King Saud University
College of Sciences
Department of Mathematics

MATH 201
MULTIVARIABLE CALCULUS

CLASS NOTES
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Chapter 1

Partial Derivatives

1.1 Functions of several variables

1.1.1 Functions of two variables

Definition: A function f of two variables is a map that assigns to each ordered pair of real numbers $(x, y) \in D \subseteq \mathbb{R}^2$ a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is $\{f(x, y) | (x, y) \in D\}$.

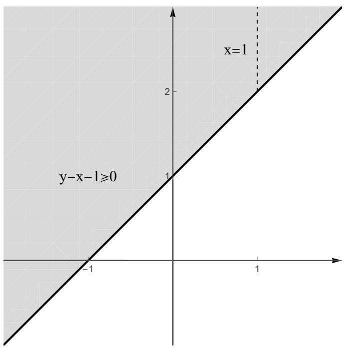
Example (1): If $f(x, y) = \dfrac{\sqrt{y - x - 1}}{x - 1}$, evaluate $f(2, 7)$, find the domain of f and sketch it.

Solution : $f(2, 7) = \dfrac{\sqrt{7 - 2 - 1}}{2 - 1} = \dfrac{\sqrt{4}}{1} = 2$.

The domain of f is the set $D = \big\{(x, y) \in \mathbb{R}^2 \mid y - x - 1 \geq 0\ ,\ x \neq 1\big\}$.

So, $D = \big\{(x, y) \in \mathbb{R}^2 \mid y \geq x + 1\ ,\ x \neq 1\big\}$.

$y - x - 1 \geq 0 \implies y \geq x + 1$ represents the points on and above the line $y = x + 1$.
$x \neq 1$ means the point on the line $x = 1$ must be excluded from the domain .

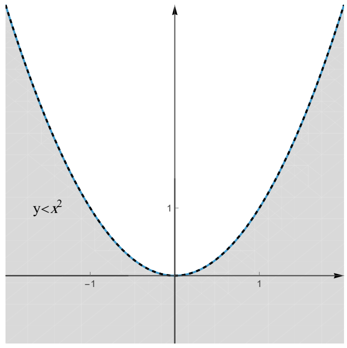


Example (2): If $f(x, y) = \ln(x^2 - y)$, find the domain of f and sketch it.

Solution :

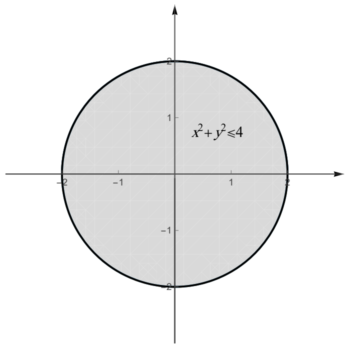
The domain of f is the set
$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y > 0\}$.
So, $D = \{(x, y) \in \mathbb{R}^2 \mid y < x^2\}$.

$x^2 - y > 0 \implies y < x^2$
represents the points below the parabola $y = x^2$.



Example (3): If $f(x, y) = \sqrt{4 - x^2 - y^2}$, find the domain and range of f.
Solution :
The domain of f is the set
$D = \{(x, y) \in \mathbb{R}^2 \mid 4 - x^2 - y^2 \geq 0\}$.
So, $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$.

$4 - x^2 - y^2 \geq 0 \implies x^2 + y^2 \leq 4$
represents the points on and inside the disk of center $(0, 0)$ and radius 2.



Note that $0 \leq \sqrt{4 - x^2 - y^2}$
and $\sqrt{4 - (x^2 + y^2)} \leq \sqrt{4} = 2$
So, the range of f is
$\{z \in \mathbb{R} \mid 0 \leq z \leq 2\} = [0, 2]$.

1.1.2 Graphs

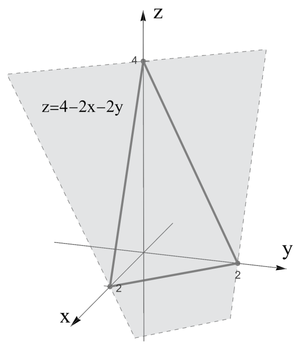
Definition: If f is a function of two variables with domain $D \subseteq \mathbb{R}^2$, then the graph of f is the set of all points $(x, y, z) \in \mathbb{R}^3$, such that $z = f(x, y)$ and $(x, y) \in D$.

Example (4): Sketch the graph of the function $f(x, y) = 4 - 2x - 2y$.
Solution:
$z = 4 - 2x - 2y \implies 2x + 2y + z = 4$
It represents a plane.

To find the x-intercept, put $y = z = 0$
$2x = 4 \implies x = 2$.
So the x-intercept is $(2, 0, 0)$.

To find the y-intercept, put $x = z = 0$
$2y = 4 \implies y = 2$.
So the y-intercept is $(0, 2, 0)$.

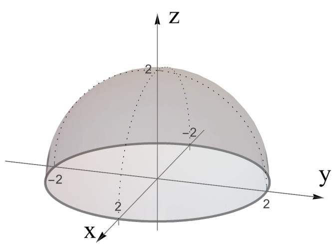
To find the z-intercept, put $x = y = 0$
$z = 4$, So the z-intercept is $(0, 0, 4)$.



Example (5): Sketch the graph of the function $f(x, y) = \sqrt{4 - x^2 - y^2}$.
Solution:

$z = \sqrt{4 - x^2 - y^2}$
$\Longrightarrow \; z^2 = 4 - x^2 - y^2$
$\Longrightarrow \; x^2 + y^2 + z^2 = 2^2$
Note that $z = \sqrt{4 - x^2 - y^2} \geq 0$.

It represents the upper half of the sphere centered at the origin and its radius is 2.

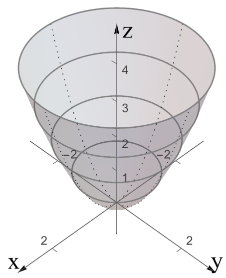


Example (6): Sketch the graph of the function $f(x, y) = x^2 + y^2$.
Solution:

The domain of f is $\mathbb{R}^2$.

$z = x^2 + y^2 \geq 0$
For each value of $z > 0$,
$x^2 + y^2 = z$ represents a circle centered at the origin and its radius is $\sqrt{z}$.

$f(x, y) = x^2 + y^2$ represents a paraboloid.



1.1.3 Level curves

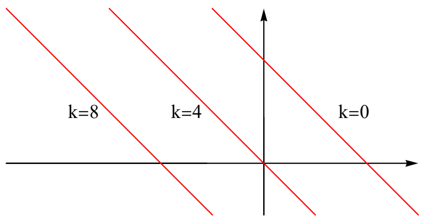
Definition: A level curve of a function $f(x, y)$ is the curve $f(x, y) = k$, where k is a constant (in the range of f).

Example (7): Sketch the level curves of the function $f(x, y) = 4 - 2x - 2y$ for the values $k = 0, 4, 8$.
Solution:
$4 - 2x - 2y = k$.
$\Longrightarrow \; 2x + 2y = 4 - k$

For $k = 0 : \; 2y + 2x = 4$
$\Longrightarrow \; y = -x + 2$

For $k = 4 : \; 2y + 2x = 0$
$\Longrightarrow \; y = -x$

For $k = 8 : \; 2y + 2x = -4$
$\Longrightarrow \; y = -x - 2$

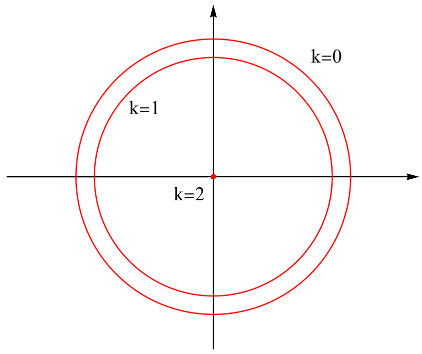


Example (8): Sketch the level curves of the function $f(x, y) = \sqrt{4 - x^2 - y^2}$ for the values $k = 0, 1, 2$.
Solution:
$\sqrt{4 - x^2 - y^2} = k$.
$\implies 4 - x^2 - y^2 = k^2$
$\implies x^2 + y^2 = 4 - k^2$

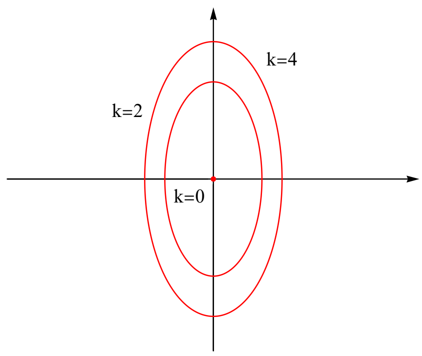
For $k = 0 : x^2 + y^2 = 4$
Circle: center is $(0, 0)$, radius $= 2$.

For $k = 1 : x^2 + y^2 = 3$
Circle: center is $(0, 0)$, radius $= \sqrt{3}$.

For $k = 2 : x^2 + y^2 = 0$
Just one point, the origin.



Example (9): Sketch the level curves of the function $f(x, y) = 4x^2 + y^2$ for the values $k = 0, 2, 4$.
Solution:



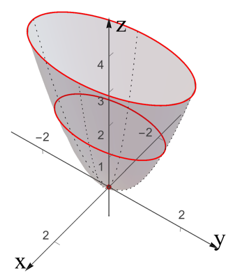
Note that the domain is $\mathbb{R}^2$.

$4x^2 + y^2 = k$.

For $k = 0 : 4x^2 + y^2 = 0$
$\implies x = 0 \; , \; y = 0$
The level curve is the origin.

For $k > 0 : 4x^2 + y^2 = k$
$\implies \dfrac{x^2}{\left(\frac{k}{4}\right)} + \dfrac{y^2}{k} = 1$
The level curve is an ellipse centered at the origin, the major axis lies on the y-axis and the minor axis lies on the x-axis.

Note that the graph of f is an elliptic paraboloid.



1.1.4 Functions of three variables

Definition: A function f of two variables is a map that assigns to each ordered pair of real numbers $(x, y, z) \in D \subseteq \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. The set D is the domain of f and its range is $\{f(x, y, z)|(x, y, z) \in D\}$.

Example (10): Find the domain of $f(x, y, z) = \ln(z - x) + yz \sin x$.
Solution : $f(x, y, z)$ is defined when $z - x > 0$.
Therefore, $D = \left\{(x, y, z) \in \mathbb{R}^3 \mid z > x\right\}$.

Example (11): Find the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$.
Solution : $x^2 + y^2 + z^2 = k$, where $k \geq 0$.
If $k = 0$, then the level surface is just the origin $(0, 0, 0) \in \mathbb{R}^3$.
If $k > 0$, then the level surface is $x^2 + y^2 + z^2 = \left(\sqrt{k}\right)^2$, which is a sphere centered at the origin, and its radius is $\sqrt{k}$.

1.1.5 EXERCISES

1. Let $f(x, y) = x^2 \ln(x + y)$

   (a). Evaluate $f(3, 1)$.

   (b). Find and sketch the domain of f.

   (c). Find the range of f.

2. Let $f(x, y, z) = \ln\left(z - \sqrt{x^2 + y^2}\right)$.

   (a). Evaluate $f(4, 23, 6)$.

   (b). Find and describe the domain of f.

3. Find and sketch the domain of the following:

   (a). $f(x, y) = \sqrt{x - 2} + \sqrt{y - 1}$.

   (b). $f(x, y) = \ln(9 - x^2 - y^2)$.

   (c). $f(x, y) = \dfrac{\ln(2 - x)}{1 - x^2 - y^2}$.

(*) Find and sketch the domain of the following:

   (a). $f(x, y) = \sqrt{25 - x^2 - y^2} + \ln(2 + x)$.

   (b). $f(x, y) = \cos(x + y) + \dfrac{x^2 - y^2 - 3}{\sqrt{x + y - 4}}$.

   (c). $f(x, y) = \sqrt{x + y^2} + \sqrt{y - x^2}$.

1.2 Limits and Continuity

1.2.1 Limits of Functions of Two Variables

Definition: Let f be a function of two variables whose domain $D \subseteq \mathbb{R}^2$ includes points arbitrarily close to (a,b). Then we say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L and we write $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$,
if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x,y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \epsilon$.

1.2.2 Showing That a Limit Does Not Exist

NOTE: If $f(x,y) \to L_1$ as $(x,y) \longrightarrow (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \longrightarrow (a,b)$ along a different path C_2, where $L_1 \neq L_2$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example (1): Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist.

Solution: Note that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2} \quad \left(\dfrac{0}{0}\right)$.
Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-m^2x^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{x^2(1-m^2)}{x^2(1+m^2)} = \frac{1-m^2}{1+m^2} .$$

Note that the limit depends only on m.
Let C_1 be the path $y = 0$ (where m=0) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2} = \dfrac{1-0}{1+0} = 1$.
Let C_2 be the path $y = x$ (where m=1) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2} = \dfrac{1-1}{1+1} = 0$.
Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist.

Example (2): Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

Solution: Note that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} \quad \left(\dfrac{0}{0}\right)$.
Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{mx^2}{x^2(1+m^2)} = \frac{m}{1+m^2} .$$

Note that the limit depends only on m.
Let C_1 be the path $y = 0$ (where m=0) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = \dfrac{0}{1+0} = 0$.
Let C_2 be the path $y = x$ (where m=1) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = \dfrac{1}{1+1} = \dfrac{1}{2}$.
Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

Example (3): Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ does not exist.

Solution: Note that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4} \quad \left(\dfrac{0}{0}\right)$.
Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$.
$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4} = \lim\limits_{x\to 0} \dfrac{x(m^2x^2)}{x^2+m^4x^4} = \lim\limits_{x\to 0} \dfrac{mx^3}{x^2(1+m^4\ x^2)} = \lim\limits_{x\to 0} \dfrac{mx}{1+m^4\ x^2}$=0 .
Note that the limit here depends on x and m .
Let C_1 be the path $y = 0$ (where m=0) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4} = 0$.
Let C_2 be the path $x = y^2$ (the parabola with vertex (0,0) and opens to the right) then $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4} = \lim\limits_{y\to 0} \dfrac{y^2\ y^2}{(y^2)^2+y^4} = \lim\limits_{y\to 0} \dfrac{y^4}{2y^4} = \dfrac{1}{2}$.
Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ does not exist.

Example (4): Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2}$ does not exist.
Solution: Note that $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2} \quad \left(\dfrac{0}{0}\right)$.
Let $y = mx$, where $m \in \mathbb{R}$, note that $x \to 0$ as $y \to 0$.
$\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{\sin(mx^2)}{x^2+m^2x^2} = \lim\limits_{x\to 0} \dfrac{\sin(mx^2)}{(1+m^2)x^2} = \dfrac{m}{1+m^2}$.
Note that the limit depends only on m.
Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2}$ does not exist.

1.2.3 Properties of Limits

If $f(x,y)$, $g(x,y)$ are two functions defined on $D \setminus \{(a,b)\}$, where $D \subseteq \mathbb{R}^2$, and $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L_1$, $\lim\limits_{(x,y)\to(a,b)} g(x,y) = L_2$, where $L_1, L_2 \in \mathbb{R}$ then
(1). $\lim\limits_{(x,y)\to(a,b)} [f(x,y) + g(x,y)] = L_1 + L_2$.
(2). $\lim\limits_{(x,y)\to(a,b)} [f(x,y) - g(x,y)] = L_1 - L_2$.
(3). $\lim\limits_{(x,y)\to(a,b)} k\ f(x,y) = k\ L_1$, where $k \in \mathbb{R}$.
(4). $\lim\limits_{(x,y)\to(a,b)} [f(x,y).g(x,y)] = L_1 L_2$.
(5). $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L_1}{L_2}$, where $L_2 \neq 0$.
(6). If $P(x,y)$ is a polynomial in x, y then $\lim\limits_{(x,y)\to(a,b)} P(x,y) = P(a,b)$.

Example (5): Evaluate $\lim\limits_{(x,y)\to(2,1)} \left(x^2y^2 - 2xy + x + y - 1\right)$.
Solution: Note that $P(x,y) = x^2y^2 - 2xy + x + y - 1$ is a polynomial.
So, $\lim\limits_{(x,y)\to(2,1)} \left(x^2y^2 - 2xy + x + y - 1\right) = P(2,1)$
$= (2)^2(1)^2 - 2(2)(1) + 2 + 1 - 1 = 2$.

Example (6): Evaluate $\lim\limits_{(x,y)\to(-1,3)} \dfrac{2x^2y+1}{xy^3-2x}$.

Solution: Note that $P(x,y) = 2x^2y+1$ and $Q(x,y) = xy^3-2x$ are both polynomials.

$$\lim_{(x,y)\to(-1,3)} \frac{2x^2y+1}{xy^3-2x} = \frac{P(-1,3)}{Q(-1,3)} = \frac{2(-1)^2(3)+1}{(-1)(3)^3-2(-1)} = \frac{7}{-25}.$$

Note that $Q(-1,3) \neq 0$.

Example (7): Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+y^2} = 0$.

