MATH 201 - Multivariable Calculus
Second Semester - 1447 H
Solution of the First Exam
Dr Tariq A. Alfadhel

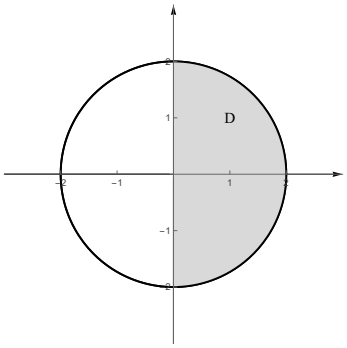
Question (1): [3 + 2 = 5 points]

1. Find and sketch the domain of the function $f(x,y) = \sqrt{x} + \sqrt{4 - x^2 - y^2}$.

   Solution :

   $\sqrt{x}$ is defined when $x \geq 0$.
   defined on the right-half plane.

   $\sqrt{4 - x^2 - y^2}$ is defined when
   $4 - x^2 - y^2 \geq 0$
   $\implies x^2 - y^2 \leq 4$.
   defined on and inside the circle, centered at the origin, with radius 2.

   The domain D is :
   $\{(x,y) \in \mathbb{R}^2 \ : \ x \geq 0 \ , \ 4 - x^2 - y^2 \geq 0\}$
   $= \{(x,y) \in \mathbb{R}^2 \ : \ x \geq 0 \ , \ x^2 + y^2 \leq 4\}$.

2. Prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^5}{x^8 + y^{10}}$ does not exist.

   Solution :

   (a). On the path $x = 0$:

   $$\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^8 + y^{10}} = \lim_{y\to 0} \frac{0 \ y^5}{0 + y^{10}} = 0 \ .$$

   (b) On the path $x = y^5$:

   $$\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^8 + y^{10}} = \lim_{y\to 0} \frac{y^5 y^5}{(y^5)^8 + y^1 0} = \lim_{y\to 0} \frac{y^{10}}{y^{40} + y^{10}}$$

   $$= \lim_{y\to 0} \frac{y^{10}}{y^{10}\,(y^{30} + 1)} = \lim_{y\to 0} \frac{1}{y^{30} + 1} = \frac{1}{0 + 1} = 1.$$

   Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^5}{x^8 + y^{10}}$ does not exist.

   Another solution : on the path $y = x$:

   $$\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^8 + y^{10}} = \lim_{x\to 0} \frac{x \ x^5}{x^8 + x^{10}} = \lim_{x\to 0} \frac{x^6}{x^6\,(x^2 + x^4)}$$

   $$= \lim_{x\to 0} \frac{1}{x^2 + x^4} = \infty.$$

Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^5}{x^8+y^{10}}$ does not exist.

Question (2): [2 + 4 = 6 points]
Let the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by :

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{\sqrt{x^2+y^2}} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

1. Study the continuity of f at $(0,0)$.

   Solution :

   $f(0,0) = 0$.

   $$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{\sqrt{x^2+y^2}}$$

   Using polar coordinates :

   $$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{\sqrt{x^2+y^2}} = \lim_{r\to 0^+} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r}$$

   $$= \lim_{r\to 0^+} \frac{r^3\left(\cos^3\theta+\sin^3\theta\right)}{r} = \lim_{r\to 0^+} r^2\left(\cos^3\theta+\sin^3\theta\right) = 0$$

   (Note that $\lim\limits_{r\to 0^+} r^2 = 0$ and $\cos^3\theta+\sin^3\theta$ is bounded)

   Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$. Hence f is continuous at $(0,0)$.

2. Study the differentiability of f at $(0,0)$.

   Solution :

   $f(0,0) = 0$.

   $$f\left(0+\Delta x, 0+\Delta y\right) = f\left(\Delta x, \Delta y\right) = \frac{\left(\Delta x\right)^3+\left(\Delta y\right)^3}{\sqrt{\left(\Delta x\right)^2+\left(\Delta y\right)^2}}$$

   $$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\left(\frac{h^3}{\sqrt{h^2}}\right)}{h} = \lim_{h\to 0} \frac{h^3}{h\,|h|}$$

   $$= \lim_{h\to 0} \frac{h^2}{|h|} = 0 \ .$$

   (Note that $\lim\limits_{h\to 0^+} \dfrac{h^2}{|h|} = \lim\limits_{h\to 0} \dfrac{h^2}{h} = \lim\limits_{h\to 0} h = 0$ and

   $\lim\limits_{h\to 0^-} \dfrac{h^2}{|h|} = \lim\limits_{h\to 0} \dfrac{h^2}{-h} = \lim\limits_{h\to 0} -h = 0$).

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\left(\frac{h^3}{\sqrt{h^2}}\right)}{h} = \lim_{h\to 0} \frac{h^3}{h\ |h|}$$

$$= \lim_{h\to 0} \frac{h^2}{|h|} = 0 \ .$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f\left(\Delta x, \Delta y\right) - f(0,0) - f_x(0,0)\ \Delta x - f_y(0,0)\ \Delta y}{\sqrt{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\left(\frac{(\Delta x)^3 + (\Delta y)^3}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}\right)}{\sqrt{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\left(\Delta x\right)^3 + \left(\Delta y\right)^3}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}$$

Using polar coordinates :

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\left(\Delta x\right)^3 + \left(\Delta y\right)^3}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2} = \lim_{r\to 0^+} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2}$$

$$= \lim_{r\to 0^+} \frac{r^3\left(\cos^3\theta + \sin^3\theta\right)}{r^2} = \lim_{r\to 0^+} r\left(\cos^3\theta + \sin^3\theta\right) = 0$$

(Note that $\lim\limits_{r\to 0^+} r = 0$ and $\cos^3\theta + \sin^3\theta$ is bounded)

Therefore, f is differentiable at $(0,0)$.

