 5} - \frac{1}{\ln 6} + \frac{1}{\ln 7} - \cdots$$

$$(3). \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n} \quad (4). \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}} \quad (5). \sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$$

$$(6). \sum_{n=1}^{\infty} (-1)^n e^{-n} \quad (7). \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4} \quad (8). \sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$$

$$(9). \sum_{n=1}^{\infty} \frac{n \cos(n\pi)}{n} \quad (10). \sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right) \quad (11). \sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$$

2. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$(1). \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \quad (2). \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[3]{n^2}} \quad (3). \sum_{n=0}^{\infty} (-1)^{n+1} \frac{n^2}{n^2+1}$$

$$(4). \sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1} \quad (5). \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} \quad (6). \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

$$(7). \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

3.6 The Ratio and Root Tests

3.6.1 The Ratio Test

Theorem : (the Ratio Test)

(1). If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(2). If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(3). If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the ratio test is inconclusive.

Example (1): Discuss the convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^4}{2^n}$.

Solution :

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^4}{2^{n+1}} \frac{2^n}{n^4} \right| = \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{n+1}{n} \right)^4 = \frac{1}{2}(1)^4 = \frac{1}{2} < 1.$$

Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{n^4}{2^n}$ is absolutely convergent.

Example (2): Discuss the convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{n!}$.

Solution :

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{5^{n+1}}{(n+1)!} \frac{n!}{5^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{5^{n+1}}{n! (n+1)} \frac{n!}{5^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{5}{n+1} = 0 < 1. \end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{n!}$ is absolutely convergent.

Example (3): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Solution :

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left[\frac{(n+1)^{n+1}}{(n+1)!} \frac{n!}{n^n} \right] = \lim_{n \rightarrow \infty} \left[\frac{(n+1)^n (n+1)}{n! (n+1)} \frac{n!}{n^n} \right] \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)^n}{n^n} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e > 1. \end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent.

Note that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$, using L'Hôpital's rule.

Example (4): Discuss the convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{3^n}$.

Solution : For $n \geq 2$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{\ln(n+1)}{3^{n+1}} \frac{3^n}{\ln(n)} \right| = \lim_{n \rightarrow \infty} \frac{1}{3} \frac{\ln(n+1)}{\ln(n)} \\ &= \lim_{n \rightarrow \infty} \frac{1}{3} \left(\frac{\frac{1}{n+1}}{\frac{1}{n}} \right) = \lim_{n \rightarrow \infty} \frac{1}{3} \frac{n}{n+1} = \frac{1}{3}(1) = \frac{1}{3} < 1. \end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{3^n}$ is convergent.

Notes :

(1). The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left(\frac{\frac{1}{n+1}}{\frac{1}{n}} \right) = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1.$$

(2). The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left(\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \right) = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^2 = (1)^2 = 1.$$

3.6.2 The Root Test

Theorem : (the Root Test)

(1). If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(2). If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(3). If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$, the root test is inconclusive.

Example (5): Discuss the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{n+3}{3n+2} \right)^n$.

Solution :

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(\frac{n+3}{3n+2} \right)^n \right|} = \lim_{n \rightarrow \infty} \left[\left(\frac{n+3}{3n+2} \right)^n \right]^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{n+3}{3n+2} = \frac{1}{3} < 1. \end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} \left(\frac{n+3}{3n+2} \right)^n$ is absolutely convergent.

Example (6): Discuss the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+1}{n+3}\right)^n$.

Solution :

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{2n+1}{n+3}\right)^n\right|} = \lim_{n \rightarrow \infty} \left[\left(\frac{2n+1}{n+3}\right)^n\right]^{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{2n+1}{n+3} = 2 > 1.$$

Therefore, $\sum_{n=1}^{\infty} \left(\frac{2n+1}{n+3}\right)^n$ is divergent.

Example (7): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n}$.

Solution :

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\frac{2^{3n+1}}{n^n}\right|} = \lim_{n \rightarrow \infty} \left(\frac{2^{3n+1}}{n^n}\right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{2^{3+\frac{1}{n}}}{n} = 0 < 1.$$

Therefore, $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n}$ is absolutely convergent.

Example (8): Discuss the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$.

Solution :

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{n+1}{n}\right)^n\right|} = \lim_{n \rightarrow \infty} \left[\left(\frac{n+1}{n}\right)^n\right]^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1.$$

So, The root test is inconclusive.

$$\text{But } \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0.$$

Therefore, $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$ is divergent (by test of divergence).

