MATH 201 - Multivariable Calculus
Second Semester - 1447 H
Solution of the First Exam
Dr Tariq A. Alfadhel

Question (1): $[3 + 2 = 5$ points$]$

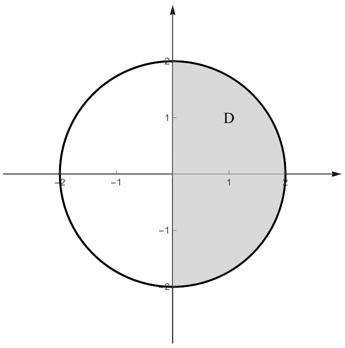
1. Find and sketch the domain of the function $f(x, y) = \sqrt{x} + \sqrt{4 - x^2 - y^2}$.

   Solution :

   $\sqrt{x}$ is defined when $x \geq 0$.
   defined on the right-half plane.

   $\sqrt{4 - x^2 - y^2}$ is defined when $4 - x^2 - y^2 \geq 0$
   $\implies x^2 - y^2 \leq 4$.
   defined on and inside the circle, centered at the origin, with radius 2.

   The domain D is :
   $\big\{(x, y) \in \mathbb{R}^2 \ : \ x \geq 0 \ , \ 4 - x^2 - y^2 \geq 0\big\}$
   $= \big\{(x, y) \in \mathbb{R}^2 \ : \ x \geq 0 \ , \ x^2 + y^2 \leq 4\big\}$.



2. Prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^5}{x^8 + y^{10}}$ does not exist.

   Solution :

   (a). On the path $x = 0$:

$$\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^8 + y^{10}} = \lim_{y\to 0} \frac{0 \ y^5}{0 + y^{10}} = 0 \ .$$

   (b) On the path $x = y^5$:

$$\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^8 + y^{10}} = \lim_{y\to 0} \frac{y^5 y^5}{(y^5)^8 + y^10} = \lim_{y\to 0} \frac{y^{10}}{y^{40} + y^{10}}$$

$$= \lim_{y\to 0} \frac{y^{10}}{y^{10}\left(y^{30} + 1\right)} = \lim_{y\to 0} \frac{1}{y^{30} + 1} = \frac{1}{0 + 1} = 1.$$

   Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^5}{x^8 + y^{10}}$ does not exist.

   Another solution : on the path $y = x$:

$$\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^8 + y^{10}} = \lim_{x\to 0} \frac{x \ x^5}{x^8 + x^{10}} = \lim_{x\to 0} \frac{x^6}{x^6\left(x^2 + x^4\right)}$$

$$= \lim_{x\to 0} \frac{1}{x^2 + x^4} = \infty.$$

Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^5}{x^8+y^{10}}$ does not exist.

Question (2): [2 + 4 = 6 points]
Let the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by :

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{\sqrt{x^2+y^2}} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

1. Study the continuity of f at $(0,0)$.

   Solution :

   $f(0,0) = 0$.

   $$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{\sqrt{x^2+y^2}}$$

   Using polar coordinates :

   $$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{\sqrt{x^2+y^2}} = \lim_{r\to0^+} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r}$$

   $$= \lim_{r\to0^+} \frac{r^3\left(\cos^3\theta+\sin^3\theta\right)}{r} = \lim_{r\to0^+} r^2\left(\cos^3\theta+\sin^3\theta\right) = 0$$

   (Note that $\lim\limits_{r\to0^+} r^2 = 0$ and $\cos^3\theta+\sin^3\theta$ is bounded)

   Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$. Hence f is continuous at $(0,0)$.