Another solution : to evaluate $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{\left(\Delta x\right)^3 + \left(\Delta y\right)^3}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}$

$$0 \le \left|\frac{\left(\Delta x\right)^3 + \left(\Delta y\right)^3}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}\right| \le \frac{\left(\Delta x\right)^2}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}\,|\Delta x| + \frac{\left(\Delta y\right)^2}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}\,|\Delta y|$$

$$0 \le \left|\frac{\left(\Delta x\right)^3 + \left(\Delta y\right)^3}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}\right| \le |\Delta x| + |\Delta y|$$

Note that $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} 0 = 0$ and $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} |\Delta x| + |\Delta y| = 0 + 0 = 0$

By Squeeze Theorem $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{\left(\Delta x\right)^3 + \left(\Delta y\right)^3}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2} = 0$.

Therefore, f is differentiable at $(0,0)$.

Question (3): [4 points]
Let $w = x^2 + y^2 + z^2$, where $x = u\cos v$, $y = u\sin v$ and $z = uv$.
Use the chain rule to find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

Solution :

(1). $\dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial u} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial u}$

$= (2x)(\cos v) + (2y)(\sin v) + (2z)(v)$

$= 2u\cos^2 v + 2u\sin^2 v + 2uv^2 = 2u(\cos^2 v + \sin^2 v + v) = 2u(1+v^2)$.

(2). $\dfrac{\partial w}{\partial v} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial v} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial v}$

$= (2x)(-u\sin v) + (2y)(u\cos v) + (2z)(u)$

$= -2u^2\sin v\cos v + 2u^2\sin v\cos v + 2u^2 v = 2u^2 v$.

Question (4): [4 points]

Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$:

$$x\ln y + y(z+1) = z^2 + 1$$

Solution :

$x\ln y + y(z+1) = z^2+1 \implies x\ln y + y(z+1) - z^2 - 1 = 0.$

Let $F(x,y,z) = x\ln y + y(z+1) - z^2 - 1$, then $F(x,y,z) = 0$.

If z is differentiable in x and y, then :

(1). $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{\ln y}{y-2z}$.

(2). $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{\frac{x}{y}+z+1}{y-2z}$.

Question (5): [3 + 3 = 6 points]

Consider the function $f(x,y) = x^4 + y^4 - 4xy + 1$,

1. Show that $(0,0)$, $(1,1)$ and $(-1,-1)$ are the critical points of f.

   Solution :

   $f_x(x,y) = 0 \implies 4x^3 - 4y = 0 \implies 4x^3 = 4y \implies x^3 = y$.

   $f_y(x,y) = 0 \implies 4y^3 - 4x = 0 \implies 4y^3 = 4x \implies y^3 = x$.

   $f_x(x,y) = f_y(x,y) \implies x = (x^3)^3 \implies x^9 = x \implies x^9 - x = 0$

   $\implies x(x^8-1) = 0 \implies x(x^4-1)(x^4+1) = 0$

   $\implies x(x^2-1)(x^2+1)(x^4+1) = 0 \implies x(x-1)(x+1)(x^2=1)(x^4+1) = 0$

   $$\implies \begin{cases} x = 0 \\ x-1 = 0 \\ x+1 = 0 \end{cases} \implies \begin{cases} x = 0 \\ x = 1 \\ x = -1 \end{cases} \implies \begin{cases} y = 0^3 = 0 \\ y = 1^3 = 1 \\ y = (-1)^3 = -1 \end{cases}$$

(Note that $x^2 + 1 > 0$ and $x^4 + 1 > 0$).

Therefore, the critical points of f are $(0, 0)$, $(1, 1)$ and $(-1, -1)$.

2. Classify each critical point as a local minimum, maximum or saddle point.

Solution :

$f_{xx}(x, y) = 12x^2$, $f_{yy}(x, y) = 12y^2$ and $f_{xy}(x, y) = -4$.

$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 = 144x^2y^2 - 16$.

(a). The critical point $(0, 0)$:

$D(0, 0) = 0 - 16 = -16 < 0$, $f(0, 0) = 0 + 0 - 0 + 1 = 1$.

Therefore, $(0, 0, 1)$ is a saddle point.

(b). The critical point $(1, 1)$:

$D(1, 1) = 144 - 16 > 0$ and $f_{xx}(1, 1) = 12 > 0$.

$f(1, 1) = 1 + 1 - 4 + 1 = -1$.

Therefore, $(1, 1, -1)$ is a local minimum.

(c). The critical point $(-1, -1)$:

$D(-1, -1) = 144 - 16 > 0$ and $f_{xx}(-1, -1) = 12 > 0$,

$f(-1, -1) = 1 + 1 - 4 + 1 = -1$.

Therefore, $(-1, -1, -1)$ is a local minimum.

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Question (1): [5 points]

Evaluate the iterated integral $\int_0^1 \int_1^2 \left(e^y e^{-x} + y^2\right)\ dx\ dy$.

