King Saud University
College of Sciences
Department of Mathematics

# MATH 201
# MULTIVARIABLE CALCULUS

## CLASS NOTES
## DRAFT - January, 2026

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# Contents

# Chapter 1

# Partial Derivatives

## 1.1 Functions of several variables

### 1.1.1 Functions of two variables

**Definition:** A function $f$ of two variables is a map that assigns to each ordered pair of real numbers $(x, y) \in D \subseteq \mathbb{R}^2$ a unique real number denoted by $f(x, y)$. The set $D$ is the domain of $f$ and its range is $\{f(x, y) | (x, y) \in D\}$.

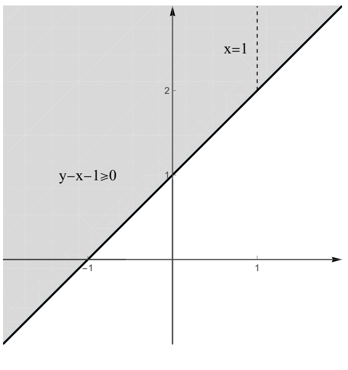
**Example (1):** If $f(x, y) = \dfrac{\sqrt{y-x-1}}{x-1}$, evaluate $f(2, 7)$, find the domain of $f$ and sketch it.

Solution : $f(2, 7) = \dfrac{\sqrt{7-2-1}}{2-1} = \dfrac{\sqrt{4}}{1} = 2.$

The domain of $f$ is the set $D = \big\{(x, y) \in \mathbb{R}^2 \mid y - x - 1 \geq 0\ ,\ x \neq 1\big\}$.

So, $D = \big\{(x, y) \in \mathbb{R}^2 \mid y \geq x + 1\ ,\ x \neq 1\big\}$.

$y - x - 1 \geq 0 \implies y \geq x + 1$ represents the points on and above the line $y = x + 1$.
$x \neq 1$ means the point on the line $x = 1$ must be excluded from the domain .

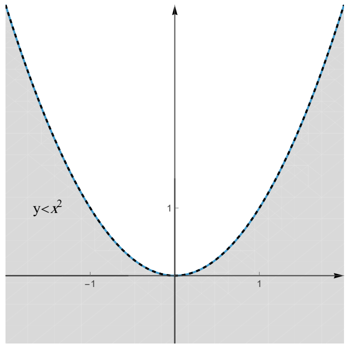



**Example (2):** If $f(x, y) = \ln(x^2 - y)$, find the domain of $f$ and sketch it.

Solution :

The domain of $f$ is the set
$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y > 0\}$.
So, $D = \{(x, y) \in \mathbb{R}^2 \mid y < x^2\}$.

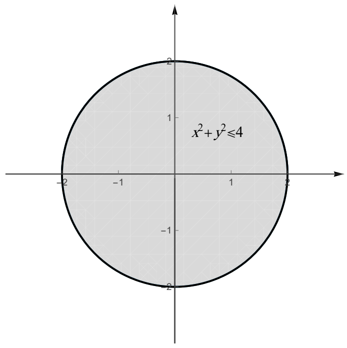
$x^2 - y > 0 \implies y < x^2$
represents the points below the parabola $y = x^2$.




**Example (3):** If $f(x, y) = \sqrt{4 - x^2 - y^2}$, find the domain and range of $f$.
Solution :
The domain of $f$ is the set
$D = \{(x, y) \in \mathbb{R}^2 \mid 4 - x^2 - y^2 \geq 0\}$.
So, $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$.

$4 - x^2 - y^2 \geq 0 \implies x^2 + y^2 \leq 4$
represents the points on and inside the disk of center $(0, 0)$ and radius 2.

Note that $0 \leq \sqrt{4 - x^2 - y^2}$
and $\sqrt{4 - (x^2 + y^2)} \leq \sqrt{4} = 2$
So, the range of $f$ is
$\{z \in \mathbb{R} \mid 0 \leq z \leq 2\} = [0, 2]$ .




### 1.1.2 Graphs

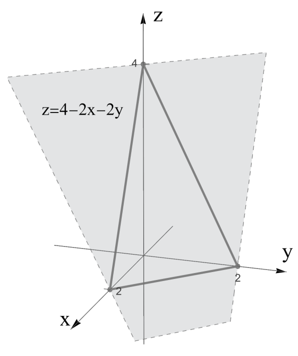
**Definition:** If $f$ is a function of two variables with domain $D \subseteq \mathbb{R}^2$, then the graph of $f$ is the set of all points $(x, y, z) \in \mathbb{R}^3$, such that $z = f(x, y)$ and $(x, y) \in D$.

**Example (4):** Sketch the graph of the function $f(x, y) = 4 - 2x - 2y$ .
Solution:
$z = 4 - 2x - 2y \implies 2x + 2y + z = 4$
It represents a plain.

To find the $x$-intercept, put $y = z = 0$
$2x = 4 \implies x = 2$.
So the $x$-intercept is $(2, 0, 0)$.

To find the $y$-intercept, put $x = z = 0$
$2y = 4 \implies y = 2$.
So the $y$-intercept is $(0, 2, 0)$.

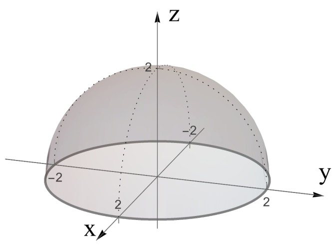
To find the $z$-intercept, put $x = y = 0$
$z = 4$, So the $z$-intercept is $(0, 0, 4)$.



**Example (5):** Sketch the graph of the function $f(x, y) = \sqrt{4 - x^2 - y^2}$ .
Solution:

$z = \sqrt{4 - x^2 - y^2}$
$\Longrightarrow\ z^2 = 4 - x^2 - y^2$
$\Longrightarrow\ x^2 + y^2 + z^2 = 2^2$
Note that $z = \sqrt{4 - x^2 - y^2} \geq 0$ .

It represents the upper half of the sphere centered at the origin and its radius is 2.

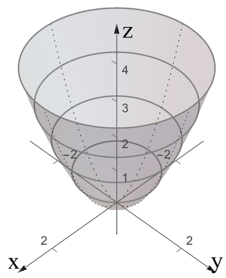



**Example (6):** Sketch the graph of the function $f(x, y) = x^2 + y^2$ .
Solution:

The domain of $f$ is $\mathbb{R}^2$.

$z = x^2 + y^2 \geq 0$
For each value of $z > 0$ ,
$x^2 + y^2 = z$ represents a circle centered at the origin and its radius is $\sqrt{z}$.

$f(x, y) = x^2 + y^2$ represents a paraboloid.




### 1.1.3 Level curves

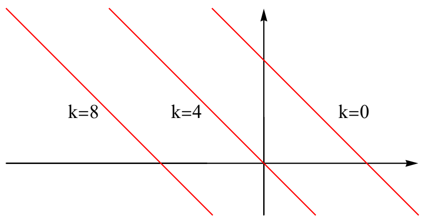
**Definition:** A level curve of a function $f(x, y)$ is the curve $f(x, y) = k$, where $k$ is a constant (in the range of $f$ ).

**Example (7):** Sketch the level curves of the function $f(x, y) = 4 - 2x - 2y$ for the values $k = 0, 4, 8$.
Solution:
$4 - 2x - 2y = k$.
$\Longrightarrow\ 2x + 2y = 4 - k$

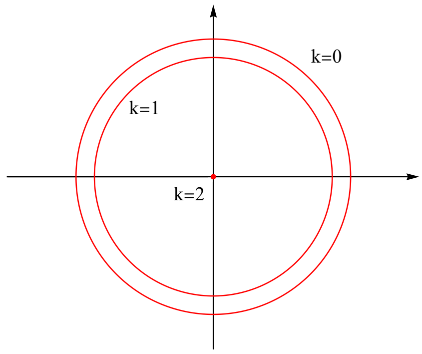
For $k = 0 :\ 2y + 2x = 4$
$\Longrightarrow\ y = -x + 2$

For $k = 4 :\ 2y + 2x = 0$
$\Longrightarrow\ y = -x$

For $k = 8 :\ 2y + 2x = -4$
$\Longrightarrow\ y = -x - 2$



**Example (8):** Sketch the level curves of the function $f(x, y) = \sqrt{4 - x^2 - y^2}$ for the values $k = 0, 1, 2$.
Solution:
$\sqrt{4 - x^2 - y^2} = k$.
$\implies 4 - x^2 - y^2 = k^2$
$\implies x^2 + y^2 = 4 - k^2$

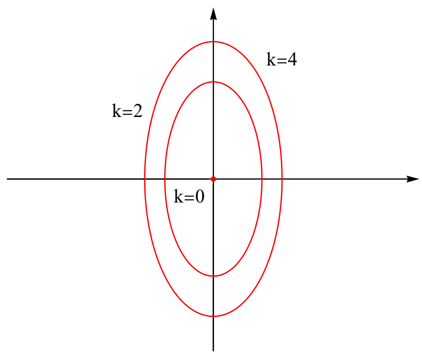



For $k = 0 : x^2 + y^2 = 4$
Circle: center is $(0, 0)$, radius $= 2$.

For $k = 1 : x^2 + y^2 = 3$
Circle: center is $(0, 0)$, radius $= \sqrt{3}$.

For $k = 2 : x^2 + y^2 = 0$
Just one point, the origin.

**Example (9):** Sketch the level curves of the function $f(x, y) = 4x^2 + y^2$ for the values $k = 0, 2, 4$.
Solution:

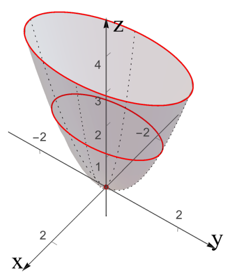



Note that the domain is $\mathbb{R}^2$.

$4x^2 + y^2 = k$.

For $k = 0 : 4x^2 + y^2 = 0$
$\implies x = 0 \ , \ y = 0$
The level curve is the origin.

For $k > 0 : 4x^2 + y^2 = k$
$\implies \dfrac{x^2}{\left(\frac{k}{4}\right)} + \dfrac{y^2}{k} = 1$
The level curve is an ellipse centered at the origin, the major axis lies on the $y$-axis and the minor axis lies on the $x$-axis.




Note that the graph of $f$ is an elliptic paraboloid.

### 1.1.4 Functions of three variables

**Definition:** A function $f$ of two variables is a map that assigns to each ordered pair of real numbers $(x, y, z) \in D \subseteq \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. The set $D$ is the domain of $f$ and its range is $\{f(x, y, z)|(x, y, z) \in D\}$.

**Example (10):** Find the domain of $f(x, y, z) = \ln(z - x) + yz \sin x$ .
Solution : $f(x, y, z)$ is defined when $z - x > 0$.
Therefore, $D = \left\{(x, y, z) \in \mathbb{R}^3 \mid z > x\right\}$ .

**Example (11):** Find the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$ .
Solution : $x^2 + y^2 + z^2 = k$, where $k \geq 0$.
If $k = 0$, then the level surface is just the origin $(0, 0, 0) \in \mathbb{R}^3$ .
If $k > 0$, then the level surface is $x^2 + y^2 + z^2 = \left(\sqrt{k}\right)^2$, which is a sphere centered at the origin, and its radius is $\sqrt{k}$ .

### 1.1.5 EXERCISES

1. Let $f(x, y) = x^2 \ln(x + y)$

   (a). Evaluate $f(3, 1)$.

   (b). Find and sketch the domain of $f$.

   (c). Find the range of $f$.

2. Let $f(x, y, z) = \ln\left(z - \sqrt{x^2 + y^2}\right)$.

   (a). Evaluate $f(4, 23, 6)$.

   (b). Find and describe the domain of $f$.

3. Find and sketch the domain of the following:

   (a). $f(x, y) = \sqrt{x - 2} + \sqrt{y - 1}$.

   (b). $f(x, y) = \ln(9 - x^2 - y^2)$.

   (c). $f(x, y) = \dfrac{\ln(2 - x)}{1 - x^2 - y^2}$.

(*) Find and sketch the domain of the following:

   (a). $f(x, y) = \sqrt{25 - x^2 - y^2} + \ln(2 + x)$.

   (b). $f(x, y) = \cos(x + y) + \dfrac{x^2 - y^2 - 3}{\sqrt{x + y - 4}}$.

   (c). $f(x, y) = \sqrt{x + y^2} + \sqrt{y - x^2}$.

# 1.2 Limits and Continuity

## 1.2.1 Limits of Functions of Two Variables

**Definition:** Let $f$ be a function of two variables whose domain $D \subseteq \mathbb{R}^2$ includes points arbitrarily close to $(a, b)$. Then we say that the limit of $f(x, y)$ as $(x, y)$ approaches $(a, b)$ is $L$ and we write $\lim\limits_{(x,y)\to(a,b)} f(x, y) = L$,
if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$ .

## 1.2.2 Showing That a Limit Does Not Exist

**NOTE:** If $f(x, y) \to L_1$ as $(x, y) \longrightarrow (a, b)$ along a path $C_1$ and $f(x, y) \to L_2$ as $(x, y) \longrightarrow (a, b)$ along a different path $C_2$, where $L_1 \neq L_2$, then $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ does not exist.

**Example (1):** Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Solution: Note that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2} \qquad \left(\dfrac{0}{0}\right)$ .
Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$ .
$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{x^2(1 - m^2)}{x^2(1 + m^2)} = \frac{1 - m^2}{1 + m^2} .$$
Note that the limit depends only on $m$.
Let $C_1$ be the path $y = 0$ (where m=0) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2} = \dfrac{1 - 0}{1 + 0} = 1$ .
Let $C_2$ be the path $y = x$ (where m=1) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2} = \dfrac{1 - 1}{1 + 1} = 0$ .
Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist.

**Example (2):** Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$ does not exist.

Solution: Note that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2} \qquad \left(\dfrac{0}{0}\right)$ .
Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$ .
$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = \lim_{x\to 0} \frac{mx^2}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{mx^2}{x^2(1 + m^2)} = \frac{m}{1 + m^2} .$$
Note that the limit depends only on $m$.
Let $C_1$ be the path $y = 0$ (where m=0) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2} = \dfrac{0}{1 + 0} = 0$ .
Let $C_2$ be the path $y = x$ (where m=1) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2} = \dfrac{1}{1 + 1} = \dfrac{1}{2}$ .
Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$ does not exist.

**Example (3):** Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^4}$ does not exist.

Solution: Note that $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} \qquad \left(\frac{0}{0}\right)$ .

Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$ .

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{x(m^2x^2)}{x^2+m^4x^4} = \lim_{x\to 0} \frac{mx^3}{x^2(1+m^4\ x^2)} = \lim_{x\to 0} \frac{mx}{1+m^4\ x^2} = 0$$

.

Note that the limit here depends on $x$ and $m$ .

Let $C_1$ be the path $y = 0$ (where m=0) then $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = 0$ .

Let $C_2$ be the path $x = y^2$ (the parabola with vertex (0,0) and opens to the right) then $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^2\ y^2}{(y^2)^2+y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$ .

Therefore, $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist.

**Example (4):** Show that $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$ does not exist.

Solution: Note that $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2} \qquad \left(\frac{0}{0}\right)$ .

Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$ .

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2} = \lim_{x\to 0} \frac{\sin(mx^2)}{x^2+m^2x^2} = \lim_{x\to 0} \frac{\sin(mx^2)}{(1+m^2)x^2} = \frac{m}{1+m^2}.$$

Note that the limit depends only on $m$.

Therefore, $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$ does not exist.

### 1.2.3 Properties of Limits

If $f(x, y)$, $g(x, y)$ are two functions defined on $D \setminus \{(a, b)\}$ , where $D \subseteq \mathbb{R}^2$, and $\lim_{(x,y)\to(a,b)} f(x, y) = L_1$, $\lim_{(x,y)\to(a,b)} g(x, y) = L_2$, where $L_1, L_2 \in \mathbb{R}$ then

(1). $\lim_{(x,y)\to(a,b)} [f(x, y) + g(x, y)] = L_1 + L_2$ .

(2). $\lim_{(x,y)\to(a,b)} [f(x, y) - g(x, y)] = L_1 - L_2$ .

(3). $\lim_{(x,y)\to(a,b)} k\ f(x, y) = k\ L_1$ , where $k \in \mathbb{R}$ .

(4). $\lim_{(x,y)\to(a,b)} [f(x, y).g(x, y)] = L_1L_2$ .

(5). $\lim_{(x,y)\to(a,b)} \frac{f(x, y)}{g(x, y)} = \frac{L_1}{L_2}$ , where $L_2 \neq 0$ .

(6). If $P(x, y)$ is a polynomial in $x, y$ then $\lim_{(x,y)\to(a,b)} P(x, y) = P(a, b)$ .

**Example (5):** Evaluate $\lim_{(x,y)\to(2,1)} \left(x^2y^2 - 2xy + x + y - 1\right)$ .

Solution: Note that $P(x, y) = x^2y^2 - 2xy + x + y - 1$ is a polynomial.

So, $\lim_{(x,y)\to(2,1)} \left(x^2y^2 - 2xy + x + y - 1\right) = P(2, 1)$

$= (2)^2(1)^2 - 2(2)(1) + 2 + 1 - 1 = 2$ .

**Example (6):** Evaluate $\displaystyle\lim_{(x,y)\to(-1,3)} \frac{2x^2y+1}{xy^3-2x}$ .

Solution: Note that $P(x,y) = 2x^2y+1$ and $Q(x,y) = xy^3 - 2x$ are both polynomials.

$$\lim_{(x,y)\to(-1,3)} \frac{2x^2y+1}{xy^3-2x} = \frac{P(-1,3)}{Q(-1,3)} = \frac{2(-1)^2(3)+1}{(-1)(3)^3-2(-1)} = \frac{7}{-25}.$$

Note that $Q(-1,3) \neq 0$ .

**Example (7):** Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = 0$ .

First Solution: $\forall\ x, y \in \mathbb{R}^*$,

$$y^2 \le x^2 + y^2 \implies \frac{y^2}{x^2+y^2} \le 1 .$$

$$0 \le \left| \frac{2xy^2}{x^2+y^2} \right| = \frac{|2x|\ |y^2|}{|x^2+y^2|} = 2|x|\ \frac{y^2}{x^2+y^2} \le 2|x| .$$

Note that $(x,y) \to (0,0) \implies x \to 0$.

Since $\displaystyle\lim_{(x,y)\to(0,0)} 0 = 0$ and $\displaystyle\lim_{(x,y)\to(0,0)} 2|x| = 2(0) = 0$ ,

By Squeeze Theorem $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = 0$ .

Second Solution: Using polar coordinates,

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = \lim_{r\to0} \frac{2r^3\cos\theta\ \sin^2\theta}{r^2} = \lim_{r\to0} 2r\cos\theta\sin^2\theta = 0.$$

Note that $\displaystyle\lim_{r\to0} 2r = 0$ and $\cos\theta\sin^2\theta$ is bounded .

### 1.2.4 Continuity

**Definition:** Let $f(x,y)$ be a function of two variables defined on a set $D \subseteq \mathbb{R}^2$ and $(a,b) \in D$, If $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, then $f$ is continuous at $(a,b)$.

If $f$ is continuous at every point in $D$, then $f$ is continuous on $D$ .

**Example (8):** Discuss the continuity of $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ .

Solution:

$f$ is not defined at $(0,0)$, so it is not continuous at $(0,0)$.

$\forall\ (a,b) \neq (0,0)$, $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} \frac{x^2-y^2}{x^2+y^2} = \frac{a^2-b^2}{a^2+b^2} = f(a,b)$.

So, $f$ is continuous on $D = \big\{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\big\} = \mathbb{R}^2 \setminus \{(0,0)\}$ .

**Example (9):** Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$ .

Solution:

$\forall\ (a,b) \neq (0,0)$, $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} \frac{x^2-y^2}{x^2+y^2} = \frac{a^2-b^2}{a^2+b^2} = f(a,b)$.

$f$ is defined at $(0,0)$ and $f(0,0) = 0$, but $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

So, $f$ is continuous on $D = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\} = \mathbb{R}^2 \setminus \{(0, 0)\}$ .

**Example (10):** Discuss the continuity of $f(x, y) = \begin{cases} \dfrac{2xy^2}{x^2 + y^2} & if \quad (x, y) \neq (0, 0) \\ 0 & if \quad (x, y) = (0, 0) \end{cases}$ .

Solution:
$\forall\ (a, b) \neq (0, 0),\ \lim\limits_{(x,y)\to(a,b)} f(x, y) = \lim\limits_{(x,y)\to(a,b)} \dfrac{2xy^2}{x^2 + y^2} = \dfrac{2ab^2}{a^2 + b^2} = f(a, b).$
$f$ is defined at $(0, 0)$ and $f(0, 0) = 0$, also $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$.
So, $f$ is continuous on $\mathbb{R}^2$.

**Example (11):** Discuss the continuity of $f(x, y) = e^{-x^2-y^2}$ .
Solution: $f(x, y) = e^{-x^2-y^2} = e^{-(x^2+y^2)}$ .
$f$ is defined on $\mathbb{R}^2$ and $\lim\limits_{(x,y)\to(a,b)} f(x, y) = e^{-(a^2+b^2)} = f(a, b)$.
Therefore, $f$ is continuous on $\mathbb{R}^2$.

**Example (12):** Discuss the continuity of $f(x, y) = \tan^{-1}\left(\dfrac{y}{x}\right)$ .
Solution:
$f$ is not defined where $x = 0$.
$\forall\ (a, b)$ where $a \neq 0$, $\lim\limits_{(x,y)\to(a,b)} f(x, y) = \tan^{-1}\left(\dfrac{b}{a}\right) = f(a, b)$.
So, $f$ is continuous on $D = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$ which is $\mathbb{R}^2$ except the $y$-axis.

### 1.2.5 Limit and Continuity of a function of three variables

**Definition:** Let $f$ be a function of three variables whose domain $D \subseteq \mathbb{R}^3$ includes points arbitrarily close to $(a, b, c)$. Then we say that the limit of $f(x, y, z)$ as $(x, y, z)$ approaches $(a, b, c)$ is $L$ and we write $\lim\limits_{(x,y,z)\to(a,b,c)} f(x, y, z) = L$,
if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that
if $(x, y, z) \in D$ and $0 < \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} < \delta$
then $|f(x, y, z) - L| < \epsilon$ .

**Definition:** Let $f(x, y, z)$ be a function of three variables defined on a set $D \subseteq \mathbb{R}^3$ and $(a, b, c) \in D$, If $\lim\limits_{(x,y,z)\to(a,b,c)} f(x, y, z) = f(a, b, c)$, then $f$ is continuous at $(a, b, c)$.
If $f$ is continuous at every point in $D$, then $f$ is continuous on $D$ .

**Example (13):** Discuss the continuity of $f(x, y, z) = \dfrac{1}{1 - x^2 - y^2 - z^2}$ .
Solution:
$f$ is not defined where $1 - x^2 - y^2 - z^2 = 0 \implies x^2 + y^2 + z^2 = 1$,
$f$ is not continuous on the unit sphere.
$\forall\ (a, b, c) \in \mathbb{R}^3$ where $a^2 + b^2 + c^2 \neq 1$,
$\lim\limits_{(x,y,z)\to(a,b,c)} f(x, y, z) = \dfrac{1}{1 - a^2 + b^2 + c^2} = f(a, b, c).$

So, $f$ is continuous on $D = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \neq 1\right\}$ which is $\mathbb{R}^3$ except the unit sphere.

### 1.2.6 EXERCISES

1. Find the limit of the following:

$$(a).\ \lim_{(x,y)\to(3,2)} \left(x^2y^3 - 4y^2\right) \quad (b).\ \lim_{(x,y)\to(2,-1)} \frac{x^2y + yx^2}{x^2 - y^2}$$

2. Show that the following limits do not exist:

$$(a).\ \lim_{(x,y)\to(0,0)} \frac{y^2}{x^2 + y^2} \qquad (b).\ \lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2 + y^2}$$

$$(c).\ \lim_{(x,y)\to(1,1)} \frac{y - x}{1 - y + \ln x}$$

3. Discuss the existence of the following limits (and find its value if exists):

$$(a).\ \lim_{(x,y)\to(2,3)} \frac{3x - 2y}{4x^2 - y^2} \quad (b).\ \lim_{(x,y)\to(0,0)} \frac{x^2y\cos y}{x^2 + y^4}$$

4. Use the Squeeze Theorem to find the limit of the following :

$$(a).\ \lim_{(x,y)\to(0,0)} xy\ \sin\left(\frac{1}{x^2 + y^2}\right) \quad (b).\ \lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2 + y^2 + z^2}$$

5. Determine the set of points of continuity of the following:

$$\text{(a)}.\ f(x,y) = \begin{cases} \dfrac{x^2y^3}{2x^2 + y^2} & if \quad (x,y) \neq (0,0) \\ 1 & if \quad (x,y) = (0,0) \end{cases}$$

$$\text{(b)}.\ f(x,y) = \begin{cases} \dfrac{xy}{x^2 + xy + y^2} & if \quad (x,y) \neq (0,0) \\ 1 & if \quad (x,y) = (0,0) \end{cases}$$

6. Use polar coordinates to find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2}$ .

(*) Show that the following limits do not exist:

$$(a).\ \lim_{(x,y)\to(0,0)} \frac{x^3+y^2}{x^2+y^2} \quad (b).\ \lim_{(x,y)\to(0,0)} \frac{xy}{x^4+y^4}$$

(*) Use Squeeze theorem to find the following limits:

$$(a).\ \lim_{(x,y)\to(-1,0)} \frac{y(x+1)^2+y^2\sin(\pi x)}{(x+1)^2+y^2} \quad (b).\ \lim_{(x,y)\to(0,0)} \frac{y^2\sin x+yx^2}{x^2+y^2}$$

(*) Show that the limit is zero in the following:

$$(a).\ \lim_{(x,y)\to(0,0)} \frac{x^5+y^5}{x^2+y^2} \quad (b).\ \lim_{(x,y)\to(0,0)} \frac{x^6+y^6}{x^4+y^4}$$

$$(c).\ \lim_{(x,y)\to(0,0)} \frac{x^2+2xy}{\sqrt{x^2+y^2}} \quad (d).\ \lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{x^2+y^2}$$

## 1.3 Partial Derivatives

### 1.3.1 Partial Derivatives of Functions of Two Variables

**Definition:** Let $f(x, y)$ be a function of two variables defined on a set $D \subseteq \mathbb{R}^2$,
If $\lim\limits_{h\to 0} \dfrac{f(x+h,y)-f(x,y)}{h}$ and $\lim\limits_{h\to 0} \dfrac{f(x,y+h)-f(x,y)}{h}$ exist, then the partial derivatives of $f$ are denoted by $f_x$ and $f_y$ and are defined as
(1). $f_x(x,y) = \lim\limits_{h\to 0} \dfrac{f(x+h,y)-f(x,y)}{h}$.
(2).$f_y(x,y) = \lim\limits_{h\to 0} \dfrac{f(x,y+h)-f(x,y)}{h}$.

**Notations for Partial Derivatives:** If $z = f(x, y)$ then
(1). $f_x(x,y) = f_x = \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x} f(x,y) = \dfrac{\partial z}{\partial x} = D_x f$ .
(2). $f_y(x,y) = f_y = \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y} f(x,y) = \dfrac{\partial z}{\partial y} = D_y f$ .

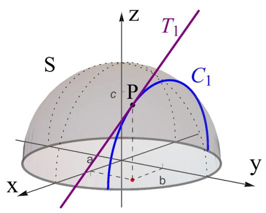
**Rule for Finding Partial Derivatives**
(1). To find $f_x$ : differentiate $f(x, y)$ with respect to $x$ regarding $y$ as a constant.
(2). To find $f_y$ : differentiate $f(x, y)$ with respect to $y$ regarding $x$ as a constant.

