

MATH 201 - Multivariable Calculus
Second Semester - 1447 H
Solution of the First Exam
Dr Tariq A. Alfadhel

Question (1): [3 + 2 = 5 points]

1. Find and sketch the domain of the function $f(x, y) = \sqrt{x} + \sqrt{4 - x^2 - y^2}$.

Solution :

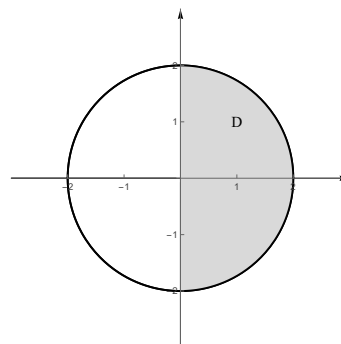
$\sqrt{x}$ is defined when $x \geq 0$.
 defined on the right-half plane.

$\sqrt{4 - x^2 - y^2}$ is defined when
 $4 - x^2 - y^2 \geq 0$
 $\implies x^2 + y^2 \leq 4$.
 defined on and inside the circle, centered at the origin, with radius 2.

The domain D is :

$$\{(x, y) \in \mathbb{R}^2 : x \geq 0, 4 - x^2 - y^2 \geq 0\}$$

$$= \{(x, y) \in \mathbb{R}^2 : x \geq 0, x^2 + y^2 \leq 4\}.$$



2. Prove that $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^5}{x^8 + y^{10}}$ does not exist.

Solution :

(a). On the path $x = 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^5}{x^8 + y^{10}} = \lim_{y \rightarrow 0} \frac{0 \cdot y^5}{0 + y^{10}} = 0 .$$

(b) On the path $x = y^5$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^5}{x^8 + y^{10}} = \lim_{y \rightarrow 0} \frac{y^5 y^5}{(y^5)^8 + y^{10}} = \lim_{y \rightarrow 0} \frac{y^{10}}{y^{40} + y^{10}}$$

$$= \lim_{y \rightarrow 0} \frac{y^{10}}{y^{10}(y^{30} + 1)} = \lim_{y \rightarrow 0} \frac{1}{y^{30} + 1} = \frac{1}{0 + 1} = 1.$$

Therefore, $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^5}{x^8 + y^{10}}$ does not exist.

Another solution : on the path $y = x$:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^5}{x^8 + y^{10}} = \lim_{x \rightarrow 0} \frac{x \cdot x^5}{x^8 + x^{10}} = \lim_{x \rightarrow 0} \frac{x^6}{x^6(x^2 + x^4)}$$

$$= \lim_{x \rightarrow 0} \frac{1}{x^2 + x^4} = \infty.$$

Therefore, $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^5}{x^8 + y^{10}}$ does not exist.

Question (2): [2 + 4 = 6 points]

Let the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by :

$$f(x, y) = \begin{cases} \frac{x^3 + y^3}{\sqrt{x^2 + y^2}} & , \quad (x, y) \neq (0, 0) \\ 0 & , \quad (x, y) = (0, 0) \end{cases}$$

1. Study the continuity of f at $(0, 0)$.

Solution :

$$f(0, 0) = 0 .$$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{\sqrt{x^2 + y^2}}$$

Using polar coordinates :

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{\sqrt{x^2 + y^2}} &= \lim_{r \rightarrow 0^+} \frac{r^3 \cos^3 \theta + r^3 \sin^3 \theta}{r} \\ &= \lim_{r \rightarrow 0^+} \frac{r^3 (\cos^3 \theta + \sin^3 \theta)}{r} = \lim_{r \rightarrow 0^+} r^2 (\cos^3 \theta + \sin^3 \theta) = 0 \end{aligned}$$

(Note that $\lim_{r \rightarrow 0^+} r^2 = 0$ and $\cos^3 \theta + \sin^3 \theta$ is bounded)

Therefore, $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = f(0, 0)$. Hence f is continuous at $(0, 0)$.

2. Study the differentiability of f at $(0, 0)$.

Solution :

$$f(0, 0) = 0 .$$

$$f(0 + \Delta x, 0 + \Delta y) = f(\Delta x, \Delta y) = \frac{(\Delta x)^3 + (\Delta y)^3}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$\begin{aligned} f_x(0, 0) &= \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\left(\frac{h^3}{\sqrt{h^2}}\right)}{h} = \lim_{h \rightarrow 0} \frac{h^3}{h |h|} \\ &= \lim_{h \rightarrow 0} \frac{h^2}{|h|} = 0 . \end{aligned}$$

(Note that $\lim_{h \rightarrow 0^+} \frac{h^2}{|h|} = \lim_{h \rightarrow 0} \frac{h^2}{h} = \lim_{h \rightarrow 0} h = 0$ and

$$\lim_{h \rightarrow 0^-} \frac{h^2}{|h|} = \lim_{h \rightarrow 0} \frac{h^2}{-h} = \lim_{h \rightarrow 0} -h = 0) .$$

$$\begin{aligned}
f_y(0,0) &= \lim_{h \rightarrow 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\left(\frac{h^3}{\sqrt{h^2}}\right)}{h} = \lim_{h \rightarrow 0} \frac{h^3}{h|h|} \\
&= \lim_{h \rightarrow 0} \frac{h^2}{|h|} = 0 . \\
\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{f(\Delta x, \Delta y) - f(0,0) - f_x(0,0) \Delta x - f_y(0,0) \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \\
&= \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{\left(\frac{(\Delta x)^3 + (\Delta y)^3}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}\right)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{(\Delta x)^3 + (\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2}
\end{aligned}$$