First Solution: $\forall\ x, y \in \mathbb{R}^*$,

$$y^2 \leq x^2+y^2 \implies \frac{y^2}{x^2+y^2} \leq 1 .$$

$$0 \leq \left| \frac{2xy^2}{x^2+y^2} \right| = \frac{|2x|\ |y^2|}{|x^2+y^2|} = 2|x|\ \frac{y^2}{x^2+y^2} \leq 2|x| .$$

Note that $(x,y) \to (0,0) \implies x \to 0$.

Since $\lim\limits_{(x,y)\to(0,0)} 0 = 0$ and $\lim\limits_{(x,y)\to(0,0)} 2|x| = 2(0) = 0$,

By Squeeze Theorem $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2+y^2} = 0$.

Second Solution: Using polar coordinates,

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = \lim_{r\to 0} \frac{2r^3\cos\theta\ \sin^2\theta}{r^2} = \lim_{r\to 0} 2r\cos\theta\sin^2\theta = 0.$$

Note that $\lim\limits_{r\to 0} 2r = 0$ and $\cos\theta\sin^2\theta$ is bounded .

1.2.4 Continuity

Definition: Let $f(x,y)$ be a function of two variables defined on a set $D \subseteq \mathbb{R}^2$ and $(a,b) \in D$, If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, then f is continuous at (a,b).

If f is continuous at every point in D, then f is continuous on D .

Example (8): Discuss the continuity of $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$.

Solution:

f is not defined at $(0,0)$, so it is not continuous at $(0,0)$.

$$\forall\ (a,b) \neq (0,0),\ \lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} \frac{x^2-y^2}{x^2+y^2} = \frac{a^2-b^2}{a^2+b^2} = f(a,b).$$

So, f is continuous on $D = \big\{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\big\} = \mathbb{R}^2 \setminus \{(0,0)\}$.

Example (9): Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$.

Solution:

$$\forall\ (a,b) \neq (0,0),\ \lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} \frac{x^2-y^2}{x^2+y^2} = \frac{a^2-b^2}{a^2+b^2} = f(a,b).$$

f is defined at $(0,0)$ and $f(0,0) = 0$, but $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

So, f is continuous on $D = \{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\} = \mathbb{R}^2 \setminus \{(0, 0)\}$.

Example (10): Discuss the continuity of $f(x, y) = \begin{cases} \dfrac{2xy^2}{x^2 + y^2} & if \quad (x, y) \neq (0, 0) \\ 0 & if \quad (x, y) = (0, 0) \end{cases}$.

Solution:
$\forall\ (a, b) \neq (0, 0),\ \lim\limits_{(x,y)\to(a,b)} f(x, y) = \lim\limits_{(x,y)\to(a,b)} \dfrac{2xy^2}{x^2 + y^2} = \dfrac{2ab^2}{a^2 + b^2} = f(a, b).$
f is defined at $(0, 0)$ and $f(0, 0) = 0$, also $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0.$
So, f is continuous on $\mathbb{R}^2$.

Example (11): Discuss the continuity of $f(x, y) = e^{-x^2-y^2}$.
Solution: $f(x, y) = e^{-x^2-y^2} = e^{-(x^2+y^2)}$.
f is defined on $\mathbb{R}^2$ and $\lim\limits_{(x,y)\to(a,b)} f(x, y) = e^{-(a^2+b^2)} = f(a, b).$
Therefore, f is continuous on $\mathbb{R}^2$.

Example (12): Discuss the continuity of $f(x, y) = \tan^{-1}\left(\dfrac{y}{x}\right)$.
Solution:
f is not defined where $x = 0$.
$\forall\ (a, b)$ where $a \neq 0$, $\lim\limits_{(x,y)\to(a,b)} f(x, y) = \tan^{-1}\left(\dfrac{b}{a}\right) = f(a, b).$
So, f is continuous on $D = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$ which is $\mathbb{R}^2$ except the y-axis.

1.2.5 Limit and Continuity of a function of three variables

Definition: Let f be a function of three variables whose domain $D \subseteq \mathbb{R}^3$ includes points arbitrarily close to (a, b, c). Then we say that the limit of $f(x, y, z)$ as (x, y, z) approaches (a, b, c) is L and we write $\lim\limits_{(x,y,z)\to(a,b,c)} f(x, y, z) = L$,
if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that
if $(x, y, z) \in D$ and $0 < \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} < \delta$
then $|f(x, y, z) - L| < \epsilon$.

Definition: Let $f(x, y, z)$ be a function of three variables defined on a set $D \subseteq \mathbb{R}^3$ and $(a, b, c) \in D$, If $\lim\limits_{(x,y,z)\to(a,b,c)} f(x, y, z) = f(a, b, c)$, then f is continuous at (a, b, c).
If f is continuous at every point in D, then f is continuous on D .

Example (13): Discuss the continuity of $f(x, y, z) = \dfrac{1}{1 - x^2 - y^2 - z^2}$.
Solution:
f is not defined where $1 - x^2 - y^2 - z^2 = 0 \implies x^2 + y^2 + z^2 = 1,$
f is not continuous on the unit sphere.
$\forall\ (a, b, c) \in \mathbb{R}^3$ where $a^2 + b^2 + c^2 \neq 1,$
$\lim\limits_{(x,y,z)\to(a,b,c)} f(x, y, z) = \dfrac{1}{1 - a^2 + b^2 + c^2} = f(a, b, c).$

So, f is continuous on $D = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \neq 1\right\}$ which is $\mathbb{R}^3$ except the unit sphere.

1.2.6 EXERCISES

1. Find the limit of the following:

$$(a).\ \lim_{(x,y)\to(3,2)} \left(x^2y^3 - 4y^2\right) \quad (b).\ \lim_{(x,y)\to(2,-1)} \frac{x^2y + yx^2}{x^2 - y^2}$$

2. Show that the following limits do not exist:

$$(a).\ \lim_{(x,y)\to(0,0)} \frac{y^2}{x^2 + y^2} \qquad (b).\ \lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2 + y^2}$$

$$(c).\ \lim_{(x,y)\to(1,1)} \frac{y - x}{1 - y + \ln x}$$

3. Discuss the existence of the following limits (and find its value if exists):

$$(a).\ \lim_{(x,y)\to(2,3)} \frac{3x - 2y}{4x^2 - y^2} \quad (b).\ \lim_{(x,y)\to(0,0)} \frac{x^2y\cos y}{x^2 + y^4}$$

4. Use the Squeeze Theorem to find the limit of the following :

$$(a).\ \lim_{(x,y)\to(0,0)} xy\ \sin\left(\frac{1}{x^2 + y^2}\right) \quad (b).\ \lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2 + y^2 + z^2}$$

5. Determine the set of points of continuity of the following:

$$\text{(a). } f(x,y) = \begin{cases} \dfrac{x^2y^3}{2x^2 + y^2} & if \quad (x,y) \neq (0,0) \\ 1 & if \quad (x,y) = (0,0) \end{cases}$$

$$\text{(b). } f(x,y) = \begin{cases} \dfrac{xy}{x^2 + xy + y^2} & if \quad (x,y) \neq (0,0) \\ 1 & if \quad (x,y) = (0,0) \end{cases}$$

6. Use polar coordinates to find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2}$.

(*) Show that the following limits do not exist:

$$(a).\ \lim_{(x,y)\to(0,0)} \frac{x^3+y^2}{x^2+y^2} \quad (b).\ \lim_{(x,y)\to(0,0)} \frac{xy}{x^4+y^4}$$

(*) Use Squeeze theorem to find the following limits:

$$(a).\ \lim_{(x,y)\to(-1,0)} \frac{y(x+1)^2+y^2\sin(\pi x)}{(x+1)^2+y^2} \quad (b).\ \lim_{(x,y)\to(0,0)} \frac{y^2\sin x+yx^2}{x^2+y^2}$$

(*) Show that the limit is zero in the following:

$$(a).\ \lim_{(x,y)\to(0,0)} \frac{x^5+y^5}{x^2+y^2} \quad (b).\ \lim_{(x,y)\to(0,0)} \frac{x^6+y^6}{x^4+y^4}$$

$$(c).\ \lim_{(x,y)\to(0,0)} \frac{x^2+2xy}{\sqrt{x^2+y^2}} \quad (d).\ \lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{x^2+y^2}$$

1.3 Partial Derivatives

1.3.1 Partial Derivatives of Functions of Two Variables

Definition: Let $f(x, y)$ be a function of two variables defined on a set $D \subseteq \mathbb{R}^2$,
If $\lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$ and $\lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}$ exist, then the partial derivatives of f are denoted by f_x and f_y and are defined as
(1). $f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$.
(2).$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}$.

Notations for Partial Derivatives: If $z = f(x, y)$ then
(1). $f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = D_x f$.
(2). $f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = D_y f$.

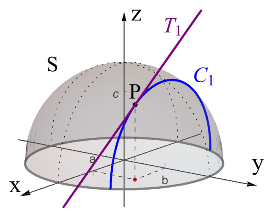
Rule for Finding Partial Derivatives
(1). To find f_x : differentiate $f(x, y)$ with respect to x regarding y as a constant.
(2). To find f_y : differentiate $f(x, y)$ with respect to y regarding x as a constant.

Example (1): If $f(x, y) = x^2 - xy^3 - 3y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$.
Solution:
(1). $f_x(x, y) = 2x - (1)y^3 - 0 = 2x - y^3$,
$f_x(1, 1) = 2(1) - (1)^3 = 1$.
(2). $f_y(x, y) = 0 - x(3y^2) - 3(2y) = -3xy^2 - 6y$,
$f_y(1, 1) = -3(1)(1)^2 - 6(1) = -9$.

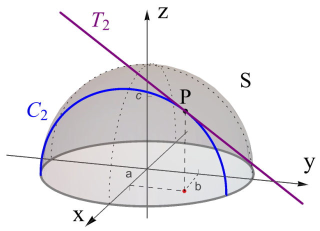
Example (2): If $f(x, y) = \sin\left(\frac{2x}{1+3y}\right)$, Evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial x}$.
Solution:
(1). $\frac{\partial f}{\partial x} = \cos\left(\frac{2x}{1+3y}\right)\left(\frac{2}{1+3y}\right)$.
(2). $\frac{\partial f}{\partial y} = \cos\left(\frac{2x}{1+3y}\right)\left((2x)(-1)(1+3y)^{-2}(3)\right) = \cos\left(\frac{2x}{1+3y}\right)\left(\frac{-6x}{(1+3y)^2}\right)$.

1.3.2 Interpretations of Partial Derivatives

Let S be the surface represented by $z = f(x, y)$ and $f(a, b) = c$.
Then $P(a, b, c)$ lies on S.
Let C_1 be the curve where the plane $y = b$ intersects the surface S,
The slope of the tangent line T_1 to the curve C_1 at $P(a, b, c)$ is $f_x(a, b)$.

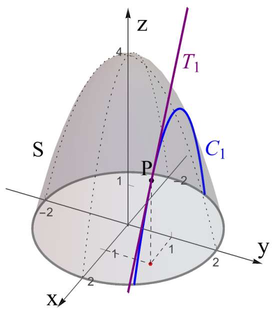


Let C_2 be the curve where the plane $x = a$ intersects the surface S,
The slope of the tangent line T_2 to the curve C_2 at $P(a, b, c)$ is $f_y(a, b)$.

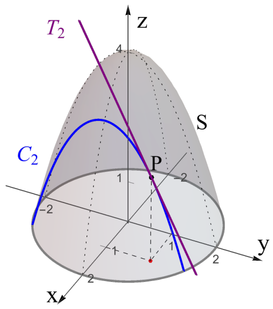


Example (3): If $f(x, y) = 4 - x^2 - y^2$, find $f_x(1, 1)$, $f_y(1, 1)$ and interpret them as slopes. Solution:

$z = 4 - x^2 - y^2$.
$f_x(x, y) = -2x$.
$f_x(1, 1) = -2(1) = -2.$
$f(1, 1) = 4 - 1 - 1 = 2$.
C_1 is the intersection of $z = 4 - x^2 - y^2$ and $y = 1$, and it is the parabola $z = 3 - x^2$.
The line T_1 is tangent to the curve C_1 at the point $P(1, 1, 2)$ and its slope is $f_x(1, 1) = -2$.



$z = 4 - x^2 - y^2$.
$f_y(x, y) = -2y$.
$f_y(1, 1) = -2(1) = -2.$
$f(1, 1) = 4 - 1 - 1 = 2$.
C_2 is the intersection of $z = 4 - x^2 - y^2$ and $x = 1$, and it is the parabola $z = 3 - y^2$.
The line T_2 is tangent to the curve C_2 at the point $P(1, 1, 2)$ and its slope is $f_y(1, 1) = -2$.



Example (4): If $x^3 + y^3 + z^3 + 6xyz + 4 = 0$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.
Solution:
Differentiating implicitly with respect to x .

$$3x^2 + 0 + 3z^2\,\frac{\partial z}{\partial x} + 6y\left(z + x\frac{\partial z}{\partial x}\right) + 0 = 0$$

$$3x^2 + 3z^2\,\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}\left(3z^2 + 6xy\right) = -3x^2 - 6yz$$

$\dfrac{\partial z}{\partial x} = \dfrac{-3x^2 - 6yz}{3z^2 + 6xy} = \dfrac{-x^2 - 2yz}{z^2 + 2xy} = -\dfrac{x^2 + 2yz}{z^2 + 2xy}$.
Differentiating implicitly with respect to y .
$0 + 3y^2 + 3z^2 \dfrac{\partial z}{\partial y} + 6x\left(z + y\dfrac{\partial z}{\partial y}\right) + 0 = 0$
$3y^2 + 3z^2 \dfrac{\partial z}{\partial y} + 6xz + 6xy\dfrac{\partial z}{\partial y} = 0$
$\dfrac{\partial z}{\partial y}\left(3z^2 + 6xy\right) = -3y^2 - 6xz$
$\dfrac{\partial z}{\partial y} = \dfrac{-3y^2 - 6xz}{3z^2 + 6xy} = \dfrac{-y^2 - 2xz}{z^2 + 2xy} = -\dfrac{y^2 + 2xz}{z^2 + 2xy}$.

1.3.3 Functions of Three Variables

Definition: Let $f(x, y, z)$ be a function of three variables defined on a set $D \subseteq \mathbb{R}^3$, If $\lim\limits_{h\to 0} \dfrac{f(x+h, y, z) - f(x, y, z)}{h}$, $\lim\limits_{h\to 0} \dfrac{f(x, y+h, z) - f(x, y, z)}{h}$ and $\lim\limits_{h\to 0} \dfrac{f(x, y, z+h) - f(x, y, z)}{h}$ exist, then the partial derivatives of f are denoted by f_x f_y and f_z and are defined as
(1). $f_x(x, y, z) = \lim\limits_{h\to 0} \dfrac{f(x+h, y, z) - f(x, y, z)}{h}$.
(2). $f_y(x, y, z) = \lim\limits_{h\to 0} \dfrac{f(x, y+h, z) - f(x, y, z)}{h}$.
(3). $f_z(x, y, z) = \lim\limits_{h\to 0} \dfrac{f(x, y, z+h) - f(x, y, z)}{h}$.

Example (5): If $f(x, y, z) = e^{xy} \ln(z^2 + 1)$, find f_x, f_y and f_z.
Solution:
$f_x(x, y, z) = (e^{xy}y) \ln(z^2 + 1) = ye^{xy} \ln(z^2 + 1)$.
$f_y(x, y, z) = (e^{xy}x) \ln(z^2 + 1) = xe^{xy} \ln(z^2 + 1)$.
$f_z(x, y, z) = e^{xy} \left(\dfrac{2z}{z^2 + 1}\right) = \dfrac{2ze^{xy}}{z^2 + 1}$.

1.3.4 Higher Derivatives

Definition: If $z = f(x, y)$, then the second partial derivatives of f are
(1) $f_{xx} = (f_x)_x = \dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial x} = \dfrac{\partial f^2}{\partial x^2}$.
(2) $f_{xy} = (f_x)_y = \dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x} = \dfrac{\partial f^2}{\partial y \partial x}$.
(3) $f_{yx} = (f_y)_x = \dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial y} = \dfrac{\partial f^2}{\partial x \partial y}$.
(y) $f_{yy} = (f_y)_y = \dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial y} = \dfrac{\partial f^2}{\partial y^2}$.

Example (6): Find the second partial derivatives of $f(x, y) = x^3 + x^2y^2 - 2y^3$.
Solution:

$f_x(x,y) = 3x^2 + (2x)y^2 - 0 = 3x^2 + 2xy^2$.
$f_{xx}(x,y) = 3(2x) + 2y^2(1) = 6x + 2y^2$.
$f_{xy}(x,y) = 0 + 2x(2y) = 4xy$.
$f_y(x,y) = 0 + x^2(2y) - 2(3y^2) = 2x^2y - 6y^2$.
$f_{yx}(x,y) = 2y(2x) - 0 = 4xy$.
$f_{yy} = 2x^2(1) - 6(2y) = 2x^2 - 12y$.
NOTE : $f_{xy}(x,y) = f_{yx}(x,y)$.

