MATH 201 - Multivariable Calculus
First Semester - 1448 H
Solution of the First Exam
Dr Tariq A. Alfadhel

Question (1): [3 + 2 = 5 points]

1. Find and sketch the domain of the function

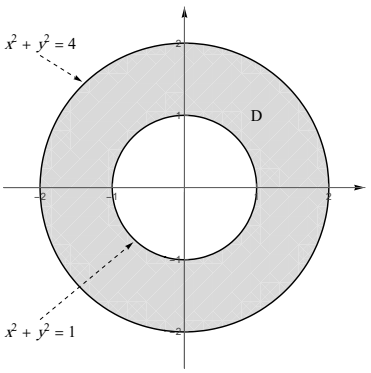
$$f(x,y) = \sqrt{x^2+y^2-1} + \sqrt{4-x^2-y^2}$$

Solution :

$\sqrt{x^2+y^2-1}$ is defined when
$x^2+y^2-1 \geq 0 \implies x^2+y^2 \geq 1$.
defined on and outside the unit circle.

$\sqrt{4-x^2-y^2}$ is defined when
$4-x^2-y^2 \geq 0 \implies x^2+y^2 \leq 4$.
defined on and inside the circle, centered at the origin, with radius 2.

The domain D is :
$\{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2 \geq 1 \ , \ x^2+y^2 \leq 4\}$
$= \{(x,y) \in \mathbb{R}^2 \ : \ 1 \leq x^2+y^2 \leq 4\}$.



2. Prove that $\lim\limits_{(x,y)\to(1,0)} \dfrac{(x-1)y}{(x-1)^2+y^2}$ does not exist.

First solution :

(a). On the path $y = 0$:

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)y}{(x-1)^2+y^2} = \lim_{x\to 1} \frac{(x-1)(0)}{(x-1)^2+(0)^2} = 0 \ .$$

(b) On the path $y = x-1$: Note that $y = x-1 \longrightarrow 0$ as $x \longrightarrow 1$,

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)y}{(x-1)^2+y^2} = \lim_{y\to 0} \frac{y\ y}{y^2+y^2} = \lim_{y\to 0} \frac{y^2}{2y^2} = \frac{1}{2} \ .$$

Therefore, $\lim\limits_{(x,y)\to(1,0)} \dfrac{(x-1)y}{(x-1)^2+y^2}$ does not exist.

Second solution : Let $y = m(x-1)$ where $m \in \mathbb{R}$,

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)y}{(x-1)^2+y^2} = \lim_{x\to 1} \frac{(x-1)[m(x-1)]}{(x-1)^2+[m(x-1)]^2}$$

$$= \lim_{x\to 1} \frac{m(x-1)^2}{(x-1)^2+m^2(x-1)^2} = \lim_{x\to 1} \frac{m(x-1)^2}{(x-1)^2\,(1+m^2)} = \frac{m}{1+m^2}.$$

The limit depends on m.

Therefore, $\lim\limits_{(x,y)\to(1,0)} \dfrac{(x-1)y}{(x-1)^2+y^2}$ does not exist.

Third solution :

Let $x-1 = r\cos\theta$, $y = r\sin\theta$, where $r>0$ and $\theta \in [0, 2\pi]$.

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)y}{(x-1)^2+y^2} = \lim_{r\to 0^+} \frac{(r\cos\theta)(r\sin\theta)}{(r\cos\theta)^2+(r\sin\theta)^2}$$

$$= \lim_{r\to 0^+} \frac{r^2\cos\theta\ \sin\theta}{r^2} = \cos\theta\ \sin\theta\ .$$

The limit depends on θ.

Therefore, $\lim\limits_{(x,y)\to(1,0)} \dfrac{(x-1)y}{(x-1)^2+y^2}$ does not exist.