3.6.3 EXERCISES

1. What can you say about the series $\sum_{n=1}^{\infty} a_n$ in each of the following cases?

$$(a). \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 8 \quad (b). \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.8 \quad (c). \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

2. Use the Ratio Test to determine whether the series is convergent or divergent.

$$(1). \sum_{n=1}^{\infty} \frac{n}{5^n} \quad (2). \sum_{n=1}^{\infty} \frac{(-2)^n}{n^2} \quad (3). \sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{2^n n^3}$$

$$(4). \sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!} \quad (5). \sum_{n=1}^{\infty} \frac{1}{n!} \quad (6). \sum_{n=1}^{\infty} \frac{n}{100^n}$$

$$(7). \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

3. Use the Root Test to determine whether the series is convergent or divergent.

$$(1). \sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1} \right)^n \quad (2). \sum_{n=1}^{\infty} \frac{(-2)^n}{n^n} \quad (3). \sum_{n=1}^{\infty} \left(1 + \frac{1}{n} \right)^{n^2}$$

3.7 Strategy for Testing Series

(1). Divergence test :

If $\lim_{n \rightarrow \infty} a_n \neq 0$, or $\lim_{n \rightarrow \infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ is divergent.

$\sum_{n=1}^{\infty} \frac{n^2 + 2n}{n^2 + 4}$ is divergent, because $\lim_{n \rightarrow \infty} \frac{n^2 + 2n}{n^2 + 4} = 1 \neq 0$.

$\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{n+2}$ is divergent, because $\lim_{n \rightarrow \infty} (-1)^n \frac{n-1}{4n+2}$ does not exist.

(2). p -series : It has the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$:

If $p > 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent, like $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$.

If $p \leq 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent, like $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

(3). Geometric series : It has the form $\sum_{n=0}^{\infty} a r^n$ where $a \neq 0$ and $r \in \mathbb{R}$:

If $|r| < 1$ then $\sum_{n=0}^{\infty} a r^n$ is convergent, like $\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$.

If $|r| \geq 1$ then $\sum_{n=0}^{\infty} a r^n$ is divergent, like $\sum_{n=1}^{\infty} \frac{3^n}{2}$.

(4). Comparison test :

It is used when the series has a form that is similar to a p -series or a geometric series, it can be used also when a_n is a rational function or an algebraic function of n .

It is used on series of positive terms, it can be used on $\sum_{n=1}^{\infty} |a_n|$ to test for absolute convergence.

$\sum_{n=1}^{\infty} \frac{3}{n+2}$ is divergent, comparing it with $\sum_{n=1}^{\infty} \frac{1}{n}$.

$\sum_{n=1}^{\infty} \frac{2}{7^n + 1}$ is convergent, comparing it with $\sum_{n=1}^{\infty} \frac{1}{7^n}$.

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 5}$ is convergent, comparing it with $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$.

(5). Alternating series test :

It is used on the series $\sum_{n=1}^{\infty} (-1)^n a_n$, where the sequence $\{a_n\}$ is decreasing and

$\lim_{n \rightarrow \infty} a_n = 0$. Note that if $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} (-1)^n a_n$ is absolutely convergent.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 + 2}$ satisfies all the conditions of the alternating series test, hence it is absolutely convergent. Also, $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3 + 2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3 + 2}$ is convergent (using comparison test), hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 + 2}$ is absolutely convergent.

(6). The Ratio test :

it is used on series involving factorial or other products like a constant raised to a power n .

It is not used on p -series or rational functions or algebraic functions of n .

$\sum_{n=1}^{\infty} \frac{2^n}{n!}$ is convergent since $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0 < 1$.

(7). The Root test :

It is used on series of the form $\sum_{n=1}^{\infty} (a_n)^n$.

$\sum_{n=1}^{\infty} \left(\frac{3n^2 + 1}{4n^2 + 5} \right)^n$ is convergent, since $\lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{3n^2 + 1}{4n^2 + 5} \right)^n} = \frac{3}{4} < 1$.

(8). The Integral test :

It is used when $a_n = f(n)$, where f is a decreasing function on $[1, \infty)$ and $\int_1^{\infty} f(x) dx$ is easy to calculate.