First Solution :

$$\int_0^1 \int_1^2 \left(e^y e^{-x} + y^2\right)\ dx\ dy = \int_0^1 \int_1^2 e^y e^{-x}\ dx\ dy + \int_0^1 \int_1^2 y^2\ dx\ dy$$
$$= \left(\int_1^2 e^{-x}\ dx\right)\left(\int_0^1 e^y\ dy\right) + \left(\int_1^2 dx\right)\left(\int_0^1 y^2\right)$$
$$= \left[-e^{-x}\right]_1^2 \left[e^y\right]_0^1 + \left[x\right]_1^2 \left[\frac{y^3}{3}\right]_0^1 = \left(-e^{-2} - \left(-e^{-1}\right)\right)(e-1) + (2-1)\left(\frac{1}{3} - 0\right)$$
$$= (e-1)\left(\frac{1}{e} - \frac{1}{e^2}\right) + \frac{1}{3} = 1 - \frac{1}{e} - \frac{1}{e} + \frac{1^2}{e} + \frac{1}{3} = \frac{4}{3} + \frac{1}{e^2} - \frac{2}{e}\ .$$

Second Solution :

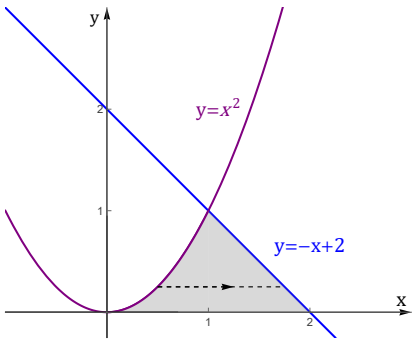
$$\int_0^1 \int_1^2 \left(e^y e^{-x} + y^2\right)\ dx\ dy = \int_0^1 \left[-e^y e^{-x} + xy^2\right]_1^2\ dy$$
$$= \int_0^1 \left[\left(-e^y e^{-2} + 2y^2\right) - \left(-e^y e^{-1} + y^2\right)\right]\ dy$$
$$= \int_0^1 \left(-e^y e^{-2} + 2y^2 + e^y e^{-1} - y^2\right)\ dy = \int_0^1 \left(e^y\left(e^{-1} - e^{-2}\right) + y^2\right)\ dy$$
$$= \left[e^y\left(e^{-1} - e^{-2}\right) + \frac{y^3}{3}\right]_0^1 = e\left(e^{-1} - e^{-2}\right) + \frac{1}{3} - \left(e^{-1} - e^{-2}\right)$$
$$= 1 - e^{-1} + \frac{1}{3} - e^{-1} + e^{-2} = \frac{4}{3} + \frac{1}{e^2} - \frac{2}{e}\ .$$

Third Solution : Using Fubini's Theorem

$$\int_0^1 \int_1^2 \left(e^y e^{-x} + y^2\right)\ dx\ dy = \int_1^2 \int_0^1 \left(e^y e^{-x} + y^2\right)\ dy\ dx$$
$$= \int_1^2 \left[e^y e^{-x} + \frac{y^3}{3}\right]_0^1\ dx = \int_1^2 \left[\left(e^1 e^{-x} + \frac{1}{3}\right) - \left(e^0 e^{-x} + 0\right)\right]\ dx$$
$$= \int_1^2 \left(e e^{-x} + \frac{1}{3} - e^{-x}\right)\ dx = \int_1^2 \left((e-1)e^{-x} + \frac{1}{3}\right)\ dx$$
$$= \left[-(e-1)e^{-x} + \frac{x}{3}\right]_1^2 = -(e-1)e^{-2} + \frac{2}{3} - \left(-(e-1)e^{-1} + \frac{1}{3}\right)$$
$$= -e^{-1} + e^{-2} + \frac{2}{3} + 1 - e^{-1} - \frac{1}{3} = \frac{4}{3} + \frac{1}{e^2} - \frac{2}{e}\ .$$

Question (2): [4 points]
Find the area of the region that is bounded below by $y = 0$ and above by the curves $x + y = 2$ and $y = x^2$.

Solution:
$y = 0$ is the x-axis.
$y = x^2$ represents a parabola opens upwards with vertex $(0, 0)$.
$y = -x + 2$ is a straight line passing through $(0, 2)$ with slope -1.



Points of intersection of $y = x^2$ and $y = -x + 2$:
$x^2 = -x + 2 \implies x^2 + x - 2 = 0$
$\implies (x + 2)(x - 1) = 0$
$\implies x = 1 \ , \ x = -2$.
Note that $x = -2$ is excluded.
If $x = 1$ then $y = 1$.
$y = -x + 2$ intersects $y = 0$ at $x = 2$.

First solution :

$$\text{Area} = \int_0^1 \int_0^{x^2} 1 \ dy \ dx + \int_1^2 \int_0^{-x+2} 1 \ dy \ dx$$

$$= \int_0^1 [y]_0^{x^2} \ dx + \int_1^2 [y]_0^{-x+2} \ dx = \int_0^1 (x^2 - 0) \ dx + \int_1^2 (-x + 2 - 0) \ dx$$

$$= \int_0^1 x^2 \ dx + \int_1^2 (-x + 2) \ dx = \left[\frac{x^3}{3}\right]_0^1 + \left[-\frac{x^2}{2} + 2x\right]_1^2$$

$$= \left(\frac{1}{3} - 0\right) + \left[\left(-\frac{4}{2} + 4\right) - \left(-\frac{1}{2} + 2\right)\right] = \frac{1}{3} - 2 + 4 - \frac{1}{2} - 2 = \frac{5}{6}$$

Second solution :
$y = x^2 \implies x = \sqrt{y}$, $x + y = 2 \implies x = -y + 2$.