Question (2): [2 + 4 = 6 points]
Let the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by :

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

1. Study the continuity of f at $(0,0)$.

   Solution :

   $f(0,0) = 0$.

   $$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^2}$$

   Using polar coordinates :

   $$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^2} = \lim_{r\to 0^+} \frac{(r\cos\theta)(r\sin\theta)^3}{r^2}$$

   $$= \lim_{r\to 0^+} \frac{r^4\cos\theta\ \sin^3\theta}{r^2} = \lim_{r\to 0^+} r^2\left(\cos\theta\sin^3\theta\right) = 0$$

   (Note that $\lim\limits_{r\to 0^+} r^2 = 0$ and $\cos\theta\sin^3\theta$ is bounded)

   Another solution : Using Squeeze theorem,

   $$0 \leq \left|\frac{xy^3}{x^2+y^2}\right| = \left|(xy)\ \frac{y^2}{x^2+y^2}\right| \leq |x\ y| = |x|\ |y|$$

   Note that $\lim\limits_{(x,y)\to(0,0)} 0 = \lim\limits_{(x,y)\to(0,0)} |x|\ |y| = 0.$

So, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^2} = 0$.

Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$. Hence f is continuous at $(0,0)$.

2. Study the differentiability of f at $(0,0)$.

Solution :

$f(0,0) = 0$.

$$f\left(0+\Delta x, 0+\Delta y\right) = f\left(\Delta x, \Delta y\right) = \frac{\Delta x \ \left(\Delta y\right)^3}{\left(\Delta x\right)^2 + \left(\Delta y\right)^2}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\left(\frac{h\ (0)^3}{h^2+0}\right)}{h} = 0.$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\left(\frac{(0)h^3}{0+h^2}\right)}{h} = 0.$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f\left(\Delta x,\Delta y\right) - f(0,0) - f_x(0,0)\ \Delta x - f_y(0,0)\ \Delta y}{\sqrt{\left(\Delta x\right)^2+\left(\Delta y\right)^2}}$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\left(\frac{\Delta x\ (\Delta y)^3}{(\Delta x)^2+(\Delta y)^2}\right)}{\sqrt{\left(\Delta x\right)^2+\left(\Delta y\right)^2}} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta x\left(\Delta y\right)^3}{\left[\left(\Delta x\right)^2+\left(\Delta y\right)^2\right]^{\frac{3}{2}}}$$

Using polar coordinates :

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta x\left(\Delta y\right)^3}{\left[\left(\Delta x\right)^2+\left(\Delta y\right)^2\right]^{\frac{3}{2}}} = \lim_{r\to 0^+} \frac{r\cos\theta\left(r\sin\theta\right)^3}{\left(r^2\right)^{\frac{3}{2}}}$$

$$= \lim_{r\to 0^+} \frac{r^4\left(\cos\theta\ \sin^3\theta\right)}{r^3} = \lim_{r\to 0^+} r\left(\cos\theta\ \sin^3\theta\right) = 0$$

(Note that $\lim\limits_{r\to 0^+} r = 0$ and $\left(\cos\theta\ \sin^3\theta\right)$ is bounded)

Therefore, f is differentiable at $(0,0)$.

Question (3): [4 points]
Let $w = \ln(xy+z^2)$, where $x = u^2+v$, $y = u+v^2$ and $z = uv$.
Use the chain rule to find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

Solution :

(1). $\dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial u} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial u}$

$$= \left(\frac{y}{xy+z^2}\right)(2u) + \left(\frac{x}{xy+z^2}\right)(1) + \left(\frac{2z}{xy+z^2}\right)(v)$$

$$= \frac{2u(u+v^2)+u^2+v+2v(uv)}{(u^2+v)(u+v^2)+u^2v^2}.$$

(2). $\dfrac{\partial w}{\partial v} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial v} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial v}$

$$= \left(\frac{y}{xy+z^2}\right)(1) + \left(\frac{x}{xy+z^2}\right)(2v) + \left(\frac{2z}{xy+z^2}\right)(u)$$

$$= \frac{u+v^2+2v(u^2+v)+2u(uv)}{(u^2+v)(u+v^2)+u^2v^2}.$$

Question (4): [4 points]

Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$:

$$x^2+y^2+z^2 = xyz+1$$

Solution :

$x^2+y^2+z^2 = xyz+1 \implies x^2+y^2+z^2-xyz-1=0.$

Let $F(x,y,z) = x^2+y^2+z^2-xyz-1$, then $F(x,y,z)=0$.

If z is differentiable in x and y, then :

(1). $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{2x-yz}{2z-xy}$.

(2). $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{2y-xz}{2z-xy}$.

Question (5): [3 + 3 = 6 points]

Consider the function $f(x,y) = 3x - x^3 + 3y - y^3$,

1. Find the critical points of f.

   Solution :

   $f_x(x,y) = 0 \implies 3-3x^2 = 0 \implies x^2 = 1 \implies x = \pm 1$.

