MATH 201 - Multivariable Calculus
Second Semester - 1447 H
Solution of the First Exam
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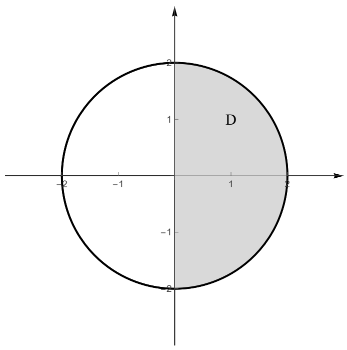
Question (1): [3 + 2 = 5 points]

1. Find and sketch the domain of the function $f(x,y) = \sqrt{x} + \sqrt{4 - x^2 - y^2}$.

   Solution :

   $\sqrt{x}$ is defined when $x \geq 0$.
   defined on the right-half plane.

   $\sqrt{4 - x^2 - y^2}$ is defined when $4 - x^2 - y^2 \geq 0$
   $\Longrightarrow \ x^2 - y^2 \leq 4$.
   defined on and inside the circle, centered at the origin, with radius 2.

   The domain D is :
   $\left\{(x,y) \in \mathbb{R}^2 \ : \ x \geq 0 \ , \ 4 - x^2 - y^2 \geq 0\right\}$
   $= \left\{(x,y) \in \mathbb{R}^2 \ : \ x \geq 0 \ , \ x^2 + y^2 \leq 4\right\}$.

2. Prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^5}{x^8 + y^{10}}$ does not exist.

   Solution :

   (a). On the path $x = 0$:

   $$\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^8 + y^{10}} = \lim_{y\to 0} \frac{0 \ y^5}{0 + y^{10}} = 0 \ .$$

   (b) On the path $x = y^5$:

   $$\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^8 + y^{10}} = \lim_{y\to 0} \frac{y^5 y^5}{(y^5)^8 + y^{1}0} = \lim_{y\to 0} \frac{y^{10}}{y^{40} + y^{10}}$$

   $$= \lim_{y\to 0} \frac{y^{10}}{y^{10}\,(y^{30} + 1)} = \lim_{y\to 0} \frac{1}{y^{30} + 1} = \frac{1}{0+1} = 1.$$

   Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^5}{x^8 + y^{10}}$ does not exist.

   Another solution : on the path $y = x$:

   $$\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^8 + y^{10}} = \lim_{x\to 0} \frac{x \ x^5}{x^8 + x^{10}} = \lim_{x\to 0} \frac{x^6}{x^6\,(x^2 + x^4)}$$

   $$= \lim_{x\to 0} \frac{1}{x^2 + x^4} = \infty.$$

Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^5}{x^8+y^{10}}$ does not exist.

Question (2): [2 + 4 = 6 points]
Let the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by :

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{\sqrt{x^2+y^2}} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

1. Study the continuity of f at $(0,0)$.

   Solution :

   $f(0,0) = 0$.

   $$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{\sqrt{x^2+y^2}}$$

   Using polar coordinates :

   $$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{\sqrt{x^2+y^2}} = \lim_{r\to0^+} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r}$$

   $$= \lim_{r\to0^+} \frac{r^3\left(\cos^3\theta+\sin^3\theta\right)}{r} = \lim_{r\to0^+} r^2\left(\cos^3\theta+\sin^3\theta\right) = 0$$

   (Note that $\lim\limits_{r\to0^+} r^2 = 0$ and $\cos^3\theta+\sin^3\theta$ is bounded)

   Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$. Hence f is continuous at $(0,0)$.

2. Study the differentiability of f at $(0,0)$.

   Solution :

   $f(0,0) = 0$.

   $$f(0+\Delta x, 0+\Delta y) = f(\Delta x, \Delta y) = \frac{(\Delta x)^3 + (\Delta y)^3}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

   $$f_x(0,0) = \lim_{h\to0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to0} \frac{\left(\frac{h^3}{\sqrt{h^2}}\right)}{h} = \lim_{h\to0} \frac{h^3}{h\,|h|}$$

   $$= \lim_{h\to0} \frac{h^2}{|h|} = 0 \ .$$

   (Note that $\lim\limits_{h\to0^+} \dfrac{h^2}{|h|} = \lim\limits_{h\to0} \dfrac{h^2}{h} = \lim\limits_{h\to0} h = 0$ and

   $\lim\limits_{h\to0^-} \dfrac{h^2}{|h|} = \lim\limits_{h\to0} \dfrac{h^2}{-h} = \lim\limits_{h\to0} -h = 0$).

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{\left(\frac{h^3}{\sqrt{h^2}}\right)}{h}=\lim_{h\to 0}\frac{h^3}{h\;|h|}$$

$$=\lim_{h\to 0}\frac{h^2}{|h|}=0\ .$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{f\left(\Delta x,\Delta y\right)-f(0,0)-f_x(0,0)\ \Delta x-f_y(0,0)\ \Delta y}{\sqrt{\left(\Delta x\right)^2+\left(\Delta y\right)^2}}$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\left(\frac{(\Delta x)^3+(\Delta y)^3}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\right)}{\sqrt{\left(\Delta x\right)^2+\left(\Delta y\right)^2}}=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\left(\Delta x\right)^3+\left(\Delta y\right)^3}{\left(\Delta x\right)^2+\left(\Delta y\right)^2}$$

Using polar coordinates :

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\left(\Delta x\right)^3+\left(\Delta y\right)^3}{\left(\Delta x\right)^2+\left(\Delta y\right)^2}=\lim_{r\to 0^+}\frac{r^3\cos^3\theta+r^3\sin^3\theta}{r^2}$$

$$=\lim_{r\to 0^+}\frac{r^3\left(\cos^3\theta+\sin^3\theta\right)}{r^2}=\lim_{r\to 0^+}r\left(\cos^3\theta+\sin^3\theta\right)=0$$

(Note that $\lim_{r\to 0^+} r=0$ and $\cos^3\theta+\sin^3\theta$ is bounded)

Therefore, f is differentiable at $(0,0)$.