Clairaut's Theorem: Suppose f is defined on a set $D \subseteq \mathbb{R}^2$ that contains the point (a,b), If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$.

Example (7): If $f(x,y,z) = \sin(2x - yz)$, find $f_{xyz}(x,y)$.
Solution:
$f_x(x,y,z) = \cos(2x - yz)(2) = 2\cos(2x - yz)$.
$f_{xy}(x,y,z) = 2\left[-\sin(2x - yz)\ (-z)\right] = 2z\sin(2x - yz)$.
$f_{xyz}(x,y,z) = (2)\sin(2x - yz) + 2z\left[\cos(2x - yz)\ (-y)\right]$
$= 2\sin(2x - yz) - 2yz\cos(2x - yz)$.

1.3.5 Partial Differential Equations

(1). **Laplace's Equation:** If $u(x,y)$ is a function of two variables and $u_{xx} + u_{yy} = 0$, then u satisfies the Laplace's equation, and u is called a harmonic function.

Example (8): Show that $u(x,y) = e^x \cos y$ is a solution of Laplace's equation.
Solution:
$u_x(x,y) = e^x \cos y$ and $u_{xx}(x,y) = e^x \cos y$.
$u_y(x,y) = -e^x \sin y$ and $u_{yy}(x,y) = -e^x \cos y$.
$u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0$.

(2). **Wave Equation:** If $u(x,t)$ is a function of two variables and $u_{tt} = a^2 u_{xx}$, then u satisfies the wave equation.

Example (9): Show that $u(x,t) = \sin(x - at)$ is a solution of the wave equation.
Solution:
$u_x(x,t) = \cos(x - at)$ and $u_{xx}(x,t) = -\sin(x - at)$.
$u_t(x,t) = -a\cos(x - at)$ and $u_{tt}(x,t) = -a^2 \sin(x - at)$.
$u_{tt} = a^2(-\sin(x - at)) = a^2 u_{xx}$.

1.3.6 Differentiability

Definition: If $z = f(x,y)$ and $(a,b) \in D_f$, if x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$, then the increment of z is
$\Delta z = f\left(a + \Delta x, b + \Delta y\right) - f(a,b)$.

Definition: If $z = f(x, y)$, then f is differentiable at (a, b), if
$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \sqrt{(\Delta x)^2 + (\Delta y)^2}\ \epsilon(\Delta x, \Delta y)$,
where $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \epsilon(\Delta x, \Delta y) = 0$.

NOTE:
If $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{f(a+\Delta x, b+\Delta y) - f(a,b) - f_x(a,b)\Delta x - f_y(a,b)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$, then
f is differentiable at (a, b).

Theorem: If f is differentiable at (a, b), then f is continuous at (a, b).

NOTE: If f is not continuous at (a, b), then f is not differentiable at (a, b).

Example (10): Show that $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$
is not differentiable at $(0, 0)$.
Solution:
$\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = \lim\limits_{r\to 0} \dfrac{r\cos\theta\ r\sin\theta}{r^2}$
$= \lim\limits_{r\to 0} \dfrac{r^2\ \cos\theta\ \sin\theta}{r^2} = \cos\theta\ \sin\theta$.
The limit depends on θ, so the limit does not exist.
f is not continuous at $(0, 0)$, hence, f is not differentiable at $(0, 0)$.

Example (11): Show that $f(x,y) = \begin{cases} \dfrac{xy^2}{\sqrt{x^2+y^2}} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$
is differentiable at $(0, 0)$.
Solution:
$f(0, 0) = 0$.
$f(0+\Delta x, 0+\Delta y) = f(\Delta x, \Delta y) = \dfrac{\Delta x\,(\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$.
$f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0}{h} = 0$.
$f_y(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,0+h) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{0}{h} = 0$.
$\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{f(\Delta x, \Delta y) - f(0,0) - f_x(0,0)\Delta x - f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$
$= \lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{\left(\frac{\Delta x(\Delta y)^2}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\right)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{\Delta x\,(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} = 0$.
Note that $0 \leq \left| \dfrac{\Delta x\,(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \right| = |\Delta x| \left| \dfrac{(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \right| \leq |\Delta x|$

By the squeeze theorem $\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\left|\dfrac{\Delta x\,(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2}\right|=0,$

So, $\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\dfrac{\Delta x\,(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2}=0.$

Therefore, f is differentiable at $(0,0)$.

Example (12): Show that $f(x,y)=\begin{cases}\dfrac{y^2\sin x+yx^2}{x^2+y^2} & \text{if} \quad (x,y)\neq(0,0)\\ 0 & \text{if} \quad (x,y)=(0,0)\end{cases}$

is not differentiable at $(0,0)$.

Solution:

$f(0,0)=0.$

$f(0+\Delta x,0+\Delta y)=f(\Delta x,\Delta y)=\dfrac{(\Delta y)^2\sin(\Delta x)+\Delta y\,(\Delta x)^2}{(\Delta x)^2+(\Delta y)^2}.$

$f_x(0,0)=\lim\limits_{h\to0}\dfrac{f(0+h,0)-f(0,0)}{h}=\lim\limits_{h\to0}\dfrac{0}{h}=0.$

$f_y(0,0)=\lim\limits_{h\to0}\dfrac{f(0,0+h)-f(0,0)}{h}=\lim\limits_{h\to0}\dfrac{0}{h}=0.$

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{f(\Delta x,\Delta y)-f(0,0)-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\left(\frac{(\Delta y)^2\sin(\Delta x)+\Delta y(\Delta x)^2}{(\Delta x)^2+(\Delta y)^2}\right)}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{(\Delta y)^2\sin(\Delta x)+\Delta y\,(\Delta x)^2}{\left[(\Delta x)^2+(\Delta y)^2\right]^{\frac{3}{2}}}$$

Taking the path $\Delta y=\Delta x$:

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{(\Delta y)^2\sin(\Delta x)+\Delta y\,(\Delta x)^2}{\left[(\Delta x)^2+(\Delta y)^2\right]^{\frac{3}{2}}}=\lim_{\Delta x\to0}\frac{(\Delta x)^2\sin(\Delta x)+\Delta x\,(\Delta x)^2}{\left[(\Delta x)^2+(\Delta x)^2\right]^{\frac{3}{2}}}$$

$$=\lim_{\Delta x\to0}\frac{(\Delta x)^2\left[\sin(\Delta x)+\Delta x\right]}{\left[2\,(\Delta x)^2\right]^{\frac{3}{2}}}=\lim_{\Delta x\to0}\frac{(\Delta x)^2\left[\sin(\Delta x)+\Delta x\right]}{\sqrt{8}\,(\Delta x)^3}=\lim_{\Delta x\to0}\frac{\sin(\Delta x)+\Delta x}{\sqrt{8}\Delta x}$$

$$=\lim_{\Delta x\to0}\frac{1}{\sqrt{8}}\left(\frac{\sin(\Delta x)}{\Delta x}+\frac{\Delta x}{\Delta x}\right)=\frac{1}{\sqrt{8}}(1+1)=\frac{2}{\sqrt{8}}=\frac{1}{\sqrt{2}}\neq0.$$

Therefore, f is not differentiable at $(0,0)$.

1.3.7 EXERCISES

1. Find the first partial derivatives of the function.

   $(a).\ f(x,y) = x^4 - 5xy^3$ $(b).\ g(x,y) = x^3 \sin y$

   $(c).\ w(u,v) = \dfrac{u}{v^2}$ $(d).\ u(r,\theta) = \sin(r\cos\theta)$

   $(e).\ w(x,y,z) = y\tan(x+2z)$

2. If $f(x,y) = y\sin^{-1}(xy)$, find $f_y\left(1, \dfrac{1}{2}\right)$.

3. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$:

   $(a).\ z = f(x) + g(y)$ $(b).\ z = f(x+y)$

4. Find all the second partial derivatives of $f(x,y) = x^4y - 2x^3y^2$.

5. Verify that the conclusion of Clairaut's Theorem holds for

   $u(x,y) = x^4y^3 - y^4$.

6. If $f(x,y) = x^4y^2 - x^3y$, find f_{xxx} and f_{xyx}.

(*) Discuss the differentiability of the following functions at the given points:

(a). $f(x,y) = \begin{cases} \dfrac{x^2(y-2)}{x^2+(y-2)^2} & \text{if } (x,y) \neq (0,2) \\ 0 & \text{if } (x,y) = (0,2) \end{cases}$ at (0,2).

(b). $f(x,y) = \begin{cases} \dfrac{x^2y - xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ at (0,0).

(c). $f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ at (0,0).

1.4 Chain Rule

1.4.1 The Chain Rule (Case 1)

Theorem: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and $\frac{dz}{dt} = \frac{\partial f}{\partial x}\,\frac{dx}{dt} + \frac{\partial f}{\partial y}\,\frac{dy}{dt}$.

Example (1): If $f(x, y) = x^3y + 2xy^3$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{dz}{dt}$ at $t = 0$.
Solution:
$\frac{\partial f}{\partial x} = 3x^2y + 2y^3$, $\frac{\partial f}{\partial y} = x^3 + 6xy^2$.
$\frac{dx}{dt} = 2\cos 2t$, $\frac{dy}{dt} = -\sin t$.
$\frac{dz}{dt} = \frac{\partial f}{\partial x}\,\frac{dx}{dt} + \frac{\partial f}{\partial y}\,\frac{dy}{dt}$
$= (3x^2y + 2y^3)(2\cos 2t) + (x^3 + 6xy^2)(-\sin t)$
$\left.\frac{dz}{dt}\right|_{t=0} = (0 + 2)(2) + (0 + 0)(0) = 4$.

1.4.2 The Chain Rule (Case 2)

Theorem: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t. Then z is a differentiable function of t and $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial s}$,
$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial t}$

Example (2): If $f(x, y) = e^x \sin y$, where $x = s^2 + t^2$ and $y = s^2t^2$, find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.
Solution:
$\frac{\partial f}{\partial x} = e^x \sin y$, $\frac{\partial f}{\partial y} = e^x \cos y$.
$\frac{\partial x}{\partial s} = 2s$, $\frac{\partial y}{\partial s} = 2st^2$.
(1). $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial s} = (e^x \sin y)\,(2s) + (e^x \cos y)\,(2st^2)$
$= 2se^{s^2+t^2} \sin(s^2t^2) + 2st^2e^{s^2+t^2} \cos(s^2t^2)$.
Also, $\frac{\partial x}{\partial t} = 2t$, $\frac{\partial y}{\partial s} = 2ts^2$.
(2). $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial t} = (e^x \sin y)\,(2t) + (e^x \cos y)\,(2ts^2)$
$= 2te^{s^2+t^2} \sin(s^2t^2) + 2ts^2e^{s^2+t^2} \cos(s^2t^2)$.

1.4.3 The Chain Rule (The General Case)

Theorem: Suppose that u is a differentiable function of the n variables $x_1, x_2, \cdots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \cdots, t_m$. Then u is a function of $t_1, t_2, \cdots, t_m$ and
$\dfrac{\partial u}{\partial t_i} = \dfrac{\partial u}{\partial x_1}\dfrac{\partial x_1}{\partial t_i} + \dfrac{\partial u}{\partial x_2}\dfrac{\partial x_2}{\partial t_i} + \cdots + \dfrac{\partial u}{\partial x_n}\dfrac{\partial x_n}{\partial t_i}$, for each $i = 1, 2, \cdots, m$.

Example (3): If $u(x, y, z) = x^2y^2 + yz^3$, where $x = r^2se^t$, $y = rse^{-t}$ and $z = rs^2 \sin t$, find $\dfrac{\partial u}{\partial s}$ when $r = 1$, $s = 1$ and $t = 0$.
Solution:
$\dfrac{\partial u}{\partial x} = 2xy^2$, $\dfrac{\partial u}{\partial y} = 2x^2y + z^3$ and $\dfrac{\partial u}{\partial z} = 3yz^2$.
$\dfrac{\partial x}{\partial s} = r^2e^t$, $\dfrac{\partial y}{\partial s} = re^{-t}$ and $\dfrac{\partial z}{\partial s} = 2rs \sin t$.
So, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial s}$
$\dfrac{\partial u}{\partial s} = (2xy^2)(r^2e^t) + (2x^2y + z^3)(re^{-t}) + (3yz^2)(2rs \sin t)$
When $r = 1, s = 1, t = 0$, $\dfrac{\partial u}{\partial s} = (2)(1) + (2)(1) + (0)(0) = 4$.

Example (4): If $w = f(x, y)$ is differentiable at (x, y) and $x = s + t$, $y = s - t$.
Show that $\dfrac{\partial w}{\partial s}\dfrac{\partial w}{\partial t} = \left(\dfrac{\partial w}{\partial x}\right)^2 - \left(\dfrac{\partial w}{\partial y}\right)^2$.
$\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial s} = f_x(1) + f_y(1) = f_x + f_y$.
$\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t} = f_x(1) + f_y(-1) = f_x - f_y$.
$\dfrac{\partial w}{\partial s}\dfrac{\partial w}{\partial t} = (f_x + f_y)(f_x - f_y) = (f_x)^2 - (f_y)^2 = \left(\dfrac{\partial w}{\partial x}\right)^2 - \left(\dfrac{\partial w}{\partial y}\right)^2$.

Example (5): If $z = f(s^2 - t^2, t^2 - s^2)$ is differentiable at (s, t),
Show that $t\,\dfrac{\partial z}{\partial s} + s\,\dfrac{\partial z}{\partial t} = 0$.
Solution:
Let $z = f(x, y)$, where $x = s^2 - t^2$ and $y = t^2 - s^2$.
$\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} = f_x(2s) + f_y(-2s) = 2sf_x - 2sf_y$.
$\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} = f_x(-2t) + f_y(2t) = -2tf_x + 2tf_y$.
$t\,\dfrac{\partial z}{\partial s} + s\,\dfrac{\partial z}{\partial t} = t\,(2sf_x - 2sf_y) + s\,(-2tf_x + 2tf_y)$
$= 2stf_x - 2stf_y - 2stf_x + 2stf_y = 0$.

Example (6): If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s$, $y = 3rs$. Find $\dfrac{\partial^2 z}{\partial r^2}$.
Solution:
$\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial r} = f_x(2r) + f_y(3s) = 2rf_x + 3sf_y$.

$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}(2rf_x + 3sf_y) = 2f_x + 2r\left(\frac{\partial f_x}{\partial r}\right) + 3s\left(\frac{\partial f_y}{\partial r}\right)$
$= 2f_x + 2r\left(\frac{\partial f_x}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial r}\right) + 3s\left(\frac{\partial f_y}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial r}\right)$
$= 2f_x + 2r\left(f_{xx}(2r) + f_{xy}(3s)\right) + 3s\left(f_{yx}(2r) + f_{yy}(3s)\right)$
$= 2f_x + 4r^2 f_{xx} + 6rsf_{xy} + 6rsf_{yx} + 9s^2 f_{yy}$.

1.4.4 Implicit Differentiation

Let $F(x,y) = 0$ where $y = f(x)$, using chain rule $\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$
$\implies F_x(1) + F_y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{F_x}{F_y}$.

Example (7): If $x^2 + y^2 = 5xy$, Find $\frac{dy}{dx}$.
Solution:
$x^2 + y^2 = 5xy \implies x^2 + y^2 - 5xy = 0$, Let $F(x,y) = x^2 + y^2 - 5xy$ then $F(x,y) = 0$.
$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x - 5y}{2y - 5x}$.

Let $F(x,y,z) = 0$ where $z = f(x,y)$, then $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$.

Example (8): If $x^3 + y^3 + z^2 + 3xyz - 1 = 0$, Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
Solution:
Let $F(x,y,z) = x^3 + y^3 + z^2 + 3xyz - 1$ then $F(x,y,z) = 0$.
$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 3yz}{2z + 3xy}$.
$\frac{\partial z}{\partial x} = -\frac{F_y}{F_z} = -\frac{3y^2 + 3xz}{2z + 3xy}$.

1.4.5 EXERCISES

1. Find $\dfrac{dw}{dt}$ of the following:

   (a). $w = xy^3 - x^2y$, where $x = t^2 + 1$ and $y = t^2 - 1$.

   (b). $w = \sin x \cos y$, where $x = \sqrt{t}$ and $y = \dfrac{1}{t}$.

   (c). $w = xe^{\frac{y}{z}}$, where $x = t^2$, $y = 1 - t$ and $z = 1 + 2t$.

   (d). $w = \ln \sqrt{x^2 + y^2 + z^2}$, where $x = \sin t$, $y = \cos t$ and $z = \tan t$.

2. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ of the following:

   (a). $z = (x - y)^5$, where $x = s^2 t$ and $y = st^2$.

   (b). $z = \tan^{-1}\left(x^2 + y^2\right)$, where $x = s\ \ln t$ and $y = t\ e^s$.

   (c). $z = \dfrac{\sin\theta}{r}$, where $r = st$ and $\theta = s^2 + t^2$.

3. If $z = x^4 + x^2y$, where $x = s + 2t - u$ and $y = stu^2$,

   Find $\dfrac{\partial z}{\partial s}, \dfrac{\partial z}{\partial t}$ and $\dfrac{\partial z}{\partial u}$ at $s = 4$, $t = 2$, $u = 1$.