   $f_y(x,y) = 0 \implies 3-3y^2 = 0 \implies y^2 = 1 \implies y = \pm 1$.

   Therefore, the critical points of f are $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$.

2. Classify each critical point as a local minimum, maximum or saddle point.

   Solution :

   $f_{xx}(x,y) = -6x$, $f_{yy}(x,y) = -6y$ and $f_{xy}(x,y) = 0$.

$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 = 36xy$.

(a). The critical point $(1,1)$:

$D(1,1) = 36(1)(1) = 36 > 0$ and $f_{xx}(1,1) = -6 < 0$.

$f(1,1) = 3 - 1 + 3 - 1 = 4.$

Therefore, $(1,1,4)$ is a local maximum.

(b). The critical point $(1,-1)$:

$D(1,-1) = 36(1)(-1) = -36 < 0$

$f(1,-1) = 3 - 1 - 3 + 1 = 0.$

Therefore, $(1,-1,0)$ is a saddle point.

(c). The critical point $(-1,1)$:

$D(-1,1) = 36(-1)(1) = -36 < 0$.

$f(-1,1) = -3 + 1 + 3 - 1 = 0.$

Therefore, $(-1,1,0)$ is a saddle point.

(d). The critical point $(-1,-1)$:

$D(-1,-1) = 36(-1)(-1) = 36 > 0$ and $f_{xx}(-1,-1) = 6 > 0$.

$f(-1,-1) = -3 + 1 - 3 + 1 = -4.$

Therefore, $(-1,-1,-4)$ is a local minimum.

MATH 201 - Multivariable Calculus
First Semester - 1448 H
Solution of the Second Exam
Dr Tariq A. Alfadhel

Question (1): [5 points]

Evaluate the iterated integral $\int_0^1 \int_0^x \int_0^{xy} 36z \; dz \; dy \; dx$.

Solution :

$$\int_0^1 \int_0^x \int_0^{xy} 36z \; dz \; dy \; dx = \int_0^1 \int_0^x \int_0^{xy} 18(2z) \; dz \; dy \; dx$$

$$= \int_0^1 \int_0^x 18 \left[z^2\right]_0^{xy} \; dy \; dx = \int_0^1 \int_0^x 18 \left[(xy)^2 - (0)^2\right] \; dy \; dx$$

$$= \int_0^1 \int_0^x 18x^2y^2 \; dy \; dx = \int_0^1 \int_0^x 6x^2 \left(3y^2\right) \; dy \; dx$$

$$= \int_0^1 6x^2 \left[y^3\right]_0^x \; dx = \int_0^1 6x^2 \left[x^3 - 0^3\right] \; dx$$

$$= \int_0^1 6x^2(x^3) \; dx = \int_0^1 6x^5 \; dx = \left[x^6\right]_0^1 = 1^6 - 0^6 = 1 \; .$$

Question (2): [4 points]

Determine the area of the region enclosed by the x-axis $y = 0$, the curve $y = \sqrt{x}$, and the vertical lines $x = 1$ and $x = 4$.

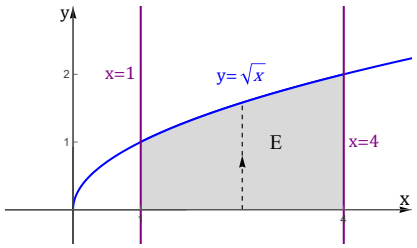
Solution:

$y = 0$ is the x-axis.

$y = \sqrt{x}$ represents the upper-half of the parabola $x = y^2$ opens to the right with vertex $(0, 0)$.

$x = 1$ is a vertical line passing through $(1, 0)$.