2. Study the differentiability of f at $(0,0)$.

   Solution :

   $f(0,0) = 0$.

   $$f(0+\Delta x, 0+\Delta y) = f(\Delta x, \Delta y) = \frac{(\Delta x)^3+(\Delta y)^3}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

   $$f_x(0,0) = \lim_{h\to0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to0} \frac{\left(\frac{h^3}{\sqrt{h^2}}\right)}{h} = \lim_{h\to0} \frac{h^3}{h\,|h|}$$

   $$= \lim_{h\to0} \frac{h^2}{|h|} = 0 \ .$$

   (Note that $\lim\limits_{h\to0^+} \dfrac{h^2}{|h|} = \lim\limits_{h\to0} \dfrac{h^2}{h} = \lim\limits_{h\to0} h = 0$ and

   $\lim\limits_{h\to0^-} \dfrac{h^2}{|h|} = \lim\limits_{h\to0} \dfrac{h^2}{-h} = \lim\limits_{h\to0} -h = 0$).

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\left(\frac{h^3}{\sqrt{h^2}}\right)}{h} = \lim_{h\to 0} \frac{h^3}{h\ |h|}$$

$$= \lim_{h\to 0} \frac{h^2}{|h|} = 0\ .$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f\left(\Delta x, \Delta y\right) - f(0,0) - f_x(0,0)\ \Delta x - f_y(0,0)\ \Delta y}{\sqrt{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\left(\frac{(\Delta x)^3+(\Delta y)^3}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\right)}{\sqrt{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\left(\Delta x\right)^3 + \left(\Delta y\right)^3}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}$$

Using polar coordinates :

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\left(\Delta x\right)^3 + \left(\Delta y\right)^3}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2} = \lim_{r\to 0^+} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2}$$

$$= \lim_{r\to 0^+} \frac{r^3\left(\cos^3\theta + \sin^3\theta\right)}{r^2} = \lim_{r\to 0^+} r\left(\cos^3\theta + \sin^3\theta\right) = 0$$

(Note that $\lim\limits_{r\to 0^+} r = 0$ and $\cos^3\theta + \sin^3\theta$ is bounded)

Therefore, f is differentiable at $(0,0)$.

Another solution : to evaluate $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{\left(\Delta x\right)^3 + \left(\Delta y\right)^3}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}$

$$0 \le \left|\frac{\left(\Delta x\right)^3 + \left(\Delta y\right)^3}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}\right| \le \frac{\left(\Delta x\right)^2}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}\ |\Delta x| + \frac{\left(\Delta y\right)^2}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}\ |\Delta y|$$

$$0 \le \left|\frac{\left(\Delta x\right)^3 + \left(\Delta y\right)^3}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}\right| \le |\Delta x| + |\Delta y|$$

Note that $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} 0 = 0$ and $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} |\Delta x| + |\Delta y| = 0 + 0 = 0$

By Squeeze Theorem $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \dfrac{\left(\Delta x\right)^3 + \left(\Delta y\right)^3}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2} = 0$.

Therefore, f is differentiable at $(0,0)$.

Question (3): [4 points]
Let $w = x^2 + y^2 + z^2$, where $x = u\cos v$, $y = u\sin v$ and $z = uv$.
Use the chain rule to find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

Solution :

(1). $\dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial u} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial u}$

$= (2x)(\cos v) + (2y)(\sin v) + (2z)(v)$

$= 2u\cos^2 v + 2u\sin^2 v + 2uv^2 = 2u(\cos^2 v + \sin^2 v + v) = 2u(1 + v^2)$.

(2). $\dfrac{\partial w}{\partial v} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial v} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial v}$

$= (2x)(-u\sin v) + (2y)(u\cos v) + (2z)(u)$

$= -2u^2\sin v\cos v + 2u^2\sin v\cos v + 2u^2 v = 2u^2 v$.

Question (4): [4 points]

Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$:

$$x\ln y + y(z+1) = z^2 + 1$$

Solution :

$x\ln y + y(z+1) = z^2 + 1 \implies x\ln y + y(z+1) - z^2 - 1 = 0.$

Let $F(x,y,z) = x\ln y + y(z+1) - z^2 - 1$, then $F(x,y,z) = 0$.

If z is differentiable in x and y, then :

(1). $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{\ln y}{y - 2z}$.

(2). $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{\frac{x}{y} + z + 1}{y - 2z}$.