**Example (1):** If $f(x, y) = x^2 - xy^3 - 3y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$.
Solution:
(1). $f_x(x,y) = 2x - (1)y^3 - 0 = 2x - y^3$ ,
$f_x(1,1) = 2(1) - (1)^3 = 1$.
(2). $f_y(x,y) = 0 - x(3y^2) - 3(2y) = -3xy^2 - 6y$ ,
$f_y(1,1) = -3(1)(1)^2 - 6(1) = -9$.

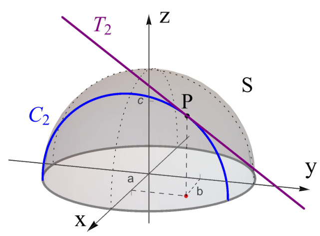
**Example (2):** If $f(x,y) = \sin\left(\dfrac{2x}{1+3y}\right)$, Evaluate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial x}$.
Solution:
(1). $\dfrac{\partial f}{\partial x} = \cos\left(\dfrac{2x}{1+3y}\right)\left(\dfrac{2}{1+3y}\right)$ .
(2). $\dfrac{\partial f}{\partial y} = \cos\left(\dfrac{2x}{1+3y}\right)\left((2x)(-1)(1+3y)^{-2}(3)\right) = \cos\left(\dfrac{2x}{1+3y}\right)\left(\dfrac{-6x}{(1+3y)^2}\right)$.

### 1.3.2 Interpretations of Partial Derivatives

Let $S$ be the surface represented by $z = f(x, y)$ and $f(a, b) = c$.
Then $P(a, b, c)$ lies on $S$.
Let $C_1$ be the curve where the plain $y = b$ intersects the surface $S$,
The slope of the tangent line $T_1$ to the curve $C_1$ at $P(a, b, c)$ is $f_x(a, b)$.

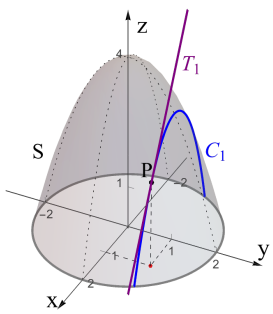


Let $C_2$ be the curve where the plain $x = a$ intersects the surface $S$,
The slope of the tangent line $T_2$ to the curve $C_2$ at $P(a, b, c)$ is $f_y(a, b)$.

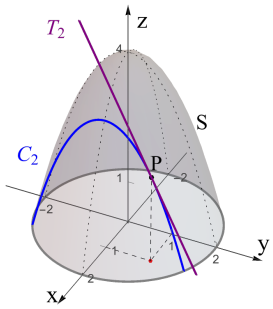



**Example (3):** If $f(x, y) = 4 - x^2 - y^2$, find $f_x(1, 1)$, $f_y(1, 1)$ and interpret them as slopes. Solution:

$z = 4 - x^2 - y^2$ .
$f_x(x, y) = -2x$ .
$f_x(1, 1) = -2(1) = -2.$
$f(1, 1) = 4 - 1 - 1 = 2$ .
$C_1$ is the intersection of $z = 4 - x^2 - y^2$ and $y = 1$, and it is the parabola $z = 3 - x^2$.
The line $T_1$ is tangent to the curve $C_1$ at the point $P(1, 1, 2)$ and its slope is $f_x(1, 1) = -2$.




$z = 4 - x^2 - y^2$ .
$f_y(x, y) = -2y$ .
$f_y(1, 1) = -2(1) = -2.$
$f(1, 1) = 4 - 1 - 1 = 2$ .
$C_2$ is the intersection of $z = 4 - x^2 - y^2$ and $x = 1$, and it is the parabola $z = 3 - y^2$.
The line $T_2$ is tangent to the curve $C_2$ at the point $P(1, 1, 2)$ and its slope is $f_y(1, 1) = -2$.




**Example (4):** If $x^3 + y^3 + z^3 + 6xyz + 4 = 0$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.
Solution:
Differentiating implicitly with respect to $x$ .

$$3x^2 + 0 + 3z^2\,\frac{\partial z}{\partial x} + 6y\left(z + x\frac{\partial z}{\partial x}\right) + 0 = 0$$

$$3x^2 + 3z^2\,\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}\left(3z^2 + 6xy\right) = -3x^2 - 6yz$$

$\dfrac{\partial z}{\partial x}=\dfrac{-3x^2-6yz}{3z^2+6xy}=\dfrac{-x^2-2yz}{z^2+2xy}=-\dfrac{x^2+2yz}{z^2+2xy}$ .
Differentiating implicitly with respect to $y$ .
$0+3y^2+3z^2\,\dfrac{\partial z}{\partial y}+6x\left(z+y\dfrac{\partial z}{\partial y}\right)+0=0$
$3y^2+3z^2\,\dfrac{\partial z}{\partial y}+6xz+6xy\dfrac{\partial z}{\partial y}=0$
$\dfrac{\partial z}{\partial y}\left(3z^2+6xy\right)=-3y^2-6xz$
$\dfrac{\partial z}{\partial y}=\dfrac{-3y^2-6xz}{3z^2+6xy}=\dfrac{-y^2-2xz}{z^2+2xy}=-\dfrac{y^2+2xz}{z^2+2xy}$ .

### 1.3.3 Functions of Three Variables

**Definition:** Let $f(x,y,z)$ be a function of three variables defined on a set $D\subseteq\mathbb{R}^3$, If $\lim\limits_{h\to 0}\dfrac{f(x+h,y,z)-f(x,y,z)}{h}$ , $\lim\limits_{h\to 0}\dfrac{f(x,y+h,z)-f(x,y,z)}{h}$ and $\lim\limits_{h\to 0}\dfrac{f(x,y,z+h)-f(x,y,z)}{h}$ exist, then the partial derivatives of $f$ are denoted by $f_x$ $f_y$ and $f_z$ and are defined as
(1). $f_x(x,y,z)=\lim\limits_{h\to 0}\dfrac{f(x+h,y,z)-f(x,y,z)}{h}$.
(2). $f_y(x,y,z)=\lim\limits_{h\to 0}\dfrac{f(x,y+h,z)-f(x,y,z)}{h}$.
(3). $f_z(x,y,z)=\lim\limits_{h\to 0}\dfrac{f(x,y,z+h)-f(x,y,z)}{h}$.

**Example (5):** If $f(x,y,z)=e^{xy}\ln(z^2+1)$, find $f_x$, $f_y$ and $f_z$.
Solution:
$f_x(x,y,z)=(e^{xy}y)\ln(z^2+1)=ye^{xy}\ln(z^2+1)$ .
$f_y(x,y,z)=(e^{xy}x)\ln(z^2+1)=xe^{xy}\ln(z^2+1)$ .
$f_z(x,y,z)=e^{xy}\,\left(\dfrac{2z}{z^2+1}\right)=\dfrac{2ze^{xy}}{z^2+1}$ .

### 1.3.4 Higher Derivatives

**Definition:** If $z=f(x,y)$, then the second partial derivatives of $f$ are
(1) $f_{xx}=(f_x)_x=\dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial x}=\dfrac{\partial f^2}{\partial x^2}$ .
(2) $f_{xy}=(f_x)_y=\dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x}=\dfrac{\partial f^2}{\partial y\partial x}$ .
(3) $f_{yx}=(f_y)_x=\dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial y}=\dfrac{\partial f^2}{\partial x\partial y}$ .
(y) $f_{yy}=(f_y)_y=\dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial y}=\dfrac{\partial f^2}{\partial y^2}$ .

**Example (6):** Find the second partial derivatives of $f(x,y)=x^3+x^2y^2-2y^3$.
Solution:

$f_x(x,y) = 3x^2 + (2x)y^2 - 0 = 3x^2 + 2xy^2$ .
$f_{xx}(x,y) = 3(2x) + 2y^2(1) = 6x + 2y^2$ .
$f_{xy}(x,y) = 0 + 2x(2y) = 4xy$ .
$f_y(x,y) = 0 + x^2(2y) - 2(3y^2) = 2x^2y - 6y^2$ .
$f_{yx}(x,y) = 2y(2x) - 0 = 4xy$ .
$f_{yy} = 2x^2(1) - 6(2y) = 2x^2 - 12y$ .
NOTE : $f_{xy}(x,y) = f_{yx}(x,y)$ .

**Clairaut's Theorem:** Suppose $f$ is defined on a set $D \subseteq \mathbb{R}^2$ that contains the point $(a,b)$, If the functions $f_{xy}$ and $f_{yx}$ are both continuous on $D$, then $f_{xy}(a,b) = f_{yx}(a,b)$.

**Example (7):** If $f(x,y,z) = \sin(2x - yz)$, find $f_{xyz}(x,y)$.
Solution:
$f_x(x,y,z) = \cos(2x - yz)(2) = 2\cos(2x - yz)$ .
$f_{xy}(x,y,z) = 2\left[-\sin(2x - yz)\ (-z)\right] = 2z\sin(2x - yz)$ .
$f_{xyz}(x,y,z) = (2)\sin(2x - yz) + 2z\left[\cos(2x - yz)\ (-y)\right]$
$= 2\sin(2x - yz) - 2yz\cos(2x - yz)$ .

### 1.3.5 Partial Differential Equations

(1). **Laplace's Equation:** If $u(x,y)$ is a function of two variables and $u_{xx} + u_{yy} = 0$, then $u$ satisfies the Laplace's equation, and $u$ is called a harmonic function.

**Example (8):** Show that $u(x,y) = e^x \cos y$ is a solution of Laplace's equation.
Solution:
$u_x(x,y) = e^x \cos y$ and $u_{xx}(x,y) = e^x \cos y$.
$u_y(x,y) = -e^x \sin y$ and $u_{yy}(x,y) = -e^x \cos y$.
$u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0$ .

(2). **Wave Equation:** If $u(x,t)$ is a function of two variables and $u_{tt} = a^2 u_{xx}$, then $u$ satisfies the wave equation.

**Example (9):** Show that $u(x,t) = \sin(x - at)$ is a solution of the wave equation.
Solution:
$u_x(x,t) = \cos(x - at)$ and $u_{xx}(x,t) = -\sin(x - at)$.
$u_t(x,t) = -a\cos(x - at)$ and $u_{tt}(x,t) = -a^2 \sin(x - at)$.
$u_{tt} = a^2(-\sin(x - at)) = a^2 u_{xx}$ .

### 1.3.6 Differentiability

**Definition:** If $z = f(x,y)$ and $(a,b) \in D_f$, if $x$ changes from $a$ to $a + \Delta x$ and $y$ changes from $b$ to $b + \Delta y$, then the increment of $z$ is
$\Delta z = f\left(a + \Delta x, b + \Delta y\right) - f(a,b)$.

**Definition:** If $z = f(x, y)$, then $f$ is differentiable at $(a, b)$, if
$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \sqrt{(\Delta x)^2 + (\Delta y)^2}\ \epsilon(\Delta x, \Delta y)$,
where $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \epsilon(\Delta x, \Delta y) = 0$.

**NOTE:**
If $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{f(a+\Delta x, b+\Delta y) - f(a,b) - f_x(a,b)\Delta x - f_y(a,b)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$, then
$f$ is differentiable at $(a, b)$.

**Theorem:** If $f$ is differentiable at $(a, b)$, then $f$ is continuous at $(a, b)$.

**NOTE:** If $f$ is not continuous at $(a, b)$, then $f$ is not differentiable at $(a, b)$.

**Example (10):** Show that $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$
is not differentiable at $(0, 0)$.
Solution:
$\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = \lim\limits_{r\to 0} \dfrac{r\cos\theta\ r\sin\theta}{r^2}$
$= \lim\limits_{r\to 0} \dfrac{r^2\ \cos\theta\ \sin\theta}{r^2} = \cos\theta\ \sin\theta$.
The limit depends on $\theta$, so the limit does not exist.
$f$ is not continuous at $(0, 0)$, hence, $f$ is not differentiable at $(0, 0)$.

**Example (11):** Show that $f(x,y) = \begin{cases} \dfrac{xy^2}{\sqrt{x^2+y^2}} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$
is differentiable at $(0, 0)$.
Solution:
$f(0, 0) = 0$.
$f(0+\Delta x, 0+\Delta y) = f(\Delta x, \Delta y) = \dfrac{\Delta x\,(\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$.
$f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0}{h} = 0$.
$f_y(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,0+h) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0}{h} = 0$.
$\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{f(\Delta x,\Delta y) - f(0,0) - f_x(0,0)\Delta x - f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$
$= \lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{\left(\frac{\Delta x(\Delta y)^2}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\right)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{\Delta x\,(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} = 0$.
Note that $0 \leq \left|\dfrac{\Delta x\,(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2}\right| = |\Delta x| \left|\dfrac{(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2}\right| \leq |\Delta x|$

By the squeeze theorem $\lim_{(\Delta x,\Delta y)\to(0,0)} \left|\dfrac{\Delta x\,(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2}\right| = 0,$

So, $\lim_{(\Delta x,\Delta y)\to(0,0)} \dfrac{\Delta x\,(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2} = 0.$

Therefore, $f$ is differentiable at $(0,0)$.

**Example (12):** Show that $f(x,y) = \begin{cases} \dfrac{y^2\sin x + yx^2}{x^2+y^2} & \text{if} \quad (x,y)\neq(0,0) \\ 0 & \text{if} \quad (x,y)=(0,0) \end{cases}$

is not differentiable at $(0,0)$.

Solution:

$f(0,0)=0.$

$f(0+\Delta x, 0+\Delta y) = f(\Delta x,\Delta y) = \dfrac{(\Delta y)^2\sin(\Delta x) + \Delta y\,(\Delta x)^2}{(\Delta x)^2+(\Delta y)^2}.$

$f_x(0,0) = \lim_{h\to 0}\dfrac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\dfrac{0}{h} = 0.$

$f_y(0,0) = \lim_{h\to 0}\dfrac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\dfrac{0}{h} = 0.$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f(\Delta x,\Delta y) - f(0,0) - f_x(0,0)\Delta x - f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\left(\frac{(\Delta y)^2\sin(\Delta x)+\Delta y(\Delta x)^2}{(\Delta x)^2+(\Delta y)^2}\right)}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{(\Delta y)^2\sin(\Delta x)+\Delta y\,(\Delta x)^2}{\left[(\Delta x)^2+(\Delta y)^2\right]^{\frac{3}{2}}}$$

Taking the path $\Delta y = \Delta x$ :

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{(\Delta y)^2\sin(\Delta x)+\Delta y\,(\Delta x)^2}{\left[(\Delta x)^2+(\Delta y)^2\right]^{\frac{3}{2}}} = \lim_{\Delta x\to 0} \frac{(\Delta x)^2\sin(\Delta x)+\Delta x\,(\Delta x)^2}{\left[(\Delta x)^2+(\Delta x)^2\right]^{\frac{3}{2}}}$$

$$= \lim_{\Delta x\to 0} \frac{(\Delta x)^2\left[\sin(\Delta x)+\Delta x\right]}{\left[2\,(\Delta x)^2\right]^{\frac{3}{2}}} = \lim_{\Delta x\to 0} \frac{(\Delta x)^2\left[\sin(\Delta x)+\Delta x\right]}{\sqrt{8}\,(\Delta x)^3} = \lim_{\Delta x\to 0} \frac{\sin(\Delta x)+\Delta x}{\sqrt{8}\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{1}{\sqrt{8}}\left(\frac{\sin(\Delta x)}{\Delta x} + \frac{\Delta x}{\Delta x}\right) = \frac{1}{\sqrt{8}}(1+1) = \frac{2}{\sqrt{8}} = \frac{1}{\sqrt{2}} \neq 0.$$

Therefore, $f$ is not differentiable at $(0,0)$.

### 1.3.7 EXERCISES

1. Find the first partial derivatives of the function.

   $(a).\ f(x,y) = x^4 - 5xy^3$ $\quad (b).\ g(x,y) = x^3 \sin y$

   $(c).\ w(u,v) = \dfrac{u}{v^2}$ $\quad (d).\ u(r,\theta) = \sin(r\cos\theta)$

   $(e).\ w(x,y,z) = y\tan(x+2z)$

2. If $f(x,y) = y\sin^{-1}(xy)$, find $f_y\left(1, \dfrac{1}{2}\right)$.

3. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ :

   $(a).\ z = f(x) + g(y) \quad (b).\ z = f(x+y)$

4. Find all the second partial derivatives of $f(x,y) = x^4y - 2x^3y^2$.

5. Verify that the conclusion of Clairaut's Theorem holds for

   $u(x,y) = x^4y^3 - y^4$.

6. If $f(x,y) = x^4y^2 - x^3y$, find $f_{xxx}$ and $f_{xyx}$.

(*) Discuss the differentiability of the following functions at the given points:

(a). $f(x,y) = \begin{cases} \dfrac{x^2(y-2)}{x^2+(y-2)^2} & \text{if} \quad (x,y) \neq (0,2) \\ 0 & \text{if} \quad (x,y) = (0,2) \end{cases}$ at (0,2).

(b). $f(x,y) = \begin{cases} \dfrac{x^2y - xy^2}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$ at (0,0).

(c). $f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$ at (0,0).

## 1.4 Chain Rule

### 1.4.1 The Chain Rule (Case 1)

**Theorem:** Suppose that $z = f(x,y)$ is a differentiable function of $x$ and $y$, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of $t$. Then $z$ is a differentiable function of $t$ and $\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$ .

**Example (1):** If $f(x,y) = x^3y + 2xy^3$, where $x = \sin 2t$ and $y = \cos t$, find $\dfrac{dz}{dt}$ at $t = 0$.

Solution:

$\dfrac{\partial f}{\partial x} = 3x^2y + 2y^3$ , $\dfrac{\partial f}{\partial y} = x^3 + 6xy^2$ .

$\dfrac{dx}{dt} = 2\cos 2t$ , $\dfrac{dy}{dt} = -\sin t$ .

$\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

$= (3x^2y + 2y^3)(2\cos 2t) + (x^3 + 6xy^2)(-\sin t)$

$\left.\dfrac{dz}{dt}\right|_{t=0} = (0 + 2)(2) + (0 + 0)(0) = 4$ .

### 1.4.2 The Chain Rule (Case 2)

**Theorem:** Suppose that $z = f(x,y)$ is a differentiable function of $x$ and $y$, where $x = g(s,t)$ and $y = h(s,t)$ are both differentiable functions of $s$ and $t$. Then $z$ is a differentiable function of $t$ and $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$ ,

$\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t}$

**Example (2):** If $f(x,y) = e^x \sin y$, where $x = s^2 + t^2$ and $y = s^2t^2$, find $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$.

Solution:

$\dfrac{\partial f}{\partial x} = e^x \sin y$ , $\dfrac{\partial f}{\partial y} = e^x \cos y$ .

$\dfrac{\partial x}{\partial s} = 2s$ , $\dfrac{\partial y}{\partial s} = 2st^2$ .

(1). $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s} = (e^x \sin y)\,(2s) + (e^x \cos y)\,(2st^2)$

$= 2se^{s^2+t^2}\sin(s^2t^2) + 2st^2e^{s^2+t^2}\cos(s^2t^2)$ .

Also, $\dfrac{\partial x}{\partial t} = 2t$ , $\dfrac{\partial y}{\partial s} = 2ts^2$ .

(2). $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t} = (e^x \sin y)\,(2t) + (e^x \cos y)\,(2ts^2)$

$= 2te^{s^2+t^2}\sin(s^2t^2) + 2ts^2e^{s^2+t^2}\cos(s^2t^2)$ .

### 1.4.3 The Chain Rule (The General Case)

**Theorem:** Suppose that $u$ is a differentiable function of the $n$ variables $x_1, x_2, \cdots, x_n$ and each $x_j$ is a differentiable function of the $m$ variables $t_1, t_2, \cdots, t_m$. Then $u$ is a function of $t_1, t_2, \cdots, t_m$ and

$\dfrac{\partial u}{\partial t_i} = \dfrac{\partial u}{\partial x_1}\dfrac{\partial x_1}{\partial t_i} + \dfrac{\partial u}{\partial x_2}\dfrac{\partial x_2}{\partial t_i} + \cdots + \dfrac{\partial u}{\partial x_n}\dfrac{\partial x_n}{\partial t_i}$ , for each $i = 1, 2, \cdots, m$ .

**Example (3):** If $u(x, y, z) = x^2y^2 + yz^3$, where $x = r^2se^t$, $y = rse^{-t}$ and $z = rs^2 \sin t$, find $\dfrac{\partial u}{\partial s}$ when $r = 1$, $s = 1$ and $t = 0$.

Solution:

$\dfrac{\partial u}{\partial x} = 2xy^2$, $\dfrac{\partial u}{\partial y} = 2x^2y + z^3$ and $\dfrac{\partial u}{\partial z} = 3yz^2$ .

$\dfrac{\partial x}{\partial s} = r^2e^t$, $\dfrac{\partial y}{\partial s} = re^{-t}$ and $\dfrac{\partial z}{\partial s} = 2rs \sin t$ .

So, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial s}$

$\dfrac{\partial u}{\partial s} = (2xy^2)(r^2e^t) + (2x^2y + z^3)(re^{-t}) + (3yz^2)(2rs \sin t)$

When $r = 1, s = 1, t = 0$, $\dfrac{\partial u}{\partial s} = (2)(1) + (2)(1) + (0)(0) = 4$.

**Example (4):** If $w = f(x, y)$ is differentiable at $(x, y)$ and $x = s + t$, $y = s - t$.

Show that $\dfrac{\partial w}{\partial s}\dfrac{\partial w}{\partial t} = \left(\dfrac{\partial w}{\partial x}\right)^2 - \left(\dfrac{\partial w}{\partial y}\right)^2$ .

$\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial s} = f_x(1) + f_y(1) = f_x + f_y$.

$\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t} = f_x(1) + f_y(-1) = f_x - f_y$.

$\dfrac{\partial w}{\partial s}\dfrac{\partial w}{\partial t} = (f_x + f_y)(f_x - f_y) = (f_x)^2 - (f_y)^2 = \left(\dfrac{\partial w}{\partial x}\right)^2 - \left(\dfrac{\partial w}{\partial y}\right)^2$ .

**Example (5):** If $z = f(s^2 - t^2, t^2 - s^2)$ is differentiable at $(s, t)$,

Show that $t\,\dfrac{\partial z}{\partial s} + s\,\dfrac{\partial z}{\partial t} = 0$.

Solution:

Let $z = f(x, y)$, where $x = s^2 - t^2$ and $y = t^2 - s^2$ .

$\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} = f_x(2s) + f_y(-2s) = 2sf_x - 2sf_y$ .

$\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} = f_x(-2t) + f_y(2t) = -2tf_x + 2tf_y$ .

$t\,\dfrac{\partial z}{\partial s} + s\,\dfrac{\partial z}{\partial t} = t(2sf_x - 2sf_y) + s(-2tf_x + 2tf_y)$

$= 2stf_x - 2stf_y - 2stf_x + 2stf_y = 0$ .

**Example (6):** If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s$ , $y = 3rs$. Find $\dfrac{\partial^2 z}{\partial r^2}$ .

Solution:

$\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial r} = f_x(2r) + f_y(3s) = 2rf_x + 3sf_y$.

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(2rf_x + 3sf_y\right) = 2f_x + 2r\left(\frac{\partial f_x}{\partial r}\right) + 3s\left(\frac{\partial f_y}{\partial r}\right)$$
$$= 2f_x + 2r\left(\frac{\partial f_x}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial r}\right) + 3s\left(\frac{\partial f_y}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial r}\right)$$
$$= 2f_x + 2r\left(f_{xx}(2r) + f_{xy}(3s)\right) + 3s\left(f_{yx}(2r) + f_{yy}(3s)\right)$$
$$= 2f_x + 4r^2 f_{xx} + 6rsf_{xy} + 6rsf_{yx} + 9s^2 f_{yy}\ .$$

### 1.4.4 Implicit Differentiation

Let $F(x,y) = 0$ where $y = f(x)$, using chain rule $\dfrac{\partial F}{\partial x}\dfrac{dx}{dx} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dx} = 0$
$\implies F_x(1) + F_y\dfrac{dy}{dx} = 0 \implies \dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$ .

**Example (7):** If $x^2 + y^2 = 5xy$, Find $\dfrac{dy}{dx}$.
Solution:
$x^2 + y^2 = 5xy \implies x^2 + y^2 - 5xy = 0$, Let $F(x,y) = x^2 + y^2 - 5xy$ then $F(x,y) = 0$.
$\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{2x - 5y}{2y - 5x}$.

Let $F(x,y,z) = 0$ where $z = f(x,y)$ , then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$.

**Example (8):** If $x^3 + y^3 + z^2 + 3xyz - 1 = 0$, Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.
Solution:
Let $F(x,y,z) = x^3 + y^3 + z^2 + 3xyz - 1$ then $F(x,y,z) = 0$.
$\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{3x^2 + 3yz}{2z + 3xy}$ .
$\dfrac{\partial z}{\partial x} = -\dfrac{F_y}{F_z} = -\dfrac{3y^2 + 3xz}{2z + 3xy}$ .

### 1.4.5 EXERCISES

1. Find $\dfrac{dw}{dt}$ of the following:

   (a). $w = xy^3 - x^2y$, where $x = t^2 + 1$ and $y = t^2 - 1$.

   (b). $w = \sin x \cos y$, where $x = \sqrt{t}$ and $y = \dfrac{1}{t}$.

   (c). $w = xe^{\frac{y}{z}}$, where $x = t^2$, $y = 1 - t$ and $z = 1 + 2t$.

   (d). $w = \ln\sqrt{x^2 + y^2 + z^2}$, where $x = \sin t$, $y = \cos t$ and $z = \tan t$.

2. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ of the following:

   (a). $z = (x - y)^5$, where $x = s^2t$ and $y = st^2$.

   (b). $z = \tan^{-1}\left(x^2 + y^2\right)$, where $x = s\ \ln t$ and $y = t\ e^s$.

   (c). $z = \dfrac{\sin\theta}{r}$, where $r = st$ and $\theta = s^2 + t^2$.

3. If $z = x^4 + x^2y$, where $x = s + 2t - u$ and $y = stu^2$,

   Find $\dfrac{\partial z}{\partial s}, \dfrac{\partial z}{\partial t}$ and $\dfrac{\partial z}{\partial u}$ at $s = 4$, $t = 2$, $u = 1$.

4. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, Show that

   $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

5. If $z = f(x + at) + g(x - at)$, Show that $\dfrac{\partial^2 z}{\partial t^2} = a^2\ \dfrac{\partial^2 z}{\partial x^2}$ .

(*) If $xe^{yz} - 2ye^{xz} + 3ze^{xy} = 1$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

## 1.5 Maximum and Minimum Values

### 1.5.1 Local Maximum and Minimum Values

**Definition:** If $f$ is a differentiable function of two variables $x$ and $y$, then the gradient of $f$ is the vector function $\nabla f$ and it is defined by
$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$ .

**Definition:** If $f$ is a differentiable function of three variables $x$,$y$ and $z$, then the gradient of $f$ is the vector function $\nabla f$ and it is defined by
$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$ .

**Definition (Critical point):** If $f(x,y)$ is a function of two variables then $(a,b) \in D_f$ is a critical point if both $f_x(a,b) = 0$ and $f_y(a,b) = 0$, or $f$ is not differentiable at $(a,b)$.

**Definition (Local maximum and Local minimum):**
(i) A function $f$ of two variables has a local maximum at $(a,b) \in D_f$ if
$f(x,y) \le f(a,b)$, for all points $(x,y)$ in some disk with center $(a,b)$.
(ii) A function $f$ of two variables has a local minimum at $(a,b) \in D_f$ if
$f(x,y) \ge f(a,b)$, for all points $(x,y)$ in some disk with center $(a,b)$.

**Theorem:** If $f(x,y)$ has a local maximum or minimum at $(a,b) \in D_f$ and the first-order partial derivatives of $f$ exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

**Second Derivatives Test:** Suppose the second partial derivatives of $f(x,y)$ are continuous on a disk with center $(a,b) \in D_f$, and suppose that
$f_x(a,b) = 0$ and $f_y(a,b) = 0$ [so $(a,b)$ is a critical point of f ].
Let $D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$.
(1). If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
(2). If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
(3). If $D < 0$, then $(a,b)$ is a saddle point.