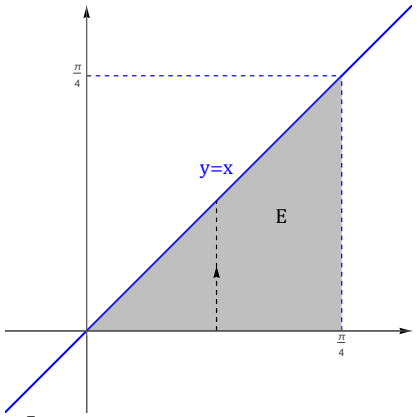
$$\text{Area} = \int_0^1 \int_{\sqrt{y}}^{-y+2} 1 \ dx \ dy = \int_0^1 (-y + 2 - \sqrt{y}) \ dy = \int_0^1 \left(-y - y^{\frac{1}{2}} + 2\right) \ dy$$

$$= \left[-\frac{y^2}{2} - \frac{2}{3}y^{\frac{3}{2}} + 2y\right]_0^1 = \left(-\frac{1}{2} - \frac{2}{3} + 2\right) - (0 - 0 + 0) = \frac{-3 - 4 + 12}{6} = \frac{5}{6}.$$

Question (3): [4 points]
Change the order of integration and then evaluate the integral

$$\int_0^{\frac{\pi}{4}} \int_y^{\frac{\pi}{4}} \left(\frac{\sec^2 x}{x}\right)\, dx\, dy$$

Solution:
Let E be the region where:
$0 \le y \le \frac{\pi}{4}$ and $y \le x \le \frac{\pi}{4}$.



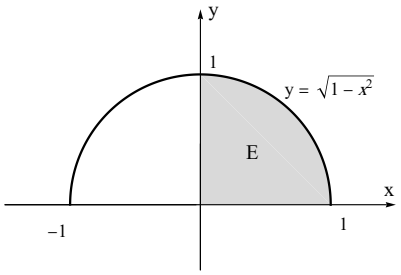
Then E can be written as :
$0 \le x \le \frac{\pi}{4}$ and $0 \le y \le x$.

$$\int_0^{\frac{\pi}{4}} \int_y^{\frac{\pi}{4}} \left(\frac{\sec^2 x}{x}\right)\, dx\, dy$$
$$= \lim_{t\to 0^+} \int_t^{\frac{\pi}{4}} \int_0^{x} \left(\frac{\sec^2 x}{x}\right)\, dy\, dx$$
$$= \lim_{t\to 0^+} \int_t^{\frac{\pi}{4}} \left(\frac{\sec^2 x}{x}\right)\, [y]_0^x\, dx$$
$$= \lim_{t\to 0^+} \int_t^{\frac{\pi}{4}} \left(\frac{\sec^2 x}{x}\right)\, (x-0)\, dx = \lim_{t\to 0^+} \int_0^{\frac{\pi}{4}} \left(\frac{\sec^2 x}{x}\right) x\, dx$$
$$= \lim_{t\to 0^+} \int_t^{\frac{\pi}{4}} \sec^2 x\, dx = \lim_{t\to 0^+} [\tan x]_t^{\frac{\pi}{4}}$$
$$= \lim_{t\to 0^+} \left[\tan\left(\frac{\pi}{4}\right) - \tan(t)\right] = 1 - \tan(0) = 1 - 0 = 1 \ .$$

Question (4): [4 points]
Evaluate the iterated integral by converting to polar coordinates

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1+x^2+y^2}\, dy\, dx$$

Solution:
Let E be the region where:
$0 \le x \le 1$ and $0 \le y \le \sqrt{1-x^2}$.



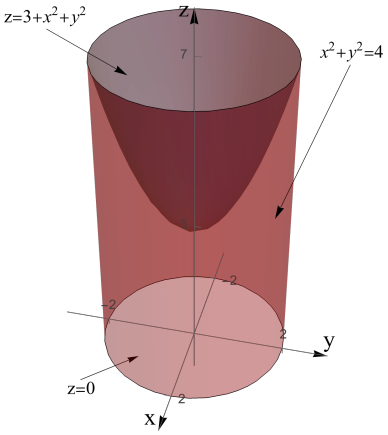
Note that $y = \sqrt{1-x^2}$ represents the upper-half of the unit circle.

Then E in polar coordinates :
$0 \le r \le 1$ and $0 \le \theta \le \frac{\pi}{2}$.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1+x^2+y^2}\, dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{1+r^2}\, r\, dr\, d\theta$$
$$= \left(\frac{1}{2}\int_0^1 (1+r^2)^{\frac{1}{2}}\, (2r)\, dr\right) \left(\int_0^{\frac{\pi}{2}} d\theta\right) = \frac{1}{2}\left[\frac{(1+r^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 [\theta]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{2}\left(\frac{2}{3}\,(2)^{\frac{3}{2}} - \frac{2}{3}\right)\left(\frac{\pi}{2} - 0\right) = \frac{1}{2}\,\frac{2}{3}\left(2\sqrt{2}-1\right)\,\frac{\pi}{2} = \frac{\pi\left(2\sqrt{2}-1\right)}{6} \ .$$

Question (5): [4 points]
Find the volume of the solid that lies inside the cylinder $x^2 + y^2 = 4$, above the plane $z = 0$, and below the paraboloid $z = 3 + x^2 + y^2$.
Solution:

Let E be the solid that lies inside the cylinder $x^2 + y^2 = 4$, above the plane $z = 0$, and below the paraboloid $z = 3 + x^2 + y^2$.



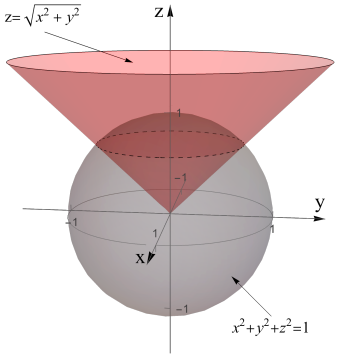
Then E in cylindrical coordinates:
$0 \le z \le 3 + x^2 + y^2 = 3 + r^2$,
$0 \le r \le 2$ and $0 \le \theta \le 2\pi$.

$$\text{Volume} = V(E) = \iiint_E 1 \; dV$$
$$= \int_0^{2\pi} \int_0^2 \int_0^{3+r^2} dz \; r \; dr \; d\theta$$
$$= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^2 \int_0^{3+r^2} dz \; r \; dr \right)$$

$$= [\theta]_0^{2\pi} \left(\int_0^2 [z]_0^{3+r^2} \; r \; dr \right) = (2\pi - 0) \left(\int_0^2 \left[(3 + r^2) - 0 \right] \; r \; dr \right)$$
$$= 2\pi \int_0^2 (3r + r^3) \; dr = 2\pi \left[\frac{3r^2}{2} + \frac{r^4}{4} \right]_0^2$$
$$= 2\pi \left[\left(\frac{3 \; (2)^2}{2} + \frac{(2)^4}{4} \right) - (0 + 0) \right] = 2\pi(6 + 4) = 20\pi \; .$$

Question (6): [4 points]
Find the volume of the solid that lies within both the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 1$.
Solution:

Let E be the solid that lies within both the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 1$.