Question (5): [3 + 3 = 6 points]

Consider the function $f(x,y) = x^4 + y^4 - 4xy + 1$,

1. Show that $(0,0)$, $(1,1)$ and $(-1,-1)$ are the critical points of f.

   Solution :

   $f_x(x,y) = 0 \implies 4x^3 - 4y = 0 \implies 4x^3 = 4y \implies x^3 = y$.

   $f_y(x,y) = 0 \implies 4y^3 - 4x = 0 \implies 4y^3 = 4x \implies y^3 = x$.

   $f_x(x,y) = f_y(x,y) \implies x = (x^3)^3 \implies x^9 = x \implies x^9 - x = 0$

   $\implies x(x^8 - 1) = 0 \implies x(x^4 - 1)(x^4 + 1) = 0$

   $\implies x(x^2-1)(x^2+1)(x^4+1) = 0 \implies x(x-1)(x+1)(x^2 = 1)(x^4+1) = 0$

   $$\implies \left\{\begin{array}{l} x = 0 \\ x - 1 = 0 \\ x + 1 = 0 \end{array}\right. \implies \left\{\begin{array}{l} x = 0 \\ x = 1 \\ x = -1 \end{array}\right. \implies \left\{\begin{array}{l} y = 0^3 = 0 \\ y = 1^3 = 1 \\ y = (-1)^3 = -1 \end{array}\right.$$

(Note that $x^2 + 1 > 0$ and $x^4 + 1 > 0$).

Therefore, the critical points of f are $(0, 0)$, $(1, 1)$ and $(-1, -1)$.

2. Classify each critical point as a local minimum, maximum or saddle point.

   Solution :

   $f_{xx}(x, y) = 12x^2$, $f_{yy}(x, y) = 12y^2$ and $f_{xy}(x, y) = -4$.

   $D(x, y) = F_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 = 144x^2y^2 - 16$.

   (a). The critical point $(0, 0)$:

   $D(0, 0) = 0 - 16 = -16 < 0$, $f(0, 0) = 0 + 0 - 0 + 1 = 1$.

   Therefore, $(0, 0, 1)$ is a saddle point.

   (b). The critical point $(1, 1)$:

   $D(1, 1) = 144 - 16 > 0$ and $f_{xx}(1, 1) = 12 > 0$.

   $f(1, 1) = 1 + 1 - 4 + 1 = -1$.

   Therefore, $(1, 1, -1)$ is a local minimum.

   (c). The critical point $(-1, -1)$:

   $D(-1, -1) = 144 - 16 > 0$ and $f_{xx}(-1, -1) = 12 > 0$,

   $f(-1, -1) = 1 + 1 - 4 + 1 = -1$.

   Therefore, $(-1, -1, -1)$ is a local minimum.

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Solution of the Second Exam

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Question (1): [5 points]

Evaluate the iterated integral $\int_0^1 \int_1^2 \left(e^y e^{-x} + y^2\right)\ dx\ dy$.

First Solution :

$$\int_0^1 \int_1^2 \left(e^y e^{-x} + y^2\right)\ dx\ dy = \int_0^1 \left[-e^y e^{-x} + xy^2\right]_1^2\ dy$$
$$= \int_0^1 \left[\left(-e^y e^{-2} + 2y^2\right) - \left(-e^y e^{-1} + y^2\right)\right]\ dy$$
$$= \int_0^1 \left(-e^y e^{-2} + 2y^2 + e^y e^{-1} - y^2\right)\ dy = \int_0^1 \left(e^y\left(e^{-1} - e^{-2}\right) + y^2\right)\ dy$$
$$= \left[e^y\left(e^{-1} - e^{-2}\right) + \frac{y^3}{3}\right]_0^1 = e\left(e^{-1} - e^{-2}\right) + \frac{1}{3} - \left(e^{-1} - e^{-2}\right)$$
$$= 1 - e^{-1} + \frac{1}{3} - e^{-1} + e^{-2} = \frac{4}{3} + \frac{1}{e^2} - \frac{2}{e}\ .$$