**NOTES:**
(1). If $D = 0$, then the test gives no information.
(2). $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}f_{yx} = f_{xx}f_{yy} - [f_{xy}]^2$.

**Example (1):** Find the local maximum and minimum values and saddle points of $f(x,y) = 6xy - 2x^3 + y^2$ .
Solution:
$f_x(x,y) = 6y - 6x^2$ and $f_y(x,y) = 6x + 2y$ .
$f_x(x,y) = 0 \implies 6y - 6x^2 = 0 \implies y = x^2$ .
$f_y(x,y) = 0 \implies 6x + 2y = 0 \implies y = -3x$ .
$f_x(x,y) = f_y(x,y) \implies x^2 = -3x \implies x^2 + 3x = 0$
$\implies x(x+3) = 0 \implies x = 0\ ,\ x = -3 \implies y = 0\ ,\ y = 9$ .
So, the critical points are $(0,0)$ and $(-3,9)$ .

$f_{xx}(x,y) = -12x,\ f_{xy}(x,y) = 6$ and $f_{yy}(x,y) = 2$ .

First- At the point $(0,0)$ :
$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - [f_{x,y}(0,0)]^2 = (0)(2) - (6)^2 = -36 < 0.$
Therefore, $(0,0)$ is a saddle point.

Second- At the point $(-3,9)$ :
$D(-3,9) = f_{xx}(-3,9)f_{yy}(-3,9) - [f_{x,y}(-3,9)]^2 = (36)(2) - (6)^2 = 36 > 0.$
Since $f_{xx}(-3,9) = 36 > 0$, then $f$ attains a local minimum at $(-3,9)$,
and its value is $f(-3,9) = 6(-3)(9) - 2(-3)^3 + (-9)^2 = -162 + 54 + 81 = -27.$

**Example (2):** Find the local maximum and minimum values and saddle points of $f(x,y) = x^3 - y^3 - 3x + 3y + 5$ .
Solution:
$f_x(x,y) = 3x^2 - 3$ and $f_y(x,y) = -3y^2 + 3$ .
$f_x(x,y) = 0 \implies 3x^2 - 3 = 0 \implies x^2 - 1 = 0 \implies x = \pm 1$ .
$f_y(x,y) = 0 \implies -3y^2 + 3 = 0 \implies y^2 - 1 = 0 \implies y = \pm 1$ .
So, the critical points are $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$ .

$f_{xx}(x,y) = 6x,\ f_{xy}(x,y) = 0$ and $f_{yy}(x,y) = -6y$ .

First- At the point $(1,1)$ :
$D(1,1) = f_{xx}(1,1)f_{yy}(1,1) - [f_{x,y}(1,1)]^2 = (6)(-6) - (0)^2 = -36 < 0.$
Therefore, $(1,1)$ is a saddle point.

Second- At the point $(1,-1)$ :
$D(1,-1) = f_{xx}(1,-1)f_{yy}(1,-1) - [f_{x,y}(1,-1)]^2 = (6)(6) - (0)^2 = 36 > 0.$
Since $f_{xx}(1,-1) = 6 > 0$, then $f$ attains a local minimum at $(1,-1)$,
and its value is $f(1,-1) = 1 + 1 - 3 - 3 + 5 = 1.$

Third- At the point $(-1,1)$ :
$D(-1,1) = f_{xx}(-1,1)f_{yy}(-1,1) - [f_{x,y}(-1,1)]^2 = (-6)(-6) - (0)^2 = 36 > 0.$
Since $f_{xx}(-1,1) = -6 < 0$, then $f$ attains a local maximum at $(-1,1)$,
and its value is $f(-1,1) = -1 - 1 + 3 + 3 + 5 = 9.$

Fourth- At the point $(-1,-1)$ :
$D(-1,-1) = f_{xx}(-1,-1)f_{yy}(-1,-1) - [f_{x,y}(-1,-1)]^2$
$= (-6)(6) - (0)^2 = -36 < 0.$
Therefore, $(-1,-1)$ is a saddle point.

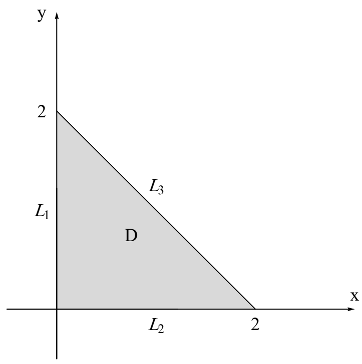
### 1.5.2 Absolute Maximum and Minimum Values

**Definition:** Let $(a,b)$ be a point in the domain $D$ of a function $f$ of two variables. Then $f(a,b)$ is the
(1). absolute maximum value of $f$ on $D$ if $f(a,b) \geq f(x,y)$ for all $(x,y)$ in $D$.
(2). absolute minimum value of $f$ on $D$ if $f(a,b) \leq f(x,y)$ for all $(x,y)$ in $D$.

**Theorem:** If $f$ is continuous on a closed, bounded set $D$ in $\mathbb{R}^2$, then $f$ attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points $(x_1, y_1)$ and $(x_2, y_2)$ in $D$.

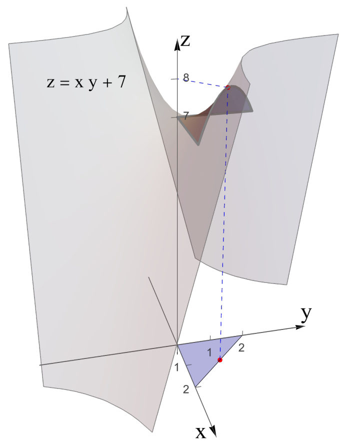
**NOTES:** To find the absolute maximum and minimum values of a continuous function $f(x, y)$ on a closed, bounded set $D \subseteq \mathbb{R}^2$:
1. Find the values of $f$ at the critical points of $f$ in the interior of $D$.
2. Find the extreme values of $f$ on the boundary of $D$.
3. The largest of the values from steps (1) and (2) is the absolute maximum value of $f$ on $D$, and the smallest of these values is the absolute minimum value of $f$ on $D$.

**Example (3):** Find the absolute maximum and minimum values of the function $f(x, y) = xy + 7$ on the plain region bounded by the graphs of the lines $x = 0$, $y = 0$ and $y + x = 2$.
Solution:




$f_x(x, y) = y$ and $f_y(x, y) = x$ .
$f_x(x, y) = 0 \implies y = 0$ .
$f_y(x, y) = 0 \implies x = 0$ .
The critical point is $(0, 0)$.
Note that the critical point is not inside the given region.

Let $L_1$ be the line $x = 0$:
then $f(x, y) = f(0, y) = 7$.
$f(x, y) = 7$ for all $(x, y) \in L_1$.



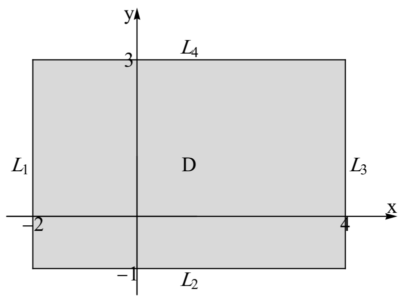

Let $L_2$ be the line $y = 0$:
then $f(x, y) = f(x, 0) = 7$.
$f(x, y) = 7$ for all $(x, y) \in L_2$.

Let $L_3$ be the line $y + x = 2$:
then $y = -x + 2$ where $0 \le x \le 2$.
$f(x, y) = f(x, -x + 2)$
$= x(-x + 2) + 7 = -x^2 + 2x + 7$ .
$f'(x) = -2x + 2$,
$f'(x) = 0 \implies -2x + 2 = 0$
$\implies x = 1$.
So, $y = -2 + 1 = 1$ .
$f(1, 1) = (1)(1) + 7 = 8$.
Also, $f(0, 2) = 7$ and $f(2, 0) = 7$.

The absolute maximum is 8, and $f$ takes it at $(1, 1)$.
The absolute minimum is 7, and $f$ takes it at any point on $L_1 \cup L_2$.

**Example (4):** Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + 2xy + 3y^2$ on the closed and bounded region
$D = \{(x, y) \in \mathbb{R}^2 : -2 \le x \le 4 \ , \ -1 \le y \le 3\}$.

Solution:




$f_x(x, y) = 2x + 2y.$
$f_y(x, y) = 2x + 6y.$
$f_x(x, y) = 0 \implies x = -y.$
$f_y(x, y) = 0 \implies x = -3y.$
$-y = -3y \implies y = 0 \implies x = 0.$
The critical point is $(0, 0)$.
$f(0, 0) = (0)^2 + 2(0)(0) + 3(0)^3 = 0.$

Let $L_1$ be the line between $(-2, -1)$ and $(-2, 3)$.
On $L_1 : x = -2$ and $-1 \le y \le 3$ .
$f(x, y) = f(-2, y) = 4 - 4y + 3y^2 \implies f'(y) = -4 + 6y$ .
$f'(y) = 0 \implies 6y = 4 \implies y = \frac{2}{3}.$
Note that $\left(-2, \frac{2}{3}\right) \in L_1.$
$f\left(-2, \frac{2}{3}\right) = (-2)^2 + 2(-2)\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)^2 = 4 - \frac{8}{3} + \frac{4}{3} = \frac{8}{3}.$

Let $L_2$ be the line between $(-2, -1)$ and $(4, -1)$.
On $L_2 : y = -1$ and $-2 \le x \le 4$ .
$f(x, y) = f(x, -1) = x^2 - 2x + 3 \implies f'(x) = 2x - 2$ .
$f'(x) = 0 \implies 2x = 2 \implies x = 1.$
Note that $(1, -1) \in L_2$.
$f(1, -1) = (1)^2 + 2(1)(-1) + 3(-1)^2 = 1 - 2 + 3 = 2.$

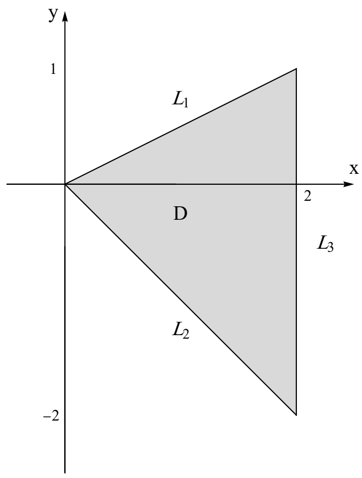
Let $L_3$ be the line between $(4, -1)$ and $(4, 3)$.
On $L_3 : x = 4$ and $-1 \le y \le 3$ .
$f(x, y) = f(4, y) = 16 + 8y + 3y^2 \implies f'(y) = 8 + 6y$ .
$f'(y) = 0 \implies 6y = -8 \implies y = -\frac{4}{3}.$
Note that $\left(4, -\frac{4}{3}\right) \notin L_3.$

Let $L_4$ be the line between $(-2, 3)$ and $(4, 3)$.
On $L_4 : y = 3$ and $-2 \le x \le 4$ .
$f(x, y) = f(x, 3) = x^2 + 6x + 27 \implies f'(x) = 2x + 6$ .
$f'(x) = 0 \implies 2x = 2 \implies x = -3.$
Note that $(-3, ) \notin L_4$.

Evaluating $f(x, y)$ at the four corners of $D$ :
$f(-2, -1) = (-2)^2 + 2(-2)(-1) + 3(-1)^2 = 4 + 4 + 3 = 11$ .
$f(-2, 3) = (-2)^2 + 2(-2)(3) + 3(3)^2 = 4 - 12 + 27 = 19$ .
$f(4, -1) = (4)^2 + 2(4)(-1) + 3(-1)^2 = 16 - 8 + 3 = 11$ .
$f(4, 3) = (4)^2 + 2(4)(3) + 3(3)^2 = 16 + 24 + 27 = 67$ .

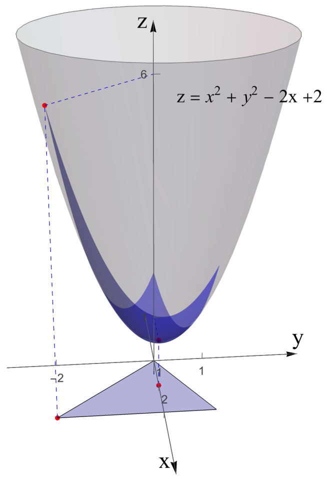
The absolute maximum is 67, and $f$ takes it at $(4, 3)$.
The absolute minimum is 0, and $f$ takes it at $(0, 0)$.

**Example (5):** Find the absolute maximum and minimum values of the function $f(x,y) = x^2 + y^2 - 2x + 2$ on the closed region with vertices $(0,0)$, $(2,1)$ and $(2,-2)$.
Solution:




$f_x(x,y) = 2x - 2$ and $f_y(x,y) = 2y$ .
$f_x(x,y) = 0 \implies x = 1$ .
$f_y(x,y) = 0 \implies y = 0$ .
The critical point is $(1,0)$.
$f(1,0) = 1 + 0 - 2 + 2 = 1$.

Let $L_1$ be the line passing through $(0,0)$ and $(2,1)$ then $y = \frac{x}{2}$.
$f\left(x, \frac{x}{2}\right) = x^2 + \frac{x^2}{4} - 2x + 2$.
$f(x) = \frac{5}{4}x^2 - 2x + 2$ .
$f'(x) = \frac{5}{2}x - 2$ .
$f'(x) = 0 \implies x = \frac{4}{5}$ and $y = \frac{2}{5}$.
$f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{16}{25} + \frac{4}{25} - \frac{8}{5} + 2$
$= \frac{16}{25} + \frac{4}{25} - \frac{40}{25} + \frac{50}{25} = \frac{30}{25} = \frac{6}{5}$.




Let $L_2$ be the line passing through $(0,0)$ and $(2,-2)$ then $y = -x$.
$f(x, -x) = x^2 + x^2 - 2x + 2$
$f(x) = 2x^2 - 2x + 2$.
$f'(x) = 4x - 2$ .
$f'(x) = 0 \implies x = \frac{1}{2}$ and $y = -\frac{1}{2}$.
$f\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} + 1 + 2 = \frac{7}{2}$.

Let $L_3$ be the line passing through $(2,-2)$ and $(2,1)$ then $x = 2$.
$f(2,y) = 4 + y^2 - 4 + 2 = y^2 + 2$, so $f(y) = y^2 + 2 \implies f'(y) = 2y$.
$f'(y) = 0 \implies y = 0$ and $x = 2$.
$f(2,0) = 4 + 0 - 4 + 2 = 2$ .

Evaluating $f(x,y)$ at the three corners of $D$ :
$f(0,0) = 0 + 0 - 0 + 2 = 2$ .
$f(2,1) = 4 + 1 - 4 + 2 = 3$ .
$f(2,-2) = 4 + 4 - 4 + 2 = 6$ .

The absolute maximum is 6, and $f$ takes it at $(2,-2)$.
The absolute minimum is 1, and $f$ takes it at $(1,0)$.

### 1.5.3 EXERCISES

1. Find the local maximum and minimum values and saddle point(s) of the functions:

   (a). $f(x, y) = x^2 + xy + y^2 + y$ .

   (b). $f(x, y) = x^3 + y^3 + 3xy$ .

   (c). $f(x, y) = 2 - x^4 + 2x^2 - y^2$ .

   (d). $f(x, y) = x^4 - 2x^2 + y^3 - 3y$ .

2. Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 - 2x$ on the closed region with vertices $(2, 0)$, $(0, 2)$ and $(0, -2)$.

(*) Find the local maximum and minimum values and saddle point(s) of the functions:

   (a). $f(x, y) = 4x^3 - 2x^2y + y^2$ .

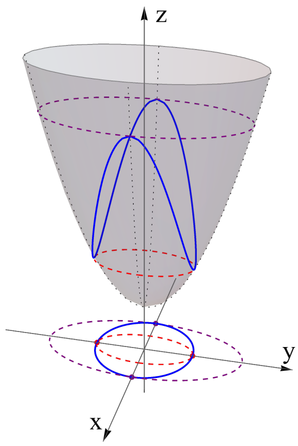
   (b). $f(x, y) = 2x^3 - 3x^2 + 3y^2 - 6xy + 2$ .

   (c). $f(x, y) = x^4 + y^3 + 32x - 3y$ .

## 1.6 Lagrange Multipliers

### 1.6.1 Lagrange Multipliers (One Constraint)

To find the extreme values of $f(x, y)$ subject to a constraint $g(x, y) = k$.
Let $z = f(x, y)$ be the gray surface, and $g(x, y)$ is the blue curve representing the constraint in the $xy$-plain.
Note that the level curves $f(x, y)$ touches $g(x, y)$ at the points where $f(x, y)$ have minimum and maximum values.
This means that $\nabla f(x, y)$ is parallel to $\nabla g(x, y)$ at these point.
Find these points by solving the equation: $\nabla f(x, y) = \lambda \nabla g(x, y)$,
where $\lambda \in \mathbb{R}$ and $\nabla g(x, y) \neq 0$.
Evaluate $f(x, y)$ at these points, The largest is the maximum value of $f$ and the smallest is the minimum value of $f$.




**Example (1):** Find the extreme values of the function $f(x, y) = 1 + xy$ on the circle $x^2 + y^2 = 1$.
Solution:
$f(x, y) = 1 + xy$ and $g(x, y) = x^2 + y^2 - 1$.
$\nabla f(x, y) = \lambda \nabla g(x, y) \implies \langle y, x\rangle = \lambda\langle 2x, 2y\rangle = \langle 2\lambda x, 2\lambda y\rangle$

$$\implies \begin{cases} y = 2\lambda x \\ x = 2\lambda y \end{cases} \implies x = 2\lambda(2\lambda x) = 4\lambda^2 x \implies x\left(1 - 4\lambda^2\right) = 0$$

$$\implies x = 0\ ,\ \lambda = \pm\frac{1}{2}\ .$$

If $x = 0$ then $y = 2\lambda(0) = 0$, but $(0, 0)$ does not lie on the unit circle, so $(0, 0)$ is excluded.

If $\lambda = \pm\frac{1}{2} \implies y = 2\left(\pm\frac{1}{2}\right)x \implies y = \pm x$.

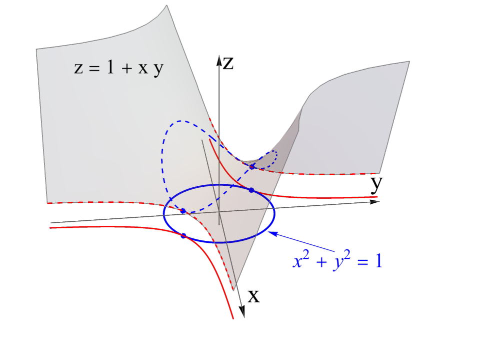
From the constraint : $x^2 + y^2 = 1 \implies 2x^2 = 1 \implies x = \pm\frac{1}{\sqrt{2}}$.

This means $\frac{1}{2} + y^2 = 1 \implies y = \pm\frac{1}{\sqrt{2}}$.

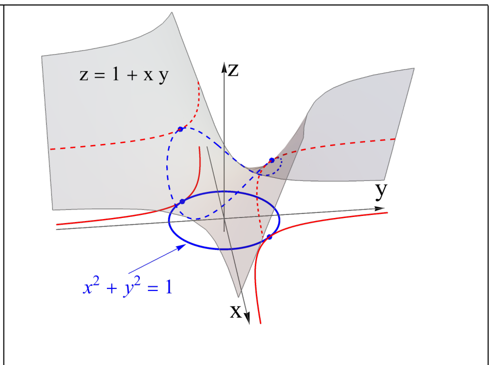
So, There are 4 points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$,$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 1 + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 1 + \frac{1}{2} = \frac{3}{2}.$$
$$f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = 1 + \frac{1}{\sqrt{2}}\frac{-1}{\sqrt{2}} = 1 - \frac{1}{2} = \frac{1}{2}.$$
$$f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 1 + \frac{-1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 1 - \frac{1}{2} = \frac{1}{2}.$$
$$f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = 1 + \frac{-1}{\sqrt{2}}\frac{-1}{\sqrt{2}} = 1 + \frac{1}{2} = \frac{3}{2}.$$

The maximum value is $\frac{3}{2}$, and $f$ takes it at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.
The minimum value is $\frac{1}{2}$, and $f$ takes it at $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.




Minimum value of $z = 1 + xy$ on $x^2 + y^2 = 1$




Maximum value of $z = 1 + xy$ on $x^2 + y^2 = 1$

**Note**: to find the extreme values of $f(x, y, z)$ subject to a constraint $g(x, y, z) = k$, where $k \in \mathbb{R}$.
(1). Find the points that satisfy the equation $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$
$f_x(x, y, z) = \lambda g_x(x, y, z)$, $f_y(x, y, z) = \lambda g_y(x, y, z)$ and $f_z(x, y, z) = \lambda g_z(x, y, z)$.
(2). Evaluate $f(x, y, z)$ at these points, The largest is the maximum value of $f$ and the smallest is the minimum value of $f$.

**Example (2):** Find the points on the sphere $x^2 + y^2 + z^2 = 1$ that are closest to and farthest from the point $(2, 2, 2)$.
Solution:
Let $f(x, y, z)$ be the function of the square of the distance between any point in the sphere and the point $(2, 2, 2)$, then $f(x, y, z) = (x-2)^2 + (y-2)^2 + (z-2)^2$.
Let $g(x, y, z) = x^2 + z^2 + y^2 - 1 = 0$ be the constraint.

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \implies \langle 2(x-2), 2(y-2), 2(z-2)\rangle = \lambda \langle 2x, 2y, 2z\rangle$$

$$\begin{cases} 2(x-2) = 2\lambda x \\ 2(y-2) = 2\lambda y \\ 2(z-2) = 2\lambda z \end{cases} \implies \begin{cases} x-2 = \lambda x \\ y-2 = \lambda y \\ z-2 = \lambda z \end{cases} \implies \begin{cases} x(1-\lambda) = 2 \\ y(1-\lambda) = 2 \\ z(1-\lambda) = 2 \end{cases}$$

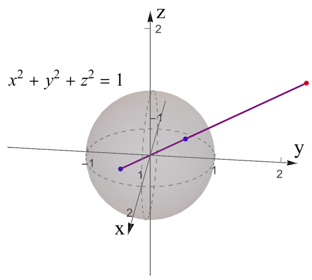
If $\lambda = 1$ then $x - 2 = x \implies -2 = 0$, so $\lambda \neq 1$.
Therefore, $x(1-\lambda) = y(1-\lambda) = z(1-\lambda) \implies x = y = z$.
So, $x^2 + y^2 + z^2 = 1 \implies 3x^2 = 1 \implies x = \pm\frac{1}{\sqrt{3}}$
Hence, $x = y = z = \pm\frac{1}{\sqrt{3}}$.
The required points are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$.

$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}} - 2\right)^2 + \left(\frac{1}{\sqrt{3}} - 2\right)^2 + \left(\frac{1}{\sqrt{3}} - 2\right)^2 = 3\left(2 - \frac{1}{\sqrt{3}}\right)^2.$$

$$f\left(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)=\left(\frac{-1}{\sqrt{3}}-2\right)^2+\left(\frac{-1}{\sqrt{3}}-2\right)^2+\left(\frac{-1}{\sqrt{3}}-2\right)^2=3\left(2+\frac{1}{\sqrt{3}}\right)^2.$$

The point $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ is the closest to $(2,2,2)$.

The point $\left(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)$ is the farthest to $(2,2,2)$.




**Example (3):** Find the maximum volume of a rectangular box without a lid, where its surface area equals 12 cm$^2$.
Solution:
Suppose the sides of the rectangular box are $x$, $y$ and $z$.
The volume of the rectangular box is $V(x,y,z)=xyz$, subject to the constraint $2xz+2yz+yz=12$.
The constraint is $g(x,y,z)=2xz+2yz+xy-12=0$.
$\nabla V(x,y,z)=\lambda\nabla g(x,y,z) \implies \langle yz,xz,xy\rangle=\lambda\langle 2z+y,2z+x+2x+2z\rangle$

$$\begin{cases} yz=\lambda(2z+y) \\ xz=\lambda(2z+x) \\ xy=\lambda(2x+2y)\end{cases} \implies \begin{cases} xyz=\lambda(2xz+xy) & \longrightarrow (1) \\ xyz=\lambda(2yz+xy) & \longrightarrow (2) \\ xyz=\lambda(2xz+2yz) & \longrightarrow (3)\end{cases}$$

If $\lambda=0$ then $V(x,y,z)=0$.
If $x=0$, $y=0$ or $z=0$, then $V(x,y,z)=0$.

If $\lambda\neq 0$, From equations (1) and (2) :
$\lambda(2xz+xy)=\lambda(2yz+xy) \implies 2xz+xy=2yz+xy \implies 2xz=2yz \implies x=y.$

From equations (2) and (3), and $x=y$ :
$\lambda(2yz+xy)=\lambda(2xz+2yz) \implies 2xz+x^2=2xz+2xz$
$\implies x^2=2xz \implies x^2-2xz=0 \implies x(x-2z)=0 \implies x=2z.$

From the equation of the constraint and $x=y=2z$ :
$4z^2+4z^2+4z^2=12 \implies z^2=1 \implies z=1$ and $x=y=2z=2$.

The sides of the rectangular box are 2, 2 and 1, and its maximum volume is 4 cm$^3$.

### 1.6.2 EXERCISES

1. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint:

   (a). $f(x, y) = x^2 - y^2$ , $x^2 + y^2 = 1$ .

   (b). $f(x, y) = xye^{-x^2-y^2}$ , $2x - y = 0$ .

   (c). $f(x, y, z) = xy^2z$ , $x^2 + y^2 + z^2 = 4$ .

   (d). $f(x, y, z) = x^4 + y^4 + z^4$ , $x^2 + y^2 + z^2 = 1$ .

2. Find the extreme values of f on the region described by the inequality:

   (a). $f(x, y) = x^2 + y^2 + 4x - 4y$ , $x^2 + y^2 \leq 9$ .

   (b). $f(x, y) = e^{-xy}$ , $x^2 + 4y^2 \leq 1$ .

3. Show that the problem of finding the minimum value of $f$ subject to the given constraint can be solved using Lagrange multipliers, but $f$ does not have a maximum value with that constraint:

   (a). $f(x, y) = x^2 + y^2$ , $xy = 1$ .

   (b). $f(x, y, z) = x^2 + 2y^2 + 3z^2$ , $x + 2y + 3z = 10$ .