4. If $z = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, Show that

   $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

5. If $z = f(x + at) + g(x - at)$, Show that $\dfrac{\partial^2 z}{\partial t^2} = a^2\ \dfrac{\partial^2 z}{\partial x^2}$.

(*) If $xe^{yz} - 2ye^{xz} + 3ze^{xy} = 1$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

1.5 Maximum and Minimum Values

1.5.1 Local Maximum and Minimum Values

Definition: If f is a differentiable function of two variables x and y, then the gradient of f is the vector function ∇f and it is defined by
$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j}$.

Definition: If f is a differentiable function of three variables x,y and z, then the gradient of f is the vector function ∇f and it is defined by
$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j} + \dfrac{\partial f}{\partial z}\vec{k}$.

Definition (Critical point): If $f(x,y)$ is a function of two variables then $(a,b) \in D_f$ is a critical point if both $f_x(a,b) = 0$ and $f_y(a,b) = 0$, or f is not differentiable at (a,b).

Definition (Local maximum and Local minimum):
(i) A function f of two variables has a local maximum at $(a,b) \in D_f$ if
$f(x,y) \le f(a,b)$, for all points (x,y) in some disk with center (a,b).
(ii) A function f of two variables has a local minimum at $(a,b) \in D_f$ if
$f(x,y) \ge f(a,b)$, for all points (x,y) in some disk with center (a,b).

Theorem: If $f(x,y)$ has a local maximum or minimum at $(a,b) \in D_f$ and the first-order partial derivatives of f exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Second Derivatives Test: Suppose the second partial derivatives of $f(x,y)$ are continuous on a disk with center $(a,b) \in D_f$, and suppose that
$f_x(a,b) = 0$ and $f_y(a,b) = 0$ [so (a,b) is a critical point of f].
Let $D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$.
(1). If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
(2). If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
(3). If $D < 0$, then (a,b) is a saddle point.

NOTES:
(1). If $D = 0$, then the test gives no information.
(2). $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}f_{yx} = f_{xx}f_{yy} - [f_{xy}]^2$.

Example (1): Find the local maximum and minimum values and saddle points of $f(x,y) = 6xy - 2x^3 + y^2$.
Solution:
$f_x(x,y) = 6y - 6x^2$ and $f_y(x,y) = 6x + 2y$.
$f_x(x,y) = 0 \implies 6y - 6x^2 = 0 \implies y = x^2$.
$f_y(x,y) = 0 \implies 6x + 2y = 0 \implies y = -3x$.
$f_x(x,y) = f_y(x,y) \implies x^2 = -3x \implies x^2 + 3x = 0$
$\implies x(x+3) = 0 \implies x = 0$, $x = -3 \implies y = 0$, $y = 9$.
So, the critical points are $(0,0)$ and $(-3,9)$.

$f_{xx}(x,y) = -12x$, $f_{xy}(x,y) = 6$ and $f_{yy}(x,y) = 2$.

First- At the point $(0,0)$:
$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - [f_{x,y}(0,0)]^2 = (0)(2) - (6)^2 = -36 < 0.$
Therefore, $(0,0)$ is a saddle point.

Second- At the point $(-3,9)$:
$D(-3,9) = f_{xx}(-3,9)f_{yy}(-3,9) - [f_{x,y}(-3,9)]^2 = (36)(2) - (6)^2 = 36 > 0.$
Since $f_{xx}(-3,9) = 36 > 0$, then f attains a local minimum at $(-3,9)$,
and its value is $f(-3,9) = 6(-3)(9) - 2(-3)^3 + (-9)^2 = -162 + 54 + 81 = -27.$

Example (2): Find the local maximum and minimum values and saddle points of $f(x,y) = x^3 - y^3 - 3x + 3y + 5$.
Solution:
$f_x(x,y) = 3x^2 - 3$ and $f_y(x,y) = -3y^2 + 3$.
$f_x(x,y) = 0 \implies 3x^2 - 3 = 0 \implies x^2 - 1 = 0 \implies x = \pm 1$.
$f_y(x,y) = 0 \implies -3y^2 + 3 = 0 \implies y^2 - 1 = 0 \implies y = \pm 1$.
So, the critical points are $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$.

$f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = 0$ and $f_{yy}(x,y) = -6y$.

First- At the point $(1,1)$:
$D(1,1) = f_{xx}(1,1)f_{yy}(1,1) - [f_{x,y}(1,1)]^2 = (6)(-6) - (0)^2 = -36 < 0.$
Therefore, $(1,1)$ is a saddle point.

Second- At the point $(1,-1)$:
$D(1,-1) = f_{xx}(1,-1)f_{yy}(1,-1) - [f_{x,y}(1,-1)]^2 = (6)(6) - (0)^2 = 36 > 0.$
Since $f_{xx}(1,-1) = 6 > 0$, then f attains a local minimum at $(1,-1)$,
and its value is $f(1,-1) = 1 + 1 - 3 - 3 + 5 = 1.$

Third- At the point $(-1,1)$:
$D(-1,1) = f_{xx}(-1,1)f_{yy}(-1,1) - [f_{x,y}(-1,1)]^2 = (-6)(-6) - (0)^2 = 36 > 0.$
Since $f_{xx}(-1,1) = -6 < 0$, then f attains a local maximum at $(-1,1)$,
and its value is $f(-1,1) = -1 - 1 + 3 + 3 + 5 = 9.$

Fourth- At the point $(-1,-1)$:
$D(-1,-1) = f_{xx}(-1,-1)f_{yy}(-1,-1) - [f_{x,y}(-1,-1)]^2$
$= (-6)(6) - (0)^2 = -36 < 0.$
Therefore, $(-1,-1)$ is a saddle point.

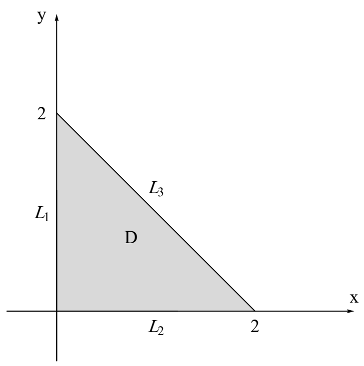
1.5.2 Absolute Maximum and Minimum Values

Definition: Let (a,b) be a point in the domain D of a function f of two variables. Then $f(a,b)$ is the
(1). absolute maximum value of f on D if $f(a,b) \geq f(x,y)$ for all (x,y) in D.
(2). absolute minimum value of f on D if $f(a,b) \leq f(x,y)$ for all (x,y) in D.

Theorem: If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

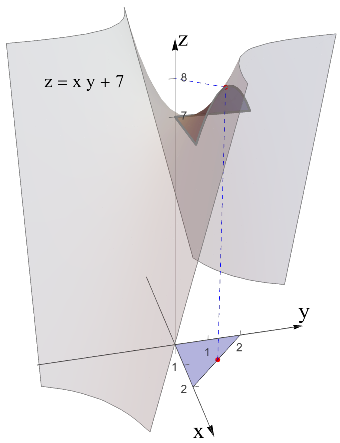
NOTES: To find the absolute maximum and minimum values of a continuous function $f(x, y)$ on a closed, bounded set $D \subseteq \mathbb{R}^2$:
1. Find the values of f at the critical points of f in the interior of D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps (1) and (2) is the absolute maximum value of f on D, and the smallest of these values is the absolute minimum value of f on D.

Example (3): Find the absolute maximum and minimum values of the function $f(x, y) = xy + 7$ on the plane region bounded by the graphs of the lines $x = 0$, $y = 0$ and $y + x = 2$.
Solution:



$f_x(x, y) = y$ and $f_y(x, y) = x$.
$f_x(x, y) = 0 \implies y = 0$.
$f_y(x, y) = 0 \implies x = 0$.
The critical point is $(0, 0)$.
Note that the critical point is not inside the given region.

Let L_1 be the line $x = 0$:
then $f(x, y) = f(0, y) = 7$.
$f(x, y) = 7$ for all $(x, y) \in L_1$.



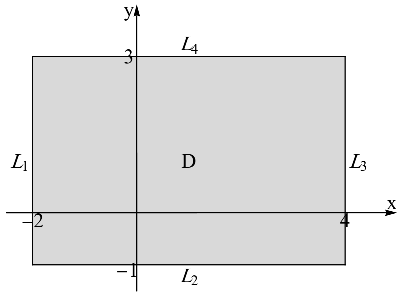
Let L_2 be the line $y = 0$:
then $f(x, y) = f(x, 0) = 7$.
$f(x, y) = 7$ for all $(x, y) \in L_2$.

Let L_3 be the line $y + x = 2$:
then $y = -x + 2$ where $0 \le x \le 2$.
$f(x, y) = f(x, -x + 2)$
$= x(-x + 2) + 7 = -x^2 + 2x + 7$.
$f'(x) = -2x + 2$,
$f'(x) = 0 \implies -2x + 2 = 0$
$\implies x = 1$.
So, $y = -2 + 1 = 1$.
$f(1, 1) = (1)(1) + 7 = 8$.
Also, $f(0, 2) = 7$ and $f(2, 0) = 7$.

The absolute maximum is 8, and f takes it at $(1, 1)$.
The absolute minimum is 7, and f takes it at any point on $L_1 \cup L_2$.

Example (4): Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + 2xy + 3y^2$ on the closed and bounded region
$D = \{(x, y) \in \mathbb{R}^2 : -2 \le x \le 4\ ,\ -1 \le y \le 3\}$.

Solution:



$f_x(x, y) = 2x + 2y$.
$f_y(x, y) = 2x + 6y$.
$f_x(x, y) = 0 \implies x = -y$.
$f_y(x, y) = 0 \implies x = -3y$.
$-y = -3y \implies y = 0 \implies x = 0$.
The critical point is $(0, 0)$.
$f(0, 0) = (0)^2 + 2(0)(0) + 3(0)^3 = 0$.

Let L_1 be the line between $(-2, -1)$ and $(-2, 3)$.
On L_1 : $x = -2$ and $-1 \leq y \leq 3$.
$f(x, y) = f(-2, y) = 4 - 4y + 3y^2 \implies f'(y) = -4 + 6y$.
$f'(y) = 0 \implies 6y = 4 \implies y = \frac{2}{3}$.
Note that $\left(-2, \frac{2}{3}\right) \in L_1$.
$f\left(-2, \frac{2}{3}\right) = (-2)^2 + 2(-2)\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)^2 = 4 - \frac{8}{3} + \frac{4}{3} = \frac{8}{3}$.

Let L_2 be the line between $(-2, -1)$ and $(4, -1)$.
On L_2 : $y = -1$ and $-2 \leq x \leq 4$.
$f(x, y) = f(x, -1) = x^2 - 2x + 3 \implies f'(x) = 2x - 2$.
$f'(x) = 0 \implies 2x = 2 \implies x = 1$.
Note that $(1, -1) \in L_2$.
$f(1, -1) = (1)^2 + 2(1)(-1) + 3(-1)^2 = 1 - 2 + 3 = 2$.

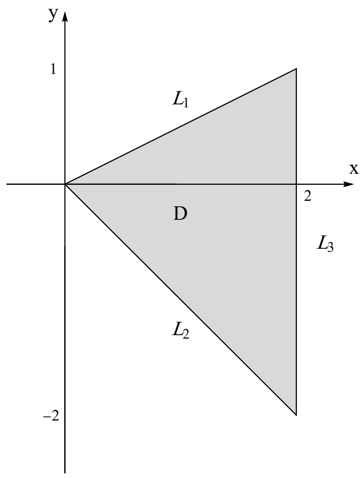
Let L_3 be the line between $(4, -1)$ and $(4, 3)$.
On L_3 : $x = 4$ and $-1 \leq y \leq 3$.
$f(x, y) = f(4, y) = 16 + 8y + 3y^2 \implies f'(y) = 8 + 6y$.
$f'(y) = 0 \implies 6y = -8 \implies y = -\frac{4}{3}$.
Note that $\left(4, -\frac{4}{3}\right) \notin L_3$.

Let L_4 be the line between $(-2, 3)$ and $(4, 3)$.
On L_4 : $y = 3$ and $-2 \leq x \leq 4$.
$f(x, y) = f(x, 3) = x^2 + 6x + 27 \implies f'(x) = 2x + 6$.
$f'(x) = 0 \implies 2x = 2 \implies x = -3$.
Note that $(-3,) \notin L_4$.

Evaluating $f(x, y)$ at the four corners of D :
$f(-2, -1) = (-2)^2 + 2(-2)(-1) + 3(-1)^2 = 4 + 4 + 3 = 11$.
$f(-2, 3) = (-2)^2 + 2(-2)(3) + 3(3)^2 = 4 - 12 + 27 = 19$.
$f(4, -1) = (4)^2 + 2(4)(-1) + 3(-1)^2 = 16 - 8 + 3 = 11$.
$f(4, 3) = (4)^2 + 2(4)(3) + 3(3)^2 = 16 + 24 + 27 = 67$.

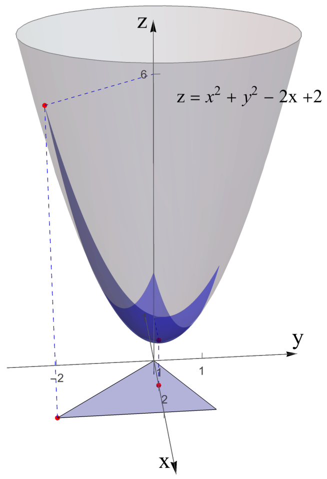
The absolute maximum is 67, and f takes it at $(4, 3)$.
The absolute minimum is 0, and f takes it at $(0, 0)$.

Example (5): Find the absolute maximum and minimum values of the function $f(x,y) = x^2 + y^2 - 2x + 2$ on the closed region with vertices $(0,0)$, $(2,1)$ and $(2,-2)$.
Solution:



$f_x(x,y) = 2x - 2$ and $f_y(x,y) = 2y$.
$f_x(x,y) = 0 \implies x = 1$.
$f_y(x,y) = 0 \implies y = 0$.
The critical point is $(1,0)$.
$f(1,0) = 1 + 0 - 2 + 2 = 1$.

Let L_1 be the line passing through $(0,0)$ and $(2,1)$ then $y = \frac{x}{2}$.
$f\left(x, \frac{x}{2}\right) = x^2 + \frac{x^2}{4} - 2x + 2$.
$f(x) = \frac{5}{4}x^2 - 2x + 2$.
$f'(x) = \frac{5}{2}x - 2$.
$f'(x) = 0 \implies x = \frac{4}{5}$ and $y = \frac{2}{5}$.
$f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{16}{25} + \frac{4}{25} - \frac{8}{5} + 2$
$= \frac{16}{25} + \frac{4}{25} - \frac{40}{25} + \frac{50}{25} = \frac{30}{25} = \frac{6}{5}$.



Let L_2 be the line passing through $(0,0)$ and $(2,-2)$ then $y = -x$.
$f(x,-x) = x^2 + x^2 - 2x + 2$
$f(x) = 2x^2 - 2x + 2$.
$f'(x) = 4x - 2$.
$f'(x) = 0 \implies x = \frac{1}{2}$ and $y = -\frac{1}{2}$.
$f\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} + 1 + 2 = \frac{7}{2}$.

Let L_3 be the line passing through $(2,-2)$ and $(2,1)$ then $x = 2$.
$f(2,y) = 4 + y^2 - 4 + 2 = y^2 + 2$, so $f(y) = y^2 + 2 \implies f'(y) = 2y$.
$f'(y) = 0 \implies y = 0$ and $x = 2$.
$f(2,0) = 4 + 0 - 4 + 2 = 2$.

Evaluating $f(x,y)$ at the three corners of D :
$f(0,0) = 0 + 0 - 0 + 2 = 2$.
$f(2,1) = 4 + 1 - 4 + 2 = 3$.
$f(2,-2) = 4 + 4 - 4 + 2 = 6$.

The absolute maximum is 6, and f takes it at $(2,-2)$.
The absolute minimum is 1, and f takes it at $(1,0)$.

1.5.3 EXERCISES

1. Find the local maximum and minimum values and saddle point(s) of the functions:

   (a). $f(x, y) = x^2 + xy + y^2 + y$.

   (b). $f(x, y) = x^3 + y^3 + 3xy$.

   (c). $f(x, y) = 2 - x^4 + 2x^2 - y^2$.

   (d). $f(x, y) = x^4 - 2x^2 + y^3 - 3y$.

2. Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 - 2x$ on the closed region with vertices $(2, 0)$, $(0, 2)$ and $(0, -2)$.

(*) Find the local maximum and minimum values and saddle point(s) of the functions:

   (a). $f(x, y) = 4x^3 - 2x^2y + y^2$.