The sphere intersects the cone at:
$x^2 + y^2 + (x^2 + y^2) = 1$
$2x^2 + 2y^2 = 1$
$$x^2 + y^2 = \frac{1}{2} = \left(\frac{1}{\sqrt{2}} \right)^2$$
$$z = \sqrt{x^2 + y^2} = r$$
$$\implies \rho \cos\phi = \rho \sin\phi$$
$$\implies \tan\phi = 1 \implies \phi = \frac{\pi}{4}.$$

E in spherical coordinates : $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \dfrac{\pi}{4}$.

$$\text{Volume} = V(E) = \iiint_E 1 \; dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^2 \; \sin\phi \; d\rho \; d\phi \; d\theta$$

$$= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\frac{\pi}{4}} \sin\phi \; d\phi \right) \left(\int_0^1 \rho^2 \; d\rho \right) = [\theta]_0^{2\pi} \;\; [-\cos\phi]_0^{\frac{\pi}{4}} \;\; \left[\frac{\rho^3}{3} \right]_0^1$$

$$= (2\pi - 0) \;\; \left(-\frac{1}{\sqrt{2}} - (-1) \right) \;\; \left(\frac{1}{3} - 0 \right) = \frac{2\pi}{3} \left(1 - \frac{1}{\sqrt{2}} \right) = \frac{\left(2 - \sqrt{2} \right) \pi}{3}.$$

MATH 201 - Multivariable Calculus
Second Semester - 1447 H
Solution of the Final Exam
Dr Tariq A. Alfadhel

Question (1): [2 + 2 + 2 = 6 points]
Let the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by :

$$f(x,y) = \begin{cases} \dfrac{x^3y^2}{x^4+y^4} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

1. Study the continuity of f at the point $(0,0)$.

   Solution :

   $f(0,0) = 0$.

   $$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^4+y^4}$$

   Using polar coordinates :

   $$\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^4+y^4} = \lim_{r\to 0^+} \frac{r^3\cos^3\theta\ r^2\sin^2\theta}{r^4\cos^4\theta + r^4\sin^4\theta}$$

   $$= \lim_{r\to 0^+} \frac{r^5\left(\cos^3\theta\sin^2\theta\right)}{r^4(\cos^4\theta+\sin^4\theta)} = \lim_{r\to 0^+} r\left(\frac{\cos^3\theta\sin^2}{\cos^4\theta+\sin^4\theta}\right) = 0$$

   (Note that $\lim\limits_{r\to 0^+} r = 0$ and $\dfrac{\cos^3\theta\sin^2\theta}{\cos^4\theta+\sin^4\theta}$ is bounded)

   Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$. Hence f is continuous at $(0,0)$.

   Another solution :

   $$0 \leq \left|\frac{x^3y^2}{x^4+y^4}\right| = \frac{\left|x^3\right|y^2}{x^4+y^4} = \frac{|x|\ x^2y^2}{x^4+y^4} \leq |x|$$

   Since $\lim\limits_{x\to 0}|x| = 0$, then by squeeze theorem $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y^2}{x^4+y^4} = 0$.

   Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$. Hence f is continuous at $(0,0)$.

2. Find the partial derivatives $f_x(0,0)$ and $f_y(0,0)$.

   Solution :

   $$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\left(\frac{h^3\ 0^2}{h^4+0^4}\right)}{h} = 0.$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\left(\frac{0^3\ h^2}{0^4+h^4}\right)}{h} = 0.$$

3. Study the differentiability of f at $(0,0)$.

   Solution :

   $f(0,0) = 0$.

   $$f\left(0+\Delta x, 0+\Delta y\right) = f\left(\Delta x, \Delta y\right) = \frac{\left(\Delta x\right)^3 \left(\Delta y\right)^2}{\left(\Delta x\right)^4 + \left(\Delta y\right)^4}$$

   $$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f\left(\Delta x, \Delta y\right) - f(0,0) - f_x(0,0)\ \Delta x - f_y(0,0)\ \Delta y}{\sqrt{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}}$$

   $$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\left(\frac{(\Delta x)^3(\Delta y)^2}{(\Delta x)^4+(\Delta y)^4}\right)}{\sqrt{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\left(\Delta x\right)^3 \left(\Delta y\right)^2}{\sqrt{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}\left[\left(\Delta x\right)^4 + \left(\Delta y\right)^4\right]}$$

   By taking the path $\Delta x = \Delta y$:

   $$\lim_{(\Delta x,\Delta y)\to(0,0)} \left| \frac{\left(\Delta x\right)^3 \left(\Delta y\right)^2}{\sqrt{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}\left[\left(\Delta x\right)^4 + \left(\Delta y\right)^4\right]} \right| = \lim_{\Delta x\to 0} \frac{|\Delta x|^5}{2\sqrt{2}\,|\Delta x^5|}$$

   $= \dfrac{1}{2\sqrt{2}} \neq 0$. Therefore, f is not differentiable at $(0,0)$.