Using polar coordinates :

$$\begin{aligned}
\lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{(\Delta x)^3 + (\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2} &= \lim_{r \rightarrow 0^+} \frac{r^3 \cos^3 \theta + r^3 \sin^3 \theta}{r^2} \\
&= \lim_{r \rightarrow 0^+} \frac{r^3 (\cos^3 \theta + \sin^3 \theta)}{r^2} = \lim_{r \rightarrow 0^+} r (\cos^3 \theta + \sin^3 \theta) = 0
\end{aligned}$$

(Note that $\lim_{r \rightarrow 0^+} r = 0$ and $\cos^3 \theta + \sin^3 \theta$ is bounded)

Therefore, f is differentiable at $(0,0)$.

$$\begin{aligned}
\text{Another solution : to evaluate } \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{(\Delta x)^3 + (\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2} \\
0 \leq \left| \frac{(\Delta x)^3 + (\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2} \right| &\leq \frac{(\Delta x)^2}{(\Delta x)^2 + (\Delta y)^2} |\Delta x| + \frac{(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} |\Delta y| \\
0 \leq \left| \frac{(\Delta x)^3 + (\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2} \right| &\leq |\Delta x| + |\Delta y| \\
\text{Note that } \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} 0 &= 0 \text{ and } \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} |\Delta x| + |\Delta y| = 0 + 0 = 0
\end{aligned}$$

$$\text{By Squeeze Theorem } \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{(\Delta x)^3 + (\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2} = 0 .$$

Therefore, f is differentiable at $(0,0)$.

Question (3): [4 points]

Let $w = x^2 + y^2 + z^2$, where $x = u \cos v$, $y = u \sin v$ and $z = uv$.

Use the chain rule to find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.

Solution :

$$\begin{aligned}
(1). \quad \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} \\
&= (2x)(\cos v) + (2y)(\sin v) + (2z)(v) \\
&= 2u \cos^2 v + 2u \sin^2 v + 2uv^2 = 2u(\cos^2 v + \sin^2 v + v) = 2u(1 + v^2) .
\end{aligned}$$

$$\begin{aligned}
(2). \quad \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v} \\
&= (2x)(-u \sin v) + (2y)(u \cos v) + (2z)(u) \\
&= -2u^2 \sin v \cos v + 2u^2 \sin v \cos v + 2u^2 v = 2u^2 v .
\end{aligned}$$

Question (4): [4 points]

Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$x \ln y + y(z + 1) = z^2 + 1$$

Solution :

$$x \ln y + y(z + 1) = z^2 + 1 \implies x \ln y + y(z + 1) - z^2 - 1 = 0.$$

Let $F(x, y, z) = x \ln y + y(z + 1) - z^2 - 1$, then $F(x, y, z) = 0$.

If z is differentiable in x and y , then :

$$\begin{aligned}
(1). \quad \frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = -\frac{\ln y}{y - 2z} . \\
(2). \quad \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = -\frac{\frac{x}{y} + z + 1}{y - 2z} .
\end{aligned}$$

Question (5): [3 + 3 = 6 points]

Consider the function $f(x, y) = x^4 + y^4 - 4xy + 1$,

1. Show that $(0, 0)$, $(1, 1)$ and $(-1, -1)$ are the critical points of f .

Solution :

$$\begin{aligned}
f_x(x, y) = 0 &\implies 4x^3 - 4y = 0 \implies 4x^3 = 4y \implies x^3 = y . \\
f_y(x, y) = 0 &\implies 4y^3 - 4x = 0 \implies 4y^3 = 4x \implies y^3 = x . \\
f_x(x, y) = f_y(x, y) &\implies x = (x^3)^3 \implies x^9 = x \implies x^9 - x = 0 \\
&\implies x(x^8 - 1) = 0 \implies x(x^4 - 1)(x^4 + 1) = 0 \\
&\implies x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0 \implies x(x - 1)(x + 1)(x^2 + 1)(x^4 + 1) = 0 \\
&\implies \begin{cases} x = 0 \\ x - 1 = 0 \\ x + 1 = 0 \end{cases} \implies \begin{cases} x = 0 \\ x = 1 \\ x = -1 \end{cases} \implies \begin{cases} y = 0^3 = 0 \\ y = 1^3 = 1 \\ y = (-1)^3 = -1 \end{cases}
\end{aligned}$$

(Note that $x^2 + 1 > 0$ and $x^4 + 1 > 0$).

Therefore, the critical points of f are $(0, 0)$, $(1, 1)$ and $(-1, -1)$.

2. Classify each critical point as a local minimum, maximum or saddle point.

Solution :

$$f_{xx}(x, y) = 12x^2 \text{ , } f_{yy}(x, y) = 12y^2 \text{ and } f_{xy}(x, y) = -4 \text{ .}$$

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 = 144x^2y^2 - 16 \text{ .}$$

(a). The critical point $(0, 0)$:

$$D(0, 0) = 0 - 16 = -16 < 0 \text{ , } f(0, 0) = 0 + 0 - 0 + 1 = 1.$$

Therefore, $(0, 0, 1)$ is a saddle point.

(b). The critical point $(1, 1)$:

$$D(1, 1) = 144 - 16 > 0 \text{ and } f_{xx}(1, 1) = 12 > 0 \text{ .}$$

$$f(1, 1) = 1 + 1 - 4 + 1 = -1.$$

Therefore, $(1, 1, -1)$ is a local minimum.

(c). The critical point $(-1, -1)$:

$$D(-1, -1) = 144 - 16 > 0 \text{ and } f_{xx}(-1, -1) = 12 > 0 \text{ ,}$$

$$f(-1, -1) = 1 + 1 - 4 + 1 = -1.$$

Therefore, $(-1, -1, -1)$ is a local minimum.

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Question (1): [5 points]

Evaluate the iterated integral $\int_0^1 \int_1^2 (e^y e^{-x} + y^2) \, dx \, dy$.