$x = 4$ is a vertical line passing through $(4, 0)$.



$$\text{Area} = \iint_E 1 \; dA = \int_1^4 \int_0^{\sqrt{x}} 1 \; dy \; dx = \int_1^4 \left[y\right]_0^{\sqrt{x}} \; dx$$

$$= \int_1^4 \left[\sqrt{x} - 0\right] \; dx = \int_1^4 x^{\frac{1}{2}} \; dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$

$$= \frac{2}{3} \; (4)^{\frac{3}{2}} - \frac{2}{3} \, (1)^{\frac{3}{2}} = \frac{2}{3} \, (8 - 1) = \frac{14}{3} \; .$$

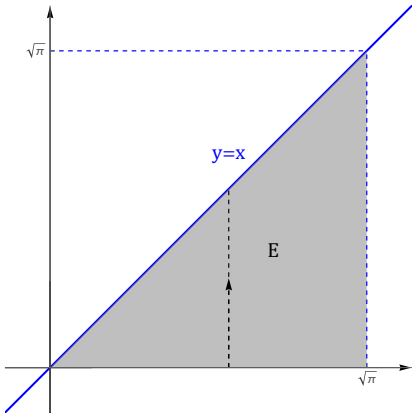
Question (3): [4 points]

Change the order of integration and then evaluate the integral

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos\left(\frac{x^2}{2}\right) \; dx \; dy$$

Solution:

Let E be the region where:
$0 \le y \le \sqrt{\pi}$ and $y \le x \le \sqrt{\pi}$.



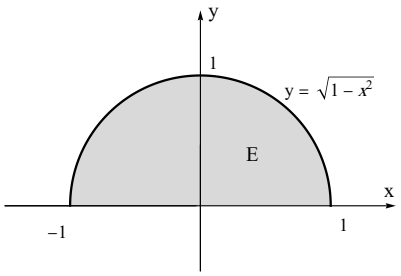
Then E can be written as :
$0 \le x \le \sqrt{\pi}$ and $0 \le y \le x$.

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos\left(\frac{x^2}{2}\right)\, dx\, dy$$
$$= \int_0^{\sqrt{\pi}} \int_0^{x} \cos\left(\frac{x^2}{2}\right)\, dy\, dx$$
$$= \int_0^{\sqrt{\pi}} \cos\left(\frac{x^2}{2}\right)\, [y]_0^x\, dx$$
$$= \int_0^{\sqrt{\pi}} \cos\left(\frac{x^2}{2}\right)\, [x-0]\, dx = \int_0^{\sqrt{\pi}} \cos\left(\frac{x^2}{2}\right)\, x\, dx$$
$$= \left[\sin\left(\frac{x^2}{2}\right)\right]_0^{\sqrt{\pi}} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1 \ .$$

Question (4): [4 points]
Evaluate the iterated integral by converting to polar coordinates

$$\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} \sin\left(x^2+y^2\right)\, dy\, dx$$

Solution:
Let E be the region where:
$-1 \le x \le 1$ and $0 \le y \le \sqrt{1-x^2}$.

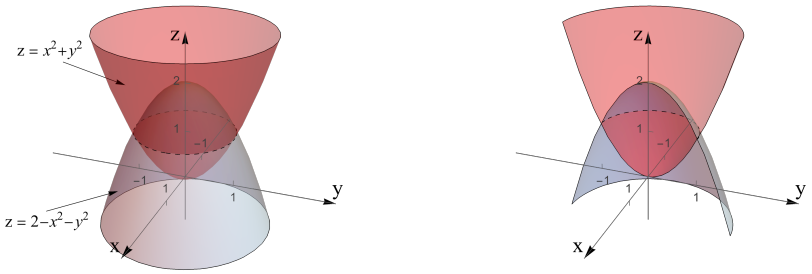


Note that $y = \sqrt{1-x^2}$ represents the upper-half of the unit circle.

Then E in polar coordinates :
$0 \le r \le 1$ and $0 \le \theta \le \pi$.

$$\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} \sin\left(x^2+y^2\right)\, dy\, dx = \int_0^{\pi} \int_0^{1} \sin(r^2)\, r\, dr\, d\theta$$
$$= \left(\int_0^{\pi} d\theta\right)\left(\int_0^{1} \sin(r^2)\, r\, dr\right) = \left(\int_0^{\pi} d\theta\right)\left(\frac{1}{2}\int_0^{1} \sin(r^2)\, (2r)\, dr\right)$$
$$= [\theta]_0^{\pi}\ \frac{1}{2}\ \left[-\cos(r^2)\right]_0^1 = (\pi - 0)\ \frac{1}{2}\ [-\cos(1) - (-\cos(0))]$$
$$= (\pi - 0)\ \frac{1}{2}\ [\cos(0) - \cos(1)] = \frac{\pi\,(1-\cos(1))}{2} \ .$$