   Another solution : Using polar coordinates

   $$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\left(\Delta x\right)^3 \left(\Delta y\right)^2}{\sqrt{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}\left[\left(\Delta x\right)^4 + \left(\Delta y\right)^4\right]} = \lim_{r\to 0^+} \frac{r^3\cos^3\theta\ r^2\sin^2\theta}{r\left(r^4\cos^4\theta + r^4\sin^4\theta\right)}$$

   $$= \lim_{r\to 0^+} \frac{r^5\cos^3\theta\sin^2\theta}{r^5\left(\cos^4\theta + \sin^4\theta\right)} = \frac{\cos^3\theta\sin^2\theta}{\cos^4\theta + \sin^4\theta} \text{ (depends on } \theta)$$

   So, The limit does not exist. Therefore, f is not differentiable at $(0,0)$.

Question (2): [4 points]
Let f be a differentiable function in one variable, and let $w = f(x^2+y^2+z^2)$.
Use the chain rule to show that $yz\dfrac{\partial w}{\partial x} + xz\dfrac{\partial w}{\partial y} = 2xy\dfrac{\partial w}{\partial z}$.

Solution :

Let $u = x^2+y^2+z^2$, then $w = f(u)$ is a function of one variable.

$$\frac{\partial w}{\partial x} = \frac{dw}{du}\frac{\partial u}{\partial x} = \frac{dw}{du}(2x) = 2x\ \frac{dw}{du}.$$

$$\frac{\partial w}{\partial y} = \frac{dw}{du}\frac{\partial u}{\partial y} = \frac{dw}{du}(2y) = 2y\ \frac{dw}{du}.$$

$$\frac{\partial w}{\partial z} = \frac{dw}{du}\frac{\partial u}{\partial z} = \frac{dw}{du}(2z) = 2z\ \frac{dw}{du}.$$

$$yz\frac{\partial w}{\partial x} + xz\frac{\partial w}{\partial y} = yz\left(2x\ \frac{dw}{du}\right) + xz\left(2y\ \frac{dw}{du}\right)$$

$$= 2xyz\ \frac{dw}{du} + 2xyz\ \frac{dw}{du} = 4xyz\ \frac{dw}{du}.$$

$$2xy\frac{\partial w}{\partial z} = 2xy\left(2z\ \frac{dw}{du}\right) = 4xyz\ \frac{dw}{du}\ .$$

Therefore, $yz\dfrac{\partial w}{\partial x} + xz\dfrac{\partial w}{\partial y} = 2xy\dfrac{\partial w}{\partial z}$.

Question (3): [5 points]
Find the maximum and minimum values of the function $f(x, y, z) = x+y-z+1$ on the sphere $x^2 + y^2 + z^2 = 12$.

Solution : Using Lagrange multipliers method.

Let $g(x, y, z) = x^2 + z^2 + y^2 - 12 = 0$ be the constraint.

$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \implies \langle 1, 1, -1\rangle = \lambda\langle 2x, 2y, 2z\rangle$

$$\begin{cases} 1 = 2\lambda x \\ 1 = 2\lambda y \\ -1 = 2\lambda z \end{cases} \implies \begin{cases} x = \dfrac{1}{2\lambda} \\ y = \dfrac{1}{2\lambda} \\ z = \dfrac{-1}{2\lambda} \end{cases} \implies x = y = -z$$

$$\text{Or : } \begin{cases} 1 = 2\lambda x \\ 1 = 2\lambda y \\ -1 = 2\lambda z \end{cases} \implies \begin{cases} \dfrac{1}{1} = \dfrac{2\lambda x}{2\lambda y} \implies x = y \\ \dfrac{1}{-1} = \dfrac{2\lambda x}{2\lambda z} \implies x = -z \end{cases}$$

Note that $\lambda \neq 0$.

From the constraint : $x^2 + y^2 + z^2 = 12 \implies x^2 + x^2 + (-x)^2 = 12$

$\implies 3x^2 = 12 \implies x^2 = 4 \implies x = \pm 2.$

If $x = 2$: $y = 2$ and $z = -2$, so the critical point is $(2, 2, -2)$.

If $x = -2$: $y = -2$ and $z = 2$, so the critical point is $(-2, -2, 2)$.

$f(2, 2, -2) = 2 + 2 - (-2) + 1 = 7$.

$f(-2,-2,2) = -2-2-2+1 = -5$.

The maximum value of the function is 7 attained at $(2,2,-2)$.

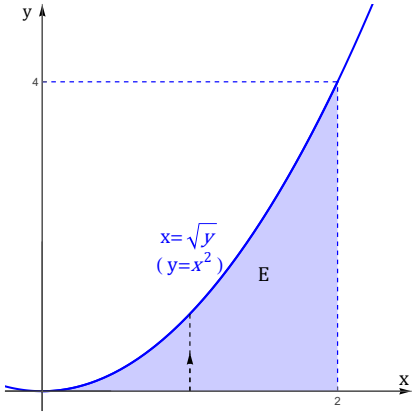
The minimum value of the function is -5 attained at $(-2,-2,2)$.

Question (4): [3.5 + 3.5 = 7 points]

1. Change the order of integration to evaluate the double integral

$$\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{x^3+1}\, dx\, dy$$

Solution :
Let E be the region where:
$0 \le y \le 4$ and $\sqrt{y} \le x \le 2$.



Then E can be written as :
$0 \le x \le 2$ and $0 \le y \le x^2$.