# Chapter 2

# Multiple Integrals

## 2.1 Double Integrals over Rectangles

### 2.1.1 Iterated Integrals

Suppose that $f(x,y)$ is integrable on the rectangle $R = [a,b] \times [c,d]$, then
$\iint_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dydx = \int_a^b \left( \int_c^d f(x,y)\ dy \right)\ dx.$

**Example (1):** Evaluate the integral:
(a). $\int_0^2 \int_1^2 x^2 y\ dydx$ , (b). $\int_1^2 \int_0^2 x^2 y\ dxdy.$
Solution:
(a). $\int_0^2 \int_1^2 x^2 y\ dydx = \int_0^2 \left( \int_1^2 x^2 y\ dy \right)\ dx = \int_0^2 \left( x^2 \int_1^2 y\ dy \right)\ dx$
$= \int_0^2 x^2\ \left[ \frac{y^2}{2} \right]_1^2\ dx = \int_0^2 x^2\ \left[ \frac{4}{2} - \frac{1}{2} \right]\ dx$
$= \frac{3}{2} \int_0^2 x^2\ dx = \frac{3}{2}\ \left[ \frac{x^3}{3} \right]_0^2 = \frac{3}{2} \left[ \frac{8}{3} - 0 \right] = 4.$
(b). $\int_1^2 \int_0^2 x^2 y\ dxdy = \int_1^2 \left( \int_0^2 x^2 y\ dx \right) dy = \int_1^2 \left( y \int_0^2 x^2\ dx \right) dy.$
$= \int_1^2 y\ \left[ \frac{x^3}{3} \right]_0^2 dy = \int_1^2 y\ \left[ \frac{8}{3} - 0 \right]_0^2 dy = \frac{8}{3} \int_1^2 y\ dy$
$= \frac{8}{3} \left[ \frac{y^2}{2} \right]_1 2 = \frac{8}{3} \left[ \frac{4}{2} - \frac{1}{2} \right] = \frac{8}{3}\ \frac{3}{2} = 4.$

**Fubini's Theorem:** If $f$ is continuous on the rectangle $R = \{(x,y) \mid a \le x \le b\ ,\ c \le y \le d\}$,
Then $\iint_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dydx = \int_c^d \int_a^b f(x,y)\ dxdy.$

**Example (2):** Evaluate the integral $\iint_R y\cos(xy)\ dA$ ,
where $R = [0,1] \times \left[0, \frac{\pi}{2}\right].$

Solution: Using Fubini's Theorem.
$\iint_R y\cos(xy)\ dA = \int_0^{\frac{\pi}{2}}\int_0^1 y\cos(xy)\ dxdy = \int_0^{\frac{\pi}{2}}\left(\int_0^1 y\cos(xy)\ dx\right)\ dy$
$= \int_0^{\frac{\pi}{2}} [\sin(xy)]_0^1\ dy = \int_0^{\frac{\pi}{2}} [\sin y - \sin(0)]\ dy = \int_0^{\frac{\pi}{2}} \sin y\ dy$
$= [-\cos y]_0^{\frac{\pi}{2}} = -\cos\left(\frac{\pi}{2}\right) - (-\cos(0)) = 0 + 1 = 1$ .
Note : Solving $\iint_R y\cos(x,y)\ dA = \int_0^1\int_0^{\frac{\pi}{2}} y\cos(xy)\ dydx$ is hard, it needs integration by parts.

### 2.1.2 Volume

If $f(x,y) \geq 0$, then the volume $V$ of the solid that lies above the rectangle $R = [a,b]\times[c,d]$ and below the surface $z = f(x,y)$ is
$\text{V} = \iint_R f(x,y)\ dA = \int_a^b\int_c^d f(x,y)\ dydx = \int_c^d\int_a^b f(x,y)\ dxdy$ .

**Example (3):** Find the volume of the solid $S$ that is bounded by $z = x^2+y^2+1$, the plains $x = 1$ and $y = 3$, and the three coordinate plains.
Solution:
Note that $S$ is the solid that lies under the surface $z = x^2+y^2+1$ and above the square $R = [0,1]\times[0,3]$.
$\text{V} = \iint_R (x^2+y^2+1)\ dA = \int_0^1\int_0^3 (x^2+y^2+1)\ dydx$
$= \int_0^1 \left[x^2y + \frac{y^3}{3} + y\right]_0^3\ dx = \int_0^1 \left[(3x^2+9+3) - (0+0+0)\right]_0^3\ dx$
$= \int_0^1 (3x^2+12)\ dx = \left[x^3+12x\right]_0^1 = (1+12) - (0+0) = 13$ .

**Corollary:** If $f(x,y) = g(x)h(y)$ is continuous on the rectangle $R = [a,b]\times[c,d]$, then $\iint_R f(x,y)\ dA = \iint_R g(x)h(y)\ dA = \int_a^b g(x)\ dx\ \int_c^d h(y)\ dy$ .

**Example (4):** Evaluate $\iint_R x\cos y\ dA$, where $R = [0,2]\times\left[0,\frac{\pi}{2}\right]$.
Solution:
$\iint_R x\cos y\ dA = \int_0^2\int_0^{\frac{\pi}{2}} x\cos y\ dydx = \left(\int_0^2 x\ dx\right)\left(\int_0^{\frac{\pi}{2}}\cos y\ dy\right)$
$= \left[\frac{x^2}{2}\right]_0^2 [\sin y]_0^{\frac{\pi}{2}} = [2-0][1-0] = 2$ .

### 2.1.3 Average Value

If $f(x, y)$ is defined on a rectangle $R$ then its average value is $f_{avg} = \dfrac{1}{A(R)} \displaystyle\iint_R f(x, y)\ dA$, where $A(R)$ is the area of the rectangle $R$.

**Example (5):** Evaluate $f_{avg}$ of $f(x, y) = x \cos y$ on $R = [0, 2] \times \left[0, \dfrac{\pi}{2}\right]$.
Solution:
$f_{avg} = \dfrac{1}{A(R)} \displaystyle\iint_R x \cos y\ dA = \dfrac{2}{(2-0)\left(\frac{\pi}{2} - 0\right)} = \dfrac{2}{\pi}$ .

### 2.1.4 EXERCISES

1. Calculate the iterated integrals :

   (a). $\int_1^4 \int_0^2 (6x^2y - 2x)\ dy\ dx$ .

   (b). $\int_{-3}^1 \int_1^2 (x^2 + y^{-2})\ dy\ dx$ .

   (c). $\int_{-3}^3 \int_0^{\frac{\pi}{2}} (y + y^2 \cos x)\ dx\ dy$ .

   (d). $\int_1^3 \int_1^5 \frac{\ln y}{xy}\ dy\ dx$ .

   (e). $\int_0^1 \int_0^1 xy\sqrt{x^2 + y^2}\ dy\ dx$ .

   (f). $\int_0^1 \int_0^1 v(u + v^2)^4\ du\ dv$ .

2. Calculate the double integrals :

   (a). $\iint_R x \sec^2 y\ dA$, where $R = \left\{(x, y) |\ 0 \leq x \leq 2\ ,\ 0 \leq y \leq \frac{\pi}{4}\right\}$ .

   (b). $\iint_R \frac{xy^2}{x^2 + 1}\ dA$, where $R = \{(x, y) |\ 0 \leq x \leq 1\ ,\ -3 \leq y \leq 3\}$ .

   (c). $\iint_R \frac{1}{1 + x + y}\ dA$, where $R = [1, 3] \times [1, 2]$ .

3. Find the volume of the solid that lies under the plain $4x+6y-2z+15 = 0$ and above the rectangle $R = \{(x, y) |\ -1 \leq x \leq 2\ ,\ -1 \leq y \leq 1\}$ .

4. Find the volume of the solid that lies under $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$ .

## 2.2 Double Integrals over General Regions

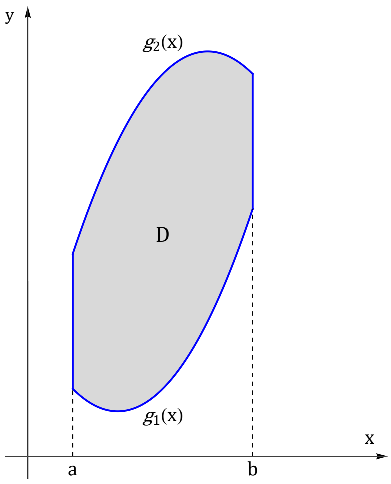
### 2.2.1 General Regions

First - Regions of Type I:

Let $D$ be the region
$\{(x, y) \mid a \leq x \leq b\ ,\ g_1(x) \leq y \leq g_2(x)\}$

If $f$ is continuous on $D$, then

$$\iint_D f(x, y)\ dA$$
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\ dydx$$
$$= \int_a^b \left( \int_{g_1(x)}^{g_2(x)} f(x, y)\ dy \right)\ dx$$

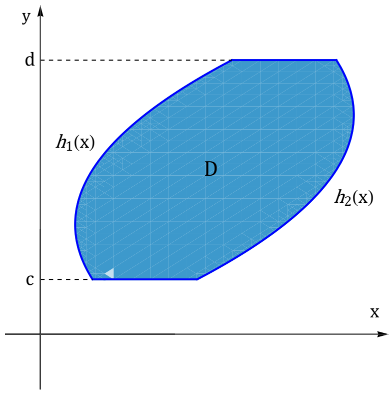



Second - Regions of Type II:

Let $D$ be the region
$\{(x, y) \mid c \leq y \leq d\ ,\ h_1(y) \leq x \leq h_2(y)\}$

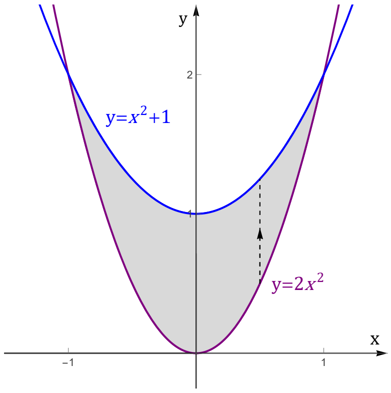
If $f$ is continuous on $D$, then

$$\iint_D f(x, y)\ dA$$
$$= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\ dxdy$$
$$= \int_c^d \left( \int_{h_1(y)}^{h_2(y)} f(x, y)\ dx \right)\ dy$$



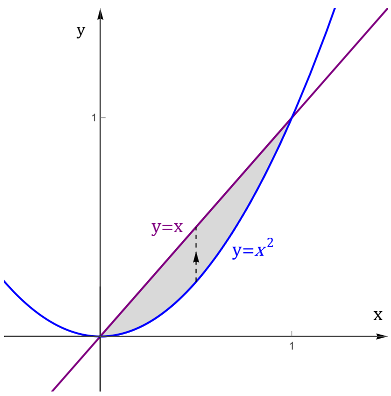

**Example (1):** Evaluate $\iint_D 2xy\ dA$, where $D$ is the region bounded by the graphs of $y = 2x^2$ and $y = x^2 + 1$.
Solution:

Points of intersection:
$2x^2 = x^2 + 1 \implies x^2 = 1$
$\implies x = \pm 1$
So, $D$ is the region where $-1 \le x \le 1$ and $2x^2 \le y \le x^2 + 1$




$$\iint_D 2xy\, dA = \int_{-1}^{1}\int_{2x^2}^{x^2+1} 2xy\, dy\, dx$$
$$= \int_{-1}^{1} x\left[y^2\right]_{2x^2}^{x^2+1} dx$$
$$= \int_{-1}^{1} x\left[(x^2+1)^2 - (2x^2)^2\right] dx$$
$$= \int_{-1}^{1} x\left(x^4 + 2x^2 + 1 - 4x^4\right) dx$$
$$= \int_{-1}^{1} x\left(-3x^4 + 2x^2 + 1\right) dx = \int_{-1}^{1} \left(-3x^5 + 2x^3 + x\right) dx$$
$$= \left[-\frac{x^6}{2} + \frac{x^4}{2} + \frac{x^2}{2}\right]_{-1}^{1}$$
$$= \left(-\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) - \left(-\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) = 0\ .$$

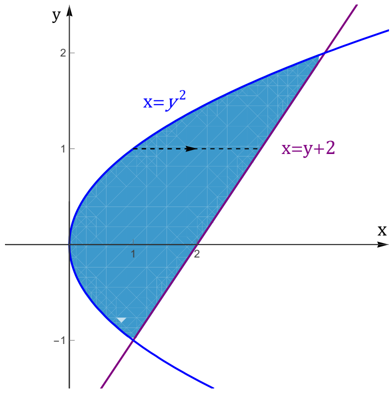
**Example (2):** Evaluate $\iint_D (x^2 + y^2)\, dA$, where $D$ is the region bounded by the graphs of $y = x^2$ and $y = x$.
Solution:
Points of intersection:
$x^2 = x \implies x^2 - x = 0$
$\implies x(x-1) = 0 \implies x = 0\ ,\ x = 1$
So, $D$ is the region where $0 \le x \le 1$ and $x^2 \le y \le x$




$$\iint_D (x^2 + y^2)\, dA$$
$$= \int_0^1 \int_{x^2}^{x} (x^2 + y^2)\, dy\, dx$$
$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_{x^2}^{x} dx$$
$$= \int_0^1 \left[\left(x^3 + \frac{x^3}{3}\right) - \left(x^4 - \frac{x^6}{3}\right)\right] dx$$
$$= \int_0^1 \left(-\frac{x^6}{3} - x^4 + \frac{4x^3}{3}\right) dx = \left[-\frac{x^7}{21} - \frac{x^5}{5} + \frac{x^4}{3}\right]_0^1$$
$$= \left(-\frac{1}{21} - \frac{1}{5} + \frac{1}{3}\right) - (0 + 0 + 0) = \frac{-5 - 21 + 35}{105} = \frac{9}{105} = \frac{3}{35}\ .$$

**Example (3):** Evaluate $\iint_D xy\, dA$, where $D$ is the region bounded by the graphs of $x = y^2$ and $x = y + 2$.

Solution:
Points of intersection:
$y^2 = y + 2 \implies y^2 - y - 2 = 0$
$\implies (y-2)(y+1) = 0 \implies y = -1\ ,\ y = 2$
So, $D$ is the region where $-1 \le y \le 2$ and $y^2 \le x \le y + 2$

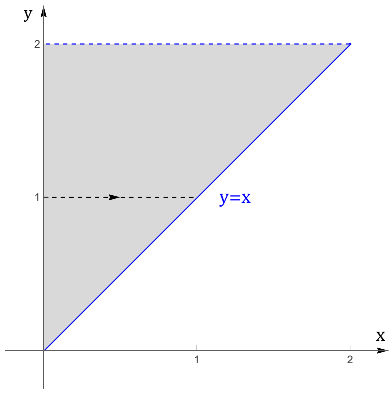



$$\iint_D xy\ dA = \int_{-1}^{2}\int_{y^2}^{y+2} xy\ dx\ dy$$
$$= \int_{-1}^{2} y\left[\frac{x^2}{2}\right]_{y^2}^{y+2} dy$$
$$= \frac{1}{2}\int_{-1}^{2} y\left[(y+2)^2 - (y^2)^2\right]\ dy$$
$$= \frac{1}{2}\int_{-1}^{2} y\left(y^2 + 4y + 4 - y^4\right)\ dy$$
$$= \frac{1}{2}\int_{-1}^{2}\left(-y^5 + y^3 + 4y^2 + 4y\right)\ dy = \frac{1}{2}\left[-\frac{y^6}{6} + \frac{y^4}{4} + \frac{4y^3}{3} + 2y^2\right]_{-1}^{2}$$
$$= \frac{1}{2}\left[\left(-\frac{64}{6} + \frac{16}{4} + \frac{32}{3} + 8\right) - \left(-\frac{1}{6} + \frac{1}{4} - \frac{4}{3} + 2\right)\right] = \frac{45}{8}\ .$$

### 2.2.2 Changing the Order of Integration

**Example (4):** Change the order of integration to evaluate $\displaystyle\int_0^2\int_x^2 e^{5+y^2}\ dy\ dx$.

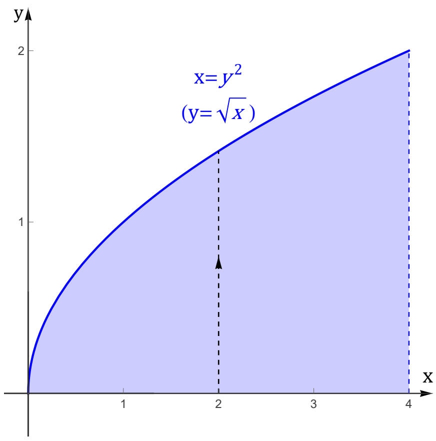
Solution:
$D$ is the region where $0 \le x \le 2$ and $x \le y \le 2$.
Or, $0 \le y \le 2$ and $0 \le x \le y$.




$$\int_0^2\int_x^2 e^{5+y^2}\ dy\ dx$$
$$= \int_0^2\int_0^y e^{5+y^2}\ dx\ dy$$
$$= \int_0^2 e^{5+y^2}[x]_0^y\ dy = \int_0^2 ye^{5+y^2}\ dy$$
$$= \frac{1}{2}\int_0^2 e^{5+y^2}(2y)\ dy$$
$$= \frac{1}{2}\left[e^{5+y^2}\right]_0^2 = \frac{1}{2}\left(e^9 - e^5\right).$$

**Example (5):** Change the order of integration to evaluate $\int_0^2 \int_{y^2}^4 \cos\left(x^{\frac{3}{2}}\right)\ dx\ dy$.

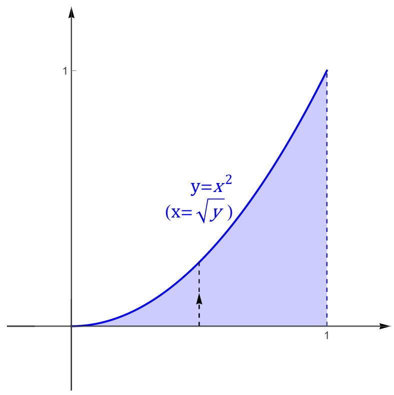
Solution:
$D$ is the region where $0 \le y \le 2$ and $y^2 \le x \le 4$.
Or, $0 \le x \le 4$ and $0 \le y \le \sqrt{x}$.




$$\int_0^2 \int_{y^2}^4 \cos\left(x^{\frac{3}{2}}\right)\ dx\ dy$$
$$= \int_0^4 \int_0^{\sqrt{x}} \cos\left(x^{\frac{3}{2}}\right)\ dy\ dx$$
$$= \int_0^4 \cos\left(x^{\frac{3}{2}}\right)\ [y]_0^{\sqrt{x}}\ dx$$
$$= \frac{2}{3}\int_0^4 \cos\left(x^{\frac{3}{2}}\right)\ \left(\frac{3}{2}x^{\frac{1}{2}}\right)\ dx$$
$$= \frac{2}{3}\left[\sin\left(x^{\frac{3}{2}}\right)\right]_0^4 = \frac{2}{3}\left[\sin(8) - \sin(0)\right] = \frac{2}{3}\sin(8)\ .$$

**Example (6):** Change the order of integration to evaluate $\int_0^1 \int_{\sqrt{y}}^1 \sin\left(x^3\right)\ dx\ dy$.

Solution:
$D$ is the region where $0 \le y \le 1$ and $\sqrt{y} \le x \le 1$.
Or, $0 \le x \le 1$ and $0 \le y \le x^2$.




$$\int_0^1 \int_{\sqrt{y}}^1 \sin\left(x^3\right)\ dx\ dy$$
$$= \int_0^1 \int_0^{x^2} \sin\left(x^3\right)\ dy\ dx$$
$$= \int_0^1 \sin\left(x^3\right)\ [y]_0^{x^2}\ dx$$
$$= \int_0^1 \sin\left(x^3\right)\ x^2\ dx$$
$$= \frac{1}{3}\int_0^1 \sin\left(x^3\right)\ (3x^2)\ dx = \frac{1}{3}\left[-\cos(x^3)\right]_0^1\ .$$
$$= \frac{1}{3}\left[-\cos(1) - (-\cos(0))\right] = \frac{1-\cos(1)}{3}\ .$$

### 2.2.3 Properties of Double Integrals

If $f(x,y)$ and $g(x,y)$ are both integrable on $D \subseteq \mathbb{R}^2$, then

(1). $\iint_D \left[f(x,y) + g(x,y)\right]\ dA = \iint_D f(x,y)\ dA + \iint_D g(x,y)\ dA$ .

(2). $\iint_D k\ f(x,y)\ dA = k\iint_D f(x,y)\ dA$, where $k \in \mathbb{R}$.

(3). If $f(x,y) \ge g(x,y)$ on $D$, then $\iint_D f(x,y)\ dA \ge \iint_D g(x,y)\ dA$.

(4). If $D = D_1 \cup D_2$, where $D_1 \cap D_2 = \phi$, then
$\iint_D f(x,y)\ dA = \iint_{D_1} f(x,y)\ dA + \iint_{D_2} f(x,y)\ dA.$

(5). $\iint_D 1\ dA = A(D)$, where $A(D)$ is the area of the region $D$.

(6). If $m \leq f(x,y) \leq M$ on $D$, then $m\ A(D) \leq \iint_D f(x,y)\ dA \leq M\ A(D)$ .

### 2.2.4 EXERCISES

1. Evaluate the iterated integrals :

   (a). $\int_1^5 \int_0^x (8x - 2y)\ dy\ dx$ (b). $\int_0^2 \int_0^{y^2} x^2 y\ dx\ dy$

   (c). $\int_0^1 \int_0^{e^x} \sqrt{1+e^x}\ dy\ dx$

2. Evaluate $\iint_D 2y\ dA$, where $D$ is the region bounded by the graphs of $y = 3x - x^2$ and $y = x$.

3. Evaluate the double integrals:

   (a). $\iint_D \frac{y}{x^2+1}\ dA$ , $D = \left\{(x, y) \mid 0 \le x \le 4\ ,\ 0 \le y \le \sqrt{x}\right\}$ .

   (b). $\iint_D (2x + y)\ dA$ , $D = \{(x, y) \mid 1 \le y \le 2\ ,\ y - 1 \le x \le 1\}$ .

   (c). $\iint_D x\ dA$ , $D$ is enclosed by the lines $y = x$ , $y = 0$ and $x = 1$ .

   (d). $\iint_D xy\ dA$ , $D$ is enclosed by the curves $y = x^2$ and $y = 3x$ .

   (e). $\iint_D x \cos y\ dA$ , $D$ is bounded by the $y = 0$ , $y = x^2$ and $x = 1$ .

   (f). $\iint_D y^2\ dA$ , $D$ is the triangular region with vertices $(0, 1)$ , $(1, 2)$ and $(4, 1)$.

4. Evaluate the integral by reversing the order of integration:

   (a). $\int_0^1 \int_{3y}^3 e^{x^2}\ dx\ dy$ (b). $\int_0^1 \int_{x^2}^1 \sqrt{y}\ \sin y\ dy\ dx$ .

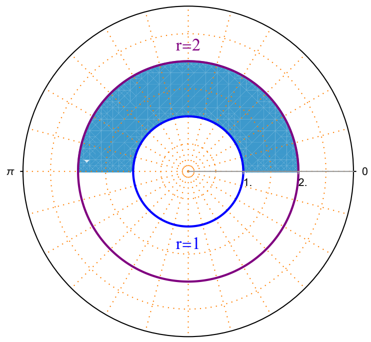
   (c). $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+1}\ dy\ dx$ (d). $\int_0^2 \int_{\frac{y}{2}}^1 y\ \cos(x^3 - 1)\ dx\ dy$

## 2.3 Double Integrals in Polar Coordinates

### 2.3.1 ouble Integrals in Polar Coordinates

If $f$ is continuous on the polar region $R$ given by $a \leq r \leq b$ and $\theta_1 \leq \theta \leq \theta_2$, where $0 \leq \theta_2 - \theta_1 \leq 2\pi$, then $\displaystyle\iint_R f(x,y)\ dA = \int_{\theta_1}^{\theta_2}\int_a^b f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta$.

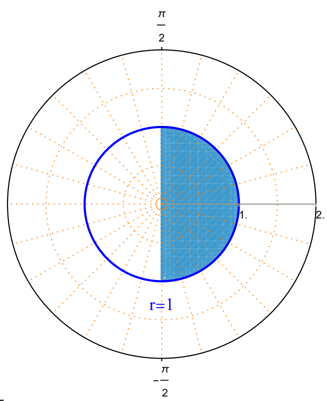
**Example (1):** Evaluate $\displaystyle\iint_R (4x^2+3y)\ dA$, where $R$ is the region in the upper half-plain bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.
Solution:
$R$ is the region where $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi$.




$$\iint_R (4x^2+3y)\ dA$$
$$= \int_0^\pi\int_1^2 (4r^2\cos^2\theta + 3r\sin\theta)\ r\ dr\ d\theta$$
$$= \int_0^\pi\int_1^2 (4r^3\cos^2\theta + 3r^2\sin\theta)\ dr\ d\theta$$
$$= \int_0^\pi \left[r^4\cos^2\theta + r^3\sin\theta\right]_1^2\ d\theta$$
$$= \int_0^\pi \left[(16\cos^2\theta + 8\sin\theta) - (\cos^2\theta + \sin\theta)\right]\ d\theta = \int_0^\pi (15\cos^2\theta + 7\sin\theta)\ d\theta$$
$$= \int_0^\pi \left[15\left(\frac{1+\cos 2\theta}{2}\right) + 7\sin\theta\right]\ d\theta = \frac{15}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^\pi + 7\left[-\cos\theta\right]_0^\pi$$
$$= \frac{15}{2}\left[(\pi+0)-(0+0)\right] + 7[-(-1)-(-1)] = \frac{15\pi}{2} + 14.$$

**Example (2):** Evaluate $\displaystyle\int_{-1}^1\int_0^{\sqrt{1-y^2}} \frac{8(x^2+y^2)}{9+(x^2+y^2)^2}\ dx\ dy$

Solution:
$R$ is the region where $-1 \leq y \leq 1$ and $0 \leq x \leq \sqrt{1-y^2}$
In polar coordinates. $0 \leq r \leq 1$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.



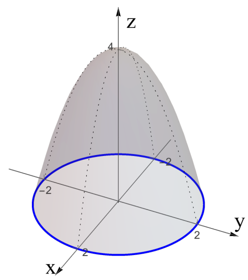

$$\int_{-1}^1\int_0^{\sqrt{1-y^2}} \frac{8(x^2+y^2)}{9+(x^2+y^2)^2}\ dx\ dy$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 \frac{8r^2}{9+r^4} r\ dr\ d\theta$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 2\left(\frac{4r^3}{9+r^4}\right)\ dr\ d\theta$$
$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\ln(9+r^4)\right]_0^1\ d\theta = 2(\ln 10 - \ln 9)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta$$

$$= 2\ln\left(\frac{10}{9}\right)[\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\ln\left(\frac{10}{9}\right)\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 2\pi\ln\left(\frac{10}{9}\right).$$

**Example (3):** Find the volume of the solid bounded by $z = 4 - x^2 - y^2$ and $z = 0$

Solution:

The surface of intersection is :
$4 - x^2 - y^2 = 0 \implies x^2 + y^2 = 4.$
$R$ is the region inside the circle centered at the origin with radius 2.
In polar coordinates, $0 \le r \le 2$ and $0 \le \theta \le 2\pi$.




$$\mathrm{V} = \int_0^{2\pi}\int_0^2 (4 - r^2)\ r\ dr\ d\theta$$
$$= \int_0^{2\pi}\int_0^2 (4r - r^3)\ dr\ d\theta$$
$$= \int_0^{2\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi}[(8-4)-(0-0)]\ d\theta$$
$$= 4\int_0^{2\pi} d\theta = 4\,[\theta]_0^{2\pi} = 4\,(2\pi - 0) = 8\pi\ .$$

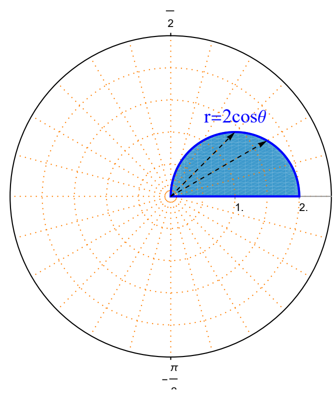
**NOTE:** If $f$ is continuous on the polar region $R = \{(r,\theta)|\theta_1 \le \theta \le \theta_2\ ,\ g_1(\theta) \le r \le g_2(\theta)\}$, then $\displaystyle\iint_R f(x,y)\ dA = \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta.$

**Example (4):** Evaluate $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)^{\frac{1}{2}}\ dy\ dx.$

Solution:

$R$ is the region where $0 \le x \le 2$ and $0 \le y \le \sqrt{2x - x^2}$.
$y = \sqrt{2x - x^2} \implies y^2 = 2x - x^2$
$\implies x^2 - 2x + y^2 = 0$
$\implies (x^2 - 2x + 1) + y^2 = 1$
$\implies (x-1)^2 + y^2 = 1.$

$y = \sqrt{2x - x^2}$ is the upper-half of the circle centered at $(1,0)$ with radius 1.




Note that : $y = \sqrt{2x - x^2} \implies y^2 = 2x - x^2 \implies x^2 + y^2 = 2x$
$\implies r^2 = 2r\cos\theta \implies r = 2\cos\theta.$

In polar coordinates, $0 \le r \le 2\cos\theta$ and $0 \le \theta \le \dfrac{\pi}{2}$.