Second Solution : Using Fubini's Theorem

$$\int_0^1 \int_1^2 \left(e^y e^{-x} + y^2\right)\ dx\ dy = \int_1^2 \int_0^1 \left(e^y e^{-x} + y^2\right)\ dy\ dx$$
$$= \int_1^2 \left[e^y e^{-x} + \frac{y^3}{3}\right]_0^1\ dx = \int_1^2 \left[\left(e^1 e^{-x} + \frac{1}{3}\right) - \left(e^0 e^{-x} + 0\right)\right]\ dx$$
$$= \int_1^2 \left(e e^{-x} + \frac{1}{3} - e^{-x}\right)\ dx = \int_1^2 \left((e-1)e^{-x} + \frac{1}{3}\right)\ dx$$
$$= \left[-(e-1)e^{-x} + \frac{x}{3}\right]_1^2 = -(e-1)e^{-2} + \frac{2}{3} - \left(-(e-1)e^{-1} + \frac{1}{3}\right)$$
$$= -e^{-1} + e^{-2} + \frac{2}{3} + 1 - e^{-1} - \frac{1}{3} = \frac{4}{3} + \frac{1}{e^2} - \frac{2}{e}\ .$$

Question (2): [4 points]

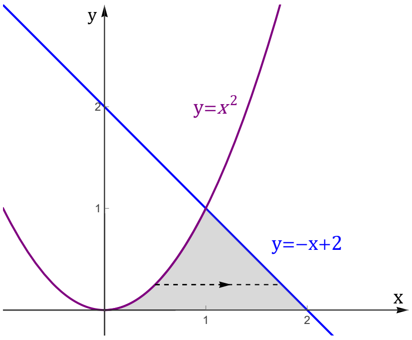
Find the area of the region that is bounded below by $y = 0$ and above by the curves $x + y = 2$ and $y = x^2$.

Solution:

$y = 0$ is the x-axis.

$y = x^2$ represents a parabola opens upwards with vertex $(0,0)$.

$y = -x + 2$ is a straight line passing through $(0,2)$ with slope -1.



Points of intersection of $y = x^2$ and $y = -x + 2$:

$x^2 = -x + 2 \implies x^2 + x - 2 = 0$

$\implies (x+2)(x-1) = 0$

$\implies x = 1\ ,\ x = -2\ .$

Note that $x = -2$ is excluded.

If $x = 1$ then $y = 1$.

$y = -x + 2$ intersects $y = 0$ at $x = 2$.

First solution :

Area $= \displaystyle\int_0^1 \int_0^{x^2} 1\ dy\ dx + \int_1^2 \int_0^{-x+2} 1\ dy\ dx$

$= \displaystyle\int_0^1 [y]_0^{x^2}\ dx + \int_1^2 [y]_0^{-x+2}\ dx = \int_0^1 (x^2 - 0)\ dx + \int_1^2 (-x + 2 - 0)\ dx$

$= \displaystyle\int_0^1 x^2\ dx + \int_1^2 (-x+2)\ dx = \left[\frac{x^3}{3}\right]_0^1 + \left[-\frac{x^2}{2} + 2x\right]_1^2$

$= \displaystyle\left(\frac{1}{3} - 0\right) + \left[\left(-\frac{4}{2} + 4\right) - \left(-\frac{1}{2} + 2\right)\right] = \frac{1}{3} - 2 + 4 - \frac{1}{2} - 2 = \frac{5}{6}$

Second solution :

$y = x^2 \implies x = \sqrt{y}$, $x + y = 2 \implies x = -y + 2$.