First Solution :

$$\begin{aligned} \int_0^1 \int_1^2 (e^y e^{-x} + y^2) \, dx \, dy &= \int_0^1 [-e^y e^{-x} + xy^2]_1^2 \, dy \\ &= \int_0^1 [(-e^y e^{-2} + 2y^2) - (-e^y e^{-1} + y^2)] \, dy \\ &= \int_0^1 (-e^y e^{-2} + 2y^2 + e^y e^{-1} - y^2) \, dy = \int_0^1 (e^y (e^{-1} - e^{-2}) + y^2) \, dy \\ &= \left[e^y (e^{-1} - e^{-2}) + \frac{y^3}{3} \right]_0^1 = e(e^{-1} - e^{-2}) + \frac{1}{3} - (e^{-1} - e^{-2}) \\ &= 1 - e^{-1} + \frac{1}{3} - e^{-1} + e^{-2} = \frac{4}{3} + \frac{1}{e^2} - \frac{2}{e}. \end{aligned}$$

Second Solution : Using Fubini's Theorem

$$\begin{aligned} \int_0^1 \int_1^2 (e^y e^{-x} + y^2) \, dx \, dy &= \int_1^2 \int_0^1 (e^y e^{-x} + y^2) \, dy \, dx \\ &= \int_1^2 \left[e^y e^{-x} + \frac{y^3}{3} \right]_0^1 \, dx = \int_1^2 \left[(e^1 e^{-x} + \frac{1}{3}) - (e^0 e^{-x} + 0) \right] \, dx \\ &= \int_1^2 \left(e e^{-x} + \frac{1}{3} - e^{-x} \right) \, dx = \int_1^2 \left((e-1)e^{-x} + \frac{1}{3} \right) \, dx \\ &= \left[-(e-1)e^{-x} + \frac{x}{3} \right]_1^2 = -(e-1)e^{-2} + \frac{2}{3} - \left(-(e-1)e^{-1} + \frac{1}{3} \right) \\ &= -e^{-1} + e^{-2} + \frac{2}{3} + 1 - e^{-1} - \frac{1}{3} = \frac{4}{3} + \frac{1}{e^2} - \frac{2}{e}. \end{aligned}$$

Question (2): [4 points]

Find the area of the region that is bounded below by $y = 0$ and above by the curves $x + y = 2$ and $y = x^2$.

Solution:

$y = 0$ is the x -axis.

$y = x^2$ represents a parabola opens upwards with vertex $(0,0)$.

$y = -x + 2$ is a straight line passing through $(0,2)$ with slope -1 .

Points of intersection of $y = x^2$ and $y = -x + 2$:

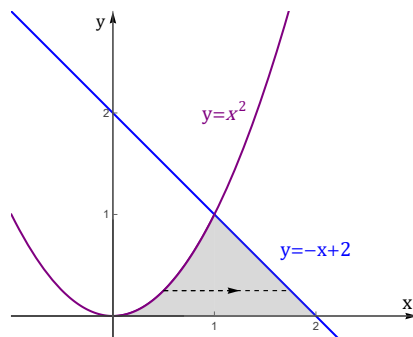
$$x^2 = -x + 2 \implies x^2 + x - 2 = 0$$

$$\implies (x+2)(x-1) = 0$$

$$\implies x = 1, x = -2.$$

Note that $x = -2$ is excluded.

If $x = 1$ then $y = 1$.



$y = -x + 2$ intersects $y = 0$ at $x = 2$.

First solution :

$$\begin{aligned}
 \text{Area} &= \int_0^1 \int_0^{x^2} 1 \, dy \, dx + \int_1^2 \int_0^{-x+2} 1 \, dy \, dx \\
 &= \int_0^1 [y]_0^{x^2} \, dx + \int_1^2 [y]_0^{-x+2} \, dx = \int_0^1 (x^2 - 0) \, dx + \int_1^2 (-x + 2 - 0) \, dx \\
 &= \int_0^1 x^2 \, dx + \int_1^2 (-x + 2) \, dx = \left[\frac{x^3}{3} \right]_0^1 + \left[-\frac{x^2}{2} + 2x \right]_1^2 \\
 &= \left(\frac{1}{3} - 0 \right) + \left[\left(-\frac{4}{2} + 4 \right) - \left(-\frac{1}{2} + 2 \right) \right] = \frac{1}{3} - 2 + 4 - \frac{1}{2} - 2 = \frac{5}{6}
 \end{aligned}$$

Second solution :

$$y = x^2 \implies x = \sqrt{y}, \quad x + y = 2 \implies x = -y + 2.$$

$$\begin{aligned}
 \text{Area} &= \int_0^1 \int_{\sqrt{y}}^{-y+2} 1 \, dx \, dy = \int_0^1 (-y + 2 - \sqrt{y}) \, dy = \int_0^1 \left(-y - y^{\frac{1}{2}} + 2 \right) \, dy \\
 &= \left[-\frac{y^2}{2} - \frac{2}{3}y^{\frac{3}{2}} + 2y \right]_0^1 = \left(-\frac{1}{2} - \frac{2}{3} + 2 \right) - (0 - 0 + 0) = \frac{-3 - 4 + 12}{6} = \frac{5}{6}.
 \end{aligned}$$

Question (3): [4 points]

Change the order of integration and then evaluate the integral

$$\int_0^{\frac{\pi}{4}} \int_y^{\frac{\pi}{4}} \left(\frac{\sec^2 x}{x} \right) \, dx \, dy$$

Solution:

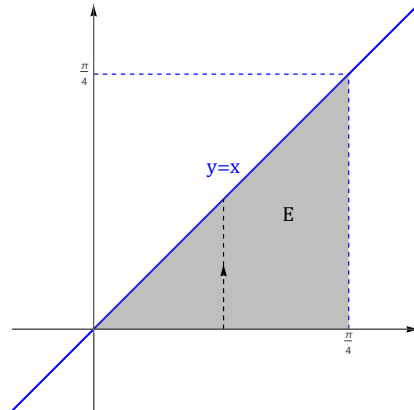
Let E be the region where:

$$0 \leq y \leq \frac{\pi}{4} \text{ and } y \leq x \leq \frac{\pi}{4}.$$

Then E can be written as :

$$0 \leq x \leq \frac{\pi}{4} \text{ and } 0 \leq y \leq x.$$

$$\begin{aligned}
 &\int_0^{\frac{\pi}{4}} \int_y^{\frac{\pi}{4}} \left(\frac{\sec^2 x}{x} \right) \, dx \, dy \\
 &= \lim_{t \rightarrow 0^+} \int_t^{\frac{\pi}{4}} \int_0^x \left(\frac{\sec^2 x}{x} \right) \, dy \, dx \\
 &= \lim_{t \rightarrow 0^+} \int_t^{\frac{\pi}{4}} \left(\frac{\sec^2 x}{x} \right) [y]_0^x \, dx \\
 &= \lim_{t \rightarrow 0^+} \int_t^{\frac{\pi}{4}} \left(\frac{\sec^2 x}{x} \right) (x - 0) \, dx = \lim_{t \rightarrow 0^+} \int_0^{\frac{\pi}{4}} \left(\frac{\sec^2 x}{x} \right) x \, dx \\
 &= \lim_{t \rightarrow 0^+} \int_t^{\frac{\pi}{4}} \sec^2 x \, dx = \lim_{t \rightarrow 0^+} [\tan x]_t^{\frac{\pi}{4}} \\
 &= \lim_{t \rightarrow 0^+} \left[\tan\left(\frac{\pi}{4}\right) - \tan(t) \right] = 1 - \tan(0) = 1 - 0 = 1.
 \end{aligned}$$



Question (4): [4 points]

Evaluate the iterated integral by converting to polar coordinates

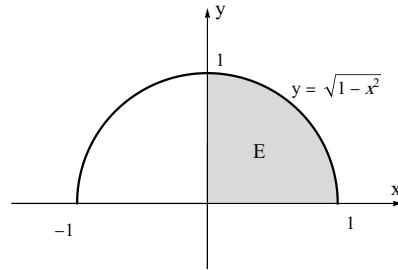
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1+x^2+y^2} \, dy \, dx$$

Solution:

Let E be the region where:

$$0 \leq x \leq 1 \text{ and } 0 \leq y \leq \sqrt{1-x^2}.$$

Note that $y = \sqrt{1-x^2}$ represents the upper-half of the unit circle.



Then E in polar coordinates :

$$0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq \frac{\pi}{2}.$$

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1+x^2+y^2} \, dy \, dx &= \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{1+r^2} \, r \, dr \, d\theta \\ &= \left(\frac{1}{2} \int_0^1 (1+r^2)^{\frac{1}{2}} (2r) \, dr \right) \left(\int_0^{\frac{\pi}{2}} d\theta \right) = \frac{1}{2} \left[\frac{(1+r^2)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 [\theta]_0^{\frac{\pi}{2}} \\ &= \frac{1}{2} \left(\frac{2}{3} (2)^{\frac{3}{2}} - \frac{2}{3} \right) \left(\frac{\pi}{2} - 0 \right) = \frac{1}{2} \cdot \frac{2}{3} (2\sqrt{2} - 1) \cdot \frac{\pi}{2} = \frac{\pi(2\sqrt{2}-1)}{6}. \end{aligned}$$

Question (5): [4 points]

Find the volume of the solid that lies inside the cylinder $x^2 + y^2 = 4$, above the plane $z = 0$, and below the paraboloid $z = 3 + x^2 + y^2$.

Solution:

Let E be the solid that lies inside the cylinder $x^2 + y^2 = 4$, above the plane $z = 0$, and below the paraboloid $z = 3 + x^2 + y^2$.

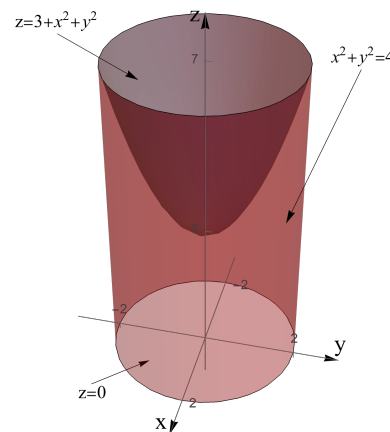
Then E in cylindrical coordinates:

$$0 \leq z \leq 3 + x^2 + y^2 = 3 + r^2,$$

$$0 \leq r \leq 2 \text{ and } 0 \leq \theta \leq 2\pi.$$

$$\begin{aligned} \text{Volume} &= V(E) = \iiint_E 1 \, dV \\ &= \int_0^{2\pi} \int_0^2 \int_0^{3+r^2} dz \, r \, dr \, d\theta \\ &= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^2 \int_0^{3+r^2} dz \, r \, dr \right) \end{aligned}$$

$$\begin{aligned} &= [\theta]_0^{2\pi} \left(\int_0^2 [z]_0^{3+r^2} r \, dr \right) = (2\pi - 0) \left(\int_0^2 [(3+r^2) - 0] r \, dr \right) \\ &= 2\pi \int_0^2 (3r + r^3) \, dr = 2\pi \left[\frac{3r^2}{2} + \frac{r^4}{4} \right]_0^2 \end{aligned}$$



$$= 2\pi \left[\left(\frac{3(2)^2}{2} + \frac{(2)^4}{4} \right) - (0+0) \right] = 2\pi(6+4) = 20\pi .$$

Question (6): [4 points]

Find the volume of the solid that lies within both the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 1$.