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)^{\frac{1}{2}} \; dy \; dx = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r \; r \; dr \; d\theta = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 \; dr \; d\theta$$
$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^{2\cos\theta} \; d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \cos^3\theta \; d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \cos^2\theta \; \cos\theta \; d\theta$$
$$= \frac{8}{3}\int_0^{\frac{\pi}{2}} \left(1-\sin^2\theta\right)\cos\theta \; d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \left(\cos\theta - \sin^2\theta \; \cos\theta\right) \; d\theta$$
$$= \frac{8}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{\frac{\pi}{2}} = \frac{8}{3}\left[\left(1-\frac{1}{3}\right) - (0-0)\right] = \frac{16}{9}.$$

### 2.3.2 EXERCISES

1. Evaluate the given integral by changing to polar coordinates.

   (a). $\iint_R (2x - y)\ dA$ , where $R$ is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.

   (b). $\iint_R e^{-x^2-y^2}\ dA$ , where $R$ is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the $y$-axis.

   (c). $\iint_R \cos\sqrt{x^2 + y^2}\ dA$ , where $R$ is the disk with center the origin and radius 2.

2. Use polar coordinates to find the volume of the given solid.

   (a). Under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leq 25$ .

   (b). Below the cone $z = \sqrt{x^2 + y^2}$ and above the ring $1 \leq x^2 + y^2 \leq 4$.

   (c). Below the plain $2x + y + z = 4$ and above the disk $x^2 + y^2 \leq 1$.

   (d). Inside the sphere $x^2+y^2+z^2 = 16$ and outside the cylinder $x^2+y^2 = 4$.

   (e). Above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

   (f). Bounded by the paraboloids $z = 6 - x^2 - y^2$ and $z = 2x^2 + 2y^2$.

3. Evaluate the iterated integral by converting to polar coordinates.

   (a). $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\ dy\ dx$ .

   (b). $\int_0^{\frac{1}{2}} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2\ dx\ dy$ .

## 2.4 Triple Integrals

### 2.4.1 Triple Integrals over Rectangular Boxes

**Fubini's Theorem:**
If $f$ is continuous on the rectangular box $E = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_E f(x,y,z)\ dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\ dx\ dy\ dz = \int_r^s \int_a^b \int_c^d f(x,y,z)\ dy\ dx\ dz$$

$$= \int_c^d \int_r^s \int_a^b f(x,y,z)\ dx\ dz\ dy = \int_c^d \int_a^b \int_r^s f(x,y,z)\ dz\ dx\ dy$$

$$= \int_a^b \int_r^s \int_c^d f(x,y,z)\ dy\ dz\ dx = \int_a^b \int_c^d \int_r^s f(x,y,z)\ dz\ dy\ dx\ .$$

**Example (1):** Evaluate $\iiint_E xyz^2\ dV$ , where $E = [0,1] \times [0,2] \times [0,3]$.

Solution:

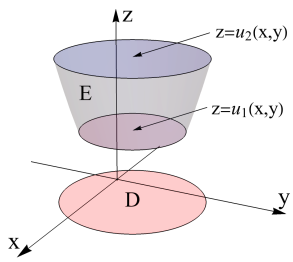
$$\iiint_E xyz^2\ dV = \int_0^1 \int_0^2 \int_0^3 xyz^2\ dz\ dy\ dx = \int_0^1 \int_0^2 xy \left[\frac{z^3}{3}\right]_0^3 dy\ dx$$

$$= \left[\frac{27}{3} - \frac{0}{3}\right] \int_0^1 \int_0^2 xy\ dy\ dx = 9 \int_0^1 x \left[\frac{y^2}{2}\right]_0^2 dx = 9 \int_0^1 x \left[\frac{2^2}{2} - \frac{0}{2}\right] dx$$

$$= 9 \int_0^1 2x\ dx = 9\left[x^2\right]_0^1 = 9[1-0] = 9\ .$$

### 2.4.2 Triple Integrals over General Regions

First - Regions of Type I:
Case (1) :
Let $E$ be the region where
$(x,y) \in D$ and
$u_1(x,y) \le z \le u_2(x,y)$.

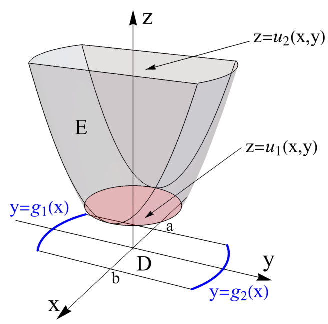



If $f$ is continuous on $E$, then

$$\iiint_E f(x,y,z)\ dV$$

$$= \iint_D \left( \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz \right) dA.$$

Case (2) :
Let $E$ be the region where
$a \le x \le b$, $g_1(x) \le y \le g_2(x)$
and $u_1(x,y) \le z \le u_2(x,y)$.




If $f$ is continuous on $E$, then

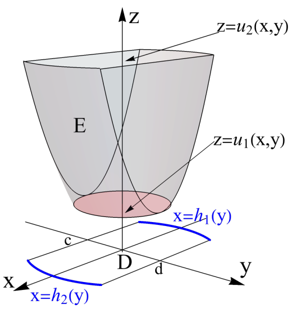
$$\iiint_E f(x,y,z)\ dV$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\ dy\ dx.$$

Case (3) :
Let $E$ be the region where $c \le y \le d$, $h_1(y) \le x \le h_2(y)$ and $u_1(x,y) \le z \le u_2(x,y)$.

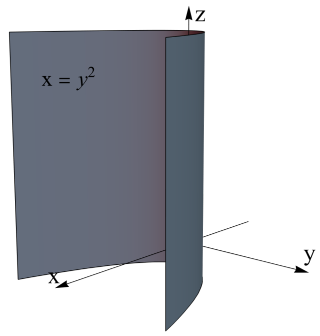
If $f$ is continuous on $E$, then

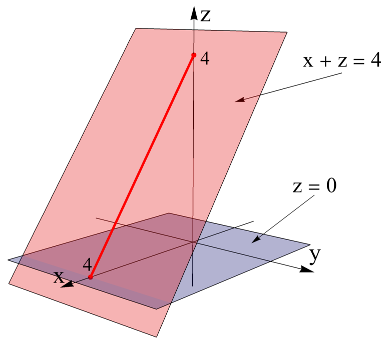
$$\iiint_E f(x,y,z)\ dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\ dx\ dy.$$

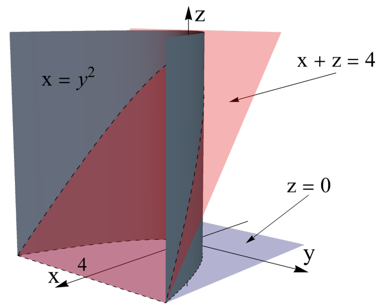


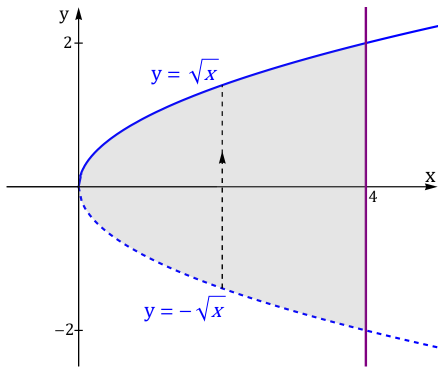

**NOTE:** $\iiint_E 1\ dV =$ V(E), where V(E) is the volume of the region $E$.

**Example (2):** Find the volume of the solid bounded by the cylinder $x = y^2$ and the plains $z = 0$ and $x + z = 4$.
Solution:




Let $E$ be the region bounded by the cylinder $x = y^2$ and the plains $z = 0$ and $x + z = 4$.
Note that $z = 0$ intersects $x + z = 4$ at the line where $x = 4$ and $z = 4$.
On $E$ : $0 \le x \le 4$ , $-\sqrt{x} \le y \le \sqrt{x}$ , $0 \le z \le 4 - x$ .

$$\text{V(E)} = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{4-x} 1\ dz\ dy\ dx = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} [z]_0^{4-x}\ dy\ dx = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} (4-x)\ dy\ dx$$

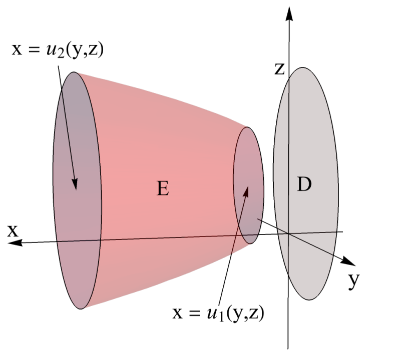
$$= \int_0^4 (4-x)\ [y]_{-\sqrt{x}}^{\sqrt{x}}\ dx = \int_0^4 2\sqrt{x}\ (4-x)\ dx = 2\int_0^4 \left(4x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)\ dx$$
$$= 2\left[\frac{8}{3}\ x^{\frac{3}{2}} - \frac{2}{5}\ x^{\frac{5}{2}}\right]_0^4 = 2\left[\frac{8}{3}\ (4)^{\frac{3}{2}} - \frac{2}{5}\ (4)^{\frac{5}{2}}\right] = 2\left[\frac{8}{3}\ (2^3) - \frac{2}{5}(2^5)\right]$$
$$2\left[\frac{2^6}{3} - \frac{2^6}{5}\right] = 2^7\ \left(\frac{1}{3} - \frac{1}{5}\right) = 2^7\ \left(\frac{2}{15}\right) = \frac{2^8}{15} = \frac{256}{15}.$$

Another Solution :
On $E:\ -2 \le y \le 2\ ,\ y^2 \le x \le 4\ ,\ 0 \le z \le 4-x.$
$$\text{V(E)} = \int_{-2}^{2}\int_{y^2}^{4}\int_0^{4-x} 1\ dz\ dx\ dy\ .$$

Second - Regions of Type II:

Let $E$ be the region where $(y,z) \in D$ and $u_1(y,z) \le x \le u_2(y,z)$.

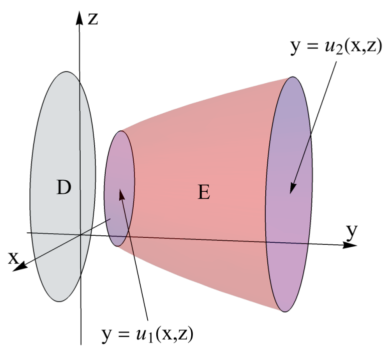
If $f$ is continuous on $E$, then
$$\iiint_E f(x,y,z)\ dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\ dx\right)\ dA.$$



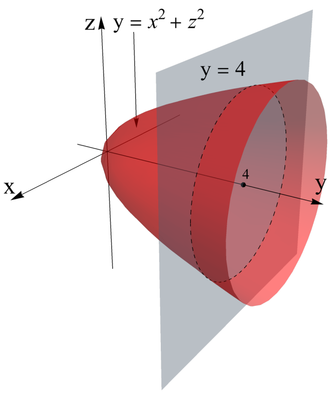

Third - Regions of Type III:

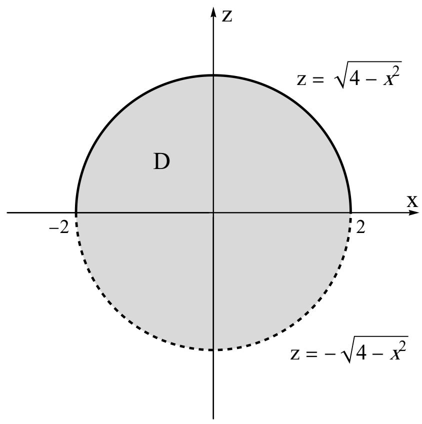
Let $E$ be the region where $(x,z) \in D$ and $u_1(x,z) \le y \le u_2(x,z)$.

If $f$ is continuous on $E$, then
$$\iiint_E f(x,y,z)\ dV = \iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\ dy\right)\ dA.$$




**Example (3):** Evaluate $\iiint_E \sqrt{x^2+z^2}\ dV$, where $E$ is the region bounded by the paraboloid $y = x^2 + z^2$ and the plain $y = 4$.
Solution:




Note that $y = x^2 + z^2$ intersects $y = 4$ at $x^z + z^2 = 4$.
So, $E$ is the region where $x^2 + z^2 \leq y \leq 4$ , $-\sqrt{4 - x^2} \leq z \leq \sqrt{4 - x^2}$ and $-2 \leq x \leq 2$.

$$\iiint_E \sqrt{x^2 + z^2}\ dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+z^2}^{4} \sqrt{x^2 + z^2}\ dy\ dz\ dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [y]_{x^2+z^2}^{4} \sqrt{x^2 + z^2}\ dz\ dx = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4 - (x^2 + z^2)\right] \sqrt{x^2 + z^2}\ dz\ dx$$

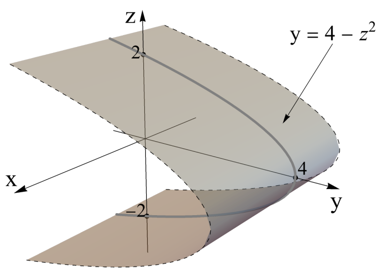
Using the polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$,

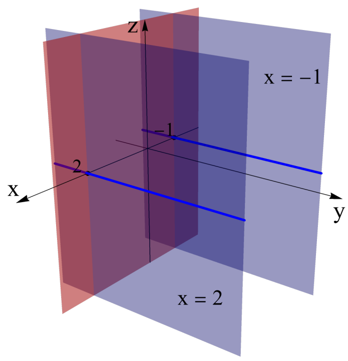
$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4 - (x^2 + z^2)\right] \sqrt{x^2 + z^2}\ dz\ dx = \int_{0}^{2\pi} \int_{0}^{2} (4 - r^2)\ r\ r\ dr\ d\theta$$

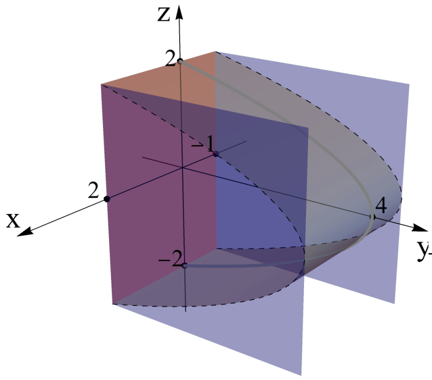
$$= \int_{0}^{2\pi} \int_{0}^{2} (4 - r^2)\ r^2\ dr\ d\theta = \int_{0}^{2\pi} \int_{0}^{2} (4r^2 - r^4)\ dr\ d\theta = \int_{0}^{2\pi} \left[4\frac{r^3}{3} - \frac{r^5}{5}\right]_0^2 d\theta$$

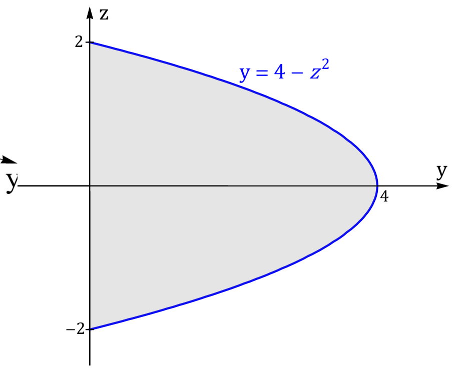
$$= \left(4\ \frac{2^3}{3} - \frac{2^5}{5}\right)\ (2\pi - 0) = 2\pi \left(\frac{2^5}{3} - \frac{2^5}{5}\right) = 2^6 \left(\frac{1}{3} - \frac{1}{5}\right) \pi = \frac{128\pi}{15}.$$

**Example (4):** Find the volume of the solid bounded by the surfaces $y = 4 - z^2$, $x = -1$, $x = 2$ and $y = 0$.
Solution:
Let $E$ be the region bounded by the surfaces $y = 4 - z^2$, $x = -1$, $x = 2$ and $y = 0$.






The surface $y = 4 - z^2$ intersects the plain $y = 0$ at the two lines passing through $z = \pm 2$.

On $E$ : $-1 \leq x \leq 2$ , $0 \leq y \leq 4 - z^2$ and $-2 \leq z \leq 2$.

$$\mathrm{V(E)} = \iiint_E dV = \int_{-1}^{2}\int_{-2}^{2}\int_{0}^{4-z^2} dy\ dz\ dx$$

$$= \int_{-1}^{2}\int_{-2}^{2} [y]_0^{4-z^2}\ dz\ dx = \int_{-1}^{2}\int_{-2}^{2} (4 - z^2) dz\ dx$$

$$= \int_{-1}^{2} \left[4z - \frac{z^3}{3}\right]_{-2}^{2}\ dx = \left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right]\ \int_{-1}^{2} dx$$

$$= \left(16 - \frac{16}{3}\right)\ (2 - (-1)) = 3\left(16 - \frac{16}{3}\right) = 48 - 16 = 32\ .$$

### 2.4.3 EXERCISES

1. Evaluate the iterated integral.

(a). $\int_0^1 \int_y^{2y} \int_0^{x+y} 6xyz \; dz \; dx \; dy$ (b). $\int_1^2 \int_0^{2z} \int_0^{\ln x} x \; e^{-y} \; dy \; dx \; dz$

2. Evaluate the triple integral.

(a). $\iiint_E y \; dV$ , where

$E = \{(x, y, z) \mid 0 \leq x \leq 3 \; , \; 0 \leq y \leq x \; , \; x - y \leq z \leq x + y\}$ .

(b). $\iiint_E \frac{1}{x^3} \; dV$ , where

$E = \left\{(x, y, z) \mid 0 \leq y \leq 1 \; , \; 0 \leq z \leq y^2 \; , \; 1 \leq x \leq z + 1\right\}$ .

(c). $\iiint_E 6xy \; dV$ , where where $E$ lies under the plain $z = 1 + x + y$ and above the region in the $xy$-plain bounded by the curves $y = \sqrt{x}$ , $y = 0$ , and $x = 1$.

3. Use a triple integral to find the volume of the given solid.

(a). The tetrahedron enclosed by the coordinate plains and the plain $2x + y + z = 4$.

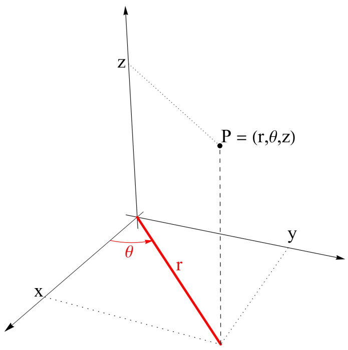
(b). The solid enclosed by the paraboloids $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$.

## 2.5 Triple Integrals in Cylindrical Coordinates

### 2.5.1 Cylindrical Coordinates

If $P(x, y, z)$ is a point in $\mathbb{R}^3 \setminus \{(0,0,0)\}$, then its cylindrical coordinates $P(r, \theta, z)$ are :
$r = \sqrt{x^2 + y^2}$,
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$, where $x \neq 0$,
and $z = z$.
Note: $r \in \mathbb{R}$ and $\theta \in [0, 2\pi]$.

If $P(r, \theta, z)$ is given, the the Cartesian coordinates are :
$x = r\cos\theta$ , $y = r\sin\theta$ and $z = z$ .

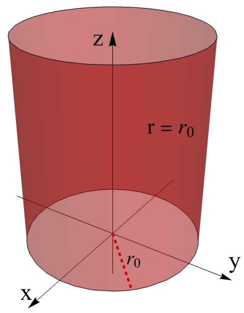



**Important equations in Cylindrical coordinates**

(1). $r = r_0$, where $r_0 \neq 0$.

$r = r_0 \implies r^2 = r_0^2$
$\implies x^2 + y^2 = r_0^2$ .

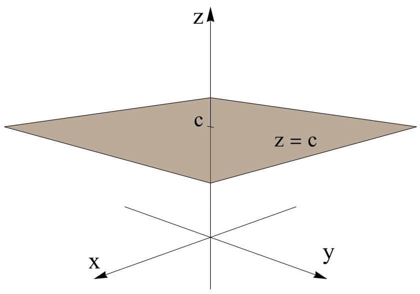
A cylinder,
centered at the origin,
and its radius is $r_0$.



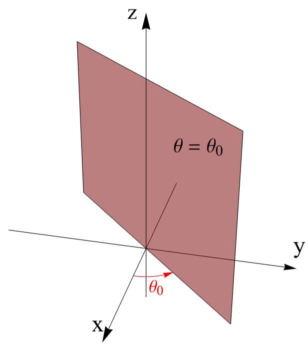

(2). $z = c$, where $c \in \mathbb{R}$.

All the points $(x, y, c)$ in $\mathbb{R}^3$, where $x, y \in \mathbb{R}$.

A horizontal plain,
parallel to the $xy$-plain,
and passes through $(0, 0, c)$.



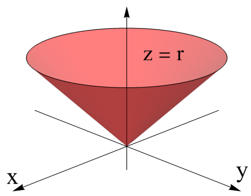
(3). $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi]$.




$$\theta = \theta_0 \implies \tan(\theta) = \tan(\theta_0),$$
$$\implies \frac{y}{x} = \tan(\theta_0)$$
$$\implies y = \tan(\theta_0)\, x.$$

A vertical plain,
passes through the origin.

(4). $z = r$, where $r \neq 0$.



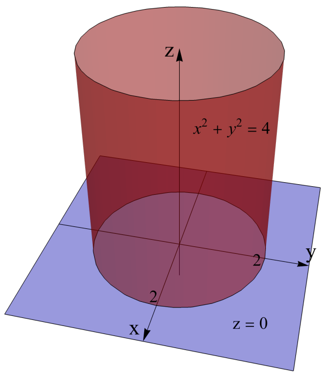

$$z = r \implies z = \sqrt{x^2 + y^2}.$$

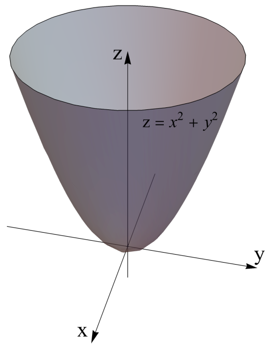
A Cone.

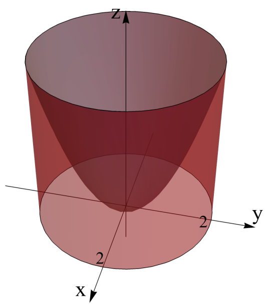
### 2.5.2 Triple Integrals in Cylindrical Coordinates

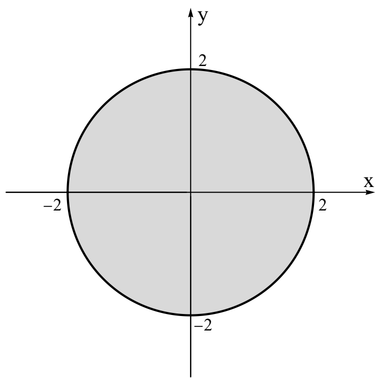
Suppose $f$ is continuous on $E = \{(x, y, z) \mid (x, y) \in D\ ,\ u_1(x, y) \le z \le u_2(x, y)\}$,
where $D$ is given in polar coordinates by $D = \{(r, \theta) \mid \theta_1 \le \theta \le \theta_2\ ,\ r_1(\theta) \le r \le r_2(\theta)\}$,
then $\iiint_E f(x, y, z)\ dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\ r\ dz\ dr\ d\theta.$

**Example (1):** Find the volume of the solid within $x^2 + y^2 = 4$, bounded above by $z = x^2 + y^2$ and below by $z = 0$.
Solution:



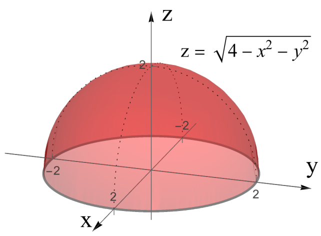



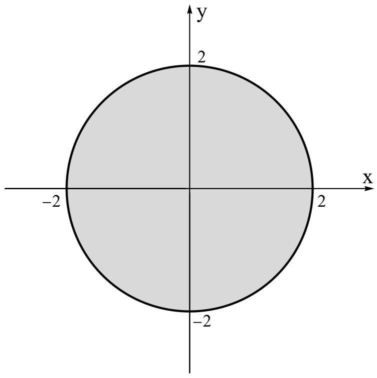




$E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi\ ,\ 0 \le r \le 2\ ,\ 0 \le z \le x^2 + y^2 = r^2\}.$

$$\text{Volume} = \iiint_E dV = \int_0^{2\pi} \int_0^2 \int_0^{x^2+y^2} r\ dz\ dr\ d\theta = \int_0^{2\pi} \int_0^2 \int_0^{r^2} r\ dz\ dr\ d\theta$$
$$= \int_0^{2\pi} \int_0^2 [z]_0^{r^2}\ r\ dr\ d\theta = \int_0^{2\pi} \int_0^2 (r^2 - 0)r\ dr\ d\theta = \int_0^{2\pi} \int_0^2 r^3\ dr\ d\theta$$
$$= \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^2 d\theta = \left[\frac{2^4}{4} - 0\right] \int_0^{2\pi} d\theta = 4(2\pi - 0) = 8\pi.$$

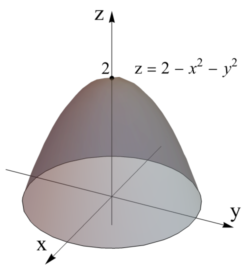
**Example (2):** Evaluate $\iiint_E z\ dx\ dy\ dz$, where $E = \left\{(x,y,z) \mid 0 \le z \le \sqrt{4-x^2-y^2}\right\}$.
Solution:
$z = \sqrt{4-x^2-y^2}$ is the upper-half of the sphere centered at the origin with radius 2.
$\sqrt{4-x^2-y^2} = 0 \implies x^2+y^2 = 4$, the upper-half of the sphere intersects the plain $z = 0$ at the circle centered at the origin with center 2.
$E = \left\{(r,\theta,z) \mid 0 \le \theta \le 2\pi\ ,\ 0 \le r \le 2\ ,\ 0 \le z \le \sqrt{4-x^2-y^2} = \sqrt{4-r^2}\right\}$.

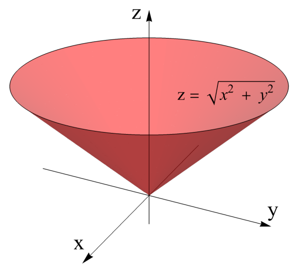


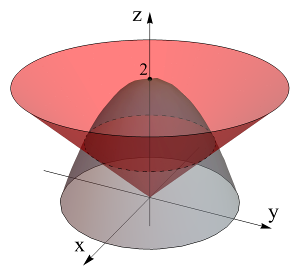



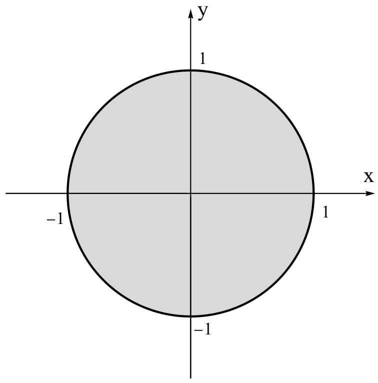

$$\iiint_E z\ dx\ dy\ dz = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{4-r^2}} z\ dz\ r\ drd\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 \left[z^2\right]_0^{\sqrt{4-r^2}} r\ dr\ d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\int_0^2 (4-r^2)r\ dr\ d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 (4r-r^3)\ dr\ d\theta = \frac{1}{2}\int_0^{2\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta$$
$$= \frac{1}{2}(8-4)\int_0^{2\pi} d\theta = 2(2\pi - 0) = 4\pi\ .$$

**Example (3):** Find the volume of the solid bounded above by $z = 2 - x^2 - y^2$ and below by $z = \sqrt{x^2+y^2}$.
Solution:
$z = 2 - x^2 - y^2 = 2 - r^2$ intersects $z = \sqrt{x^2+y^2} = r$ at :
$2 - r^2 = r \implies r^2 + r - 2 = 0 \implies (r+2)(r-1) = 0 \implies r = 1.$
(Note that $r = -2$ is excluded because $r \ge 0$).






$E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi\ ,\ 0 \le r \le 1\ ,\ r \le z \le 2 - r^2\}$.