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent, since $f(x) = \frac{1}{x(\ln x)^2}$ is decreasing on $[2, \infty)$ and

$$\begin{aligned} \int_2^{\infty} \frac{1}{x(\ln x)^2} &= \lim_{t \rightarrow \infty} \int_2^t (\ln x)^{-2} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \left[\frac{(\ln x)^{-1}}{-1} \right]_2^t = \lim_{t \rightarrow \infty} \left[\frac{-1}{\ln x} \right]_2^t \\ &= \lim_{t \rightarrow \infty} \left[\frac{-1}{\ln t} - \left(\frac{-1}{\ln 2} \right) \right] = 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2}. \end{aligned}$$

3.8 Power Series

3.8.1 Power Series

Definition : A power series is a series of the form $\sum_{n=1}^{\infty} c_n x^n$, where x is a variable and c_n 's are real constants called the coefficients.

Notes :

(1). If $c_n = 1$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} c_n x^n = \sum_{n=1}^{\infty} x^n$ is the geometric series and it converges when $|x| < 1$ and diverges when $|x| > 1$.

If $x = \frac{1}{2}$, then $\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges to 2.

If $x = 2$, then $\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} 2^n$ diverges.

(2). $\sum_{n=1}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$ is called a power series in $(x-a)$ or a power series centered at a or a power series about a .

Example (1): For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$ converge?

Solution :

Let $a_n = \frac{(x-2)^n}{n}$, using the ratio test :

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{n+1} \frac{n}{(x-2)^n} \right| = \lim_{n \rightarrow \infty} \frac{n}{n+1} |x-2| = |x-2|.$$

The series converges when $|x-2| < 1 \implies -1 < x-2 < 1 \implies 1 < x < 3$.

The series diverges when $|x-2| > 1 \implies x-2 < -1$ or $x-2 > 1$

$\implies x < 1$ or $x > 3$.

If $x = 1$ then $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is the convergent alternating harmonic series.

If $x = 3$ then $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Therefore, $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$ converges when $x \in [1, 3)$ or $1 \leq x < 3$.

Example (2): For what values of x does the series $\sum_{n=1}^{\infty} n! x^n$ converge?

Solution :

Let $a_n = n! x^n$, using the ratio test :

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = \lim_{n \rightarrow \infty} (n+1) |x|$$

If $x \neq 0$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$. Therefore the series diverges when $x \neq 0$.

If $x = 0$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$. Therefore the series converges when $x = 0$.

Example (3): For what values of x does the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converge?

Solution :

Let $a_n = \frac{x^n}{n!}$, using the ratio test :

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0 < 1.$$

Therefore, the series converges for all $x \in \mathbb{R}$.

3.8.2 Interval of Convergence

Theorem : For a power series $\sum_{n=1}^{\infty} c_n(x-a)^n$, there are only three possibilities:

- (1). The series converges only when $x = a$.
- (2). The series converges for all $x \in \mathbb{R}$.
- (3). There is a positive real number R such that the series converges when $|x - a| < R$ and diverges when $|x - a| > R$.

Notes :

(1). The positive real number R is called the radius of convergence. If the series converges only when $x = a$ then $R = 0$, and if the series converges for all $x \in \mathbb{R}$ then $R = \infty$.

(2). The interval of convergence of a power series is the interval that consists of all values of x for which the series converges.

If the series converges only when $x = a$ then the interval of convergence consists of only one point a .

If the series converges for all $x \in \mathbb{R}$ then the interval of convergence is $(-\infty, \infty)$.

If the radius of convergence is R , then interval of convergence is $(a - R, a + R)$.

Example (4): Find the radius of convergence and interval of convergence of

the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$.

Solution :

Let $a_n = \frac{(x-2)^n}{n^2+1}$, using the ratio test :

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2+1} \frac{n^2+1}{(x-2)^n} \right| = \lim_{n \rightarrow \infty} \frac{n^2+1}{(n+1)^2+1} |x-2| \\ &= \lim_{n \rightarrow \infty} \frac{n^2+1}{n^2+2n+2} |x-2| = (1) |x-2| = |x-2|. \end{aligned}$$

The series converges when $|x-2| < 1$, so the radius of convergence is $R = 1$.

Therefore, the series converges on $(1, 3)$.

If $x = 1$, then $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ which is convergent.

If $x = 3$, then $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1} = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ which is convergent.