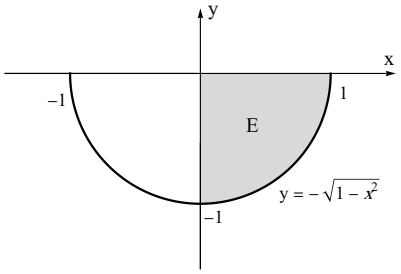
$$\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{x^3+1}\, dx\, dy$$
$$= \int_0^2 \int_0^{x^2} \frac{1}{x^3+1}\, dy\, dx$$
$$= \int_0^2 \frac{1}{x^3+1}\, [y]_0^{x^2}\, dx$$
$$= \int_0^2 \frac{x^2}{x^3+1}\, dx = \frac{1}{3}\int_0^2 \frac{3x^2}{x^3+1}\, dx = \frac{1}{3}\left[\ln\left|x^3+1\right|\right]_0^2$$
$$= \frac{1}{3}\left[\ln\left|2^3+1\right| - \ln\left|0^3+1\right|\right] = \frac{1}{3}\left[\ln(9) - \ln(1)\right] = \frac{\ln(9)}{3} = \frac{2\ln(3)}{3}.$$

2. Evaluate the iterated integral by converting to polar coordinates

$$\int_0^1 \int_{-\sqrt{1-x^2}}^0 xy\, dy\, dx$$

Solution :

Let E be the region where:
$0 \le x \le 1$ and $-\sqrt{1-x^2} \le y \le 0$.



Note that $y = -\sqrt{1-x^2}$ represents the lower-half of the unit circle.

Then E in polar coordinates :
$0 \le r \le 1$ and $-\frac{\pi}{2} \le \theta \le 0$.

$$\int_0^1 \int_{-\sqrt{1-x^2}}^0 xy\, dy\, dx = \int_{-\frac{\pi}{2}}^0 \int_0^1 (r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta$$

$$= \left(\int_0^1 r^3 \, dr\right)\left(\int_{-\frac{\pi}{2}}^0 \sin\theta\cos\theta \, d\theta\right) = \left[\frac{r^4}{4}\right]_0^1 \left[\frac{\sin^2\theta}{2}\right]_{-\frac{\pi}{2}}^0$$

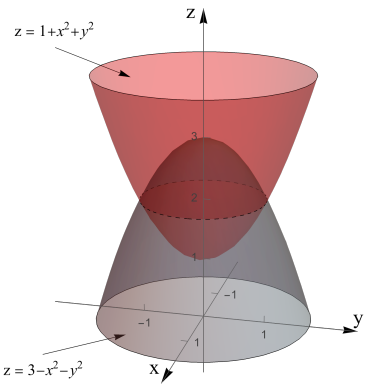
$$= \left(\frac{1}{4} - 0\right)\left(0 - \frac{1}{2}\right) = -\frac{1}{8} \, .$$

Question (5): [3 + 3 = 6 points]

1. Find the volume of the solid that lies within the two paraboloids $z = x^2+y^2+1$ and $z = 3 - x^2 - y^2$.

Solution :

Let E be the solid that lies within the two paraboloids $z = x^2 + y^2 + 1$ and $z = 3 - x^2 - y^2$.



The two paraboloids intersect at
$x^2 + y^2 + 1 = 3 - x^2 - y^2$
$\implies 2x^2 + 2y^2 = 2$
$\implies x^2 + y^2 = 1$
They intersect at the unit circle.

Then E in cylindrical coordinates:
$x^2 + y^2 + 1 \le z \le 3 - x^2 - y^2$,
$1 + r^2 \le z \le 3 - r^2$,
$0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

$$\text{Volume} = V(E) = \iiint_E 1 \, dV = \int_0^{2\pi}\int_0^1\int_{1+r^2}^{3-r^2} dz \; r \; dr \; d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^1\int_{1+r^2}^{3-r^2} dz \; r \; dr\right) = [\theta]_0^{2\pi}\left(\int_0^1 [z]_{1+r^2}^{3-r^2} \; r \; dr\right)$$

$$= (2\pi - 0)\int_0^1 (3 - r^2 - (1 + r^2)) \; r \; dr = 2\pi \int_0^1 (2 - 2r^2) \; r \; dr$$

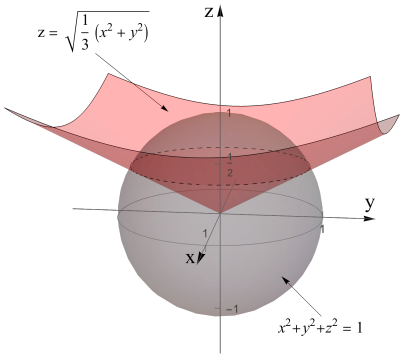
$$= 2\pi \left(-\frac{1}{4}\int_0^1 (2 - 2r^2)\,(-4r) \; dr\right) = 2\pi \left[\frac{(2 - 2r^2)^2}{2}\right]_0^1$$

$$= 2\pi \; \frac{-1}{4} \; \left(0 - \frac{4}{2}\right) = 2\pi \; \frac{-1}{4} \; (-2) = \pi \, .$$

2. Find the volume of the solid that lies within both the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{\frac{1}{3}(x^2 + y^2)}$.

Solution :

Let E be the solid that lies within both the cone $z = \sqrt{\frac{1}{3}(x^2+y^2)}$ and the sphere $x^2+y^2+z^2=1$.