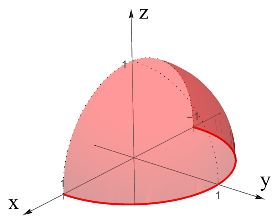
Volume $= \iiint_E dV = \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} dz\ r\ dr\ d\theta = \int_0^{2\pi} \int_0^1 [z]_r^{2-r^2}\ r\ dr\ d\theta$

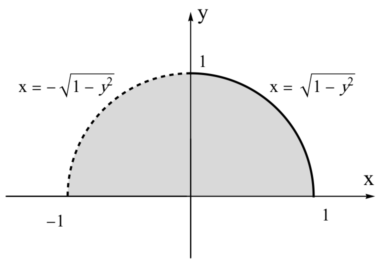
$= \int_0^{2\pi} \int_0^1 (2 - r^2 - r) r\ dr\ d\theta = \int_0^{2\pi} \int_0^1 (-r^3 - r^2 + 2r)\ dr\ d\theta$

$= \int_0^{2\pi} \left[ -\frac{r^4}{4} - \frac{r^3}{3} + r^2 \right]_0^1 d\theta = \left( -\frac{1}{4} - \frac{1}{3} + 1 \right) \int_0^{2\pi} d\theta = \frac{5}{12}(2\pi - 0) = \frac{5\pi}{6}.$

**Example (4):** Evaluate the integral $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\ dx\ dy$

Solution:
Note that $0 \le y \le 1$, $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$ and $0 \le z \le \sqrt{1-x^2-y^2}$.
$z = \sqrt{1-x^2-y^2} \implies x^2 + y^2 + z^2 = 1$ represents the upper half of the unit sphere.
$x = \sqrt{1-y^2} \implies x^2 + y^2 = 1$ represents the right half of the unit circle.
$x = -\sqrt{1-y^2} \implies x^2 + y^2 = 1$ represents the left half of the unit circle.




In cylindrical coordinates: $0 \le \theta \le \pi$, $0 \le r \le 1$ and $0 \le z \le \sqrt{1-r^2}$.

$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\ dx\ dy = \int_0^{\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} dz\ r\ dr\ d\theta$

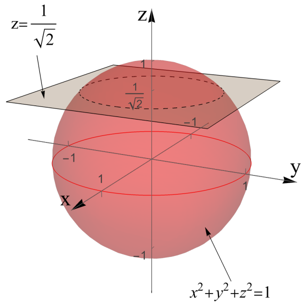
$= \int_0^{\pi} \int_0^1 \sqrt{1-r^2}\ r\ dr\ d\theta = -\frac{1}{2} \int_0^{\pi} \int_0^1 (1-r^2)^{\frac{1}{2}}\ (-2r)\ dr\ d\theta$

$= -\frac{1}{2} \left[ \frac{2}{3} (1-r^2)^{\frac{3}{2}} \right]_0^1 \int_0^{\pi} d\theta = -\frac{1}{2} \left( 0 - \frac{2}{3} \right) (\pi - 0) = \frac{\pi}{3}.$

**Example (5):** Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 1$ and above the plain $z = \dfrac{1}{\sqrt{2}}$.

Solution:

Note that $x^2 + y^2 + z^2 = 1$ intersects $z = \dfrac{1}{\sqrt{2}}$ at $x^2 + y^2 = \dfrac{1}{2}$ which is a circle centered at the origin and its radius is $\dfrac{1}{\sqrt{2}}$.




The solid is bounded above by the upper-half of the sphere, and below by the plain.

$x^2 + y^2 + z^2 = 1$

$\Longrightarrow \; z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - r^2}.$

The cylindrical coordinates of the solid : $0 \leq r \leq \dfrac{1}{\sqrt{2}}$ , $0 \leq \theta \leq 2\pi$ and $\dfrac{1}{\sqrt{2}} \leq z \leq \sqrt{1 - r^2}$ .

$$\text{V} = \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \int_{\frac{1}{\sqrt{2}}}^{\sqrt{1-r^2}} dz \; r \; dr \; d\theta = \left( \int_0^{2\pi} d\theta \right) \left( \int_0^{\frac{1}{\sqrt{2}}} \int_{\frac{1}{\sqrt{2}}}^{\sqrt{1-r^2}} dz \; r \; dr \right)$$

$$= [\theta]_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} [z]_{\frac{1}{\sqrt{2}}}^{\sqrt{1-r^2}} \; r \; dr = (2\pi - 0) \int_0^{\frac{1}{\sqrt{2}}} \left( \sqrt{1 - r^2} - \frac{1}{\sqrt{2}} \right) r \; dr$$

$$= 2\pi \int_0^{\frac{1}{\sqrt{2}}} \left( r\sqrt{1 - r^2} - \frac{r}{\sqrt{2}} \right) \; dr = 2\pi \left[ -\frac{1}{2} \, \frac{(1 - r^2)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{r^2}{2\sqrt{2}} \right]_0^{\frac{1}{\sqrt{2}}}$$

$$= 2\pi \left[ \left( -\frac{1}{2}\frac{2}{3} \left( 1 - \frac{1}{2} \right)^{\frac{3}{2}} - \frac{1}{4\sqrt{2}} \right) - \left( -\frac{1}{2}\frac{2}{3} - 0 \right) \right]$$

$$= 2\pi \left( -\frac{1}{3}\frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{2}} + \frac{1}{3} \right) = \frac{2\pi}{3} \left( -\frac{1}{2\sqrt{2}} - \frac{3}{4\sqrt{2}} + 1 \right)$$

$$= \frac{2\pi}{3} \left( 1 - \frac{5}{4\sqrt{2}} \right).$$

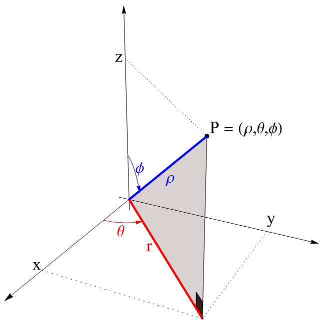
### 2.5.3 EXERCISES

1. Evaluate $\iiint_E \sqrt{x^2+y^2}\, dV$ , where $E$ is the region that lies inside the cylinder $x^2+y^2=16$ and between the plains $z=-5$ and $z=4$.
2. Evaluate $\iiint_E z\, dV$ , where $E$ is enclosed by the paraboloid $z=x^2+y^2$ and the plain $z=4$.
3. Evaluate $\iiint_E (x+y+z)\, dV$ , where $E$ is the solid in the first octant that lies under the paraboloid $z=4-x^2-y^2$.
4. Find the volume of the solid that lies within both the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$.
5. Find the volume of the solid that is enclosed by the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=2$.
6. Find the volume of the solid that lies between the paraboloid $z=x^2+y^2$ and the sphere $x^2+y^2+z^2=2$.
7. Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{2} xz\; dz\; dx\; dy.$
8. Evaluate $\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}}\int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\; dz\; dy\; dx.$

## 2.6 Triple Integrals in Spherical Coordinates

### 2.6.1 Spherical Coordinates

If $P(x, y, z)$ is a point in $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$,
then its spherical coordinates
$P(\rho, \theta, \phi)$ are :
$\rho = \sqrt{x^2 + y^2 + z^2}$,
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$, where $x \neq 0$,
and $\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$, where $\rho \neq 0$.
Note: $\rho \geq 0$, $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$.

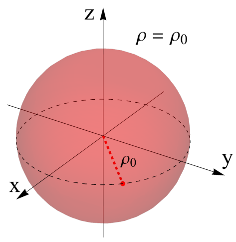



Note that if $P(\rho, \theta, \phi)$ is given then :
$\sin\phi = \frac{r}{\rho} \implies r = \rho \sin\phi$ and $\cos\phi = \frac{z}{\rho} \implies z = \rho\cos\phi$ .
So, $x = r\cos\theta = \rho\cos\theta\sin\phi$ and $y = r\sin\theta = \rho\sin\theta\sin\phi$ .
**Important equations in Spherical coordinates**

(1). $\rho = \rho_0$, where $\rho_0 > 0$.

$\rho = \rho_0 \implies \rho^2 = \rho_0^2$
$\implies x^2 + y^2 + z^2 = r_0^2$ .

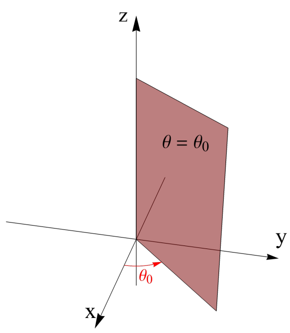
A sphere,
centered at the origin,
and its radius is $\rho_0$.




(2). $\theta = \theta_0$, where $\theta_0 \in [0, 2\pi]$.

Since $\phi \in [0, \pi]$ then
$\sin\phi \geq 0$ .
If $\theta \in [0, \pi]$ then :
$y = \rho\sin\theta\sin\phi \geq 0$
$\theta = \theta_0$ represents a half-plain.

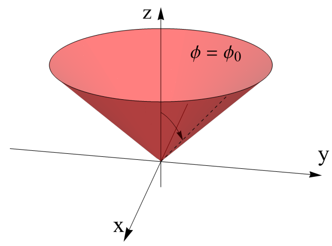
If $\theta \in [\pi, 2\pi]$ then :
$y = \rho\sin\theta\sin\phi \leq 0$
$\theta = \theta_0$ represents the other half-plain.



(3). $\phi = \phi_0$, where $0 < \phi_0 < \dfrac{\pi}{2}$.

$\cot\phi = \dfrac{z}{r} \implies z = r\ \cot\phi,$
$\implies z = \cot\phi\ \sqrt{x^2 + y^2}$

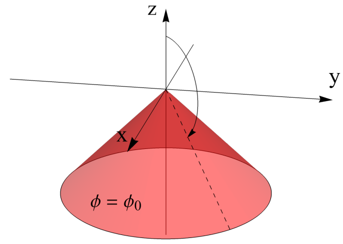
Since $\phi \in \left(0, \dfrac{\pi}{2}\right)$ then
$\cot\phi > 0$.
$\phi = \phi_0$ represents an upper cone.




(4). $\phi = \phi_0$, where $\dfrac{\pi}{2} < \phi_0 < \pi$.

$\cot\phi = \dfrac{z}{r} \implies z = r\ \cot\phi,$
$\implies z = \cot\phi\ \sqrt{x^2 + y^2}$

Since $\phi \in \left(\dfrac{\pi}{2}, \pi\right)$ then
$\cot\phi < 0$.
$\phi = \phi_0$ represents a lower cone.



### 2.6.2 Triple Integrals in Spherical Coordinates

$$\iiint_E f(x,y,z)\ dV = \int_{\phi_1}^{\phi_2}\int_{\theta_1}^{\theta_2}\int_a^b f(\rho,\theta,\phi)\ \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi\ ,$$

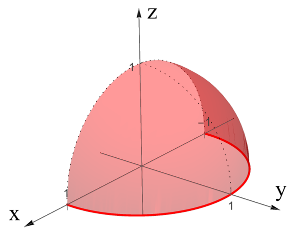
where $E = \{(\rho,\theta,\phi) \mid a \le \rho \le b\ ,\ \theta_1 \le \theta \le \theta_2\ ,\ \phi_1 \le \phi \le \phi_2\}$ .

**Example (1):** Evaluate the integral $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\ dx\ dy$

Solution: Referring to Example(4) in section (2.5).

In Spherical coordinates :

$0 \le \rho \le 1$ , $0 \le \theta \le \pi$ , and $0 \le \phi \le \dfrac{\pi}{2}$ .

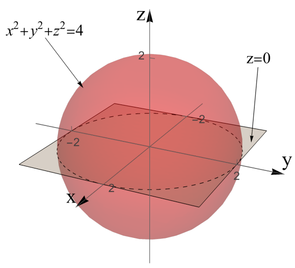



$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\ dx\ dy$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^1 \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$
$$= \left(\int_0^1 \rho^2\ d\rho\right)\left(\int_0^{\frac{\pi}{2}} \sin\phi\ d\phi\right)\left(\int_0^{\pi} d\theta\right)$$
$$= \left[\frac{\rho^3}{3}\right]_0^1\ \left[-\cos\phi\right]_0^{\frac{\pi}{2}}\ \left[\theta\right]_0^{\pi} = \left(\frac{1}{3} - 0\right)\ \left(-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right)\ (\pi - 0) = \frac{\pi}{3}\ .$$

**Example (2):** Evaluate the integral $\displaystyle\iiint_E 4z\ dV$ , where $E$ is the region bounded above by $x^2+y^2+z^2 = 4$ and below by $z = 0$.

Solution:

Note that $x^2+y^2+z^2 = 4$ intersects $z = 0$ at $x^2+y^2 = 4$ which is a circle centered at the origin and its radius is 2.




$E$ in spherical coordinates:

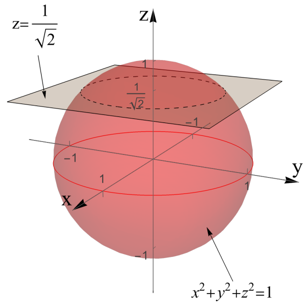
$0 \le \rho \le 2$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \dfrac{\pi}{2}$.

$z = \rho\cos\phi \implies 4z = 4\rho\cos\phi$ .

$$\iiint_E 4z\ dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^2 (4\rho\cos\phi)\ \rho^2\sin\phi\ d\rho\ d\phi\ d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^2 (4\rho^3)\ (\cos\phi\ \sin\phi)\ d\rho\ d\phi\ d\theta$$
$$= \left(\int_0^2 4\rho^3\ d\rho\right)\left(\int_0^{\frac{\pi}{2}} \cos\phi\ \sin\phi\ d\phi\right)\left(\int_0^{2\pi} d\theta\right) = \left[\rho^4\right]_0^2\ \left[\frac{\sin^2\phi}{2}\right]_0^{\frac{\pi}{2}}\ \left[\theta\right]_0^{2\pi}$$
$$= (16-0)\ \left(\frac{1}{2} - 0\right)\ (2\pi - 0) = 16\pi\ .$$

**Example (3):** Find the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 1$ and above the plain $z = \frac{1}{\sqrt{2}}$.
Solution:




Note that $x^2 + y^2 + z^2 = 1$ intersects $z = \frac{1}{\sqrt{2}}$ at $x^2+y^2 = \frac{1}{2}$ which is a circle centered at the origin and its radius is $\frac{1}{\sqrt{2}}$.

$z = \rho\cos\phi \implies \cos\phi = \frac{1}{\sqrt{2}}$

$\implies \phi = \frac{\pi}{4}$.

$$x^2 + y^2 + z^2 = 1 \implies \rho = 1 \;,\; z = \frac{1}{\sqrt{2}} \implies \rho\cos\phi = \frac{1}{\sqrt{2}} \implies \rho = \frac{1}{\sqrt{2}\cos\phi}.$$

The spherical coordinates of the solid : $\frac{1}{\sqrt{2}\cos\phi} \le \rho \le 1 \;,\; 0 \le \theta \le 2\pi$ and $0 \le \phi \le \frac{\pi}{4}$ .

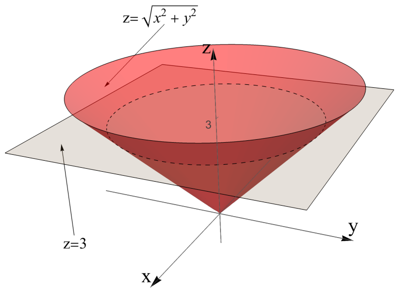
$$\text{V} = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\sqrt{2}\cos\phi}}^{1} \rho^2\sin\phi\; d\rho\; d\phi\; d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{4}}\int_{\frac{1}{\sqrt{2}\cos\phi}}^{1} \rho^2\sin\phi\; d\rho\; d\phi\right)$$

$$= [\theta]_0^{2\pi}\int_0^{\frac{\pi}{4}}\left[\frac{\rho^3}{3}\right]_{\frac{1}{\sqrt{2}\cos\phi}}^{1}\sin\phi\; d\phi = \frac{2\pi}{3}\int_0^{\frac{\pi}{4}}\left[1 - \frac{1}{2\sqrt{2}\cos^3\phi}\right]\sin\phi\; d\phi$$

$$= \frac{2\pi}{3}\int_0^{\frac{\pi}{4}}\left[\sin\phi - \frac{(\cos\phi)^{-3}\sin\phi}{2\sqrt{2}}\right]\; d\phi = \frac{2\pi}{3}\left[-\cos\phi + \frac{1}{2\sqrt{2}}\frac{(\cos\phi)^{-2}}{-2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{2\pi}{3}\left[\left(-\frac{1}{\sqrt{2}} - \frac{2}{4\sqrt{2}}\right) - \left(-1 - \frac{1}{4\sqrt{2}}\right)\right] = \frac{2\pi}{3}\left(1 - \frac{5}{4\sqrt{2}}\right).$$

**Example (4):** Evaluate $\iiint_E \sqrt{x^2+y^2+z^2}\; dV$, where $E$ is the region bounded above by the palne $z = 3$ and below by the cone $z = \sqrt{x^2+y^2}$.
Solution:




Note that $z = \sqrt{x^2+y^2}$ intersects $z = 3$ at $x^2 + y^2 = 9$ which is a circle centered at the origin and its radius is 3.

$z = \sqrt{x^2+y^2} \implies$

$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right)}$

$\implies \rho\cos\phi = \rho\sin\phi$

$\implies \phi = \frac{\pi}{4}$.

$z = 3 \implies \rho\cos\phi = 3 \implies \rho = \dfrac{3}{\cos\phi}$.

The spherical coordinates of $E$ : $0 \le \rho \le \dfrac{3}{\cos\phi}$ , $0 \le \theta \le 2\pi$ and $0 \le \phi \le \dfrac{\pi}{4}$.

$$\iiint_E \sqrt{x^2+y^2+z^2}\, dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\frac{3}{\cos\phi}} \rho\, \rho^2\, \sin\phi\, d\rho\, d\phi\, d\theta$$

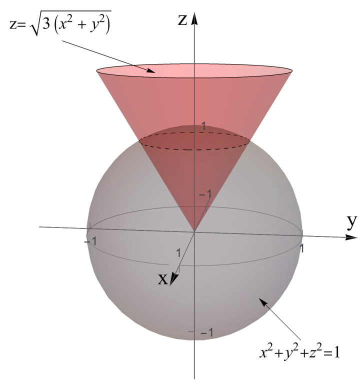
$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{4}}\int_0^{\frac{3}{\cos\phi}} \rho^3 \sin\phi\, d\rho\, d\phi\right) = [\theta]_0^{2\pi}\int_0^{\frac{\pi}{4}} \left[\frac{\rho^4}{4}\right]_0^{\frac{3}{\cos\phi}} \sin\phi\, d\phi$$

$$= (2\pi - 0)\,\frac{3^4}{4}\int_0^{\frac{\pi}{4}} (\cos\phi)^{-4}\sin\phi\, d\phi = \frac{81\pi}{2}\left[\frac{-(\cos\phi)^{-3}}{-3}\right]_0^{\frac{\pi}{4}} = \frac{27\pi}{2}\left(2\sqrt{2}-1\right)$$

.

**Example (5):** Find the volume of the solid inside the sphere $x^2+y^2+z^2 = 1$ and inside the cone $z = \sqrt{3(x^2+y^2)}$.
Solution:

Note that $x^2+y^2+z^2 = 1$ intersects $z = \sqrt{3(x^2+y^2)}$ at
$3x^2+3y^2 = 1 - x^2 - y^2$
$\implies 4x^2+4y^2 = 1$
$\implies x^2+y^2 = \dfrac{1}{4}$
which is a circle centered at the origin and its radius is $\dfrac{1}{2}$.




$z = \sqrt{3(x^2+y^2)} \implies$
$\rho\cos\phi = \sqrt{3}\rho\sin\phi \implies \tan\phi = \dfrac{1}{\sqrt{3}}$
$\implies \phi = \dfrac{\pi}{6}$.

The spherical coordinates of the solid : $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \dfrac{\pi}{6}$.

$$\text{V} = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{1} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \left(\int_0^1 \rho^2\, d\rho\right)\left(\int_0^{\frac{\pi}{6}} \sin\phi\, d\phi\right)\left(\int_0^{2\pi} d\theta\right)$$

$$= \left[\frac{\rho^3}{3}\right]_0^1 [-\cos\phi]_0^{\frac{\pi}{6}}\, [\theta]_0^{2\pi} = \left(\frac{1}{3} - 0\right)\left(-\frac{\sqrt{3}}{2} - (-1)\right)(2\pi - 0)$$

$$= \frac{2\pi}{3}\left(1 - \frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}\left(2 - \sqrt{3}\right) .$$

### 2.6.3 EXERCISES

1. Change from rectangular to spherical coordinates:

   (a). $(3,3,0)$ . (b). $\left(1,-\sqrt{3},2\sqrt{3}\right)$.

2. Identify the surface whose equation is $\rho = \cos\phi$.

3. Sketch the solid described by the given inequalities:

   (a). $\rho \le 1$ , $0 \le \theta \le \pi$ , $0 \le \phi \le \dfrac{\pi}{6}$. (b). $\rho \le 2$ , $\rho \le \csc\phi$ .

4. Evaluate $\displaystyle\iiint_E \left(x^2+y^2+z^2\right)^2 \, dV$ , where $E$ is the ball with center the origin and radius 5.

5. Evaluate $\displaystyle\iiint_E y^2z^2 \, dV$ , where $E$ lies above the cone $\phi = \dfrac{\pi}{3}$ and below the sphere $\rho = 1$.

6. Evaluate $\displaystyle\iiint_E xe^{x^2+y^2+z^2} \, dV$ , where $E$ is the portion of the unit ball $x^2+y^2+z^2 \le 1$ that lies in the first octant.

7. Find the volume of the part of the ball $\rho \le a$ that lies between the cones $\phi = \dfrac{\pi}{6}$ and $\phi = \dfrac{\pi}{3}$.

# Chapter 3

# Sequences, Series, and Power Series

## 3.1 Sequences

### 3.1.1 Infinite Sequences

**Definition:** An infinite sequence is a function $f : \mathbb{N} \longrightarrow \mathbb{R}$, where $f(n) = a_n$.

**Notation:** The sequence $\{a_1, a_2, \cdots\}$ is also denoted by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

**Example (1):** Some sequences can be defined by giving a formula for the n$^{\text{th}}$ term.
(a). $a_n = \dfrac{1}{2^n}$ is the n$^{\text{th}}$ term of the sequence $\left\{\dfrac{1}{2^n}\right\} = \left\{\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \cdots\right\}$.
(b). $\left\{\dfrac{(-1)^n(n+2)}{5^n}\right\} = \left\{-\dfrac{3}{5}, \dfrac{4}{25}, -\dfrac{5}{125}, \cdots\right\}$.

**Example (2):** The Fibonacci sequences is defined recursively by
$f_1 = 1$ , $f_2 = 1$ , $f_n = f_{n-1} + f_{n-2}$ , $n \geq 3$.
$\{f_n\} = \{1, 1, 2, 3, 5, 8, 13, \cdots\}$ .

### 3.1.2 The Limit of a Sequence

**Definition:** A sequence $\{a_n\}$ has the limit $L \in \mathbb{R}$, if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that : if $n \geq N$ then $|a_n - L| < \epsilon$, and we write $\lim\limits_{n\to\infty} a_n = L$ .

**Definition:** A sequence $\{a_n\}$ goes to $\infty$, if for every positive real number $M$ there exists an $N \in \mathbb{N}$ such that : if $n \geq N$ then $a_n \geq M$, and we write $\lim\limits_{n\to\infty} a_n = \infty$.

### 3.1.3 Properties of Convergent Sequences

**Theorem:** If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ where $n \in \mathbb{N}$ then $\lim_{x\to\infty} a_n = L$.

**Example (3):** Calculate $\lim_{n\to\infty} \frac{\ln n}{n}$ .

Solution : Let $f(x) = \frac{\ln x}{x}$.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$ (By L'Hôpital's Rule).

Therefore, $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ .

**Theorem:** Suppose $\lim_{n\to\infty} a_n = L_1$ , $\lim_{n\to\infty} b_n = L_2$ and $c \in \mathbb{R}$, then:

(1). $\lim_{n\to\infty} c = c$ .

(2). $\lim_{n\to\infty} a_n \pm b_n = L_1 \pm L_2$ .

(3). $\lim_{n\to\infty} c\ a_n = cL_1$ .

(4). $\lim_{n\to\infty} a_n.b_n = L_1.L_2$ .

(5). $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L_1}{L_2}$ , where $L_2 \neq 0$ .

**Power Law:** $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ , if $p > 0$ and $a_n > 0$.

**Squeeze Theorem for sequences:** If $a_n \leq b_n \leq c_n$ for $n \geq N_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ , then $\lim_{n\to\infty} b_n = L$ .

**Example (4):** Calculate $\lim_{n\to\infty} \frac{n!}{n^n}$ .

Solution:

$0 \leq \frac{n!}{n^n} = \frac{1.2.3\cdots n}{n.n.n\cdots n} = \frac{1}{n}\left(\frac{2.3\cdots n}{n.n\cdots n}\right) \leq \frac{1}{n}(1) = \frac{1}{n}$

Since $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0$, then $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ (By Squeeze Theorem).

**Theorem:** If $\lim_{n\to\infty} |a_n| = 0$ , then $\lim_{n\to\infty} a_n = 0$ .

**Example (5):** Calculate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ .

Solution: Let $a_n = \frac{(-1)^n}{n}$ .

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{|(-1)^n|}{|n|} = \lim_{n\to\infty} \frac{1}{n} = 0$ .

Therefore $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ .

**Theorem:** If $\lim_{n\to\infty} a_n = L$ , and the function $f$ is continuous at $L$ , then $\lim_{n\to\infty} f(a_n) = f(L)$ .

**Example (6):** Calculate $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)$ .
Solution:
$\lim_{n\to\infty} \frac{\pi}{n} = \lim_{n\to\infty} \pi \, \frac{1}{n} = \pi(0) = 0$ .
Since the cosine function is continuous at 0 , then $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$ .

**Important Note:** The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$, and divergent for all other values of $r$.
$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad -1 < r < 1 \\ 1 & \text{if} \quad r = 1 \end{cases} .$$

### 3.1.4 Monotonic and bounded sequences

**Definition:**
(1). A sequence $\{a_n\}$ is called increasing if $a_{n+1} \geq a_n$ for all $n \geq 1$ .
(2). A sequence $\{a_n\}$ is called decreasing if $a_{n+1} \leq a_n$ for all $n \geq 1$ .
(3). A sequence is called monotonic if it is either increasing or decreasing.

**Definition:**
(1). A sequence $\{a_n\}$ is bounded above if there exists a real number $M$ such that $a_n \leq M$ for all $n \geq 1$ .
(2). A sequence $\{a_n\}$ is bounded below if there exists a real number $m$ such that $a_n \geq m$ for all $n \geq 1$ .
(3). If a sequence is bounded above and below, then it is called a bounded sequence.

**Monotonic Sequence Theorem**
Every bounded, monotonic sequence is convergent. In particular :
(1). A sequence that is increasing and bounded above converges.
(2). A sequence that is decreasing and bounded below converges.

**Important Notes :**
(1). If $a_n \leq b_n$ for some $n \geq N_0$, $\{a_n\}$ is an increasing sequence and $\{b_n\}$ is convergent, then $\{a_n\}$ is convergent.
(2). If $a_n \leq b_n$ for some $n \geq N_0$, if $\lim_{n\to\infty} a_n = \infty$ , then $\lim_{n\to\infty} b_n = \infty$ .

## 3.2 Series

### 3.2.1 Infinite Series

**Definition:**
(1). The sum of the terms of the sequence $\{a_n\}$ is called an infinite series (or a series) and it is denoted by $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$ .
(2). $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$ is called the n$^{\text{th}}$ partial sum of the series.

**Definition:**
If the sequence $\{s_n\}$ of the partial sums of the series $\sum_{n=1}^{\infty} a_n$ is convergent, and $\lim_{n\to\infty} s_n = s \in \mathbb{R}$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} a_n = s$ .
The number $s$ is called the sum of the series.
If the sequence $\{s_n\}$ is divergent, then the series is divergent.