Therefore, the interval of convergence is $[1, 3]$.

Example (5): Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n 5^n}$.

Solution :

Let $a_n = \frac{(2x-6)^n}{n 5^n}$, using the ratio test :

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(2x-6)^{n+1}}{(n+1) 5^{n+1}} \frac{n 5^n}{(2x-6)^n} \right| = \lim_{n \rightarrow \infty} \frac{n}{5(n+1)} |2x-6| \\ &= \lim_{n \rightarrow \infty} \frac{n}{5n+5} |2x-6| = \frac{1}{5} |2x-6| = \frac{|2x-6|}{5}. \end{aligned}$$

The series converges when $\frac{|2x-6|}{5} < 1 \implies 2|x-3| < 5 \implies |x-3| < \frac{5}{2}$, so the radius of convergence is $R = \frac{5}{2}$.

Therefore, the series converges on $\left(3 - \frac{5}{2}, 3 + \frac{5}{2}\right) = \left(\frac{1}{2}, \frac{11}{2}\right)$.

If $x = \frac{1}{2}$, then $\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n 5^n} = \sum_{n=1}^{\infty} \frac{(-5)^n}{n 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is convergent.

If $x = \frac{11}{2}$, then $\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n 5^n} = \sum_{n=1}^{\infty} \frac{5^n}{n 5^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent.

Therefore, the interval of convergence is $\left[\frac{1}{2}, \frac{11}{2}\right)$.

Example (6): Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{n}{7^n} (x+3)^n$.

Solution :

Let $a_n = \frac{n}{7^n} (x+3)^n$, using the ratio test :

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x+3)^{n+1}}{7^{n+1}} \frac{7^n}{n(x+3)^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{7n} |x+3| = \frac{1}{7} |x+3| = \frac{|x+3|}{7}. \end{aligned}$$

The series converges when $\frac{|x+3|}{7} < 1 \implies |x+3| < 7$, so the radius of convergence is $R = 7$.

Therefore, the series converges on $(-3-7, -3+7) = (-10, 4)$.

If $x = -10$, then $\sum_{n=1}^{\infty} \frac{n}{7^n} (x+3)^n = \sum_{n=1}^{\infty} (-1)^n n$ which is divergent.

If $x = 4$, then $\sum_{n=1}^{\infty} \frac{n}{7^n} (x+3)^n = \sum_{n=1}^{\infty} n$ which is divergent.

Therefore, the interval of convergence is $(-10, 4)$.

3.8.3 EXERCISES

1. Find the radius of convergence and interval of convergence of the power series.

$$(1). \sum_{n=1}^{\infty} \frac{x^n}{n!} \quad (2). \sum_{n=1}^{\infty} (-1)^n n x^n \quad (3). \sum_{n=1}^{\infty} \sqrt{n} x^n$$

$$(4). \sum_{n=1}^{\infty} \frac{n}{5^n} x^n \quad (5). \sum_{n=1}^{\infty} \frac{5^n}{n} x^n \quad (6). \sum_{n=1}^{\infty} \frac{x^n}{n 3^n}$$

$$(7). \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2} \quad (8). \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (9). \sum_{n=1}^{\infty} n^n x^n$$

$$(10). \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 5^n} x^n \quad (11). \sum_{n=1}^{\infty} \frac{x^{2n}}{n!} \quad (12). \sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$$

$$(13). \sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n} \quad (14). \sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$$

3.9 Representations of Functions as Power Series

3.9.1 Representations of Functions using Geometric Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}, \text{ when } |x| < 1.$$

Example (1): Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

Solution :

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \text{ where } |-x^2| < 1.$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - \dots, \text{ where } |-x^2| < 1.$$

$$|-x^2| < 1 \implies |x^2| < 1 \implies |x| < 1.$$

If $x = \pm 1$, the series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ diverge.

Therefore, The interval of convergence is $(-1, 1)$.

Example (2): Express $\frac{x^2}{1-x^6}$ as the sum of a power series and find the interval of convergence.

Solution :

$$\frac{x^2}{1-x^6} = x^2 \frac{1}{1-x^6} = x^2 \sum_{n=0}^{\infty} (x^6)^n = x^2 \sum_{n=0}^{\infty} x^{6n} = \sum_{n=0}^{\infty} x^{6n+2}, \text{ where } |x^6| < 1.$$

$$|x^6| < 1 \implies |x| < 1.$$

If $x = \pm 1$, the series $\sum_{n=0}^{\infty} x^{6n+2}$ diverge.