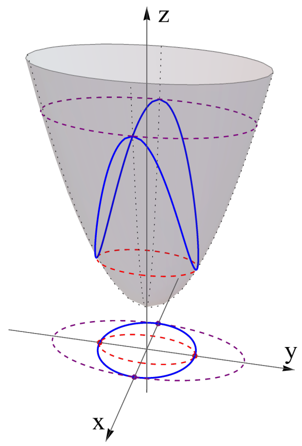
   (b). $f(x, y) = 2x^3 - 3x^2 + 3y^2 - 6xy + 2$.

   (c). $f(x, y) = x^4 + y^3 + 32x - 3y$.

1.6 Lagrange Multipliers

1.6.1 Lagrange Multipliers (One Constraint)

To find the extreme values of $f(x, y)$ subject to a constraint $g(x, y) = k$.
Let $z = f(x, y)$ be the gray surface, and $g(x, y)$ is the blue curve representing the constraint in the xy-plane.
Note that the level curves $f(x, y)$ touches $g(x, y)$ at the points where $f(x, y)$ have minimum and maximum values.
This means that $\nabla f(x, y)$ is parallel to $\nabla g(x, y)$ at these point.
Find these points by solving the equation: $\nabla f(x, y) = \lambda \nabla g(x, y)$,
where $\lambda \in \mathbb{R}$ and $\nabla g(x, y) \neq 0$.
Evaluate $f(x, y)$ at these points, The largest is the maximum value of f and the smallest is the minimum value of f.



Example (1): Find the extreme values of the function $f(x, y) = 1 + xy$ on the circle $x^2 + y^2 = 1$.
Solution:
$f(x, y) = 1 + xy$ and $g(x, y) = x^2 + y^2 - 1$.
$\nabla f(x, y) = \lambda \nabla g(x, y) \implies \langle y, x\rangle = \lambda\langle 2x, 2y\rangle = \langle 2\lambda x, 2\lambda y\rangle$

$$\implies \begin{cases} y = 2\lambda x \\ x = 2\lambda y \end{cases} \implies x = 2\lambda(2\lambda x) = 4\lambda^2 x \implies x\left(1 - 4\lambda^2\right) = 0$$

$$\implies x = 0\ ,\ \lambda = \pm\frac{1}{2}\ .$$

If $x = 0$ then $y = 2\lambda(0) = 0$, but $(0, 0)$ does not lie on the unit circle, so $(0, 0)$ is excluded.

If $\lambda = \pm\frac{1}{2} \implies y = 2\left(\pm\frac{1}{2}\right)x \implies y = \pm x.$

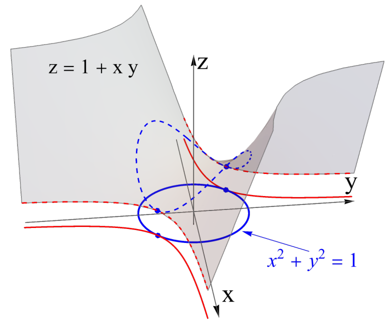
From the constraint : $x^2 + y^2 = 1 \implies 2x^2 = 1 \implies x = \pm\frac{1}{\sqrt{2}}.$

This means $\frac{1}{2} + y^2 = 1 \implies y = \pm\frac{1}{\sqrt{2}}.$

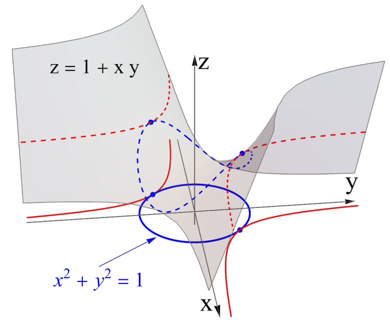
So, There are 4 points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$,$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 1 + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 1 + \frac{1}{2} = \frac{3}{2}.$$
$$f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = 1 + \frac{1}{\sqrt{2}}\frac{-1}{\sqrt{2}} = 1 - \frac{1}{2} = \frac{1}{2}.$$
$$f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 1 + \frac{-1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 1 - \frac{1}{2} = \frac{1}{2}.$$
$$f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = 1 + \frac{-1}{\sqrt{2}}\frac{-1}{\sqrt{2}} = 1 + \frac{1}{2} = \frac{3}{2}.$$

The maximum value is $\frac{3}{2}$, and f takes it at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.
The minimum value is $\frac{1}{2}$, and f takes it at $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.



Minimum value of $z = 1 + xy$ on $x^2 + y^2 = 1$



Maximum value of $z = 1 + xy$ on $x^2 + y^2 = 1$

Note: to find the extreme values of $f(x, y, z)$ subject to a constraint $g(x, y, z) = k$, where $k \in \mathbb{R}$.
(1). Find the points that satisfy the equation $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$
$f_x(x, y, z) = \lambda g_x(x, y, z)$, $f_y(x, y, z) = \lambda g_y(x, y, z)$ and $f_z(x, y, z) = \lambda g_z(x, y, z)$.
(2). Evaluate $f(x, y, z)$ at these points, The largest is the maximum value of f and the smallest is the minimum value of f.

Example (2): Find the points on the sphere $x^2 + y^2 + z^2 = 1$ that are closest to and farthest from the point $(2, 2, 2)$.
Solution:
Let $f(x, y, z)$ be the function of the square of the distance between any point in the sphere and the point $(2, 2, 2)$, then $f(x, y, z) = (x-2)^2 + (y-2)^2 + (z-2)^2$.
Let $g(x, y, z) = x^2 + z^2 + y^2 - 1 = 0$ be the constraint.

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \implies \langle 2(x-2), 2(y-2), 2(z-2) \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

$$\begin{cases} 2(x-2) = 2\lambda x \\ 2(y-2) = 2\lambda y \\ 2(z-2) = 2\lambda z \end{cases} \implies \begin{cases} x - 2 = \lambda x \\ y - 2 = \lambda y \\ z - 2 = \lambda z \end{cases} \implies \begin{cases} x(1-\lambda) = 2 \\ y(1-\lambda) = 2 \\ z(1-\lambda) = 2 \end{cases}$$

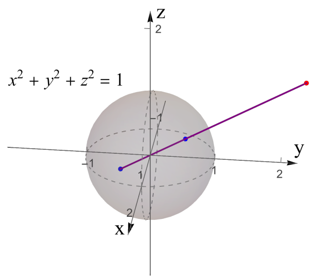
If $\lambda = 1$ then $x - 2 = x \implies -2 = 0$, so $\lambda \neq 1$.
Therefore, $x(1-\lambda) = y(1-\lambda) = z(1-\lambda) \implies x = y = z$.
So, $x^2 + y^2 + z^2 = 1 \implies 3x^2 = 1 \implies x = \pm\frac{1}{\sqrt{3}}$
Hence, $x = y = z = \pm\frac{1}{\sqrt{3}}$.
The required points are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$.

$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}} - 2\right)^2 + \left(\frac{1}{\sqrt{3}} - 2\right)^2 + \left(\frac{1}{\sqrt{3}} - 2\right)^2 = 3\left(2 - \frac{1}{\sqrt{3}}\right)^2.$$

$$f\left(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)=\left(\frac{-1}{\sqrt{3}}-2\right)^2+\left(\frac{-1}{\sqrt{3}}-2\right)^2+\left(\frac{-1}{\sqrt{3}}-2\right)^2=3\left(2+\frac{1}{\sqrt{3}}\right)^2.$$

The point $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ is the closest to $(2,2,2)$.

The point $\left(\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)$ is the farthest to $(2,2,2)$.



Example (3): Find the maximum volume of a rectangular box without a lid, where its surface area equals 12 cm^2.
Solution:
Suppose the sides of the rectangular box are x, y and z.
The volume of the rectangular box is $V(x,y,z)=xyz$, subject to the constraint $2xz+2yz+yz=12$.
The constraint is $g(x,y,z)=2xz+2yz+xy-12=0$.
$\nabla V(x,y,z)=\lambda\nabla g(x,y,z) \implies \langle yz,xz,xy\rangle=\lambda\langle 2z+y,2z+x+2x+2z\rangle$

$$\begin{cases} yz=\lambda(2z+y) \\ xz=\lambda(2z+x) \\ xy=\lambda(2x+2y)\end{cases} \implies \begin{cases} xyz=\lambda(2xz+xy) & \longrightarrow (1) \\ xyz=\lambda(2yz+xy) & \longrightarrow (2) \\ xyz=\lambda(2xz+2yz) & \longrightarrow (3)\end{cases}$$

If $\lambda=0$ then $V(x,y,z)=0$.
If $x=0$, $y=0$ or $z=0$, then $V(x,y,z)=0$.

If $\lambda\neq 0$, From equations (1) and (2) :
$\lambda(2xz+xy)=\lambda(2yz+xy) \implies 2xz+xy=2yz+xy \implies 2xz=2yz \implies x=y.$

From equations (2) and (3), and $x=y$:
$\lambda(2yz+xy)=\lambda(2xz+2yz) \implies 2xz+x^2=2xz+2xz$
$\implies x^2=2xz \implies x^2-2xz=0 \implies x(x-2z)=0 \implies x=2z.$

From the equation of the constraint and $x=y=2z$:
$4z^2+4z^2+4z^2=12 \implies z^2=1 \implies z=1$ and $x=y=2z=2$.

The sides of the rectangular box are 2, 2 and 1, and its maximum volume is 4 cm^3.

1.6.2 EXERCISES

1. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint:

   (a). $f(x,y) = x^2 - y^2$, $x^2 + y^2 = 1$.

   (b). $f(x,y) = xye^{-x^2-y^2}$, $2x - y = 0$.

   (c). $f(x,y,z) = xy^2z$, $x^2 + y^2 + z^2 = 4$.

   (d). $f(x,y,z) = x^4 + y^4 + z^4$, $x^2 + y^2 + z^2 = 1$.

2. Find the extreme values of f on the region described by the inequality:

   (a). $f(x,y) = x^2 + y^2 + 4x - 4y$, $x^2 + y^2 \leq 9$.

   (b). $f(x,y) = e^{-xy}$, $x^2 + 4y^2 \leq 1$.

3. Show that the problem of finding the minimum value of f subject to the given constraint can be solved using Lagrange multipliers, but f does not have a maximum value with that constraint:

   (a). $f(x,y) = x^2 + y^2$, $xy = 1$.

   (b). $f(x,y,z) = x^2 + 2y^2 + 3z^2$, $x + 2y + 3z = 10$.

Chapter 2

Multiple Integrals

2.1 Double Integrals over Rectangles

2.1.1 Iterated Integrals

Suppose that $f(x, y)$ is integrable on the rectangle $R = [a, b] \times [c, d]$, then
$\iint_R f(x, y)\ dA = \int_a^b \int_c^d f(x, y)\ dydx = \int_a^b \left(\int_c^d f(x, y)\ dy \right)\ dx.$

Example (1): Evaluate the integral:
(a). $\int_0^2 \int_1^2 x^2 y\ dydx$, (b). $\int_1^2 \int_0^2 x^2 y\ dxdy.$
Solution:
(a). $\int_0^2 \int_1^2 x^2 y\ dydx = \int_0^2 \left(\int_1^2 x^2 y\ dy \right)\ dx = \int_0^2 \left(x^2 \int_1^2 y\ dy \right)\ dx$
$= \int_0^2 x^2\ \left[\frac{y^2}{2} \right]_1^2\ dx = \int_0^2 x^2\ \left[\frac{4}{2} - \frac{1}{2} \right]\ dx$
$= \frac{3}{2} \int_0^2 x^2\ dx = \frac{3}{2}\ \left[\frac{x^3}{3} \right]_0^2 = \frac{3}{2} \left[\frac{8}{3} - 0 \right] = 4.$
(b). $\int_1^2 \int_0^2 x^2 y\ dxdy = \int_1^2 \left(\int_0^2 x^2 y\ dx \right) dy = \int_1^2 \left(y \int_0^2 x^2\ dx \right) dy.$
$= \int_1^2 y\ \left[\frac{x^3}{3} \right]_0^2 dy = \int_1^2 y\ \left[\frac{8}{3} - 0 \right]_0^2 dy = \frac{8}{3} \int_1^2 y\ dy$
$= \frac{8}{3} \left[\frac{y^2}{2} \right]_1 2 = \frac{8}{3} \left[\frac{4}{2} - \frac{1}{2} \right] = \frac{8}{3}\ \frac{3}{2} = 4.$

Fubini's Theorem: If f is continuous on the rectangle $R = \{(x, y) \mid a \leq x \leq b\ ,\ c \leq y \leq d\}$,
Then $\iint_R f(x, y)\ dA = \int_a^b \int_c^d f(x, y)\ dydx = \int_c^d \int_a^b f(x, y)\ dxdy.$

Example (2): Evaluate the integral $\iint_R y \cos(xy)\ dA$,
where $R = [0, 1] \times \left[0, \frac{\pi}{2} \right].$

Solution: Using Fubini's Theorem.
$\iint_R y\cos(xy)\ dA = \int_0^{\frac{\pi}{2}} \int_0^1 y\cos(xy)\ dxdy = \int_0^{\frac{\pi}{2}} \left(\int_0^1 y\cos(xy)\ dx \right)\ dy$
$= \int_0^{\frac{\pi}{2}} [\sin(xy)]_0^1\ dy = \int_0^{\frac{\pi}{2}} [\sin y - \sin(0)]\ dy = \int_0^{\frac{\pi}{2}} \sin y\ dy$
$= [-\cos y]_0^{\frac{\pi}{2}} = -\cos\left(\frac{\pi}{2}\right) - (-\cos(0)) = 0 + 1 = 1$.
Note : Solving $\iint_R y\cos(x,y)\ dA = \int_0^1 \int_0^{\frac{\pi}{2}} y\cos(xy)\ dydx$ is hard, it needs integration by parts.

2.1.2 Volume

If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle $R = [a,b] \times [c,d]$ and below the surface $z = f(x,y)$ is
$\text{V} = \iint_R f(x,y)\ dA = \int_a^b \int_c^d f(x,y)\ dydx = \int_c^d \int_a^b f(x,y)\ dxdy$.

Example (3): Find the volume of the solid S that is bounded by $z = x^2+y^2+1$, the planes $x = 1$ and $y = 3$, and the three coordinate planes.
Solution:
Note that S is the solid that lies under the surface $z = x^2 + y^2 + 1$ and above the square $R = [0,1] \times [0,3]$.
$\text{V} = \iint_R (x^2 + y^2 + 1)\ dA = \int_0^1 \int_0^3 (x^2 + y^2 + 1)\ dydx$
$= \int_0^1 \left[x^2 y + \frac{y^3}{3} + y \right]_0^3\ dx = \int_0^1 \left[(3x^2 + 9 + 3) - (0 + 0 + 0) \right]_0^3\ dx$
$= \int_0^1 (3x^2 + 12)\ dx = \left[x^3 + 12x \right]_0^1 = (1 + 12) - (0 + 0) = 13$.

Corollary: If $f(x,y) = g(x)h(y)$ is continuous on the rectangle $R = [a,b]\times[c,d]$, then $\iint_R f(x,y)\ dA = \iint_R g(x)h(y)\ dA = \int_a^b g(x)\ dx\ \int_c^d h(y)\ dy$.

Example (4): Evaluate $\iint_R x\cos y\ dA$, where $R = [0,2] \times \left[0, \frac{\pi}{2}\right]$.
Solution:
$\iint_R x\cos y\ dA = \int_0^2 \int_0^{\frac{\pi}{2}} x\cos y\ dydx = \left(\int_0^2 x\ dx \right) \left(\int_0^{\frac{\pi}{2}} \cos y\ dy \right)$
$= \left[\frac{x^2}{2} \right]_0^2 [\sin y]_0^{\frac{\pi}{2}} = [2-0][1-0] = 2$.

2.1.3 Average Value

If $f(x, y)$ is defined on a rectangle R then its average value is $f_{avg} = \dfrac{1}{A(R)} \displaystyle\iint_R f(x, y)\ dA$, where $A(R)$ is the area of the rectangle R.

Example (5): Evaluate f_{avg} of $f(x, y) = x \cos y$ on $R = [0, 2] \times \left[0, \dfrac{\pi}{2}\right]$.
Solution:
$f_{avg} = \dfrac{1}{A(R)} \displaystyle\iint_R x \cos y\ dA = \dfrac{2}{(2-0)\left(\frac{\pi}{2}-0\right)} = \dfrac{2}{\pi}$.

2.1.4 EXERCISES

1. Calculate the iterated integrals :

   (a). $\displaystyle\int_1^4 \int_0^2 (6x^2y - 2x)\ dy\ dx$.

   (b). $\displaystyle\int_{-3}^1 \int_1^2 (x^2 + y^{-2})\ dy\ dx$.

   (c). $\displaystyle\int_{-3}^3 \int_0^{\frac{\pi}{2}} (y + y^2 \cos x)\ dx\ dy$.

   (d). $\displaystyle\int_1^3 \int_1^5 \frac{\ln y}{xy}\ dy\ dx$.

   (e). $\displaystyle\int_0^1 \int_0^1 xy\sqrt{x^2 + y^2}\ dy\ dx$.

   (f). $\displaystyle\int_0^1 \int_0^1 v(u + v^2)^4\ du\ dv$.