Solution:

Let E be the solid that lies within both the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 1$.

The sphere intersects the cone at:

$$x^2 + y^2 + (x^2 + y^2) = 1$$

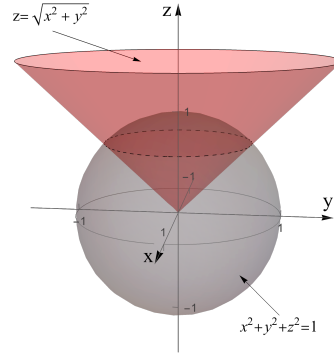
$$2x^2 + 2y^2 = 1$$

$$x^2 + y^2 = \frac{1}{2} = \left(\frac{1}{\sqrt{2}} \right)^2$$

$$z = \sqrt{x^2 + y^2} = r$$

$$\implies \rho \cos \phi = \rho \sin \phi$$

$$\implies \tan \phi = 1 \implies \phi = \frac{\pi}{4}.$$



E in spherical coordinates : $0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{\pi}{4}$.

$$\begin{aligned} \text{Volume} &= V(E) = \iiint_E 1 \, dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\frac{\pi}{4}} \sin \phi \, d\phi \right) \left(\int_0^1 \rho^2 \, d\rho \right) = [\theta]_0^{2\pi} [-\cos \phi]_0^{\frac{\pi}{4}} \left[\frac{\rho^3}{3} \right]_0^1 \\ &= (2\pi - 0) \left(-\frac{1}{\sqrt{2}} - (-1) \right) \left(\frac{1}{3} - 0 \right) = \frac{2\pi}{3} \left(1 - \frac{1}{\sqrt{2}} \right) = \frac{(2 - \sqrt{2})\pi}{3}. \end{aligned}$$

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Question (1): [2 + 2 + 2 = 6 points]

Let the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by :

$$f(x, y) = \begin{cases} \frac{x^3 y^2}{x^4 + y^4} & , \quad (x, y) \neq (0, 0) \\ 0 & , \quad (x, y) = (0, 0) \end{cases}$$

1. Study the continuity of f at the point $(0, 0)$.

Solution :

$$f(0, 0) = 0 .$$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 y^2}{x^4 + y^4}$$

Using polar coordinates :

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 y^2}{x^4 + y^4} &= \lim_{r \rightarrow 0^+} \frac{r^3 \cos^3 \theta \, r^2 \sin^2 \theta}{r^4 \cos^4 \theta + r^4 \sin^4 \theta} \\ &= \lim_{r \rightarrow 0^+} \frac{r^5 (\cos^3 \theta \sin^2 \theta)}{r^4 (\cos^4 \theta + \sin^4 \theta)} = \lim_{r \rightarrow 0^+} r \left(\frac{\cos^3 \theta \sin^2 \theta}{\cos^4 \theta + \sin^4 \theta} \right) = 0 \end{aligned}$$

(Note that $\lim_{r \rightarrow 0^+} r = 0$ and $\frac{\cos^3 \theta \sin^2 \theta}{\cos^4 \theta + \sin^4 \theta}$ is bounded)

Therefore, $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = f(0, 0)$. Hence f is continuous at $(0, 0)$.

Another solution :

$$0 \leq \left| \frac{x^3 y^2}{x^4 + y^4} \right| = \frac{|x^3| y^2}{x^4 + y^4} = \frac{|x| x^2 y^2}{x^4 + y^4} \leq |x|$$

Since $\lim_{x \rightarrow 0} |x| = 0$, then by squeeze theorem $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 y^2}{x^4 + y^4} = 0$.

Therefore, $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = f(0, 0)$. Hence f is continuous at $(0, 0)$.

2. Find the partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$.

Solution :

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\left(\frac{h^3 \cdot 0^2}{h^4 + 0^4} \right)}{h} = 0.$$

$$f_y(0,0) = \lim_{h \rightarrow 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\left(\frac{0^3 h^2}{0^4 + h^4}\right)}{h} = 0.$$

3. Study the differentiability of f at $(0,0)$.

Solution :

$$f(0,0) = 0.$$

$$f(0 + \Delta x, 0 + \Delta y) = f(\Delta x, \Delta y) = \frac{(\Delta x)^3 (\Delta y)^2}{(\Delta x)^4 + (\Delta y)^4}$$

$$\begin{aligned} & \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{f(\Delta x, \Delta y) - f(0,0) - f_x(0,0) \Delta x - f_y(0,0) \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \\ &= \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{\left(\frac{(\Delta x)^3 (\Delta y)^2}{(\Delta x)^4 + (\Delta y)^4}\right)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{(\Delta x)^3 (\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2} [(\Delta x)^4 + (\Delta y)^4]} \end{aligned}$$

By taking the path $\Delta x = \Delta y$:

$$\begin{aligned} & \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \left| \frac{(\Delta x)^3 (\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2} [(\Delta x)^4 + (\Delta y)^4]} \right| = \lim_{\Delta x \rightarrow 0} \frac{|\Delta x|^5}{2\sqrt{2} |\Delta x|^5} \\ &= \frac{1}{2\sqrt{2}} \neq 0. \text{ Therefore, } f \text{ is not differentiable at } (0,0). \end{aligned}$$

Another solution : Using polar coordinates

$$\begin{aligned} & \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{(\Delta x)^3 (\Delta y)^2}{\sqrt{(\Delta x)^2 + (\Delta y)^2} [(\Delta x)^4 + (\Delta y)^4]} = \lim_{r \rightarrow 0^+} \frac{r^3 \cos^3 \theta r^2 \sin^2 \theta}{r (r^4 \cos^4 \theta + r^4 \sin^4 \theta)} \\ &= \lim_{r \rightarrow 0^+} \frac{r^5 \cos^3 \theta \sin^2 \theta}{r^5 (\cos^4 \theta + \sin^4 \theta)} = \frac{\cos^3 \theta \sin^2 \theta}{\cos^4 \theta + \sin^4 \theta} \text{ (depends on } \theta) \end{aligned}$$

So, The limit does not exist. Therefore, f is not differentiable at $(0,0)$.