Area $= \displaystyle\int_0^1 \int_{\sqrt{y}}^{-y+2} 1\ dx\ dy = \int_0^1 (-y + 2 - \sqrt{y})\ dy = \int_0^1 \left(-y - y^{\frac{1}{2}} + 2\right)\ dy$

$= \displaystyle\left[-\frac{y^2}{2} - \frac{2}{3}y^{\frac{3}{2}} + 2y\right]_0^1 = \left(-\frac{1}{2} - \frac{2}{3} + 2\right) - (0 - 0 + 0) = \frac{-3 - 4 + 12}{6} = \frac{5}{6}.$

Question (3): [4 points]

Change the order of integration and then evaluate the integral

$$\int_0^{\frac{\pi}{4}} \int_y^{\frac{\pi}{4}} \left(\frac{\sec^2 x}{x}\right)\ dx\ dy$$

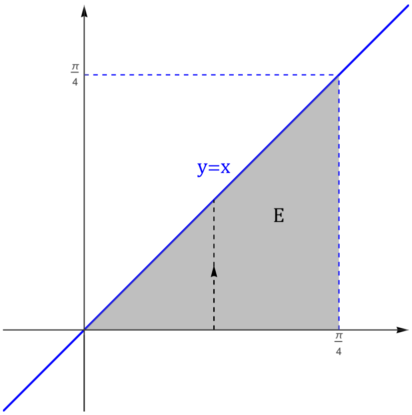
Solution:

Let E be the region where:

$0 \leq y \leq \dfrac{\pi}{4}$ and $y \leq x \leq \dfrac{\pi}{4}$.

Then E can be written as :

$0 \leq x \leq \dfrac{\pi}{4}$ and $0 \leq y \leq x$.



$\displaystyle\int_0^{\frac{\pi}{4}} \int_y^{\frac{\pi}{4}} \left(\frac{\sec^2 x}{x}\right)\ dx\ dy$

$= \displaystyle\lim_{t \to 0^+} \int_t^{\frac{\pi}{4}} \int_0^x \left(\frac{\sec^2 x}{x}\right)\ dy\ dx$

$= \displaystyle\lim_{t \to 0^+} \int_t^{\frac{\pi}{4}} \left(\frac{\sec^2 x}{x}\right)\ [y]_0^x\ dx$

$= \displaystyle\lim_{t \to 0^+} \int_t^{\frac{\pi}{4}} \left(\frac{\sec^2 x}{x}\right)\ (x - 0)\ dx = \lim_{t \to 0^+} \int_0^{\frac{\pi}{4}} \left(\frac{\sec^2 x}{x}\right) x\ dx$

$= \displaystyle\lim_{t \to 0^+} \int_t^{\frac{\pi}{4}} \sec^2 x\ dx = \lim_{t \to 0^+} [\tan x]_t^{\frac{\pi}{4}}$

$= \displaystyle\lim_{t \to 0^+} \left[\tan\left(\frac{\pi}{4}\right) - \tan(t)\right] = 1 - \tan(0) = 1 - 0 = 1$.

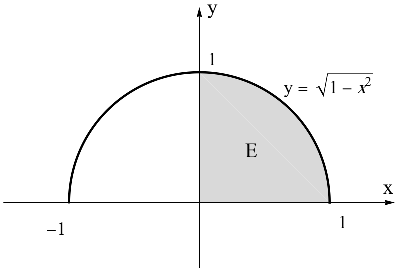
Question (4): [4 points]
Evaluate the iterated integral by converting to polar coordinates

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1+x^2+y^2}\ dy\ dx$$

Solution:
Let E be the region where:
$0 \leq x \leq 1$ and $0 \leq y \leq \sqrt{1-x^2}$.

Note that $y = \sqrt{1-x^2}$ represents the upper-half of the unit circle.

Then E in polar coordinates :
$0 \leq r \leq 1$ and $0 \leq \theta \leq \frac{\pi}{2}$.