2. Calculate the double integrals :

   (a). $\displaystyle\iint_R x \sec^2 y\ dA$, where $R = \left\{(x, y)|\ 0 \le x \le 2\ ,\ 0 \le y \le \frac{\pi}{4}\right\}$.

   (b). $\displaystyle\iint_R \frac{xy^2}{x^2 + 1}\ dA$, where $R = \{(x, y)|\ 0 \le x \le 1\ ,\ -3 \le y \le 3\}$.

   (c). $\displaystyle\iint_R \frac{1}{1 + x + y}\ dA$, where $R = [1, 3] \times [1, 2]$.

3. Find the volume of the solid that lies under the plane $4x+6y-2z+15 = 0$ and above the rectangle $R = \{(x, y)|\ -1 \le x \le 2\ ,\ -1 \le y \le 1\}$.

4. Find the volume of the solid that lies under $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

2.2 Double Integrals over General Regions

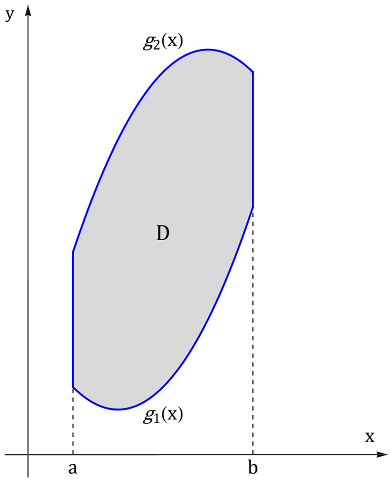
2.2.1 General Regions

First - Regions of Type I:

Let D be the region
$\{(x,y) \mid a \leq x \leq b \ , \ g_1(x) \leq y \leq g_2(x)\}$

If f is continuous on D, then

$$\iint_D f(x,y)\ dA$$
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dydx$$
$$= \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\ dy \right)\ dx$$

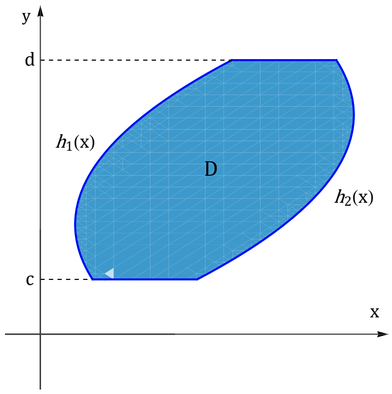


Second - Regions of Type II:

Let D be the region
$\{(x,y) \mid c \leq y \leq d \ , \ h_1(y) \leq x \leq h_2(y)\}$

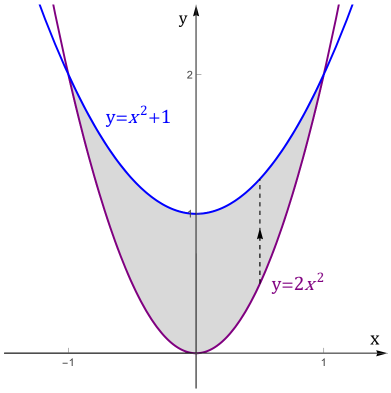
If f is continuous on D, then

$$\iint_D f(x,y)\ dA$$
$$= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dxdy$$
$$= \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\ dx \right)\ dy$$



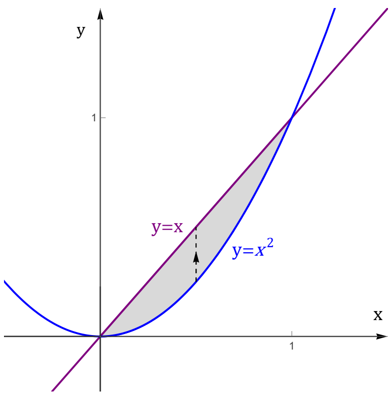
Example (1): Evaluate $\iint_D 2xy\ dA$, where D is the region bounded by the graphs of $y = 2x^2$ and $y = x^2 + 1$.
Solution:

Points of intersection:
$2x^2 = x^2 + 1 \implies x^2 = 1$
$\implies x = \pm 1$
So, D is the region where $-1 \le x \le 1$ and $2x^2 \le y \le x^2 + 1$



$$\iint_D 2xy\; dA = \int_{-1}^{1} \int_{2x^2}^{x^2+1} 2xy\; dy\; dx$$
$$= \int_{-1}^{1} x \left[y^2 \right]_{2x^2}^{x^2+1} \; dx$$
$$= \int_{-1}^{1} x \left[(x^2+1)^2 - (2x^2)^2 \right] \; dx$$
$$= \int_{-1}^{1} x \left(x^4 + 2x^2 + 1 - 4x^4 \right) \; dx$$
$$= \int_{-1}^{1} x \left(-3x^4 + 2x^2 + 1 \right) \; dx = \int_{-1}^{1} \left(-3x^5 + 2x^3 + x \right) dx$$
$$= \left[-\frac{x^6}{2} + \frac{x^4}{2} + \frac{x^2}{2} \right]_{-1}^{1}$$
$$= \left(-\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) - \left(-\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) = 0 \; .$$

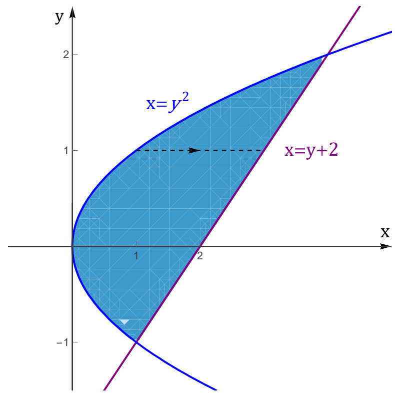
Example (2): Evaluate $\iint_D (x^2 + y^2)\; dA$, where D is the region bounded by the graphs of $y = x^2$ and $y = x$.
Solution:
Points of intersection:
$x^2 = x \implies x^2 - x = 0$
$\implies x(x-1) = 0 \implies x = 0\;,\; x = 1$
So, D is the region where $0 \le x \le 1$ and $x^2 \le y \le x$



$$\iint_D (x^2 + y^2)\; dA$$
$$= \int_0^1 \int_{x^2}^{x} (x^2 + y^2)\; dy\; dx$$
$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_{x^2}^{x} \; dx$$
$$= \int_0^1 \left[\left(x^3 + \frac{x^3}{3} \right) - \left(x^4 - \frac{x^6}{3} \right) \right] \; dx$$
$$= \int_0^1 \left(-\frac{x^6}{3} - x^4 + \frac{4x^3}{3} \right) \; dx = \left[-\frac{x^7}{21} - \frac{x^5}{5} + \frac{x^4}{3} \right]_0^1$$
$$= \left(-\frac{1}{21} - \frac{1}{5} + \frac{1}{3} \right) - (0 + 0 + 0) = \frac{-5 - 21 + 35}{105} = \frac{9}{105} = \frac{3}{35} \; .$$

Example (3): Evaluate $\iint_D xy\; dA$, where D is the region bounded by the graphs of $x = y^2$ and $x = y + 2$.

Solution:
Points of intersection:
$y^2 = y + 2 \implies y^2 - y - 2 = 0$
$\implies (y-2)(y+1) = 0 \implies y = -1\ ,\ y = 2$
So, D is the region where $-1 \leq y \leq 2$ and $y^2 \leq x \leq y + 2$

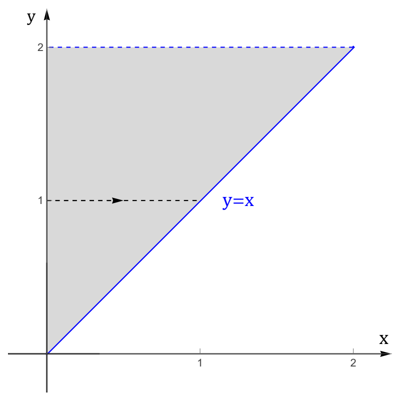


$$\iint_D xy\ dA = \int_{-1}^{2}\int_{y^2}^{y+2} xy\ dx\ dy$$
$$= \int_{-1}^{2} y\left[\frac{x^2}{2}\right]_{y^2}^{y+2}\ dy$$
$$= \frac{1}{2}\int_{-1}^{2} y\left[(y+2)^2 - (y^2)^2\right]\ dy$$
$$= \frac{1}{2}\int_{-1}^{2} y\left(y^2 + 4y + 4 - y^4\right)\ dy$$
$$= \frac{1}{2}\int_{-1}^{2}\left(-y^5 + y^3 + 4y^2 + 4y\right)\ dy = \frac{1}{2}\left[-\frac{y^6}{6} + \frac{y^4}{4} + \frac{4y^3}{3} + 2y^2\right]_{-1}^{2}$$
$$= \frac{1}{2}\left[\left(-\frac{64}{6} + \frac{16}{4} + \frac{32}{3} + 8\right) - \left(-\frac{1}{6} + \frac{1}{4} - \frac{4}{3} + 2\right)\right] = \frac{45}{8}\ .$$

2.2.2 Changing the Order of Integration

Example (4): Change the order of integration to evaluate $\displaystyle\int_0^2\int_x^2 e^{5+y^2}\ dy\ dx$.

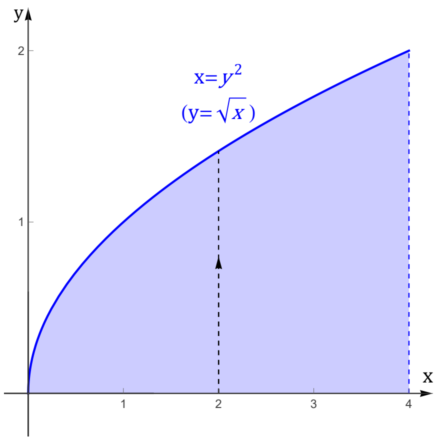
Solution:
D is the region where $0 \leq x \leq 2$ and $x \leq y \leq 2$.
Or, $0 \leq y \leq 2$ and $0 \leq x \leq y$.



$$\int_0^2\int_x^2 e^{5+y^2}\ dy\ dx$$
$$= \int_0^2\int_0^y e^{5+y^2}\ dx\ dy$$
$$= \int_0^2 e^{5+y^2}[x]_0^y\ dy = \int_0^2 ye^{5+y^2}\ dy$$
$$= \frac{1}{2}\int_0^2 e^{5+y^2}(2y)\ dy$$
$$= \frac{1}{2}\left[e^{5+y^2}\right]_0^2 = \frac{1}{2}\left(e^9 - e^5\right).$$

Example (5): Change the order of integration to evaluate $\int_0^2 \int_{y^2}^4 \cos\left(x^{\frac{3}{2}}\right)\ dx\ dy$.

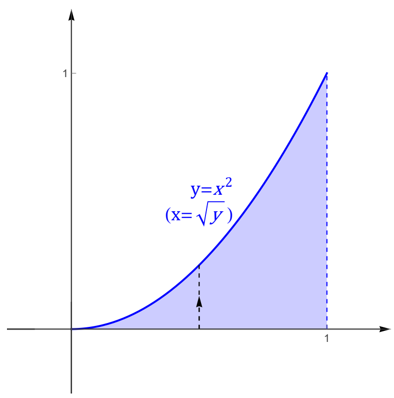
Solution:
D is the region where $0 \le y \le 2$ and $y^2 \le x \le 4$.
Or, $0 \le x \le 4$ and $0 \le y \le \sqrt{x}$.



$$\int_0^2 \int_{y^2}^4 \cos\left(x^{\frac{3}{2}}\right)\ dx\ dy$$
$$= \int_0^4 \int_0^{\sqrt{x}} \cos\left(x^{\frac{3}{2}}\right)\ dy\ dx$$
$$= \int_0^4 \cos\left(x^{\frac{3}{2}}\right)\ [y]_0^{\sqrt{x}}\ dx$$
$$= \frac{2}{3}\int_0^4 \cos\left(x^{\frac{3}{2}}\right)\ \left(\frac{3}{2}x^{\frac{1}{2}}\right)\ dx$$
$$= \frac{2}{3}\left[\sin\left(x^{\frac{3}{2}}\right)\right]_0^4 = \frac{2}{3}\left[\sin(8) - \sin(0)\right] = \frac{2}{3}\sin(8)\ .$$

Example (6): Change the order of integration to evaluate $\int_0^1 \int_{\sqrt{y}}^1 \sin\left(x^3\right)\ dx\ dy$.

Solution:
D is the region where $0 \le y \le 1$ and $\sqrt{y} \le x \le 1$.
Or, $0 \le x \le 1$ and $0 \le y \le x^2$.



$$\int_0^1 \int_{\sqrt{y}}^1 \sin\left(x^3\right)\ dx\ dy$$
$$= \int_0^1 \int_0^{x^2} \sin\left(x^3\right)\ dy\ dx$$
$$= \int_0^1 \sin\left(x^3\right)\ [y]_0^{x^2}\ dx$$
$$= \int_0^1 \sin\left(x^3\right)\ x^2\ dx$$
$$= \frac{1}{3}\int_0^1 \sin\left(x^3\right)\ (3x^2)\ dx = \frac{1}{3}\left[-\cos(x^3)\right]_0^1 .$$
$$= \frac{1}{3}\left[-\cos(1) - (-\cos(0))\right] = \frac{1-\cos(1)}{3}\ .$$

2.2.3 Properties of Double Integrals

If $f(x,y)$ and $g(x,y)$ are both integrable on $D \subseteq \mathbb{R}^2$, then

(1). $\iint_D \left[f(x,y) + g(x,y)\right]\ dA = \iint_D f(x,y)\ dA + \iint_D g(x,y)\ dA$.

(2). $\iint_D k\ f(x,y)\ dA = k\iint_D f(x,y)\ dA$, where $k \in \mathbb{R}$.

(3). If $f(x,y) \ge g(x,y)$ on D, then $\iint_D f(x,y)\ dA \ge \iint_D g(x,y)\ dA$.

(4). If $D = D_1 \cup D_2$, where $D_1 \cap D_2 = \phi$, then
$\iint_D f(x,y)\ dA = \iint_{D_1} f(x,y)\ dA + \iint_{D_2} f(x,y)\ dA.$

(5). $\iint_D 1\ dA = A(D)$, where $A(D)$ is the area of the region D.

(6). If $m \leq f(x,y) \leq M$ on D, then $m\ A(D) \leq \iint_D f(x,y)\ dA \leq M\ A(D)$.

2.2.4 EXERCISES

1. Evaluate the iterated integrals :

   (a). $\displaystyle\int_1^5 \int_0^x (8x - 2y)\ dy\ dx$ (b). $\displaystyle\int_0^2 \int_0^{y^2} x^2 y\ dx\ dy$

   (c). $\displaystyle\int_0^1 \int_0^{e^x} \sqrt{1+e^x}\ dy\ dx$

2. Evaluate $\displaystyle\iint_D 2y\ dA$, where D is the region bounded by the graphs of $y = 3x - x^2$ and $y = x$.

3. Evaluate the double integrals:

   (a). $\displaystyle\iint_D \frac{y}{x^2+1}\ dA$, $D = \left\{(x,y) \mid 0 \le x \le 4\ ,\ 0 \le y \le \sqrt{x}\right\}$.

   (b). $\displaystyle\iint_D (2x+y)\ dA$, $D = \{(x,y) \mid 1 \le y \le 2\ ,\ y-1 \le x \le 1\}$.

   (c). $\displaystyle\iint_D x\ dA$, D is enclosed by the lines $y = x$, $y = 0$ and $x = 1$.

   (d). $\displaystyle\iint_D xy\ dA$, D is enclosed by the curves $y = x^2$ and $y = 3x$.

   (e). $\displaystyle\iint_D x\cos y\ dA$, D is bounded by the $y = 0$, $y = x^2$ and $x = 1$.

   (f). $\displaystyle\iint_D y^2\ dA$, D is the triangular region with vertices $(0,1)$, $(1,2)$ and $(4,1)$.

4. Evaluate the integral by reversing the order of integration:

   (a). $\displaystyle\int_0^1 \int_{3y}^3 e^{x^2}\ dx\ dy$ (b). $\displaystyle\int_0^1 \int_{x^2}^1 \sqrt{y}\ \sin y\ dy\ dx$

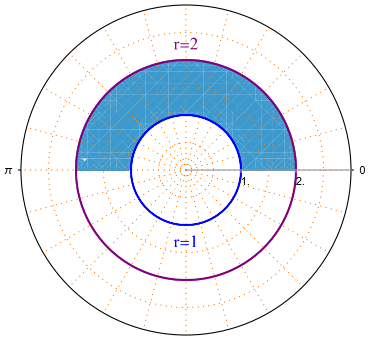
   (c). $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+1}\ dy\ dx$ (d). $\displaystyle\int_0^2 \int_{\frac{y}{2}}^1 y\ \cos(x^3-1)\ dx\ dy$

2.3 Double Integrals in Polar Coordinates

If f is continuous on the polar region R given by $a \le r \le b$ and $\theta_1 \le \theta \le \theta_2$, where $0 \le \theta_2 - \theta_1 \le 2\pi$, then $\iint_R f(x,y)\; dA = \int_{\theta_1}^{\theta_2} \int_a^b f(r\cos\theta, r\sin\theta)\; r\; dr\; d\theta$.