Question (2): [4 points]

Let f be a differentiable function in one variable, and let $w = f(x^2 + y^2 + z^2)$.

Use the chain rule to show that $yz \frac{\partial w}{\partial x} + xz \frac{\partial w}{\partial y} = 2xy \frac{\partial w}{\partial z}$.

Solution :

Let $u = x^2 + y^2 + z^2$, then $w = f(u)$ is a function of one variable.

$$\frac{\partial w}{\partial x} = \frac{dw}{du} \frac{\partial u}{\partial x} = \frac{dw}{du} (2x) = 2x \frac{dw}{du}.$$

$$\frac{\partial w}{\partial y} = \frac{dw}{du} \frac{\partial u}{\partial y} = \frac{dw}{du} (2y) = 2y \frac{dw}{du}.$$

$$\frac{\partial w}{\partial z} = \frac{dw}{du} \frac{\partial u}{\partial z} = \frac{dw}{du} (2z) = 2z \frac{dw}{du}.$$

$$yz \frac{\partial w}{\partial x} + xz \frac{\partial w}{\partial y} = yz \left(2x \frac{dw}{du} \right) + xz \left(2y \frac{dw}{du} \right)$$

$$= 2xyz \frac{dw}{du} + 2xyz \frac{dw}{du} = 4xyz \frac{dw}{du}.$$

$$2xy \frac{\partial w}{\partial z} = 2xy \left(2z \frac{dw}{du} \right) = 4xyz \frac{dw}{du}.$$

$$\text{Therefore, } yz \frac{\partial w}{\partial x} + xz \frac{\partial w}{\partial y} = 2xy \frac{\partial w}{\partial z}.$$

Question (3): [5 points]

Find the maximum and minimum values of the function $f(x, y, z) = x + y - z + 1$ on the sphere $x^2 + y^2 + z^2 = 12$.

Solution : Using Lagrange multipliers method.

Let $g(x, y, z) = x^2 + y^2 + z^2 - 12 = 0$ be the constraint.

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \implies \langle 1, 1, -1 \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

$$\begin{cases} 1 = 2\lambda x \\ 1 = 2\lambda y \\ -1 = 2\lambda z \end{cases} \implies \begin{cases} x = \frac{1}{2\lambda} \\ y = \frac{1}{2\lambda} \\ z = \frac{-1}{2\lambda} \end{cases} \implies x = y = -z$$

$$\text{Or : } \begin{cases} 1 = 2\lambda x \\ 1 = 2\lambda y \\ -1 = 2\lambda z \end{cases} \implies \begin{cases} \frac{1}{1} = \frac{2\lambda x}{2\lambda y} \implies x = y \\ \frac{1}{-1} = \frac{2\lambda x}{2\lambda z} \implies x = -z \end{cases}$$

Note that $\lambda \neq 0$.

$$\text{From the constraint : } x^2 + y^2 + z^2 = 12 \implies x^2 + x^2 + (-x)^2 = 12$$

$$\implies 3x^2 = 12 \implies x^2 = 4 \implies x = \pm 2.$$

If $x = 2$: $y = 2$ and $z = -2$, so the critical point is $(2, 2, -2)$.

If $x = -2$: $y = -2$ and $z = 2$, so the critical point is $(-2, -2, 2)$.

$$f(2, 2, -2) = 2 + 2 - (-2) + 1 = 7.$$

$$f(-2, -2, 2) = -2 - 2 - 2 + 1 = -5 .$$

The maximum value of the function is 7 attained at $(2, 2, -2)$.

The minimum value of the function is -5 attained at $(-2, -2, 2)$.

Question (4): [3.5 + 3.5 = 7 points]

1. Change the order of integration to evaluate the double integral

$$\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{x^3 + 1} dx dy$$

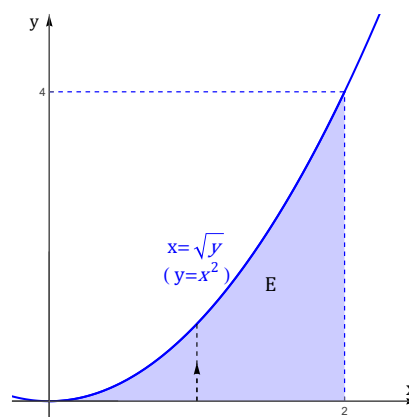
Solution :

Let E be the region where:
 $0 \leq y \leq 4$ and $\sqrt{y} \leq x \leq 2$.

Then E can be written as :
 $0 \leq x \leq 2$ and $0 \leq y \leq x^2$.

$$\begin{aligned} & \int_0^4 \int_{\sqrt{y}}^2 \frac{1}{x^3 + 1} dx dy \\ &= \int_0^2 \int_0^{x^2} \frac{1}{x^3 + 1} dy dx \\ &= \int_0^2 \frac{1}{x^3 + 1} [y]_0^{x^2} dx \end{aligned}$$

$$\begin{aligned} &= \int_0^2 \frac{x^2}{x^3 + 1} dx = \frac{1}{3} \int_0^2 \frac{3x^2}{x^3 + 1} dx = \frac{1}{3} [\ln |x^3 + 1|]_0^2 \\ &= \frac{1}{3} [\ln |2^3 + 1| - \ln |0^3 + 1|] = \frac{1}{3} [\ln(9) - \ln(1)] = \frac{\ln(9)}{3} = \frac{2\ln(3)}{3} . \end{aligned}$$



2. Evaluate the iterated integral by converting to polar coordinates

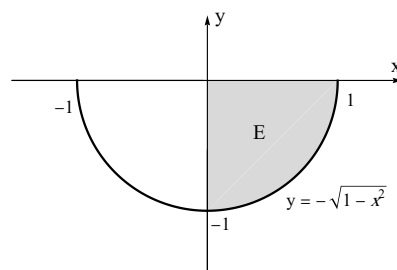
$$\int_0^1 \int_{-\sqrt{1-x^2}}^0 xy dy dx$$

Solution :

Let E be the region where:
 $0 \leq x \leq 1$ and $-\sqrt{1-x^2} \leq y \leq 0$.