Therefore, The interval of convergence is $(-1, 1)$.

Example (3): Express $\frac{x}{5-x}$ as the sum of a power series and find the interval of convergence.

Solution :

$$\frac{x}{5-x} = \frac{x}{5(1-\frac{x}{5})} = \frac{x}{5} \frac{1}{1-\frac{x}{5}} = \frac{x}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n = \frac{x}{5} \sum_{n=0}^{\infty} \frac{x^n}{5^n} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}},$$

where $\left|\frac{x}{5}\right| < 1$.

$$\left|\frac{x}{5}\right| < 1 \implies \frac{|x|}{5} < 1 \implies |x| < 5.$$

If $x = \pm 5$, the series $\sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}}$ diverge.

Therefore, The interval of convergence is $(-5, 5)$.

Example (4): Express $\frac{1}{2-x}$ as power series in $x-1$, and find the interval of convergence.

Solution :

$$\frac{1}{2-x} = \frac{1}{1-x+1} = \frac{1}{1-(x-1)} = \sum_{n=0}^{\infty} (x-1)^n, \text{ where } |x-1| < 1.$$

$$|x-1| < 1 \implies -1 < x-1 < 1 \implies 0 < x < 2.$$

If $x=0$ or $x=2$, the series $\sum_{n=0}^{\infty} (x-1)^n$ diverge.

Therefore, The interval of convergence is $(0, 2)$.

3.9.2 Differentiation and Integration of Power Series

Theorem : If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$(1). f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$$

$$(2). \int f(x) dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \dots$$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}, \text{ where } C \text{ is a constant.}$$

The radii of convergence of the power series in (1) and (2) are both R .

Notes : Equations (1) and (2) in the last theorem can be rewritten as :

$$(1). \frac{d}{dx} \left[\sum_{n=0}^{\infty} c_n(x-a)^n \right] = \sum_{n=0}^{\infty} \left[\frac{d}{dx} c_n(x-a)^n \right].$$

$$(2). \int \left[\sum_{n=0}^{\infty} c_n(x-a)^n \right] dx = \sum_{n=0}^{\infty} \left[\int c_n(x-a)^n dx \right].$$

Example (5): Express $\frac{1}{(1-x)^2}$ as a power series, and find its radius of convergence.

Solution :

$$\frac{d}{dx} \left[\frac{1}{1-x} \right] = \frac{d}{dx} [(1-x)^{-1}] = (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}.$$

Note that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$, where $|x| < 1$.

$$\frac{1}{(1-x)^2} = \frac{d}{dx} \left[\frac{1}{1-x} \right] = \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{d}{dx} [x^n] = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}.$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

Since the radius of convergence of $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ is $R = 1$ then the radius of convergence of $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ is also $R = 1$.

Example (6): Express $\ln(1+x)$ as a power series, and find its radius of convergence.

Solution :

$$\int \frac{1}{1+x} dx = \ln(1+x) + c$$

Note that $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$, where $|x| < 1$.

$$\begin{aligned} \ln(1+x) &= \int \frac{1}{1+x} dx = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n \right] dx = \sum_{n=0}^{\infty} \left[\int (-1)^n x^n dx \right] \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C, \text{ where } |x| < 1. \end{aligned}$$

Put $x = 0$ in the last equation : $\ln(1+0) = 0 + C \implies C = 0$.

$$\text{Therefore, } \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

The radius of convergence of $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ is $R = 1$.

Example (7): Express $\tan^{-1} x$ as a power series, and find its radius of convergence.

Solution :

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$$

From Example (1), $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, where $|x| < 1$.

$$\begin{aligned} \tan^{-1} x &= \int \frac{1}{1+x^2} dx = \int \left[\sum_{n=0}^{\infty} (-1)^n x^{2n} \right] dx = \sum_{n=0}^{\infty} \int [(-1)^n x^{2n}] dx \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C, \text{ where } |x| < 1. \end{aligned}$$

Put $x = 0$ in the last equation : $\tan^{-1}(0) = 0 + C \implies C = 0$.

$$\text{Therefore, } \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

The radius of convergence of $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ is $R = 1$.