Example (1): Evaluate $\iint_R (4x^2 + 3y)\; dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
Solution:

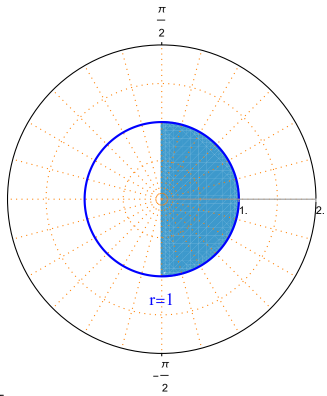
R is the region where $1 \le r \le 2$ and $0 \le \theta \le \pi$.



$$\iint_R (4x^2 + 3y)\; dA$$
$$= \int_0^{\pi} \int_1^2 (4r^2\cos^2\theta + 3r\sin\theta)\; r\; dr\; d\theta$$
$$= \int_0^{\pi} \int_1^2 (4r^3\cos^2\theta + 3r^2\sin\theta)\; dr\; d\theta$$
$$= \int_0^{\pi} \left[r^4\cos^2\theta + r^3\sin\theta\right]_1^2\; d\theta$$
$$= \int_0^{\pi} \left[(16\cos^2\theta + 8\sin\theta) - (\cos^2\theta + \sin\theta)\right]\; d\theta = \int_0^{\pi} (15\cos^2\theta + 7\sin\theta)\; d\theta$$
$$= \int_0^{\pi} \left[15\left(\frac{1+\cos 2\theta}{2}\right) + 7\sin\theta\right]\; d\theta = \frac{15}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi} + 7\left[-\cos\theta\right]_0^{\pi}$$
$$= \frac{15}{2}\left[(\pi + 0) - (0 + 0)\right] + 7[-(-1) - (-1)] = \frac{15\pi}{2} + 14.$$

Example (2): Evaluate $\int_{-1}^{1} \int_0^{\sqrt{1-y^2}} \frac{8(x^2+y^2)}{9 + (x^2+y^2)^2}\; dx\; dy$

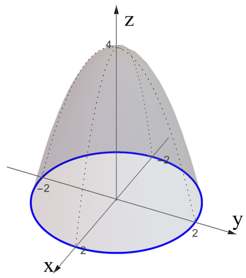
Solution:
R is the region where $-1 \le y \le 1$ and $0 \le x \le \sqrt{1-y^2}$
In polar coordinates. $0 \le r \le 1$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.



$$\int_{-1}^{1} \int_0^{\sqrt{1-y^2}} \frac{8(x^2+y^2)}{9 + (x^2+y^2)^2}\; dx\; dy$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \frac{8r^2}{9 + r^4} r\; dr\; d\theta$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 2\left(\frac{4r^3}{9 + r^4}\right)\; dr\; d\theta$$
$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\ln(9 + r^4)\right]_0^1\; d\theta = 2\left(\ln 10 - \ln 9\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta$$
$$= 2\ln\left(\frac{10}{9}\right)\left[\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\ln\left(\frac{10}{9}\right)\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 2\pi\ln\left(\frac{10}{9}\right).$$

Example (3): Find the volume of the solid bounded by $z = 4 - x^2 - y^2$ and $z = 0$
Solution:

The surface of intersection is :
$4 - x^2 - y^2 = 0 \implies x^2 + y^2 = 4.$
R is the region inside the circle centered at the origin with radius 2.
In polar coordinates, $0 \le r \le 2$ and $0 \le \theta \le 2\pi$.

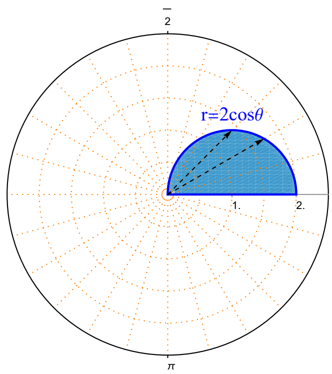


$$V = \int_0^{2\pi} \int_0^2 (4 - r^2)\ r\ dr\ d\theta$$
$$= \int_0^{2\pi} \int_0^2 (4r - r^3)\ dr\ d\theta$$
$$= \int_0^{2\pi} \left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi} [(8-4) - (0-0)]\ d\theta$$
$$= 4 \int_0^{2\pi} d\theta = 4\,[\theta]_0^{2\pi} = 4\,(2\pi - 0) = 8\pi\ .$$

NOTE: If f is continuous on the polar region $R = \{(r,\theta) | \theta_1 \le \theta \le \theta_2\ ,\ g_1(\theta) \le r \le g_2(\theta)\}$, then $\displaystyle\iint_R f(x,y)\ dA = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta.$

Example (4): Evaluate $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)^{\frac{1}{2}}\ dy\ dx.$

Solution:
R is the region where $0 \le x \le 2$ and $0 \le y \le \sqrt{2x - x^2}$.
$y = \sqrt{2x - x^2} \implies y^2 = 2x - x^2$
$\implies x^2 - 2x + y^2 = 0$
$\implies (x^2 - 2x + 1) + y^2 = 1$
$\implies (x-1)^2 + y^2 = 1.$
$y = \sqrt{2x - x^2}$ is the upper-half of the circle centered at $(1, 0)$ with radius 1.



In polar coordinates, $0 \le r \le 2\cos\theta$ and $0 \le \theta \le \dfrac{\pi}{2}$.

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)^{\frac{1}{2}}\ dy\ dx = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r\ r\ dr\ d\theta = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\ dr\ d\theta$$
$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \cos^3\theta\ d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \cos^2\theta\ \cos\theta\ d\theta$$
$$= \frac{8}{3}\int_0^{\frac{\pi}{2}} (1 - \sin^2\theta)\cos\theta\ d\theta = \frac{8}{3}\int_0^{\frac{\pi}{2}} \left(\cos\theta - \sin^2\theta\ \cos\theta\right)\ d\theta$$
$$= \frac{8}{3}\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{\frac{\pi}{2}} = \frac{8}{3}\left[\left(1 - \frac{1}{3}\right) - (0-0)\right] = \frac{16}{9}.$$

2.4 Triple Integrals

2.4.1 Triple Integrals over Rectangular Boxes

Fubini's Theorem:
If f is continuous on the rectangular box $E = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_E f(x,y,z)\ dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\ dx\ dy\ dz = \int_r^s \int_a^b \int_c^d f(x,y,z)\ dy\ dx\ dz$$
$$= \int_c^d \int_r^s \int_a^b f(x,y,z)\ dx\ dz\ dy = \int_c^d \int_a^b \int_r^s f(x,y,z)\ dz\ dx\ dy$$
$$= \int_a^b \int_r^s \int_c^d f(x,y,z)\ dy\ dz\ dx = \int_a^b \int_c^d \int_r^s f(x,y,z)\ dz\ dy\ dx\ .$$

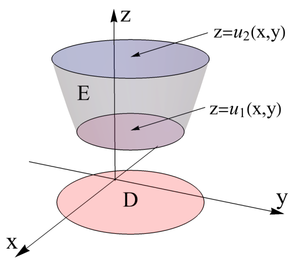
Example (1): Evaluate $\iiint_E xyz^2\ dV$, where $E = [0, 1] \times [0, 2] \times [0, 3]$.

Solution:

$$\iiint_E xyz^2\ dV = \int_0^1 \int_0^2 \int_0^3 xyz^2\ dz\ dy\ dx = \int_0^1 \int_0^2 xy \left[\frac{z^3}{3}\right]_0^3 dy\ dx$$
$$= \left[\frac{27}{3} - \frac{0}{3}\right] \int_0^1 \int_0^2 xy\ dy\ dx = 9 \int_0^1 x \left[\frac{y^2}{2}\right]_0^2 dx = 9 \int_0^1 x \left[\frac{2^2}{2} - \frac{0}{2}\right] dx$$
$$= 9 \int_0^1 2x\ dx = 9 \left[x^2\right]_0^1 = 9[1 - 0] = 9\ .$$

2.4.2 Triple Integrals over General Regions

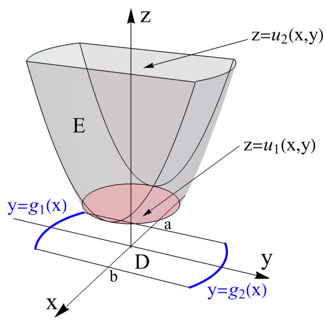
First - Regions of Type I:
Case (1) :
Let E be the region where
$(x, y) \in D$ and
$u_1(x, y) \le z \le u_2(x, y)$.



If f is continuous on E, then

$$\iiint_E f(x,y,z)\ dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz \right) dA.$$

Case (2) :
Let E be the region where
$a \le x \le b$, $g_1(x) \le y \le g_2(x)$
and $u_1(x, y) \le z \le u_2(x, y)$.



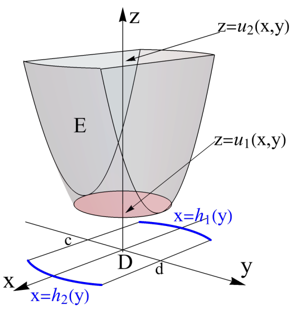
If f is continuous on E, then

$$\iiint_E f(x,y,z)\ dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\ dy\ dx.$$

Case (3) :
Let E be the region where
$c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$
and $u_1(x,y) \leq z \leq u_2(x,y)$.

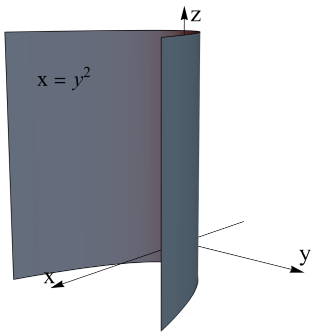
If f is continuous on E, then

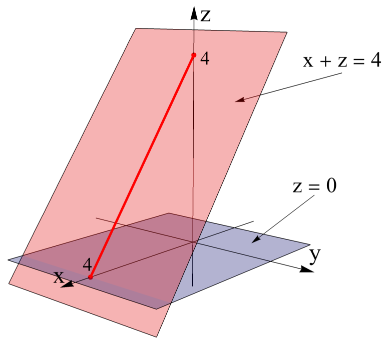
$$\iiint_E f(x,y,z)\ dV$$
$$= \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\ dx\ dy.$$

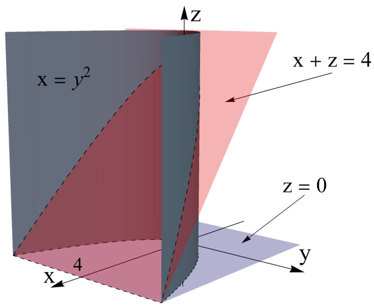


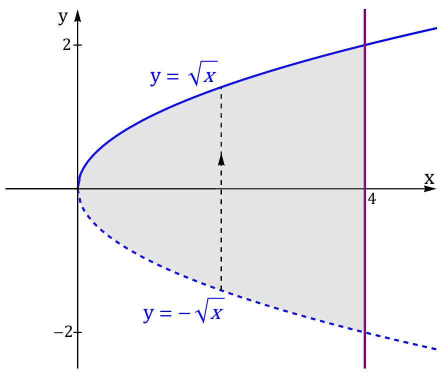
NOTE: $\iiint_E 1\ dV =$ V(E), where V(E) is the volume of the region E.

Example (2): Find the volume of the solid bounded by the cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 4$.
Solution:









Let E be the region bounded by the cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 4$.
Note that $z = 0$ intersects $x + z = 4$ at the line where $x = 4$ and $z = 4$.
On E : $0 \leq x \leq 4$, $-\sqrt{x} \leq y \leq \sqrt{x}$, $0 \leq z \leq 4 - x$.

$$\text{V(E)} = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{4-x} 1\ dz\ dy\ dx = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} [z]_0^{4-x}\ dy\ dx = \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} (4-x)\ dy\ dx$$

$= \int_0^4 (4-x)\ [y]_{-\sqrt{x}}^{\sqrt{x}}\ dx = \int_0^4 2\sqrt{x}\ (4-x)\ dx = 2\int_0^4 \left(4x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)\ dx$

$= 2\left[\frac{8}{3}\ x^{\frac{3}{2}} - \frac{2}{5}\ x^{\frac{5}{2}}\right]_0^4 = 2\left[\frac{8}{3}\ (4)^{\frac{3}{2}} - \frac{2}{5}\ (4)^{\frac{5}{2}}\right] = 2\left[\frac{8}{3}\ (2^3) - \frac{2}{5}(2^5)\right]$

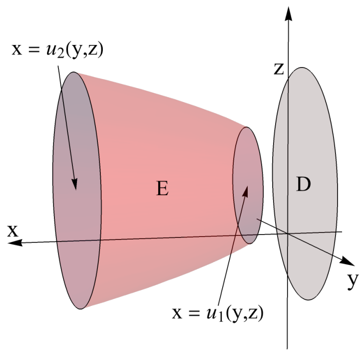
$2\left[\frac{2^6}{3} - \frac{2^6}{5}\right] = 2^7\ \left(\frac{1}{3} - \frac{1}{5}\right) = 2^7\ \left(\frac{2}{15}\right) = \frac{2^8}{15} = \frac{256}{15}.$

Another Solution :
On E : $-2 \le y \le 2\ ,\ y^2 \le x \le 4\ ,\ 0 \le z \le 4 - x.$
V(E) $= \int_{-2}^{2}\int_{y^2}^{4}\int_0^{4-x} 1\ dz\ dx\ dy$.

Second - Regions of Type II:

Let E be the region where $(y,z) \in D$ and $u_1(y,z) \le x \le u_2(y,z)$.

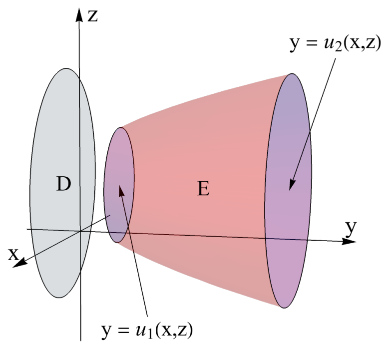
If f is continuous on E, then
$\iiint_E f(x,y,z)\ dV$
$= \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\ dx\right)\ dA.$



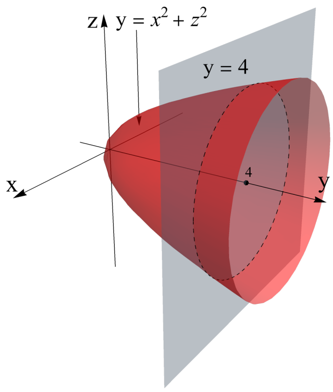
Third - Regions of Type III:

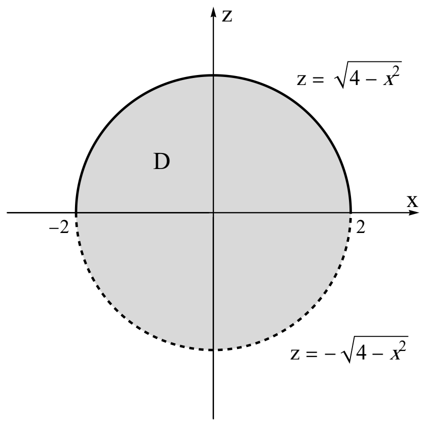
Let E be the region where $(x,z) \in D$ and $u_1(x,z) \le y \le u_2(x,z)$.