Note that $y = -\sqrt{1-x^2}$ represents the lower-half of the unit circle.

Then E in polar coordinates :
 $0 \leq r \leq 1$ and $-\frac{\pi}{2} \leq \theta \leq 0$.



$$\int_0^1 \int_{-\sqrt{1-x^2}}^0 xy dy dx = \int_{-\frac{\pi}{2}}^0 \int_0^1 (r \cos \theta)(r \sin \theta) r dr d\theta$$

$$\begin{aligned}
&= \left(\int_0^1 r^3 dr \right) \left(\int_{-\frac{\pi}{2}}^0 \sin \theta \cos \theta d\theta \right) = \left[\frac{r^4}{4} \right]_0^1 \left[\frac{\sin^2 \theta}{2} \right]_{-\frac{\pi}{2}}^0 \\
&= \left(\frac{1}{4} - 0 \right) \left(0 - \frac{1}{2} \right) = -\frac{1}{8} .
\end{aligned}$$

Question (5): [3 + 3 = 6 points]

1. Find the volume of the solid that lies within the two paraboloids $z = x^2 + y^2 + 1$ and $z = 3 - x^2 - y^2$.

Solution :

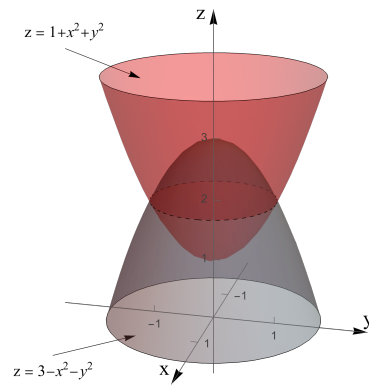
Let E be the solid that lies within the two paraboloids $z = x^2 + y^2 + 1$ and $z = 3 - x^2 - y^2$.

The two paraboloids intersect at
 $x^2 + y^2 + 1 = 3 - x^2 - y^2$
 $\implies 2x^2 + 2y^2 = 2$
 $\implies x^2 + y^2 = 1$

They intersect at the unit circle.

Then E in cylindrical coordinates:

$$\begin{aligned}
&x^2 + y^2 + 1 \leq z \leq 3 - x^2 - y^2, \\
&1 + r^2 \leq z \leq 3 - r^2, \\
&0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq 2\pi.
\end{aligned}$$



$$\begin{aligned}
\text{Volume} &= V(E) = \iiint_E 1 dV = \int_0^{2\pi} \int_0^1 \int_{1+r^2}^{3-r^2} dz r dr d\theta \\
&= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^1 \int_{1+r^2}^{3-r^2} dz r dr \right) = [\theta]_0^{2\pi} \left(\int_0^1 [z]_{1+r^2}^{3-r^2} r dr \right) \\
&= (2\pi - 0) \int_0^1 (3 - r^2 - (1 + r^2)) r dr = 2\pi \int_0^1 (2 - 2r^2) r dr \\
&= 2\pi \left(-\frac{1}{4} \int_0^1 (2 - 2r^2) (-4r) dr \right) = 2\pi \left[\frac{(2 - 2r^2)^2}{2} \right]_0^1 \\
&= 2\pi \frac{-1}{4} \left(0 - \frac{4}{2} \right) = 2\pi \frac{-1}{4} (-2) = \pi .
\end{aligned}$$

2. Find the volume of the solid that lies within both the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{\frac{1}{3}(x^2 + y^2)}$.

Solution :

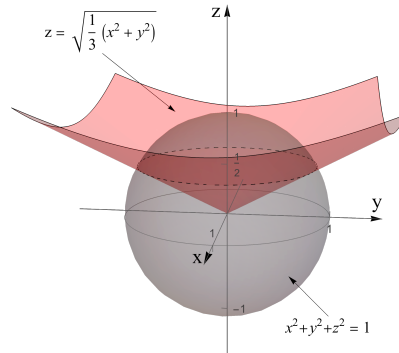
Let E be the solid that lies within both the cone $z = \sqrt{\frac{1}{3}(x^2 + y^2)}$ and the sphere $x^2 + y^2 + z^2 = 1$.

The sphere intersects the cone at:

$$x^2 + y^2 + \frac{1}{3}(x^2 + y^2) = 1$$

$$\frac{4}{3}(x^2 + y^2) = 1$$

$$x^2 + y^2 = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2.$$



$$z = \sqrt{\frac{1}{3}(x^2 + y^2)} = \frac{1}{\sqrt{3}}r \implies \rho \cos \phi = \frac{1}{\sqrt{3}}\rho \sin \phi$$

$$\implies \tan \phi = \sqrt{3} \implies \phi = \frac{\pi}{3}.$$

E in spherical coordinates : $0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{\pi}{3}$.

$$\begin{aligned} \text{Volume} = V(E) &= \iiint_E 1 \, dV = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\frac{\pi}{3}} \sin \phi \, d\phi \right) \left(\int_0^1 \rho^2 \, d\rho \right) = [\theta]_0^{2\pi} [-\cos \phi]_0^{\frac{\pi}{3}} \left[\frac{\rho^3}{3} \right]_0^1 \\ &= (2\pi - 0) \left(-\frac{1}{2} - (-1) \right) \left(\frac{1}{3} - 0 \right) = 2\pi \frac{1}{2} \frac{1}{3} = \frac{\pi}{3}. \end{aligned}$$

Question (6): [1.5 + (1.5 + 1.5 + 1.5) = 6 points]

1. Use the Integral Test to determine whether the series $\sum_{n=0}^{\infty} \frac{2n+1}{n^2+n+1}$ converges or diverges.