3.9.3 EXERCISES

1. Find a power series representation for the function and determine the interval of convergence.

$$(1). f(x) = \frac{x}{x+1} \quad (2). f(x) = \frac{1}{1-x^2} \quad (3). f(x) = \frac{5}{1-4x^2}$$

$$(4). f(x) = \frac{2}{3-x} \quad (5). f(x) = \frac{4}{2x+3} \quad (6). f(x) = \frac{x^2}{x^4+16}$$

$$(7). f(x) = \frac{x}{2x^2+1} \quad (8). f(x) = \frac{x-1}{x+2}$$

2. (a). Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(x+1)^2}, \text{ What is the radius of convergence?}$$

(b). Use part (a) to find a power series for $f(x) = \frac{1}{(x+1)^3}$.

(c). Use part (b) to find a power series for $f(x) = \frac{x^2}{(x+1)^3}$.

3. Find a power series representation for the function and determine the radius of convergence.

(1). $f(x) = \ln(5-x)$

(2). $f(x) = x^2 \tan^{-1}(x^3)$.

3.10 Taylor and Maclaurin Series

3.10.1 Definitions of Taylor Series and Maclaurin Series

Theorem : If f has a power series representation at a , that is $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$,

where $|x-a| < R$, then the coefficient c_n is given by $c_n = \frac{f^{(n)}(a)}{n!}$.

Therefore, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$,

or $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$.

Definition : $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the Taylor series of the function f at a (or about a , or centered at a).

Definition : If $a = 0$ in the Taylor series of f , then $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ is

called the Maclaurin series of f . In this case :

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$.

Example (1): Find the Maclaurin Series and its radius of convergence for the function $f(x) = \frac{1}{1-x}$.

Solution :

$$f(x) = \frac{1}{1-x} \implies f(0) = 1 = 0! .$$

$$f'(x) = \frac{1}{(1-x)^2} \implies f'(0) = 1 = 1! .$$

$$f''(x) = \frac{2}{(1-x)^3} \implies f''(0) = 2 = 2! .$$

$$f'''(x) = \frac{6}{(1-x)^4} \implies f'''(0) = 6 = 3! .$$

$$f^{(n)}(x) = \frac{1 \cdot 2 \cdot 3 \cdots n}{(1-x)^{n+1}} \implies f^{(n)}(0) = n! .$$

The Maclaurin series for $f(x) = \frac{1}{1-x}$ is $\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{n!}{n!}x^n = \sum_{n=0}^{\infty} x^n$.

Let $a_n = x^n$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{x^n} \right| = |x|$.

Therefore, the Maclaurin series converges when $|x| < 1$. Hence $R = 1$.

Example (2): Find the Maclaurin Series and its radius of convergence for the function $f(x) = e^x$.

Solution :

Note that $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$.

The Maclaurin series for $f(x) = e^x$ is $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Let $a_n = \frac{x^n}{n!}$, then $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0 < 1$.

Therefore, the Maclaurin series converges for all $x \in \mathbb{R}$. Hence $R = \infty$.

3.10.2 Remainder of a Taylor series

Definition :

Suppose $f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!} (x-a)^i$ then $f(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i + \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!} (x-a)^i$,

$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i$ is called the n^{th} -degree polynomial of f at a , and

$R_n(x) = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!} (x-a)^i$ is called the remainder of the Taylor series.

Theorem : If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the n^{th} -degree polynomial of f at a , and If $\lim_{n \rightarrow \infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Theorem : (Taylor's Inequality)

Suppose $f(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i + \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!} (x-a)^i = T_n(x) + R_n(x)$.

If $|f^{(i)}(x)| \leq M$ for $|x-a| \leq d$ and for all $i \geq n+1$, where M and d are positive real numbers, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality : $|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$, for $|x-a| \leq d$.

In this case, $\lim_{n \rightarrow \infty} |R_n(x)| = 0$, hence $\lim_{n \rightarrow \infty} R_n(x) = 0$ for all values of x .

Example (3): Show that e^x is equal to the sum of its Maclaurin series.

Solution :

$f(x) = e^x \implies f^{(i)}(x) = e^x$ for all $i \geq 1$.

If $|x| \leq d$ where $d > 0$, then $|f^{(i)}(x)| = e^x \leq e^d$, for all $i \geq n+1$.

So Taylor's inequality, with $a = 0$ and $M = e^d$ is

$|R_n(x)| \leq \frac{e^d}{(n+1)!} |x|^{n+1}$ for $|x| \leq d$.