If f is continuous on E, then
$\iiint_E f(x,y,z)\ dV$
$= \iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\ dy\right)\ dA.$



Example (3): Evaluate $\iiint_E \sqrt{x^2+z^2}\ dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.
Solution:





Note that $y = x^2 + z^2$ intersects $y = 4$ at $x^z + z^2 = 4$.
So, E is the region where $x^2 + z^2 \leq y \leq 4$, $-\sqrt{4-x^2} \leq z \leq \sqrt{4-x^2}$ and $-2 \leq x \leq 2$.

$$\iiint_E \sqrt{x^2+z^2}\, dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\, dz\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [y]_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dz\, dx = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4-(x^2+z^2)\right] \sqrt{x^2+z^2}\, dz\, dx$$

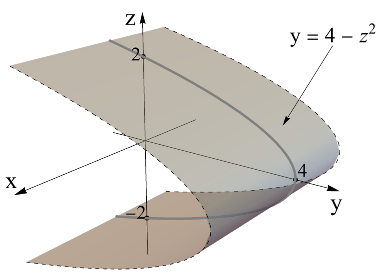
Using the polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$,

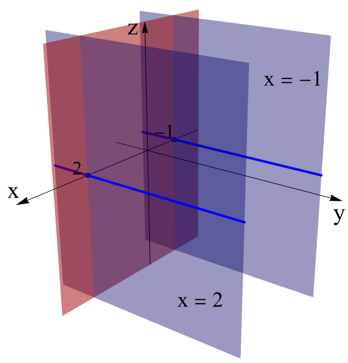
$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4-(x^2+z^2)\right] \sqrt{x^2+z^2}\, dz\, dx = \int_{0}^{2\pi} \int_{0}^{2} (4-r^2)\, r\, r\, dr\, d\theta$$

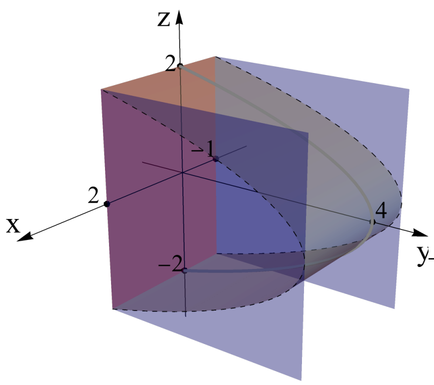
$$= \int_{0}^{2\pi} \int_{0}^{2} (4-r^2)\, r^2\, dr\, d\theta = \int_{0}^{2\pi} \int_{0}^{2} (4r^2 - r^4)\, dr\, d\theta = \int_{0}^{2\pi} \left[4\frac{r^3}{3} - \frac{r^5}{5}\right]_0^2 d\theta$$

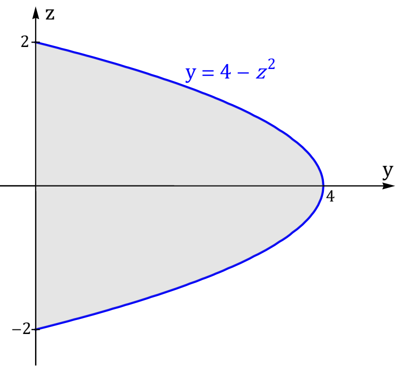
$$= \left(4\,\frac{2^3}{3} - \frac{2^5}{5}\right)(2\pi - 0) = 2\pi\left(\frac{2^5}{3} - \frac{2^5}{5}\right) = 2^6\left(\frac{1}{3} - \frac{1}{5}\right)\pi = \frac{128\pi}{15}.$$

Example (4): Find the volume of the solid bounded by the surfaces $y = 4 - z^2$, $x = -1$, $x = 2$ and $y = 0$.
Solution:
Let E be the region bounded by the surfaces $y = 4 - z^2$, $x = -1$, $x = 2$ and $y = 0$.









The surface $y = 4 - z^2$ intersects the plane $y = 0$ at the two lines passing through $z = \pm 2$.
On E : $-1 \le x \le 2$, $0 \le y \le 4 - z^2$ and $-2 \le z \le 2$.

$$\text{V(E)} = \iiint_E dV = \int_{-1}^{2}\int_{-2}^{2}\int_{0}^{4-z^2} dy\ dz\ dx$$

$$= \int_{-1}^{2}\int_{-2}^{2} [y]_0^{4-z^2}\ dz\ dx = \int_{-1}^{2}\int_{-2}^{2} (4 - z^2) dz\ dx$$

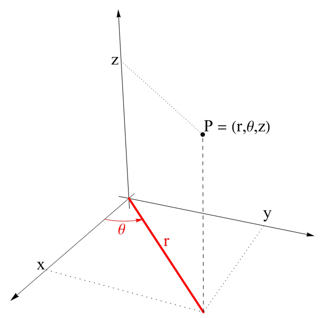
$$= \int_{-1}^{2} \left[4z - \frac{z^3}{3}\right]_{-2}^{2}\ dx = \left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right]\ \int_{-1}^{2} dx$$

$$= \left(16 - \frac{16}{3}\right)\ (2 - (-1)) = 3\left(16 - \frac{16}{3}\right) = 48 - 16 = 32\ .$$

2.5 Triple Integrals in Cylindrical Coordinates

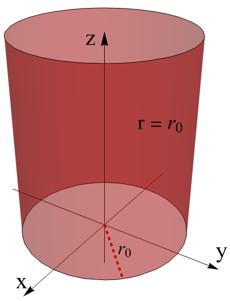
2.5.1 Cylindrical Coordinates

If $P(x, y, z)$ is a point in $\mathbb{R}^3 \setminus \{(0,0,0)\}$, then its cylindrical coordinates $P(r, \theta, z)$ are :
$r = \sqrt{x^2 + y^2}$,
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$, where $x \neq 0$,
and $z = z$.
Note: $r > 0$ and $\theta \in [0, 2\pi]$.

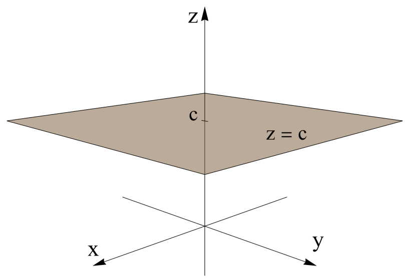


Important equations in Cylindrical coordinates

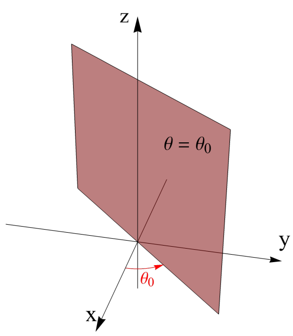
(1). $r = r_0$
A cylinder with radius r_0 .



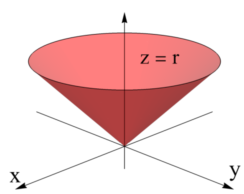
(2). $z = c$
A horizontal plane.



(3). $\theta = \theta_0$
A vertical plane.



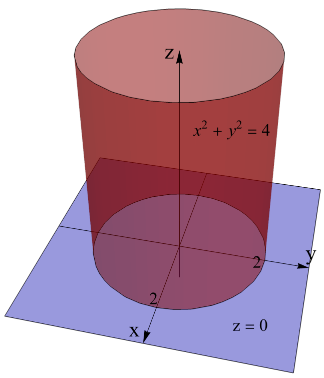
(4). $z = r = \sqrt{x^2 + y^2}$
A Cone.

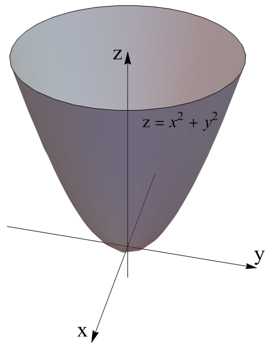


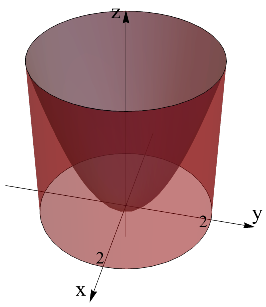
2.5.2 Triple Integrals in Cylindrical Coordinates

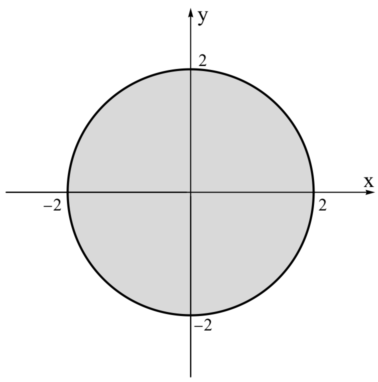
Suppose f is continuous on $E = \{(x,y,z) \mid (x,y) \in D\ ,\ u_1(x,y) \le z \le u_2(x,y)\}$, where D is given in polar coordinates by $D = \{(r,\theta) \mid \theta_1 \le \theta \le \theta_2\ ,\ r_1(\theta) \le r \le r_2(\theta)\}$, then $\displaystyle\iiint_E f(x,y,z)\ dV = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\ r\ dz\ dr\ d\theta.$

Example (1): Find the volume of the solid within $x^2+y^2=4$, bounded above by $z=x^2+y^2$ and below by $z=0$.
Solution:









$E = \left\{(r,\theta,z) \mid 0 \le \theta \le 2\pi\ ,\ 0 \le r \le 2\ ,\ 0 \le z \le x^2+y^2 = r^2\right\}$.

$$\text{Volume} = \iiint_E dV = \int_0^{2\pi}\int_0^2\int_0^{x^2+y^2} r\ dz\ dr\ d\theta = \int_0^{2\pi}\int_0^2\int_0^{r^2} r\ dz\ dr\ d\theta$$

$$= \int_0^{2\pi}\int_0^2 [z]_0^{r^2}\ r\ dr\ d\theta = \int_0^{2\pi}\int_0^2 (r^2-0)r\ dr\ d\theta = \int_0^{2\pi}\int_0^2 r^3\ dr\ d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta = \left[\frac{2^4}{4}-0\right]\int_0^{2\pi} d\theta = 4(2\pi-0) = 8\pi.$$

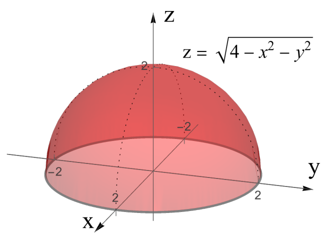
Example (2): Evaluate $\iiint_E z\ dx\ dy\ dz$, where $E = \left\{(x,y,z) \mid 0 \le z \le \sqrt{4-x^2-y^2}\right\}$.

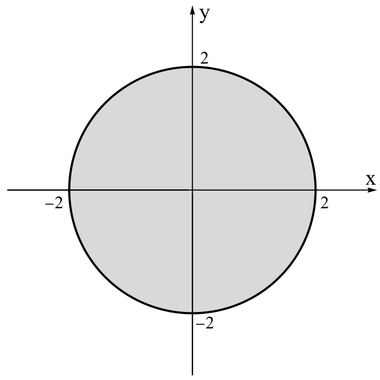
Solution:

$z = \sqrt{4-x^2-y^2}$ is the upper-half of the sphere centered at the origin with radius 2.

$\sqrt{4-x^2-y^2} = 0 \implies x^2+y^2 = 4$, the upper-half of the sphere intersects the plane $z = 0$ at the circle centered at the origin with center 2.

$E = \left\{(r,\theta,z) \mid 0 \le \theta \le 2\pi\ ,\ 0 \le r \le 2\ ,\ 0 \le z \le \sqrt{4-x^2-y^2} = \sqrt{4-r^2}\right\}$.





$$\iiint_E z\ dx\ dy\ dz = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{4-r^2}} z\ dz\ r\ dr d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 \left[z^2\right]_0^{\sqrt{4-r^2}} r\ dr\ d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\int_0^2 (4-r^2)r\ dr\ d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 (4r-r^3)\ dr\ d\theta = \frac{1}{2}\int_0^{2\pi}\left[2r^2 - \frac{r^4}{4}\right]_0^2 d\theta$$
$$= \frac{1}{2}(8-4)\int_0^{2\pi} d\theta = 2(2\pi - 0) = 4\pi\ .$$

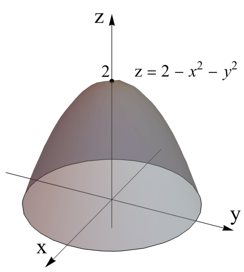
Example (3): Find the volume of the solid bounded above by $z = 2-x^2-y^2$ and below by $z = \sqrt{x^2+y^2}$.

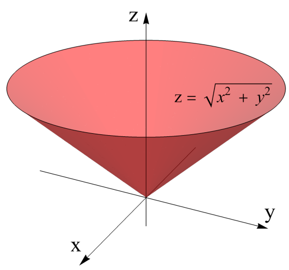
Solution:

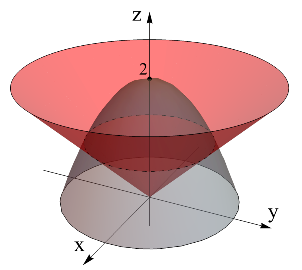
$z = 2-x^2-y^2 = 2-r^2$ intersects $z = \sqrt{x^2+y^2} = r$ at :

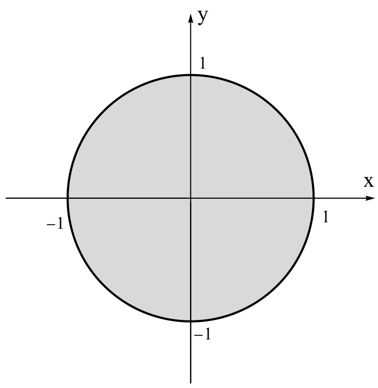
$2-r^2 = r \implies r^2+r-2 = 0 \implies (r+2)(r-1) = 0 \implies r = 1$.

(Note that $r = -2$ is excluded because $r \ge 0$).









$E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi\ ,\ 0 \le r \le 1\ ,\ r \le z \le 2 - r^2\}$.

Volume $= \iiint_E dV = \int_0^{2\pi}\int_0^1\int_r^{2-r^2} dz\ r\ dr\ d\theta = \int_0^{2\pi}\int_0^1 [z]_r^{2-r^2}\ r\ dr\ d\theta$

$= \int_0^{2\pi}\int_0^1 (2 - r^2 - r)r\ dr\ d\theta = \int_0^{2\pi}\int_0^1 (-r^3 - r^2 + 2r)\ dr\ d\theta$

$= \int_0^{2\pi} \left[-\frac{r^4}{4} - \frac{r^3}{3} + r^2\right]_0^1 d\theta = \left(-\frac{1}{4} - \frac{1}{3} + 1\right)\int_0^{2\pi} d\theta = \frac{5}{12}(2\pi - 0) = \frac{5\pi}{6}.$

Example (4): Evaluate the integral $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\ dx\ dy$

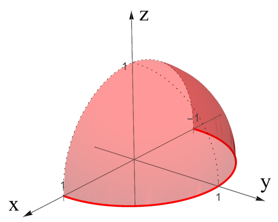
Solution:

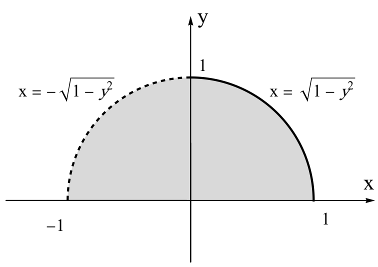
Note that $0 \le y \le 1$, $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$ and $0 \le z \le \sqrt{1-x^2-y^2}$.

$z = \sqrt{1-x^2-y^2} \implies x^2 + y^2 + z^2 = 1$ represents the upper half of the unit sphere.

$x = \sqrt{1-y^2} \implies x^2 + y^2 = 1$ represents the right half of the unit circle.

$x = -\sqrt{1-y^2} \implies x^2 + y^2 = 1$ represents the left half of the unit circle.





In polar coordinates: $0 \le \theta \le \pi$, $0 \le r \le 1$ and $0 \le z \le \sqrt{1-r^2}$.

$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\ dx\ dy = \int_0^{\pi}\int_0^1\int_0^{\sqrt{1-r^2}} dz\ r\ dr\ d\theta$

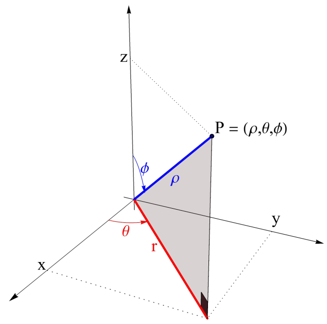
$= \int_0^{\pi}\int_0^1 \sqrt{1-r^2}\ r\ dr\ d\theta = -\frac{1}{2}\int_0^{\pi}\int_0^1 (1-r^2)^{\frac{1}{2}}\ (-2r)\ dr\ d\theta$

$= -\frac{1}{2}\left[\frac{2}{3}(1-r^2)^{\frac{3}{2}}\right]_0^1 \int_0^{\pi} d\theta = -\frac{1}{2}\left(0 - \frac{2}{3}\right)\ (\pi - 0) = \frac{\pi}{3}.$

2.6 Triple Integrals in Spherical Coordinates

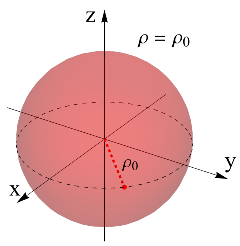
2.6.1 Spherical Coordinates

If $P(x, y, z)$ is a point in $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$, then its spherical coordinates $P(\rho, \theta, \phi)$ are :
$\rho = \sqrt{x^2 + y^2 + z^2}$,
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$, where $x \neq 0$,
and $\phi = \sin^{-1}\left(\frac{z}{\rho}\right)$, where $\rho \neq 0$.
Note: $\rho > 0$, $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$.

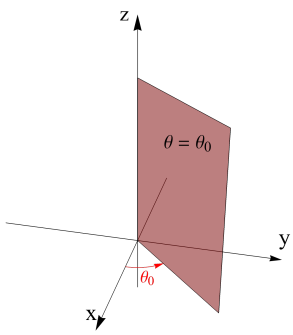


Important equations in Spherical coordinates

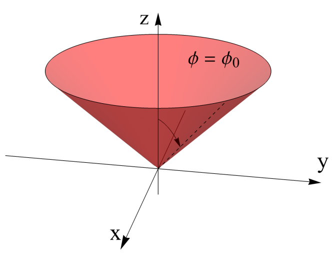
(1). $\rho = \rho_0$
A sphere with radius ρ_0 .



(2). $\theta = \theta_0$
A half-plane.



(3). $\phi = \phi_0 \quad \left(0 < \phi_0 < \frac{\pi}{2}\right)$
A Cone.



(4). $\phi = \phi_0 \quad \left(\frac{\pi}{2} < \phi_0 < \pi\right)$
A Cone.