Solution :

Let $f(x) = \frac{2x+1}{x^2+x+1}$ defined on the interval $[1, \infty)$.

(i). f is positive and continuous on $[1, \infty)$.

$$(ii). f'(x) = \frac{2(x^2+x+1) - (2x+1)(x+1)}{(x^2+x+1)^2}$$

$$= \frac{2x^2+2x+2-4x^2-8x-1}{(x^2+x+1)^2} = \frac{-2x^2-6x+1}{(x^2+x+1)^2} < 0, \forall x \geq 1.$$

So, f is decreasing on the interval $[1, \infty)$.

$$\begin{aligned}\int_1^\infty \frac{2x+1}{x^2+x+1} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{2x+1}{x^2+x+1} dx = \lim_{t \rightarrow \infty} [\ln |x^2+x+1|]_1^t \\ &= \lim_{t \rightarrow \infty} (\ln |t^2+t+1| - \ln(3)) = \infty.\end{aligned}$$

Since the improper integral diverges, then $\sum_{n=0}^{\infty} \frac{2n+1}{n^2+n+1}$ diverges.

2. Test the following series and determine the type of convergence (absolute or conditional):

$$(a). \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \quad (b). \sum_{n=0}^{\infty} \frac{(-1)^n}{n^4+2} \quad (c). \sum_{n=1}^{\infty} \frac{\sin n + \cos n}{n^2}$$

Solution :

$$(a). \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

$$(i). \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges (a } p\text{-series with } p = \frac{1}{2} \leq 1).$$

$$(ii). \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ is an alternating series with } \{a_n\} = \left\{ \frac{1}{\sqrt{n}} \right\}.$$

$$a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n. \text{ So } \left\{ \frac{1}{\sqrt{n}} \right\} \text{ is decreasing.}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0.$$

$$\text{So, } \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ is convergent.}$$

$$\text{Therefore, } \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ is conditionally convergent.}$$

$$(b). \sum_{n=0}^{\infty} \frac{(-1)^n}{n^4+2}$$

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n^4+2} \right| = \sum_{n=0}^{\infty} \frac{1}{n^4+2}$$

$$\left| \frac{(-1)^n}{n^4+2} \right| = \frac{1}{n^4+2} \leq \frac{1}{n^4}.$$

$$\text{Since } \sum_{n=0}^{\infty} \frac{1}{n^4} \text{ is convergent, then } \sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n^4+2} \right| \text{ is convergent.}$$

Therefore, $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^4 + 2}$ is absolutely convergent.

$$(c). \sum_{n=1}^{\infty} \frac{\sin n + \cos n}{n^2}$$

$$\left| \frac{\sin n + \cos n}{n^2} \right| = \frac{|\sin n + \cos n|}{n^2} \leq \frac{2}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{2}{n^2}$ is convergent, then $\sum_{n=1}^{\infty} \left| \frac{\sin n + \cos n}{n^2} \right|$ is convergent.

Therefore, $\sum_{n=1}^{\infty} \frac{\sin n + \cos n}{n^2}$ is absolutely convergent.

Question (7): [3 + 3 = 6 points]

1. Find the radius of convergence and the interval of convergence of the power

series $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^{n+1}}$.

Solution :

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+1}}{2^{n+2}} \cdot \frac{2^{n+1}}{(x-1)^n} \right| = \frac{|x-1|}{2}$$

$$\frac{|x-1|}{2} < 1 \implies |x-1| < 2.$$

The radius of convergence is $R = 2$.

$$|x-1| < 2 \implies -2 < x-1 < 2 \implies -1 < x < 3.$$

$$(i). \text{ If } x = 3 \text{ then } \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{2^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2}$$

Since $\lim_{n \rightarrow \infty} \frac{1}{2} = \frac{1}{2} \neq 0$, then $\sum_{n=0}^{\infty} \frac{1}{2}$ is divergent.

$$(ii) \text{ If } x = -1 \text{ then } \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-2)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2}$$

$$\text{Since } \lim_{n \rightarrow \infty} \frac{(-1)^n}{2} = \begin{cases} -\frac{1}{2} & , \text{ n is odd} \\ \frac{1}{2} & , \text{ n is even} \end{cases}, \text{ then } \lim_{n \rightarrow \infty} \frac{(-1)^n}{2} \text{ does not}$$

exist. Therefore, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2}$ is divergent.

The interval of convergence is $(-1, 3)$.

2. Use the Maclaurin series of the function $f(x) = e^x$ to write the function $g(x) = e^{2x+1}$ as a power series in $x - 1$, stating the interval of convergence.

Solution :

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} , \text{ where } |x| < \infty .$$

$$e^{2x+1} = e^{2x-2+3} = e^3 e^{2x-2} = e^3 e^{2(x-1)}$$

$$= e^3 \sum_{n=0}^{\infty} \frac{[2(x-1)]^n}{n!} = \sum_{n=0}^{\infty} \frac{e^3 2^n (x-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^3 2^n}{n!} (x-1)^n .$$

$$2|x-1| < \infty \implies |x| < \infty .$$

The interval of convergence is $(-\infty, \infty)$.