Therefore, $\lim_{n \rightarrow \infty} |R_n(x)| = \lim_{n \rightarrow \infty} \frac{e^d}{(n+1)!} |x|^{n+1} = 0$.

So, $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \dots$, for all $x \in \mathbb{R}$.

Note : If $x = 1$ then $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$.

Example (4): Find the Taylor series of $f(x) = e^x$ at $a = 2$.

Solution :

$$f(x) = e^x \implies f^{(n)}(x) = e^x \implies f^{(n)}(2) = e^2, \text{ for all } n.$$

$$\text{So, } e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n = \sum_{n=0}^{\infty} \frac{e^2(x-2)^n}{n!} \text{ for all } x \in \mathbb{R}.$$

$$\text{Another Solution : } e^{x-2} = \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} \implies \frac{e^x}{e^2} = \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$$

$$\implies e^x = e^2 \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^2(x-2)^n}{n!}.$$

Example (5): Write the function $f(x) = e^{x^2-1}$ as a Taylor series of x .

Solution :

$$f(x) = e^{x^2-1} = e^{-1}e^{x^2} = e^{-1} \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{e} \frac{x^{2n}}{n!} \text{ for all } x.$$

3.10.3 Taylor Series of Important Functions

Example (6): Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x .

Solution :

$$f(x) = \sin x \implies f(0) = \sin(0) = 0.$$

$$f'(x) = \cos x \implies f'(0) = \cos(0) = 1.$$

$$f''(x) = -\sin x \implies f''(0) = -\sin(0) = 0.$$

$$f'''(x) = -\cos x \implies f'''(0) = -\cos(0) = -1.$$

$$f^{(4)}(x) = \sin x \implies f^{(4)}(0) = \sin(0) = 0.$$

$$\text{Therefore, } \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Since $f^{(i)}(x) = \pm \sin x$ or $\pm \cos x$ for all $i \geq 1$, then $|f^{(i)}(x)| \leq 1$.

Taylor's inequality, with $a = 0$ and $M = 1$ is

$$|R_n(x)| \leq \frac{1}{(n+1)!} |x|^{n+1} \text{ for } |x| \leq d, \text{ where } d > 0.$$

$$\text{Therefore, } \lim_{n \rightarrow \infty} |R_n(x)| = \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

$$\text{So, } \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \text{ for all } x.$$

Example (7): Find the Maclaurin series for $\cos x$ and prove that it represents $\cos x$ for all x .

Solution :

$$f(x) = \cos x \implies f(0) = \cos(0) = 1.$$

$$f'(x) = -\sin x \implies f'(0) = -\sin(0) = 0.$$

$$f''(x) = -\cos x \implies f''(0) = -\cos(0) = -1.$$

$$f'''(x) = \sin x \implies f'''(0) = \sin(0) = 0.$$

$$f^{(4)}(x) = \cos x \implies f^{(4)}(0) = \cos(0) = 1.$$

$$\text{Therefore, } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} .$$

Since $f^{(i)}(x) = \pm \sin x$ or $\pm \cos x$ for all $i \geq 1$, then $|f^{(i)}(x)| \leq 1$.

Taylor's inequality, with $a = 0$ and $M = 1$ is

$$|R_n(x)| \leq \frac{1}{(n+1)!} |x|^{n+1} \text{ for } |x| \leq d, \text{ where } d > 0.$$

$$\text{Therefore, } \lim_{n \rightarrow \infty} |R_n(x)| = \lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0 .$$

$$\text{So, } \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \text{ for all } x .$$

Example (8): Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

Solution :

$$f(x) = (1+x)^k \implies f(0) = 1 .$$

$$f'(x) = k(1+x)^{k-1} \implies f'(0) = k .$$

$$f''(x) = k(k-1)(1+x)^{k-2} \implies f''(0) = k(k-1) .$$

$$f'''(x) = k(k-1)(k-2)(1+x)^{k-3} \implies f'''(0) = k(k-1)(k-2) .$$

$$f^{(n)}(x) = k(k-1)\dots(k-n+1)(1+x)^{k-n} \implies f^{(n)}(0) = k(k-1)\dots(k-n+1) .$$

$$\text{So, } (1+x)^k = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\dots(k-n+1)}{n!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n .$$

Let $a_n = \frac{k(k-1)\dots(k-n+1)}{n!} x^n$ for all $n \geq 1$.