Question (5): [4 points]
Calculate the volume of the solid enclosed between the two paraboloids $z = x^2 + y^2$ and $x = 2 - x^2 - y^2$.
Solution:



Let E be the solid enclosed between the two paraboloids.
The two paraboloids intersects at :
$x^2 + y^2 = 2 - x^2 - y^2 \implies 2x^2 + 2y^2 = 2 \implies x^2 + y^2 = 1$.
The two paraboloids intersects at the unit circle.
E in cylindrical coordinates : $r^2 = x^2 + y^2 \le z \le 2 - x^2 - y^2 = 2 - r^2$, $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

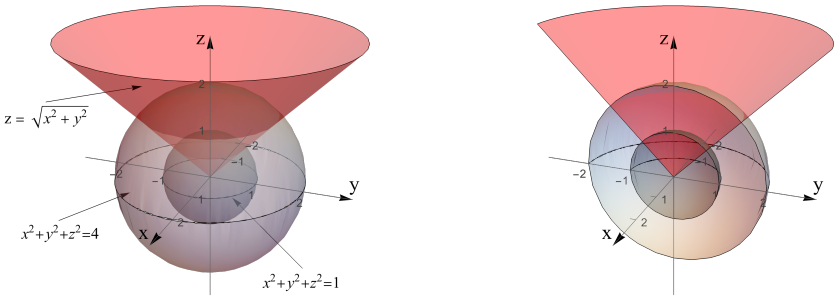
$$\text{Volume} = V(E) = \iiint_E 1 \; dV = \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} dz \; r \; dr \; d\theta$$

$$= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^1 \int_{r^2}^{2-r^2} dz \; r \; dr \right) = [\theta]_0^{2\pi} \left(\int_0^2 [z]_{r^2}^{2-r^2} \; r \; dr \right)$$

$$= (2\pi - 0) \left(\int_0^1 \left[(2 - r^2) - r^2 \right] \; r \; dr \right) = 2\pi \int_0^1 (2 - 2r^2) \; r \; dr$$

$$= 2\pi \int_0^1 (2r - 2r^3) \; dr = 2\pi \left[r^2 - \frac{r^4}{2} \right]_0^1 = 2\pi \left[\left(1 - \frac{1}{2} \right) - (0 - 0) \right] = \pi \; .$$

Question (6): [4 points]
Determine the volume of the region enclosed by the cone $z = \sqrt{x^2 + y^2}$ lying inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the sphere $x^2 + y^2 + z^2 = 1$.
Solution:



Let E be the region enclosed by the cone $z = \sqrt{x^2 + y^2}$ lying inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the sphere $x^2 + y^2 + z^2 = 1$.
$x^2 + y^2 + z^2 = 1 \implies \rho = 1$.

$x^2 + y^2 + z^2 = 4 \implies \rho = 2$.
$z = \sqrt{x^2 + y^2} \implies \rho\cos\phi = r = \rho\sin\phi \implies \cos\phi = \sin\phi \implies \phi = \dfrac{\pi}{4}$.
The cone intersects the sphere $x^2 + y^2 + z^2 = 4$ on the circle $x^2 + y^2 = 2$.
The cone intersects the sphere $x^2 + y^2 + z^2 = 1$ on the circle $x^2 + y^2 = \dfrac{1}{2}$.

E in spherical coordinates : $1 \le \rho \le 1$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \dfrac{\pi}{4}$.

Volume $= V(E) = \iiint_E 1 \; dV = \displaystyle\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_1^2 \rho^2 \; \sin\phi \; d\rho \; d\phi \; d\theta$

$= \left(\displaystyle\int_0^{2\pi} d\theta\right) \left(\int_0^{\frac{\pi}{4}} \sin\phi \; d\phi\right) \left(\int_1^2 \rho^2 \; d\rho\right) = [\theta]_0^{2\pi} \;\; [-\cos\phi]_0^{\frac{\pi}{4}} \;\; \left[\dfrac{\rho^3}{3}\right]_1^2$

$= (2\pi - 0) \;\; \left(-\dfrac{1}{\sqrt{2}} - (-1)\right) \;\; \left(\dfrac{8}{3} - \dfrac{1}{3}\right) = 2\pi \;\; \left(\dfrac{7}{3}\right) \left(1 - \dfrac{1}{\sqrt{2}}\right) = \dfrac{7\left(2 - \sqrt{2}\right)\pi}{3}$.