$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1+x^2+y^2}\ dy\ dx = \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{1+r^2}\ r\ dr\ d\theta$$

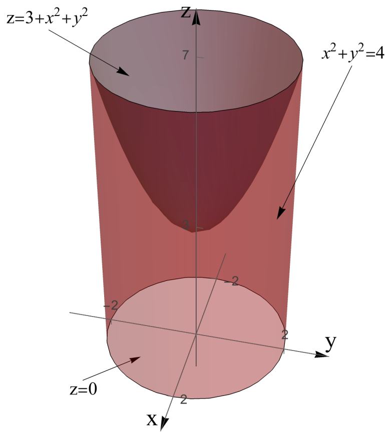
$$= \left(\frac{1}{2} \int_0^1 (1+r^2)^{\frac{1}{2}}\ (2r)\ dr\right) \left(\int_0^{\frac{\pi}{2}} d\theta\right) = \frac{1}{2} \left[\frac{(1+r^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 \ [\theta]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \left(\frac{2}{3}\ (2)^{\frac{3}{2}} - \frac{2}{3}\right) \left(\frac{\pi}{2} - 0\right) = \frac{1}{2}\ \frac{2}{3} \left(2\sqrt{2} - 1\right)\ \frac{\pi}{2} = \frac{\pi\left(2\sqrt{2}-1\right)}{6} .$$

Question (5): [4 points]
Find the volume of the solid that lies inside the cylinder $x^2 + y^2 = 4$, above the plane $z = 0$, and below the paraboloid $z = 3 + x^2 + y^2$.
Solution:

Let E be the solid that lies inside the cylinder $x^2 + y^2 = 4$, above the plane $z = 0$, and below the paraboloid $z = 3 + x^2 + y^2$.

Then E in cylindrical coordinates:
$0 \leq z \leq 3 + x^2 + y^2 = 3 + r^2$,
$0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$.



$$\text{Volume} = V(E) = \iiint_E 1\ dV$$

$$= \int_0^{2\pi} \int_0^2 \int_0^{3+r^2} dz\ r\ dr\ d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right) \left(\int_0^2 \int_0^{3+r^2} dz\ r\ dr\right)$$

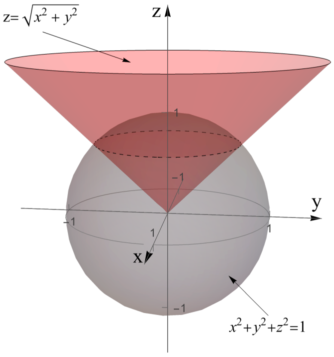
$$= [\theta]_0^{2\pi} \ \left(\int_0^2 [z]_0^{3+r^2}\ r\ dr\right) = (2\pi - 0) \ \left(\int_0^2 \left[(3+r^2) - 0\right]\ r\ dr\right)$$

$$= 2\pi \int_0^2 (3r + r^3)\ dr = 2\pi \left[\frac{3r^2}{2} + \frac{r^4}{4}\right]_0^2$$

$$= 2\pi \left[\left(\frac{3\ (2)^2}{2} + \frac{(2)^4}{4} \right) - (0+0) \right] = 2\pi(6+4) = 20\pi\ .$$

Question (6): [4 points]
Find the volume of the solid that lies within both the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 1$.
Solution:

Let E be the solid that lies within both the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 1$.



The sphere intersects the cone at:
$x^2+y^2+(x^2+y^2) = 1$
$2x^2+2y^2 = 1$
$x^2+y^2 = \frac{1}{2} = \left(\frac{1}{\sqrt{2}}\right)^2$
$z = \sqrt{x^2+y^2} = r$
$\implies\ \rho\cos\phi = \rho\sin\phi$
$\implies\ \tan\phi = 1 \implies\ \phi = \frac{\pi}{4}$.

E in spherical coordinates : $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \frac{\pi}{4}$.

$$\text{Volume} = V(E) = \iiint_E 1\ dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^2\ \sin\phi\ d\rho\ d\phi\ d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right) \left(\int_0^{\frac{\pi}{4}} \sin\phi\ d\phi\right) \left(\int_0^1 \rho^2\ d\rho\right) = [\theta]_0^{2\pi}\ \ [-\cos\phi]_0^{\frac{\pi}{4}}\ \ \left[\frac{\rho^3}{3}\right]_0^1$$

$$= (2\pi - 0)\ \left(-\frac{1}{\sqrt{2}} - (-1)\right)\ \left(\frac{1}{3} - 0\right) = \frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{(2-\sqrt{2})\,\pi}{3}.$$