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{k(k-1)\dots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \frac{n!}{k(k-1)\dots(k-n+1)x^n} \right|$$

$$= \left| \frac{k-n}{n+1} x \right| = \left| \frac{1 - \frac{k}{n}}{1 + \frac{1}{n}} \right| |x| .$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1 - \frac{k}{n}}{1 + \frac{1}{n}} \right| |x| = |x| .$$

So, $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ converges when $|x| < 1$. Therefore, $R = 1$.

Note : $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ is called the binomial series.

Example (9): For $f(x) = \frac{1}{\sqrt{4-x}}$, find the Maclaurin series and its radius of convergence.

Solution :

$$f(x) = \frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4(1-\frac{x}{4})}} = \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2} \left(1 + \left(-\frac{x}{4} \right) \right)^{-\frac{1}{2}} .$$

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2} \left(1 + \left(-\frac{x}{4} \right) \right)^{-\frac{1}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(-\frac{x}{4} \right)^n, \text{ where } \left| -\frac{x}{4} \right| < 1 .$$

$$\left| -\frac{x}{4} \right| < 1 \implies \left| \frac{x}{4} \right| < 1 \implies |x| < 4 .$$

The radius of convergence of the Maclaurin series is $R = 4$.

Table of Important Series :

No.	Series	R
1	$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$	1
2	$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$	1
3	$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$	1
4	$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \dots$	1
5	$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	∞
6	$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$	∞
7	$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$	∞

3.10.4 New Taylor Series from Old

Example (10): Find the Maclaurin series for $f(x) = x^2 \cos x$.

Solution :

$$f(x) = x^2 \cos x = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n)!} \text{ for all } x.$$

Example (11): Find the Maclaurin series for $f(x) = \ln(1 + 4x^2)$.

Solution :

$$f(x) = \ln(1 + 4x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(4x^2)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4^n x^{2n}}{n},$$

Where $|4x^2| < 1 \implies |x|^2 < \frac{1}{4} \implies |x| < \frac{1}{2}$.

Example (12): Find the function represented by the power series $\sum_{n=0}^{\infty} (-1)^n \frac{3^n x^n}{n!}$.

Solution :

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = e^{-3x}.$$

Example (13): Find the sum of the series $\frac{1}{1(2)} - \frac{1}{2(2^2)} + \frac{1}{3(2^3)} - \dots$.

Solution :

$$\begin{aligned} \frac{1}{1(2)} - \frac{1}{2(2^2)} + \frac{1}{3(2^3)} - \dots &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n(2^n)} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n} \\ &= \ln\left(1 + \frac{1}{2}\right) = \ln\left(\frac{3}{2}\right). \end{aligned}$$

Example (14): Use series to evaluate $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$.

Solution : For all $x \neq 0$,

$$\begin{aligned} \frac{e^x - 1 - x}{x^2} &= \frac{1}{x^2} \left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right) - 1 - x \right] \\ &= \frac{1}{x^2} \left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right) = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \\ \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x \rightarrow 0} \left(\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \right) = \frac{1}{2}. \end{aligned}$$

3.10.5 EXERCISES

1. Use the definition of a Taylor series to find the first four nonzero terms of the series for $f(x)$ centered at the given value of a .

$$(1). f(x) = xe^x, \quad a = 0 \quad (2). f(x) = \frac{1}{1+x}, \quad a = 2$$

$$(3). f(x) = \ln x, \quad a = 1 \quad (4). f(x) = \cos^2 x, \quad a = 0$$

2. Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. Also find the associated radius of convergence.

$$(1). f(x) = (1-x)^{-2} \quad (2). f(x) = \ln(1+x) \quad (3). f(x) = \cos x$$

$$(4). f(x) = 2x^4 - 3x^2 + 3 \quad (5). f(x) = x \cos x \quad (6). f(x) = \sinh x$$

3. Find the Taylor series for $f(x)$ centered at the given value of a . Also find the associated radius of convergence.

$$(1). f(x) = x^5 + 2x^3 + x, \quad a = 2 \quad (2). f(x) = \ln x, \quad a = 2$$

$$(3). f(x) = \sin x, \quad a = \pi \quad (4). f(x) = \cos x, \quad a = \frac{\pi}{2}$$

4. Show that : (a). $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ (b). $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

$$\text{Hint : use } \sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2} .$$