The sphere intersects the cone at:

$$x^2+y^2+\frac{1}{3}(x^2+y^2)=1$$

$$\frac{4}{3}(x^2+y^2)=1$$

$$x^2+y^2=\frac{3}{4}=\left(\frac{\sqrt{3}}{2}\right)^2.$$

$$z=\sqrt{\frac{1}{3}(x^2+y^2)}=\frac{1}{\sqrt{3}}r \implies \rho\cos\phi=\frac{1}{\sqrt{3}}\rho\sin\phi$$

$$\implies \tan\phi=\sqrt{3} \implies \phi=\frac{\pi}{3}.$$

E in spherical coordinates : $0\le\rho\le 1$, $0\le\theta\le 2\pi$ and $0\le\phi\le\frac{\pi}{3}$.

$$\text{Volume} = V(E) = \iiint_E 1\ dV = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 \rho^2\ \sin\phi\ d\rho\ d\phi\ d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{3}} \sin\phi\ d\phi\right)\left(\int_0^1 \rho^2\ d\rho\right) = [\theta]_0^{2\pi}\ \ [-\cos\phi]_0^{\frac{\pi}{3}}\ \ \left[\frac{\rho^3}{3}\right]_0^1$$

$$= (2\pi-0)\ \left(-\frac{1}{2}-(-1)\right)\ \left(\frac{1}{3}-0\right) = 2\pi\ \frac{1}{2}\ \frac{1}{3} = \frac{\pi}{3}.$$

Question (6): [1.5 + (1.5 + 1.5 + 1.5) = 6 points]

1. Use the Integral Test to determine whether the series $\sum_{n=0}^{\infty}\frac{2n+1}{n^2+n+1}$ converges or diverges.

Solution :

Let $f(x)=\frac{2x+1}{x^2+x+1}$ defined on the interval $[1,\infty)$.

(i). f is positive and continuous on $[1,\infty)$.

(ii). $f'(x)=\frac{2(x^2+x+1)-(2x+1)(2x+1)}{(x^2+x+1)^2}$

$$=\frac{2x^2+2x+2-4x^2-8x-1}{(x^2+x+1)^2}=\frac{-2x^2-6x+1}{(x^2+x+1)^2}<0\ ,\ \forall x\ge 1\ .$$

So, f is decreasing on the interval $[1,\infty)$.

$$\int_1^{\infty} \frac{2x+1}{x^2+x+1}\,dx = \lim_{t\to\infty}\int_1^{t} \frac{2x+1}{x^2+x+1}\,dx = \lim_{t\to\infty}\left[\ln|x^2+x+1|\right]_1^t$$

$$= \lim_{t\to\infty}\left(\ln|t^2+t+1| - \ln(3)\right) = \infty.$$

Since the improper integral diverges, then $\sum_{n=0}^{\infty} \frac{2n+1}{n^2+n+1}$ diverges.

2. Test the following series and determine the type of convergence (absolute or conditional):

(a). $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ (b). $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^4+2}$ (c). $\sum_{n=1}^{\infty} \frac{\sin n + \cos n}{n^2}$

Solution :

(a). $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

(i). $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (a p-series with $p = \frac{1}{2} \le 1$).

(ii). $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is an alternating series with $\{a_n\} = \left\{\frac{1}{\sqrt{n}}\right\}$.

$a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$. So $\left\{\frac{1}{\sqrt{n}}\right\}$ is decreasing.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

So, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is conditionally convergent.

(b). $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^4+2}$

$\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{n^4+2}\right| = \sum_{n=0}^{\infty} \frac{1}{n^4+2}$

$\left|\frac{(-1)^n}{n^4+2}\right| = \frac{1}{n^4+2} \le \frac{1}{n^4}$.

Since $\sum_{n=0}^{\infty} \frac{1}{n^4}$ is convergent, then $\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{n^4+2}\right|$ is convergent.

Therefore, $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^4+2}$ is absolutely convergent.

(c). $\sum_{n=1}^{\infty} \frac{\sin n + \cos n}{n^2}$

$$\left|\frac{\sin n + \cos n}{n^2}\right| = \frac{|\sin n + \cos n|}{n^2} \leq \frac{2}{n^2} .$$

Since $\sum_{n=1}^{\infty} \frac{2}{n^2}$ is convergent, then $\sum_{n=1}^{\infty} \left|\frac{\sin n + \cos n}{n^2}\right|$ is convergent.

Therefore, $\sum_{n=1}^{\infty} \frac{\sin n + \cos n}{n^2}$ is absolutely convergent.

Question (7): [3 + 3 = 6 points]

1. Find the radius of convergence and the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^{n+1}}$.

Solution :

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{2^{n+2}} \frac{2^{n+1}}{(x-1)^n}\right| = \frac{|x-1|}{2}$$

$$\frac{|x-1|}{2} < 1 \implies |x-1| < 2.$$

The radius of convergence is $R = 2$.

$$|x-1| < 2 \implies -2 < x-1 < 2 \implies -1 < x < 3 .$$

(i). If $x = 3$ then $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{2^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2}$

Since $\lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} \neq 0$, then $\sum_{n=0}^{\infty} \frac{1}{2}$ is divergent.

(ii) If $x = -1$ then $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-2)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2}$

Since $\lim_{n\to\infty} \frac{(-1)^n}{2} = \begin{cases} -\frac{1}{2} & , \text{ n is odd} \\ \frac{1}{2} & , \text{ n is even} \end{cases}$, then $\lim_{n\to\infty} \frac{(-1)^n}{2}$ does not exist. Therefore, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2}$ is divergent.

The interval of convergence is $(-1, 3)$.

2. Use the Maclaurin series of the function $f(x) = e^x$ to write the function $g(x) = e^{2x+1}$ as a power series in $x - 1$, stating the interval of convergence.

Solution :

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, where $|x| < \infty$.

$e^{2x+1} = e^{2x-2+3} = e^3 \; e^{2x-2} = e^3 \; e^{2(x-1)}$

$= e^3 \; \sum_{n=0}^{\infty} \frac{[2(x-1)]^n}{n!} = \sum_{n=0}^{\infty} \frac{e^3 \; 2^n \; (x-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^3 \; 2^n}{n!} \; (x-1)^n$.

$2|x-1| < \infty \implies |x| < \infty$.

The interval of convergence is $(-\infty, \infty)$.