Another solution : to evaluate $\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\left(\Delta x\right)^3+\left(\Delta y\right)^3}{\left(\Delta x\right)^2+\left(\Delta y\right)^2}$

$$0\le\left|\frac{\left(\Delta x\right)^3+\left(\Delta y\right)^3}{\left(\Delta x\right)^2+\left(\Delta y\right)^2}\right|\le\frac{\left(\Delta x\right)^2}{\left(\Delta x\right)^2+\left(\Delta y\right)^2}\,|\Delta x|+\frac{\left(\Delta y\right)^2}{\left(\Delta x\right)^2+\left(\Delta y\right)^2}\,|\Delta y|$$

$$0\le\left|\frac{\left(\Delta x\right)^3+\left(\Delta y\right)^3}{\left(\Delta x\right)^2+\left(\Delta y\right)^2}\right|\le|\Delta x|+|\Delta y|$$

Note that $\lim_{(\Delta x,\Delta y)\to(0,0)} 0=0$ and $\lim_{(\Delta x,\Delta y)\to(0,0)} |\Delta x|+|\Delta y|=0+0=0$

By Squeeze Theorem $\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\left(\Delta x\right)^3+\left(\Delta y\right)^3}{\left(\Delta x\right)^2+\left(\Delta y\right)^2}=0$.

Therefore, f is differentiable at $(0,0)$.

Question (3): [4 points]
Let $w=x^2+y^2+z^2$, where $x=u\cos v$, $y=u\sin v$ and $z=uv$.
Use the chain rule to find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.

Solution :

(1). $\dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial u} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial u}$

$= (2x)(\cos v) + (2y)(\sin v) + (2z)(v)$

$= 2u\cos^2 v + 2u\sin^2 v + 2uv^2 = 2u(\cos^2 v + \sin^2 v + v) = 2u(1+v^2)$.

(2). $\dfrac{\partial w}{\partial v} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial v} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial v}$

$= (2x)(-u\sin v) + (2y)(u\cos v) + (2z)(u)$

$= -2u^2\sin v\cos v + 2u^2\sin v\cos v + 2u^2 v = 2u^2 v$.

Question (4): [4 points]
Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$:

$$x\ln y + y(z+1) = z^2 + 1$$

Solution :

$x\ln y + y(z+1) = z^2 + 1 \implies x\ln y + y(z+1) - z^2 - 1 = 0.$

Let $F(x,y,z) = x\ln y + y(z+1) - z^2 - 1$, then $F(x,y,z) = 0$.

If z is differentiable in x and y, then :

(1). $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{\ln y}{y-2z}$.

(2). $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{\frac{x}{y}+z+1}{y-2z}$.

Question (5): [3 + 3 = 6 points]
Consider the function $f(x,y) = x^4 + y^4 - 4xy + 1$,

1. Show that $(0,0)$, $(1,1)$ and $(-1,-1)$ are the critical points of f.

   Solution :

   $f_x(x,y) = 0 \implies 4x^3 - 4y = 0 \implies 4x^3 = 4y \implies x^3 = y$.

   $f_y(x,y) = 0 \implies 4y^3 - 4x = 0 \implies 4y^3 = 4x \implies y^3 = x$.

   $f_x(x,y) = f_y(x,y) \implies x = (x^3)^3 \implies x^9 = x \implies x^9 - x = 0$

   $\implies x(x^8-1) = 0 \implies x(x^4-1)(x^4+1) = 0$

   $\implies x(x^2-1)(x^2+1)(x^4+1) = 0 \implies x(x-1)(x+1)(x^2=1)(x^4+1) = 0$

   $$\implies \begin{cases} x = 0 \\ x-1 = 0 \\ x+1 = 0 \end{cases} \implies \begin{cases} x = 0 \\ x = 1 \\ x = -1 \end{cases} \implies \begin{cases} y = 0^3 = 0 \\ y = 1^3 = 1 \\ y = (-1)^3 = -1 \end{cases}$$

(Note that $x^2 + 1 > 0$ and $x^4 + 1 > 0$).

Therefore, the critical points of f are $(0,0)$, $(1,1)$ and $(-1,-1)$.

2. Classify each critical point as a local minimum, maximum or saddle point.

   Solution :

   $f_{xx}(x,y) = 12x^2$, $f_{yy}(x,y) = 12y^2$ and $f_{xy}(x,y) = -4$.

   $D(x,y) = F_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 = 144x^2y^2 - 16$.

   (a). The critical point $(0,0)$:

   $D(0,0) = 0 - 16 = -16 < 0$, $f(0,0) = 0 + 0 - 0 + 1 = 1$.

   Therefore, $(0,0,1)$ is a saddle point.

   (b). The critical point $(1,1)$:

   $D(1,1) = 144 - 16 > 0$ and $f_{xx}(1,1) = 12 > 0$.

   $f(1,1) = 1 + 1 - 4 + 1 = -1$.

   Therefore, $(1,1,-1)$ is a local minimum.

   (c). The critical point $(-1,-1)$:

   $D(-1,-1) = 144 - 16 > 0$ and $f_{xx}(-1,-1) = 12 > 0$,

   $f(-1,-1) = 1 + 1 - 4 + 1 = -1$.

   Therefore, $(-1,-1,-1)$ is a local minimum.