**Example (1):** Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.
Solution : Note that $\frac{1}{n} - \frac{1}{n+1} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n(n+1)}$ .
$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$
$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$
So, $s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1}$ .
$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$ .
Therefore, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ .

### 3.2.2 Sum of a Geometric Series

**Definition :** $\sum_{n=0}^{\infty} ar^n$ (where $a \neq 0$ and $r \in \mathbb{R}$) is called a geometric series.

Note that $s_n = a + ar + ar^2 + \cdots + ar^n$ , $rs_n = ra + ar^2 + ar^3 + \cdots + ar^n + ar^{n+1}$.
$s_n - rs_n = a - ar^{n+1} \implies (1-r)s_n = a\left(1 - a^{n+1}\right) \implies s_n = \frac{a\left(1 - a^{n+1}\right)}{1-r}$
where $r \neq 1$.

**Sum of a geometric series :**

(1). If $|r| < 1$ , then $\sum_{n=0}^{\infty} ar^n$ is convergent and its sum is $\frac{a}{1-r}$.

(2). If $|r| \geq 1$ , then $\sum_{n=0}^{\infty} ar^n$ is divergent.

**Example (2):** Find the sum of the geometric series $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \cdots$ .

Solution : $a = 3$ .

$r = \frac{a_2}{a_1} = \frac{-\frac{3}{2}}{3} = -\frac{1}{2}$.

the sum of the geometric series is $\frac{a}{1-r} = \frac{3}{1-(-\frac{1}{2})} = \frac{3}{(\frac{3}{2})} = 2$ .

**Example (3):** Does the series $\sum_{n=1}^{\infty} 3^{2n} 5^{1-n}$ converge or diverge?.

Solution :

$$\sum_{n=1}^{\infty} 3^{2n} 5^{1-n} = \sum_{n=1}^{\infty} 9^n 5^{1-n} = \sum_{n=1}^{\infty} \frac{9^n}{5^{n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{9\ (9^{n-1})}{5^{n-1}} = \sum_{n=1}^{\infty} 9\ \left(\frac{9}{5}\right)^{n-1} = \sum_{n=0}^{\infty} 9\ \left(\frac{9}{5}\right)^{n} .$$

Since $|r| = \left|\frac{9}{5}\right| = \frac{9}{5} > 1$ , then the geometric series diverges.

### 3.2.3 Test for Divergence

**Example (4):** Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Solution :

$s_2 = s_{2^1} = 1 + \frac{1}{2}$ .

$s_4 = s_{2^2} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$ .

$s_8 = s_{2^3} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$

$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}$ .

So, $s_{2^n} > 1 + \frac{n}{2}$ (By induction).

$\lim_{n\to\infty} s_{2^n} \geq \lim_{n\to\infty} \left(1 + \frac{2}{n}\right) = \infty$.

Therefore, The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

**Theorem :** If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$ .

**Important Note :** The converse of the last theorem is not true.

In other words, if $\lim_{n\to\infty} a_n = 0$ then this does not mean that $\sum_{n=1}^{\infty} a_n$ is convergent.

For example $\lim_{n\to\infty} \frac{1}{n} = 0$ , while $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

**Test of divergence :** If $\lim_{n\to\infty} a_n$ does not exist, or $\lim_{n\to\infty} a_n \neq 0$ , then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

**Example (5):** Show that the series $\sum_{n=1}^{\infty} \frac{n^3}{2n^3+1}$ is divergent.

Solution :

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3}{2n^3+1} = \frac{1}{2} \neq 0$ .

Therefore, the series $\sum_{n=1}^{\infty} \frac{n^3}{2n^3+1}$ is divergent.

### 3.2.4 Properties of Convergent Series

**Theorem :** If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent series and $c \in \mathbb{R}$ then :

(1). The series $\sum_{n=1}^{\infty} (ca_n)$ is convergent, and $\sum_{n=1}^{\infty} (ca_n) = c\sum_{n=1}^{\infty} a_n$ .

(2). The series $\sum_{n=1}^{\infty} (a_n + b_n)$ is convergent, and $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

(3). The series $\sum_{n=1}^{\infty} (a_n - b_n)$ is convergent, and $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

**Example (6):** Find the sum of the series $\sum_{n=1}^{\infty} \left( \frac{4}{n(n+1)} + \frac{2}{3^n} \right)$ .

Solution :

From Example (1), $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ .

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = \sum_{n=1}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^{n-1} = \sum_{n=0}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^{n} = \frac{2}{3} \, \frac{1}{1-\frac{1}{3}} = \frac{2}{3} \, \frac{3}{2} = 1 \ .$$

$$\sum_{n=1}^{\infty} \left( \frac{4}{n(n+1)} + \frac{2}{3^n} \right) = \sum_{n=1}^{\infty} \frac{4}{n(n+1)} + \sum_{n=1}^{\infty} \frac{2}{3^n}$$

$$= 4 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{2}{3^n} = 4(1) + 1 = 5 \ .$$

### 3.2.5 EXERCISES

1. Let $a_n = \dfrac{2n}{3n+1}$ .

   (a). Determine whether $\{a_n\}$ is convergent.

   (b). Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

2. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

   (1). $\sum_{n=1}^{\infty} \dfrac{5}{\pi^n}$ (2). $\sum_{n=1}^{\infty} \dfrac{(-3)^{n-1}}{4^n}$ (3). $\sum_{n=0}^{\infty} \dfrac{3^{n+1}}{(-2)^n}$

   (4). $\sum_{n=1}^{\infty} \dfrac{e^{2n}}{6^{n-1}}$ (5). $\sum_{n=1}^{\infty} \dfrac{6\ 2^{2n-1}}{3^n}$

3. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

   (1). $\sum_{n=1}^{\infty} \dfrac{2+n}{1-2n}$ (2). $\sum_{n=1}^{\infty} \dfrac{n^2}{n^2-2k+5}$ (3). $\sum_{n=1}^{\infty} \dfrac{1}{4+e^{-n}}$

   (4). $\sum_{n=1}^{\infty} \dfrac{2^n+4^n}{e^n}$ (5). $\sum_{n=1}^{\infty} (\sin 100)^n$ (6). $\sum_{n=1}^{\infty} \dfrac{1}{1+\left(\frac{2}{3}\right)^n}$

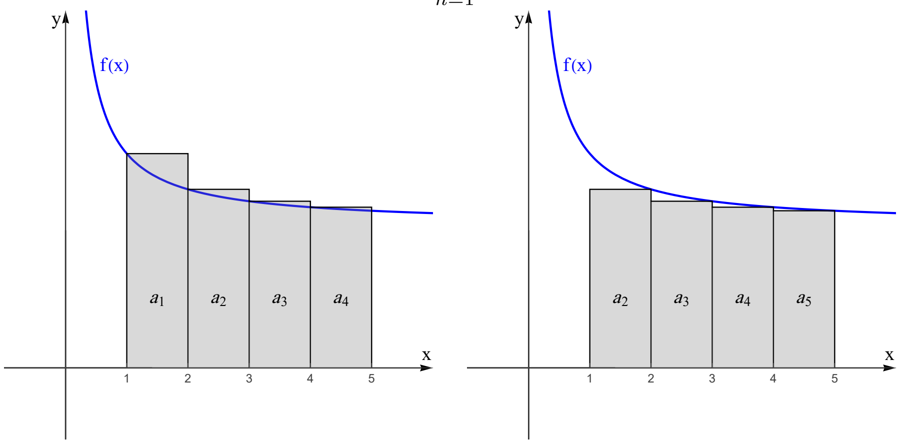
   (7). $\sum_{n=1}^{\infty} \ln\left(\dfrac{n^2+1}{2n^2+1}\right)$

## 3.3 The Integral Test

**Theorem (The Integral Test)** :
Suppose $f$ is a positive, decreasing continuous function defined on $[1, \infty)$, and let $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\ dx$ is convergent. In other words:

(1). If $\int_1^{\infty} f(x)\ dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent .

(2). If $\int_1^{\infty} f(x)\ dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent .




**Example (1):** Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ .

Solution :
Let $f(x) = \frac{1}{x^2+1}$ , the $f$ is a positive continuous function on $[1, \infty)$ .
$f'(x) = \frac{-2x}{(x^2+1)^2} < 0$ on the interval $[1, \infty)$, then $f$ is decreasing on $[1, \infty)$ .
$\int_1^{\infty} \frac{1}{x^2+1}\ dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\ dx = \lim_{t\to\infty} \left[\tan^{-1} x\right]_1^t$
$= \lim_{t\to\infty} \left[\tan^{-1}(t) - \tan^{-1}(1)\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$
So, $\int_1^{\infty} \frac{1}{x^2+1}\ dx$ is convergent, then $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent.

**Example (2):** Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n\ \ln n}$ .

Solution :
Let $f(x) = \frac{1}{x\ \ln x}$ , the $f$ is a positive continuous function on $[2, \infty)$.
$f'(x) = \frac{-(1+\ln x)}{(x\ \ln x)^2} < 0$ on $[2, \infty)$, then $f$ is decreasing on $[2, \infty)$ .

$\displaystyle\int_2^\infty \frac{1}{x\ \ln x}\ dx = \lim_{t\to\infty}\int_2^t \frac{\left(\frac{1}{x}\right)}{\ln x}\ dx = \lim_{t\to\infty}\left[\ln(\ln x)\right]_2^t$
$\displaystyle= \lim_{t\to\infty}\left[\ln(\ln t) - \ln(\ln 2)\right] = \infty.$ (Note that $\displaystyle\lim_{t\to\infty}\ln(t) = \infty$).
So, $\displaystyle\int_2^\infty \frac{1}{x\ \ln x}\ dx$ is divergent, then $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ \ln n}$ is divergent.

**Definition :** (The $p$-series)
The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ where $p\in\mathbb{R}$ is called a $p$-series.

**Theorem :**
The $p$-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent if $p>1$, and divergent if $p\le 1$.
Proof:
(1). If $p<0$, then $\displaystyle\lim_{n\to\infty}a_n = \lim_{n\to\infty}\frac{1}{n^p} = \lim_{n\to\infty}n^{-p} = \infty.$ (Note: $-p>0$).
Since $\displaystyle\lim_{n\to\infty}a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ is divergent.
(2). If $p=0$, then $\displaystyle a_n = \frac{1}{n^0} = 1$, hence $\displaystyle\lim_{n\to\infty}a_n = 1\neq 0$.
Therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ is divergent.
(3). If $0<p<1$, then using the integral test:
Note that $\displaystyle f(x)=\frac{1}{x^p}$ is a positive, decreasing continuous function on $[1,\infty)$.
$\displaystyle\int_1^\infty \frac{1}{x^p}\ dx = \lim_{t\to\infty}\int_1^t x^{-p}\ dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^t$
$\displaystyle= \lim_{t\to\infty}\left[\frac{t^{1-p}}{1-p} - \frac{1}{1-p}\right] = \infty.$ (Note : $1-p>0$).
Since $\displaystyle\int_1^\infty \frac{1}{x^p}\ dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ is divergent.
(4). If $p=1$, then using the integral test:
Note that $\displaystyle f(x)=\frac{1}{x}$ is a positive, decreasing continuous function on $[1,\infty)$.
$\displaystyle\int_1^\infty \frac{1}{x}\ dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\ dx = \lim_{t\to\infty}\left[\ln x\right]_1^t = \lim_{t\to\infty}\left[\ln(t)-\ln(1)\right] = \infty$ .
Since $\displaystyle\int_1^\infty \frac{1}{x}\ dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.
(5). If $p>1$, then using the integral test:
Note that $\displaystyle f(x)=\frac{1}{x^p}$ is a positive, decreasing continuous function on $[1,\infty)$.
$\displaystyle\int_1^\infty \frac{1}{x^p}\ dx = \lim_{t\to\infty}\int_1^t x^{-p}\ dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty}\left[\frac{1}{(1-p)x^{p-1}}\right]_1^t$
$\displaystyle= \lim_{t\to\infty}\left[\frac{1}{(1-p)\ t^{p-1}} - \frac{1}{1-p}\right] = 0 - \frac{1}{1-p} = \frac{1}{p-1}$ .

Since $\displaystyle\int_1^{\infty} \frac{1}{x^p}\ dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

**Example (3):** Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ .

Solution :
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ is a $p$-series, where $p = 4 > 1$, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent.

**Example (4):** Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ .

Solution :
$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is a $p$-series, where $p = \dfrac{1}{2} < 1$, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

**Example (5):** Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} n\ e^{-n^2}$ .

Solution : Let $f(x) = x\ e^{-x^2}$ then $f$ is a positive continuous function on $[1, \infty)$.
$f'(x) = e^{-x^2} - 2x^2\ e^{-x^2} = \big(1 - 2x^2)\big)\ e^{-x^2} < 0$ on $[1, \infty)$.
Therefore, $f$ is decreasing on $[1, \infty)$. Using the integral test :

$$\int_1^{\infty} x\ e^{-x^2}\ dx = \lim_{t\to\infty} \left( -\frac{1}{2} \int_1^t e^{-x^2}\ (-2x)\ dx \right) = \lim_{t\to\infty} \left( -\frac{1}{2} \left[ e^{-x^2} \right]_1^t \right)$$
$$= \lim_{t\to\infty} \left( -\frac{1}{2} \left[ e^{-t^2} - e^{-1} \right] \right) = \lim_{t\to\infty} \left( -\frac{1}{2} \left[ \frac{1}{e^{t^2}} - \frac{1}{e} \right] \right) = -\frac{1}{2} \left( 0 - \frac{1}{e} \right) = \frac{1}{2e} \ .$$

Since $\displaystyle\int_1^{\infty} x\ e^{-x^2}\ dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} n\ e^{-n^2}$ is convergent.

# 3.4 The Comparison Tests

## 3.4.1 The Direct Comparison Test

**Theorem :** (The Direct Comparison Test)

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(1). If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \le b_n$ for all $n$, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

(2). If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \ge b_n$ for all $n$, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

**Example (1):** Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\sin n}{2+3^n}$ .

Solution :

For all $n \ge 1$ : $\frac{\sin n}{2+3^n} \le \frac{1}{2+3^n} < \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$ .

Since $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is convergent (a geometric series with $r = \frac{1}{3} < 1$), then $\sum_{n=1}^{\infty} \frac{\sin n}{2+3^n}$ is also convergent (by The direct comparison test).

**Example (2):** Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{2}{\sqrt{n}-1}$ .

Solution :

For all $n \ge 2$ : $\sqrt{n} - 1 < \sqrt{n} \implies \frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$

$\implies \frac{2}{\sqrt{n}-1} > \frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$ .

Since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is divergent ( a $p$-series, with $p = \frac{1}{2} \le 1$), then $\sum_{n=2}^{\infty} \frac{2}{\sqrt{n}-1}$ is also divergent (by The direct comparison test) .

**Example (3):** Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^4+1}$ .

Solution :

For all $n \ge 1$ : $e^{-n} = \frac{1}{e^n} < 1 \implies \frac{e^{-n}}{n^4+1} < \frac{1}{n^4+1} < \frac{1}{n^4}$ .

Since $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent( a $p$-series, with $p = 4 > 1$), then $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^4+1}$ is also convergent.

**Example (4):** Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ .

Solution :
Since the function $f(x) = \ln(x)$ is increasing on $[1, \infty)$, then for all $n \geq 3$ :
$\ln(n) \geq \ln(3) > \ln(e) = 1 \implies \dfrac{\ln n}{n} > \dfrac{1}{n}$ .
Since $\sum_{n=1}^{\infty} \dfrac{1}{n}$ is divergent ( a $p$-series, with $p = 1 \leq 1$), then $\sum_{n=1}^{\infty} \dfrac{\ln n}{n}$ is also divergent.

### 3.4.2 Limit Comparison Test

**Theorem:** (Limit comparison test)
Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.
If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = c$ , where $c \in \mathbb{R}$ and $c > 0$, then either both series converge or both series diverge.

**Example (1):** Discuss the convergence of the series $\sum_{n=1}^{\infty} \dfrac{1}{2^n - 1}$ .

Solution :
Note that $a_n = \dfrac{1}{2^n - 1}$. Let $b_n = \dfrac{1}{2^n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{2^n-1}\right)}{\left(\frac{1}{2^n}\right)} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = 1 > 0 \ .$$

Since $\sum_{n=1}^{\infty} \dfrac{1}{2^n} = \sum_{n=1}^{\infty} \left(\dfrac{1}{2}\right)^n$ is convergent (a geometric series with $r = \dfrac{1}{2} < 1$), then $\sum_{n=1}^{\infty} \dfrac{1}{2^n - 1}$ is convergent (by limit comparison test).

**Example (2):** Discuss the convergence of the series $\sum_{n=2}^{\infty} \dfrac{1}{\sqrt[3]{n^2 - 1}}$ .

Solution :
Note that $a_n = \dfrac{1}{\sqrt[3]{n^2 - 1}}$. Let $b_n = \dfrac{1}{\sqrt[3]{n^2}} = \dfrac{1}{n^{\frac{3}{2}}}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt[3]{n^2}}{\sqrt[3]{n^2 - 1}} = \lim_{n\to\infty} \sqrt[3]{\frac{n^2}{n^2 - 1}} = \sqrt[3]{1} = 1 > 0 \ .$$

Since $\sum_{n=2}^{\infty} \dfrac{1}{\sqrt[3]{n^2}} = \sum_{n=2}^{\infty} \dfrac{1}{n^{\frac{3}{2}}}$ is convergent (a $p$-series, with $p = \dfrac{3}{2} > 1$), then $\sum_{n=2}^{\infty} \dfrac{1}{\sqrt[3]{n^2 - 1}}$ is convergent (by limit comparison test).

**Example (3):** Discuss the convergence of the series $\sum_{n=1}^{\infty} \dfrac{5n^2 + 3n}{3^n(n^2 + 2)}$ .

Solution :

Note that $a_n = \dfrac{5n^2+3n}{3^n(n^2+2)}$. Let $b_n = \dfrac{5n^2}{3^n\ n^2} = \dfrac{5}{3^n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left( \frac{5n^2+3n}{3^n(n^2+2)} . \frac{3^n}{5} \right) = \lim_{n\to\infty} \frac{5n^2+3n}{5n^2+10} = 1 > 0 \ .$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{5}{3^n} = \sum_{n=1}^{\infty} 5 \left(\frac{1}{3}\right)^n$ is convergent (a geometric series with $r = \dfrac{1}{3} < 1$), then $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2+3n}{3^n(n^2+2)}$ is convergent.

**Example (4):** Discuss the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+3n^2}{\sqrt{1+n^5}}$ .

Solution :

Note that $a_n = \dfrac{3n^2+n}{\sqrt{n^5+1}}$. Let $b_n = \dfrac{3n^2}{\sqrt{n^5}} = \dfrac{3n^2}{n^{\frac{5}{2}}} = \dfrac{3}{n^{\frac{1}{2}}}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left( \frac{3n^2+n}{\sqrt{n^5+1}} . \frac{n^{\frac{1}{2}}}{3} \right) = \lim_{n\to\infty} \frac{3n^{\frac{5}{2}}+n^{\frac{3}{2}}}{3\sqrt{n^5+1}} = 1 > 0 \ .$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^{\frac{1}{2}}}$ is divergent (a $p$-series, with $p = \dfrac{1}{2} < 1$), then $\displaystyle\sum_{n=1}^{\infty} \frac{n+3n^2}{\sqrt{1+n^5}}$ is divergent.

## 3.5 Alternating Series and Absolute Convergence

### 3.5.1 Alternating Series

**Definition :** If $a_n \geq 0$, the the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is called an alternating series.

**Theorem :** (Alternating Series Test)
If $a_n \geq 0$ and the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ satisfies :
(1). The sequence $\{a_n\}$ is decreasing, i.e. $a_{n+1} \leq a_n$ for all $n \geq 1$ .
(2). $\lim_{n\to\infty} a_n = 0$ .
Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.

**Example (1):** Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ .
Solution :
(1). For all $n \geq 1$ : $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$.
The sequence $\{a_n\} = \left\{\frac{1}{n}\right\}$ is decreasing.
(2). $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ .
By the alternating series test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent .

**Example (2):** Discuss the convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{4n^2-1}$ .
Solution :
(1). Let $f(x) = \frac{2x}{4x^2-1}$ , then for $x \geq 1$ :
$f'(x) = \frac{2(4x^2-1) - 2x(8x)}{(4x^2-1)^2} = \frac{8x^2 - 2 - 16x^2}{(4x^2-1)^2} = \frac{-2-8x^2}{(4x^2-1)^2} < 0$ .
The sequence $\{a_n\} = \left\{\frac{2n}{4n^2-1}\right\}$ is decreasing.
(2). $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{4n^2-1} = 0$ .
By the alternating series test, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n}{4n^2-1}$ is convergent .

**Example (3):** Discuss the convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n+1}$ .
Solution :
$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$ .

So, the alternating series test does not apply.
Note that if $b_n = (-1)^{n+1}\dfrac{n}{2n+1}$, then $\lim\limits_{n\to\infty} b_{2n} = \dfrac{1}{2}$ and $\lim\limits_{n\to\infty} b_{2n+1} = -\dfrac{1}{2}$.
So, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} (-1)^{n+1}\dfrac{n}{2n+1}$ does not exist.
Therefore, $\sum\limits_{n=1}^{\infty} (-1)^{n+1}\dfrac{n}{2n+1}$ is divergent (by test of divergence).

### 3.5.2 Absolute Convergence and Conditional Convergence

**Definition :** (Absolute Convergence)
A series $\sum\limits_{n=1}^{\infty} a_n$ is called absolutely convergent if the series of absolute values $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent .

**Example (4):** The series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^p}$ , where $p > 1$ is absolutely convergent.
Solution :
$\sum\limits_{n=1}^{\infty} \left|\dfrac{(-1)^{n+1}}{n^p}\right| = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ is convergent (a $p$-series, with $p > 1$).
Therefore, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^p}$ is absolutely convergent.

**Definition :** (Conditional Convergence)
A series $\sum\limits_{n=1}^{\infty} a_n$ is called conditionally convergent if it is convergent but not absolutely convergent. That is, if $\sum\limits_{n=1}^{\infty} a_n$ is convergent but $\sum\limits_{n=1}^{\infty} |a_n|$ is divergent.

**Example (5):** The series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}$ is conditionally convergent.
Solution :
Note that $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}$ is convergent (see Example (1)).
But $\sum\limits_{n=1}^{\infty} \left|\dfrac{(-1)^{n+1}}{n}\right| = \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ is divergent.
Therefore, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}$ is conditionally convergent.

**Theorem :**
If a series is absolutely convergent then it is convergent.

**Example (6):** Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ .

Solution :

$\left|\frac{\cos n}{n^2}\right| = \frac{|\cos n|}{|n^2|} \leq \frac{1}{n^2}$ .

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent then by direct comparison test $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ is convergent.

Therefore, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent and hence convergent.

**Example (7):** Discuss the convergence of the following series :

(a). $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1}$ (b). $\sum_{n=1}^{\infty} (-1)^n \frac{1}{1+n\sqrt{n}}$ (c). $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{(2n-3)^2}$ .

Solution :

(a). $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{n+1}{n^2+1}\right| = \sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$

Let $a_n = \frac{n+1}{n^2+1}$, put $b_n = \frac{1}{n}$, then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n+1}{n^2+1} \frac{n}{1}\right) = \lim_{n\to\infty} \frac{n^2+n}{n^2+1} = 1 > 0$ .

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, then $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{n+1}{n^2+1}\right|$ is divergent.

(1). Let $f(x) = \frac{x+1}{x^2+1}$, then for $x \geq 1$ :

$f'(x) = \frac{(1)(x^2+1) - (x+1)(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2-2x}{(x^2+1)^2} = \frac{-x^2-2x1}{(x^2+1)^2} < 0.$

Therefore, the sequence $\left\{\frac{n+1}{n^2+1}\right\}$ is decreasing.

(2). $\lim_{n\to\infty} \frac{n+1}{n^2+1} = 0.$

From (1) and (2), the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1}$ is convergent.

Hence, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+1}$ is conditionally convergent.

(b). $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{1+n\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{1+n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{1+n^{\frac{3}{2}}}$ .

Let $a_n = \frac{1}{1+n^{\frac{3}{2}}}$, put $b_n = \frac{1}{n^{\frac{3}{2}}}$, then

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{\frac{3}{2}}}{n^{\frac{3}{2}}+1} = 1 > 0.$

Since $\sum_{n=1}^{\infty} \frac{1}{1+n^{\frac{3}{2}}}$ is convergent, then $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{1+n\sqrt{n}}\right|$ is convergent.

Therefore, $\sum_{n=1}^{\infty}(-1)^n \frac{1}{1+n\sqrt{n}}$ is absolutely convergent.

(c). Let $a_n = (-1)^n \frac{n^2+1}{(2n-3)^2} = (-1)^n \frac{n^2+1}{4n^2-12n+9}$

$\lim_{n\to\infty} a_{2n} = \frac{1}{4}$ and $\lim_{n\to\infty} a_{2n+1} = -\frac{1}{4}$.

Hence $\lim_{n\to\infty} a_n$ does not exist.

Therefore, $\sum_{n=1}^{\infty}(-1)^n \frac{n^2+1}{(2n-3)^2}$ is divergent.

## 3.6 The Ratio and Root Tests

### 3.6.1 The Ratio Test

**Theorem :** (the Ratio Test)
(1). If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
(2). If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$ , then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
(3). If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the ratio test is inconclusive.

**Example (1):** Discuss the convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^4}{2^n}$ .
Solution :
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^4}{2^{n+1}}\ \frac{2^n}{n^4}\right| = \lim_{n\to\infty}\frac{1}{2}\left(\frac{n+1}{n}\right)^4 = \frac{1}{2}(1)^4 = \frac{1}{2} < 1 \ .$$
Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{n^4}{2^n}$ is absolutely convergent.

**Example (2):** Discuss the convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{n!}$ .
Solution :
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5^{n+1}}{(n+1)!}\ \frac{n!}{5^n}\right| = \lim_{n\to\infty}\left|\frac{5^{n+1}}{n!\ (n+1)}\ \frac{n!}{5^n}\right|$$
$$= \lim_{n\to\infty}\frac{5}{n+1} = 0 < 1 \ .$$
Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{n!}$ is absolutely convergent.

**Example (3):** Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ .
Solution :
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{(n+1)!}\ \frac{n!}{n^n}\right] = \lim_{n\to\infty}\left[\frac{(n+1)^n\ (n+1)}{n!\ (n+1)}\ \frac{n!}{n^n}\right]$$
$$= \lim_{n\to\infty}\frac{(n+1)^n}{n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1 \ .$$
Therefore, $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent.
Note that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ , using L'Hôpital's rule.

**Example (4):** Discuss the convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{3^n}$ .

Solution : For $n \geq 2$ ,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\ln(n+1)}{3^{n+1}} \; \frac{3^n}{\ln(n)} \right| = \lim_{n\to\infty} \frac{1}{3} \; \frac{\ln(n+1)}{\ln(n)}$$

$$= \lim_{n\to\infty} \frac{1}{3} \left( \frac{\frac{1}{n+1}}{\frac{1}{n}} \right) = \lim_{n\to\infty} \frac{1}{3} \; \frac{n}{n+1} = \frac{1}{3}(1) = \frac{1}{3} < 1 \; .$$

Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{3^n}$ is convergent.

**Notes :**

(1). The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left( \frac{\frac{1}{n+1}}{\frac{1}{n}} \right) = \lim_{n\to\infty} \frac{n}{n+1} = 1 \; .$$

(2). The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left( \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \right) = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \left( \frac{n}{n+1} \right)^2 = (1)^2 = 1 \; .$$

### 3.6.2 The Root Test

**Theorem :** (the Root Test)

(1). If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(2). If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(3). If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the root test is inconclusive.

**Example (5):** Discuss the convergence of the series $\sum_{n=1}^{\infty} \left( \frac{n+3}{3n+2} \right)^n$ .

Solution :

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left| \left( \frac{n+3}{3n+2} \right)^n \right|} = \lim_{n\to\infty} \left[ \left( \frac{n+3}{3n+2} \right)^n \right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n+3}{3n+2} = \frac{1}{3} < 1 \; .$$

Therefore, $\sum_{n=1}^{\infty} \left( \frac{n+3}{3n+2} \right)^n$ is absolutely convergent.

**Example (6):** Discuss the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+1}{n+3}\right)^n$ .

Solution :

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n+1}{n+3}\right)^n\right|} = \lim_{n\to\infty} \left[\left(\frac{2n+1}{n+3}\right)^n\right]^{\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{2n+1}{n+3} = 2 > 1$ .

Therefore, $\sum_{n=1}^{\infty} \left(\frac{2n+1}{n+3}\right)^n$ is divergent.

**Example (7):** Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n}$ .

Solution :

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{2^{3n+1}}{n^n}\right|} = \lim_{n\to\infty} \left(\frac{2^{3n+1}}{n^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2^{3+\frac{1}{n}}}{n} = 0 < 1$ .

Therefore, $\sum_{n=1}^{\infty} \frac{2^{3n+1}}{n^n}$ is absolutely convergent.

**Example (8):** Discuss the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$ .

Solution :

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n+1}{n}\right)^n\right|} = \lim_{n\to\infty} \left[\left(\frac{n+1}{n}\right)^n\right]^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n+1}{n} = 1.$

So, The root test is inconclusive.

But $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$ .

Therefore, $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$ is divergent (by test of divergence).

## 3.7 Strategy for Testing Series

(1). Divergence test :

If $\lim_{n\to\infty} a_n \neq 0$ , or $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ is divergent.

$\sum_{n=1}^{\infty} \frac{n^2+2n}{n^2+4}$ is divergent, because $\lim_{n\to\infty} \frac{n^2+2n}{n^2+4} = 1 \neq 0.$

$\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{n+2}$ is divergent, because $\lim_{n\to\infty} (-1)^n \frac{n-1}{4n+2}$ does not exist.

(2). $p$-series : It has the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ :

If $p > 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent, like $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$.

If $p \leq 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent, like $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

(3). Geometric series : It has the form $\sum_{n=0}^{\infty} a\ r^n$ where $a \neq 0$ and $r \in \mathbb{R}$ :

If $|r| < 1$ then $\sum_{n=0}^{\infty} a\ r^n$ is convergent, like $\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$.

If $|r| \geq 1$ then $\sum_{n=0}^{\infty} a\ r^n$ is divergent, like $\sum_{n=1}^{\infty} \frac{3^n}{2}$.

(4). Comparison test :
It is used when the series has a form that is similar to a $p$-series or a geometric series, it can be used also when $a_n$ is a rational function or an algebraic function of $n$.

It is used on series of positive terms, it can be used on $\sum_{n=1}^{\infty} |a_n|$ to test for absolute convergence.

$\sum_{n=1}^{\infty} \frac{3}{n+2}$ is divergent, comparing it with $\sum_{n=1}^{\infty} \frac{1}{n}$.

$\sum_{n=1}^{\infty} \frac{2}{7^n+1}$ is convergent, comparing it with $\sum_{n=1}^{\infty} \frac{1}{7^n}$.

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+5}$ is convergent, comparing it with $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ .

(5). Alternating series test :

It is used on the series $\sum_{n=1}^{\infty} (-1)^n a_n$, where the sequence $\{a_n\}$ is decreasing and

$\lim_{n\to\infty} a_n = 0$. Note that if $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} (-1)^n a_n$ is absolutely convergent.
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+2}$ satisfies all the conditions of the alternating series test, hence it is absolutely convergent. Also, $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3+2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3+2}$ is convergent (using comparison test), hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+2}$ is absolutely convergent.

(6). The Ratio test :
it is used on series involving factorial or other products like a constant raised to a power $n$.
It is not used on $p$-series or rational functions or algebraic functions of $n$.
$\sum_{n=1}^{\infty} \frac{2^n}{n!}$ is convergent since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$.

(7). The Root test :
It is used on series of the form $\sum_{n=1}^{\infty} (a_n)^n$.
$\sum_{n=1}^{\infty} \left( \frac{3n^2+1}{4n^2+5} \right)^n$ is convergent, since $\lim_{n\to\infty} \sqrt[n]{\left( \frac{3n^2+1}{4n^2+5} \right)^n} = \frac{3}{4} < 1$.

(8). The Integral test :
It is used when $a_n = f(n)$, where $f$ is a decreasing function on $[1,\infty)$ and $\int_1^{\infty} f(x)\, dx$ is easy to calculate.
$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent, since $f(x) = \frac{1}{x(\ln x)^2}$ is decreasing on $[2,\infty)$ and

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} = \lim_{t\to\infty} \int_2^t (\ln x)^{-2} \frac{1}{x}\, dx = \lim_{t\to\infty} \left[ \frac{(\ln x)^{-1}}{-1} \right]_2^t = \lim_{t\to\infty} \left[ \frac{-1}{\ln x} \right]_2^t$$
$$= \lim_{t\to\infty} \left[ \frac{-1}{\ln t} - \left( \frac{-1}{\ln 2} \right) \right] = 0 + \frac{1}{\ln 2} = \frac{1}{\ln 2} .$$

# 3.8 Power Series

## 3.8.1 Power Series

**Definition :** A power series is a series of the form $\sum_{n=1}^{\infty} c_n x^n$, where $x$ is a variable and $c_n$'s are real constants called the coefficients.

**Notes :**

(1). If $c_n = 1$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} c_n x^n = \sum_{n=1}^{\infty} x^n$ is the geometric series and it converges when $|x| < 1$ and diverges when $|x| > 1$.

If $x = \frac{1}{2}$, then $\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges to 2.

If $x = 2$, then $\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} 2^n$ diverges.

(2). $\sum_{n=1}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-c)^2 + \cdots$ is called a power series in $(x-a)$ or a power series centered at $a$ or a power series about $a$.

**Example (1):** For what values of $x$ does the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$ converge?

Solution :

Let $a_n = \frac{(x-2)^n}{n}$, using the ratio test :

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-2)^{n+1}}{n+1} \; \frac{n}{(x-2)^n}\right| = \lim_{n\to\infty} \frac{n}{n+1}|x-2| = |x-2| \; .$$

The series converges when $|x-2| < 1 \implies -1 < x-2 < 1 \implies 1 < x < 3$ .

The series diverges when $|x-2| > 1 \implies x-2 < -1$ or $x-2 > 1$

$\implies x < 1$ or $x > 3$ .

If $x = 1$ then $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is the convergent alternating harmonic series.

If $x = 3$ then $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Therefore, $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$ converges when $x \in [1,3)$ or $1 \leq x < 3$.

**Example (2):** For what values of $x$ does the series $\sum_{n=1}^{\infty} n! \; x^n$ converge?

Solution :

Let $a_n = n! \; x^n$, using the ratio test :

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)! \; x^{n+1}}{n! \; x^n}\right| = \lim_{n\to\infty} (n+1) \; |x|$$

If $x \neq 0$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$. Therefore the series diverges when $x \neq 0$.

If $x = 0$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$. Therefore the series converges when $x = 0$.

**Example (3):** For what values of $x$ does the series $\sum_{n=1}^{\infty}\frac{x^n}{n!}$ converge?
Solution :
Let $a_n = \frac{x^n}{n!}$, using the ratio test :
$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\ \frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0 < 1.$
Therefore, the series converges for all $x \in \mathbb{R}$.

### 3.8.2 Interval of Convergence

**Theorem :** For a power series $\sum_{n=1}^{\infty} c_n(x-a)^n$, there are only three possibilities:
(1). The series converges only when $x = a$.
(2). The series converges for all $x \in \mathbb{R}$.
(3). There is a positive real number $R$ such that the series converges when $|x-a| < R$ and diverges when $|x-a| > R$.

**Notes :**
(1). The positive real number $R$ is called the radius of convergence. If the series converges only when $x = a$ then $R = 0$, and if the series converges for all $x \in \mathbb{R}$ then $R = \infty$.
(2). The interval of convergence of a power series is the interval that consists of all values of $x$ for which the series converges.
If the series converges only when $x = a$ then the interval of convergence consists of only one point $a$.
If the series converges for all $x \in \mathbb{R}$ then the interval of convergence is $(-\infty, \infty)$.
If the radius of convergence is $R$, then interval of convergence is $(a-R, a+R)$.

**Example (4):** Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^2+1}$.
Solution :
Let $a_n = \frac{(x-2)^n}{n^2+1}$, using the ratio test :
$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2+1}\ \frac{n^2+1}{(x-2)^n}\right| = \lim_{n\to\infty}\frac{n^2+1}{(n+1)^2+1}\ |x-2|$
$= \lim_{n\to\infty}\frac{n^2+1}{n^2+2n+2}\ |x-2| = (1)\ |x-2| = |x-2|.$
The series converges when $|x-2| < 1$, so the radius of convergence is $R = 1$ .
Therefore, the series converges on $(1, 3)$.
If $x = 1$, then $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^2+1} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$ which is convergent.

If $x = 3$, then $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1} = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ which is convergent.
Therefore, the interval of convergence is $[1, 3]$.

**Example (5):** Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n\ 5^n}$.
Solution :
Let $a_n = \frac{(2x-6)^n}{n\ 5^n}$, using the ratio test :

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2x-6)^{n+1}}{(n+1)\ 5^{n+1}} \ \frac{n\ 5^n}{(2x-6)^n}\right| = \lim_{n\to\infty} \frac{n}{5(n+1)}\ |2x-6|$$

$$= \lim_{n\to\infty} \frac{n}{5n+5}\ |2x-6| = \frac{1}{5}\ |2x-6| = \frac{|2x-6|}{5}\ .$$

The series converges when $\frac{|2x-6|}{5} < 1 \implies 2|x-3| < 5 \implies |x-3| < \frac{5}{2}$, so the radius of convergence is $R = \frac{5}{2}$ .
Therefore, the series converges on $\left(3 - \frac{5}{2}, 3 + \frac{5}{2}\right) = \left(\frac{1}{2}, \frac{11}{2}\right)$.
If $x = \frac{1}{2}$, then $\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n\ 5^n} = \sum_{n=1}^{\infty} \frac{(-5)^n}{n\ 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is convergent.
If $x = \frac{11}{2}$, then $\sum_{n=1}^{\infty} \frac{(2x-6)^n}{n\ 5^n} = \sum_{n=1}^{\infty} \frac{5^n}{n\ 5^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is divergent.
Therefore, the interval of convergence is $\left[\frac{1}{2}, \frac{11}{2}\right)$.

**Example (6):** Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{n}{7^n}(x+3)^n$.
Solution :
Let $a_n = \frac{n}{7^n}(x+3)^n$, using the ratio test :

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(x+3)^{n+1}}{7^{n+1}} \quad \frac{7^n}{n(x+3)^n}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{7n}|x+3| = \frac{1}{7}|x+3| = \frac{|x+3|}{7}\ .$$

The series converges when $\frac{|x+3|}{7} < 1 \implies |x+3| < 7$, so the radius of convergence is $R = 7$ .
Therefore, the series converges on $(-3-7, -3+7) = (-10, 4)$.
If $x = -10$, then $\sum_{n=1}^{\infty} \frac{n}{7^n}(x+3)^n = \sum_{n=1}^{\infty} (-1)^n n$ which is divergent.
If $x = 4$, then $\sum_{n=1}^{\infty} \frac{n}{7^n}(x+3)^n = \sum_{n=1}^{\infty} n$ which is divergent.
Therefore, the interval of convergence is $(-10, 4)$.

## 3.9 Representations of Functions as Power Series

### 3.9.1 Representations of Functions using Geometric Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \text{ when } |x| < 1.$$

**Example (1):** Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.
Solution :

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \text{ , where } \left|-x^2\right| < 1.$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - \cdots \text{ , where } \left|-x^2\right| < 1 \ .$$

$\left|-x^2\right| < 1 \implies \left|x^2\right| < 1 \implies |x| < 1$ .

If $x = \pm 1$, the series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ diverge.
Therefore, The interval of convergence is $(-1, 1)$.

**Example (2):** Express $\frac{x^2}{1-x^6}$ as the sum of a power series and find the interval of convergence.
Solution :

$$\frac{x^2}{1-x^6} = x^2 \, \frac{1}{1-x^6} = x^2 \sum_{n=0}^{\infty} \left(x^6\right)^n = x^2 \sum_{n=0}^{\infty} x^{6n} = \sum_{n=0}^{\infty} x^{6n+2} \text{ , where } \left|x^6\right| < 1.$$

$\left|x^6\right| < 1 \implies |x| < 1$ .

If $x = \pm 1$, the series $\sum_{n=0}^{\infty} x^{6n+2}$ diverge.
Therefore, The interval of convergence is $(-1, 1)$.

**Example (3):** Express $\frac{x}{5-x}$ as the sum of a power series and find the interval of convergence.
Solution :

$$\frac{x}{5-x} = \frac{x}{5\left(1-\frac{x}{5}\right)} = \frac{x}{5} \, \frac{1}{1-\frac{x}{5}} = \frac{x}{5} \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n = \frac{x}{5} \sum_{n=0}^{\infty} \frac{x^n}{5^n} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}} \ ,$$

where $\left|\frac{x}{5}\right| < 1$.

$\left|\frac{x}{5}\right| < 1 \implies \frac{|x|}{5} < 1 \implies |x| < 5$ .

If $x = \pm 5$, the series $\sum_{n=0}^{\infty} \frac{x^{n+1}}{5^{n+1}}$ diverge.
Therefore, The interval of convergence is $(-5, 5)$.

**Example (4):** Express $\dfrac{1}{2-x}$ as power series in $x-1$, and find the interval of convergence.
Solution :
$\dfrac{1}{2-x} = \dfrac{1}{1-x+1} = \dfrac{1}{1-(x-1)} = \sum_{n=0}^{\infty}(x-1)^n$ , where $|x-1|<1$.
$|x-1|<1 \implies -1<x-1<1 \implies 0<x<2$ .
If $x=0$ or $x=2$, the series $\sum_{n=0}^{\infty}(x-1)^n$ diverge.
Therefore, The interval of convergence is $(0,2)$.

### 3.9.2 Differentiation and Integration of Power Series

**Theorem :** If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R>0$, then the function $f$ defined by
$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$
is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and
(1). $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ .
(2). $\int f(x)\ dx = C + c_0(x-a) + c_1\dfrac{(x-a)^2}{2} + c_2\dfrac{(x-a)^3}{3} + \cdots$
$= C + \sum_{n=0}^{\infty} c_n \dfrac{(x-a)^{n+1}}{n+1}$ , where $C$ is a constant.
The radii of convergence of the power series in (1) and (2) are both $R$.

**Notes :** Equations (1) and (2) in the last theorem can be rewritten as :
(1). $\dfrac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty}\left[\dfrac{d}{dx}c_n(x-a)^n\right]$.
(2). $\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]\ dx = \sum_{n=0}^{\infty}\left[\int c_n(x-a)^n\ dx\right]$.

**Example (5):** Express $\dfrac{1}{(1-x)^2}$ as a power series, and find its radius of convergence.
Solution :
$\dfrac{d}{dx}\left[\dfrac{1}{1-x}\right] = \dfrac{d}{dx}\left[(1-x)^{-1}\right] = (-1)(1-x)^{-2}(-1) = \dfrac{1}{(1-x)^2}$.
Note that $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$, where $|x|<1$.
$\dfrac{1}{(1-x)^2} = \dfrac{d}{dx}\left[\dfrac{1}{1-x}\right] = \dfrac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\dfrac{d}{dx}[x^n] = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}$.

$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$ .

Since the radius of convergence of $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ is $R = 1$ then the radius of convergence of $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ is also $R = 1$.

**Example (6):** Express $\ln(1+x)$ as a power series, and find its radius of convergence.
Solution :
$\int \frac{1}{1+x}\, dx = \ln(1+x) + c$
Note that $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$, where $|x| < 1$.
$\ln(1+x) = \int \frac{1}{1+x}\, dx = \int \left[\sum_{n=0}^{\infty} (-1)^n x^n\right] dx = \sum_{n=0}^{\infty} \left[\int (-1)^n x^n\, dx\right]$
$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C$ , where $|x| < 1$.
Put $x = 0$ in the last equation : $\ln(1+0) = 0 + C \implies C = 0$.
Therefore, $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$.
The radius of convergence of $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ is $R = 1$.

**Example (7):** Express $\tan^{-1} x$ as a power series, and find its radius of convergence.
Solution :
$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + c$ .
From Example (1), $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ , where $|x| < 1$.
$\tan^{-1} x = \int \frac{1}{1+x^2}\, dx = \int \left[\sum_{n=0}^{\infty} (-1)^n x^{2n}\right] dx = \sum_{n=0}^{\infty} \int \left[(-1)^n x^{2n}\right] dx$
$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$ , where $|x| < 1$.
Put $x = 0$ in the last equation : $\tan^{-1}(0) = 0 + C \implies C = 0$.
Therefore, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ .
The radius of convergence of $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ is $R = 1$.

# 3.10 Taylor and Maclaurin Series

## 3.10.1 Definitions of Taylor Series and Maclaurin Series

**Theorem :** If $f$ has a power series representation at $a$, that is $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, where $|x-a| < R$, then the coefficient $c_n$ is given by $c_n = \dfrac{f^{(n)}(a)}{n!}$ .

Therefore, $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$,

or $f(x) = f(a) + \dfrac{f'(a)}{1!}(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \dfrac{f'''(a)}{3!}(x-a)^3 + \cdots$ .

**Definition :** $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$ is called the Taylor series of the function $f$ at $a$ ( or about $a$, or centered at $a$).

**Definition :** If $a = 0$ in the Taylor series of $f$, then $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!} x^n$ is called the Maclaurin series of $f$. In this case :

$f(x) = f(0) + \dfrac{f'(0)}{1!}x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 + \cdots$ .

**Example (1):** Find the Maclaurin Series and its radius of convergence for the function $f(x) = \dfrac{1}{1-x}$ .

Solution :

$f(x) = \dfrac{1}{1-x} \implies f(0) = 1 = 0!$ .

$f'(x) = \dfrac{1}{(1-x)^2} \implies f'(0) = 1 = 1!$ .

$f''(x) = \dfrac{2}{(1-x)^3} \implies f''(0) = 2 = 2!$ .

$f'''(x) = \dfrac{6}{(1-x)^4} \implies f'''(0) = 6 = 3!$ .

$f^{(n)}(x) = \dfrac{1.2.3\cdots n}{(1-x)^{n+1}} \implies f^{(n)}(0) = n!$ .

The Maclaurin series for $f(x) = \dfrac{1}{1-x}$ is $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} \dfrac{n!}{n!} x^n = \sum_{n=0}^{\infty} x^n$ .

Let $a_n = x^n$, then $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{x^{n+1}}{x^n}\right| = |x|$.

Therefore, the Maclaurin series converges when $|x| < 1$. Hence $R = 1$ .

**Example (2):** Find the Maclaurin Series and its radius of convergence for the function $f(x) = e^x$ .

Solution :

Note that $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = e^0 = 1$ .

The Maclaurin series for $f(x) = e^x$ is $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ .

Let $a_n = \frac{x^n}{n!}$, then $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$.
Therefore, the Maclaurin series converges for all $x \in \mathbb{R}$. Hence $R = \infty$ .

### 3.10.2 Remainder of a Taylor series

**Definition :**
Suppose $f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$ then $f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i + \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$,
$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$ is called the n$^{th}$-degree polynomial of $f$ at $a$, and
$R_n(x) = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$ is called the remainder of the Taylor series.

**Theorem :** If $f(x) = T_n(x) + R_n(x)$ , where $T_n(x)$ is the n$^{th}$-degree polynomial of $f$ at $a$ , and If $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then $f$ is equal to the sum of its Taylor series on the interval $|x-a| < R$ .

**Theorem :** (Taylor's Inequality)
Suppose $f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i + \sum_{i=n+1}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i = T_n(x) + R_n(x)$ .
If $\left| f^{(i)}(x) \right| \le M$ for $|x-a| \le d$ and for all $i \ge n+1$, where $M$ and $d$ are positive real numbers, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality : $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$ , for $|x-a| \le d$ .
In this case, $\lim_{n\to\infty} |R_n(x)| = 0$, hence $\lim_{n\to\infty} R_n(x) = 0$ for all values of $x$.

**Example (3):** Show that $e^x$ is equal to the sum of its Maclaurin series.
Solution :
$f(x) = e^x \implies f^{(i)}(x) = e^x$ for all $i \ge 1$.
If $|x| \le d$ where $d > 0$, then $\left| f^{(i)}(x) \right| = e^x \le e^d$ , for all $i \ge n+1$.
So Taylor's inequality, with $a = 0$ and $M = e^d$ is
$|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1}$ for $|x| \le d$.
Therefore, $\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = 0$ .
So, $e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \cdots$ , for all $x \in \mathbb{R}$.
Note : If $x = 1$ then $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + x + \frac{1}{2!} + \frac{1}{3!} + \cdots$ .

**Example (4):** Find the Taylor series of $f(x) = e^x$ at $a = 2$.
Solution :
$f(x) = e^x \implies f^{(n)}(x) = e^x \implies f^{(n)}(2) = e^2$ , for all $n$.
So, $e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{e^2(x-2)^n}{n!}$ for all for all $x \in \mathbb{R}$.
Another Solution : $e^{x-2} = \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} \implies \frac{e^x}{e^2} = \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$
$\implies e^x = e^2 \sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^2(x-2)^n}{n!}$ .

**Example (5):** Write the function $f(x) = e^{x^2-1}$ as a Taylor series of $x$.
Solution :
$f(x) = e^{x^2-1} = e^{-1}e^{x^2} = e^{-1} \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{e} \frac{x^{2n}}{n!}$ for all $x$ .

### 3.10.3 Taylor Series of Important Functions

**Example (6):** Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all $x$.
Solution :
$f(x) = \sin x \implies f(0) = \sin(0) = 0$ .
$f'(x) = \cos x \implies f'(0) = \cos(0) = 1$ .
$f''(x) = -\sin x \implies f''(0) = -\sin(0) = 0$ .
$f'''(x) = -\cos x \implies f'(0) = -\cos(0) = -1$ .
$f^{(4)}(x) = \sin x \implies f^{(4)}(0) = \sin(0) = 0$ .
Therefore, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ .
Since $f^{(i)}(x) = \pm \sin x$ or $\pm \cos x$ for all $i \geq 1$, then $\left| f^{(i)}(x) \right| \leq 1$.
Taylor's inequality, with $a = 0$ and $M = 1$ is
$|R_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1}$ for $|x| \leq d$ , where $d > 0$.
Therefore, $\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ .
So, $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for all $x$.

**Example (7):** Find the Maclaurin series for $\cos x$ and prove that it represents $\sin x$ for all $x$.
Solution :
$f(x) = \cos x \implies f(0) = \cos(0) = 1$ .
$f'(x) = -\sin x \implies f'(0) = -\sin(0) = 0$ .
$f''(x) = -\cos x \implies f''(0) = -\cos(0) = -1$ .
$f'''(x) = \sin x \implies f'(0) = \sin(0) = 0$ .
$f^{(4)}(x) = \cos x \implies f^{(4)}(0) = \cos(0) = 1$ .

Therefore, $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ .

Since $f^{(i)}(x) = \pm \sin x$ or $\pm \cos x$ for all $i \geq 1$, then $\left| f^{(i)}(x) \right| \leq 1$.

Taylor's inequality, with $a = 0$ and $M = 1$ is

$|R_n(x)| \leq \dfrac{1}{(n+1)!} |x|^{n+1}$ for $|x| \leq d$ , where $d > 0$.

Therefore, $\displaystyle\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ .

So, $\cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for all $x$ .

**Example (8):** Find the Maclaurin series for $f(x) = (1+x)^k$ , where $k$ is any real number.

Solution :

$f(x) = (1+x)^k \implies f(0) = 1$ .

$f'(x) = k(1+x)^{k-1} \implies f(0) = k$ .

$f''(x) = k(k-1)(1+x)^{k-2} \implies f(0) = k(k-1)$ .

$f'''(x) = k(k-1)(k-2)(1+x)^{k-3} \implies f(0) = k(k-1)(k-2)$ .

$f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n} \implies f(0) = k(k-1)\cdots(k-n+1)$.

So, $(1+x)^k = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n$.

Let $a_n = \dfrac{k(k-1)\cdots(k-n+1)}{n!} x^n$ for all $n \geq 1$.

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \frac{n!}{k(k-1)\cdots(k-n+1)x^n} \right|$$

$$= \left| \frac{k-n}{n+1}\, x \right| = \frac{\left|1 - \frac{k}{n}\right|}{1 + \frac{1}{n}} |x|.$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\left|1 - \frac{k}{n}\right|}{1 + \frac{1}{n}} |x| = |x| .$$

So, $(1+x)^k = \displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n$ converges when $|x| < 1$. Therefore, $R = 1$.

Note : $(1+x)^k = \displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n$ is called the binomial series.

**Example (9):** For $f(x) = \dfrac{1}{\sqrt{4-x}}$ , find the Maclaurin series and its radius of convergence.

Solution :

$$f(x) = \frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1 - \frac{x}{4}\right)}} = \frac{1}{2\sqrt{1 - \frac{x}{4}}} = \frac{1}{2}\left(1 + \left(-\frac{x}{4}\right)\right)^{-\frac{1}{2}}.$$

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1 + \left(-\frac{x}{4}\right)\right)^{-\frac{1}{2}} = \frac{1}{2}\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(-\frac{x}{4}\right)^n \text{, where } \left|-\frac{x}{4}\right| < 1.$$

$\left|-\dfrac{x}{4}\right| < 1 \implies \dfrac{|x|}{4} < 1 \implies |x| < 4$.

The radius of convergence of the Maclaurin series is $R = 4$.

**Table of Important Series :**

| No. | Series | R |
|---|---|---|
| 1 | $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$ | 1 |
| 2 | $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots$ | 1 |
| 3 | $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots$ | 1 |
| 4 | $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \dfrac{k(k-1)}{2!} x^2 + \cdots$ | 1 |
| 5 | $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ | $\infty$ |
| 6 | $\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$ | $\infty$ |
| 7 | $\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots$ | $\infty$ |

### 3.10.4 New Taylor Series from Old

**Example (10):** Find the Maclaurin series for $f(x) = x^2 \cos x$.
Solution :

$$f(x) = x^2 \cos x = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n)!} \text{ for all } x.$$

**Example (11):** Find the Maclaurin series for $f(x) = \ln(1 + 4x^2)$.
Solution :

$$f(x) = \ln(1 + 4x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(4x^2)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4^n x^{2n}}{n} ,$$

Where $\left|4x^2\right| < 1 \implies |x|^2 < \dfrac{1}{4} \implies |x| < \dfrac{1}{2}$ .

**Example (12):** Find the function represented by the power series $\sum_{n=0}^{\infty} (-1)^n \dfrac{3^n x^n}{n!}$.

Solution :

$$\sum_{n=0}^{\infty} (-1)^n \frac{3^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = e^{-3x} .$$

**Example (13):** Find the sum of the series $\dfrac{1}{1\ (2)} - \dfrac{1}{2\ (2^2)} + \dfrac{1}{3\ (2^3)} - \cdots$.

Solution :

$$\frac{1}{1\ (2)} - \frac{1}{2\ (2^2)} + \frac{1}{3\ (2^3)} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n\ (2^n)} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n}$$

$$= \ln\left(1 + \frac{1}{2}\right) = \ln\left(\frac{3}{2}\right) .$$

**Example (14):** Use series to evaluate $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

Solution : For all $x \neq 0$ ,

$$\frac{e^x - 1 - x}{x^2} = \frac{1}{x^2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - 1 - x\right]$$

$$= \frac{1}{x^2}\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots$$

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0}\left(\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots\right) = \frac{1}{2}.$$