

MATH 201 - Multivariable Calculus
First Semester - 1448 H
Solution of the First Exam
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Question (1): [3 + 2 = 5 points]

1. Find and sketch the domain of the function

$$f(x, y) = \sqrt{x^2 + y^2 - 1} + \sqrt{4 - x^2 - y^2}$$

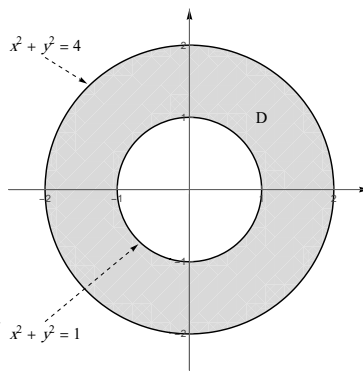
Solution :

$\sqrt{x^2 + y^2 - 1}$ is defined when
 $x^2 + y^2 - 1 \geq 0 \implies x^2 + y^2 \geq 1$.
 defined on and outside the unit circle.

$\sqrt{4 - x^2 - y^2}$ is defined when
 $4 - x^2 - y^2 \geq 0 \implies x^2 + y^2 \leq 4$.
 defined on and inside the circle, centered at the origin, with radius 2.

The domain D is :

$$\begin{aligned} & \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1, x^2 + y^2 \leq 4\} \\ &= \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}. \end{aligned}$$



2. Prove that $\lim_{(x,y) \rightarrow (1,0)} \frac{(x-1)y}{(x-1)^2 + y^2}$ does not exist.

First solution :

(a). On the path $y = 0$:

$$\lim_{(x,y) \rightarrow (1,0)} \frac{(x-1)y}{(x-1)^2 + y^2} = \lim_{x \rightarrow 1} \frac{(x-1)(0)}{(x-1)^2 + (0)^2} = 0.$$

(b) On the path $y = x - 1$: Note that $y = x - 1 \implies 0$ as $x \implies 1$,

$$\lim_{(x,y) \rightarrow (1,0)} \frac{(x-1)y}{(x-1)^2 + y^2} = \lim_{y \rightarrow 0} \frac{y y}{y^2 + y^2} = \lim_{y \rightarrow 0} \frac{y^2}{2y^2} = \frac{1}{2}.$$

Therefore, $\lim_{(x,y) \rightarrow (1,0)} \frac{(x-1)y}{(x-1)^2 + y^2}$ does not exist.

Second solution : Let $y = m(x - 1)$ where $m \in \mathbb{R}$,

$$\begin{aligned} \lim_{(x,y) \rightarrow (1,0)} \frac{(x-1)y}{(x-1)^2 + y^2} &= \lim_{x \rightarrow 1} \frac{(x-1)[m(x-1)]}{(x-1)^2 + [m(x-1)]^2} \\ &= \lim_{x \rightarrow 1} \frac{m(x-1)^2}{(x-1)^2 + m^2(x-1)^2} = \lim_{x \rightarrow 1} \frac{m(x-1)^2}{(x-1)^2(1+m^2)} = \frac{m}{1+m^2}. \end{aligned}$$

The limit depends on m .

Therefore, $\lim_{(x,y) \rightarrow (1,0)} \frac{(x-1)y}{(x-1)^2 + y^2}$ does not exist.

Third solution :

Let $x - 1 = r \cos \theta$, $y = r \sin \theta$, where $r > 0$ and $\theta \in [0, 2\pi]$.

$$\begin{aligned} \lim_{(x,y) \rightarrow (1,0)} \frac{(x-1)y}{(x-1)^2 + y^2} &= \lim_{r \rightarrow 0^+} \frac{(r \cos \theta)(r \sin \theta)}{(r \cos \theta)^2 + (r \sin \theta)^2} \\ &= \lim_{r \rightarrow 0^+} \frac{r^2 \cos \theta \sin \theta}{r^2} = \cos \theta \sin \theta . \end{aligned}$$

The limit depends on θ .

Therefore, $\lim_{(x,y) \rightarrow (1,0)} \frac{(x-1)y}{(x-1)^2 + y^2}$ does not exist.

Question (2): [2 + 4 = 6 points]

Let the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by :

$$f(x, y) = \begin{cases} \frac{xy^3}{x^2 + y^2} & , \quad (x, y) \neq (0, 0) \\ 0 & , \quad (x, y) = (0, 0) \end{cases}$$

1. Study the continuity of f at $(0, 0)$.

Solution :

$$f(0, 0) = 0 .$$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{x^2 + y^2}$$

Using polar coordinates :

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{x^2 + y^2} &= \lim_{r \rightarrow 0^+} \frac{(r \cos \theta)(r \sin \theta)^3}{r^2} \\ &= \lim_{r \rightarrow 0^+} \frac{r^4 \cos \theta \sin^3 \theta}{r^2} = \lim_{r \rightarrow 0^+} r^2 (\cos \theta \sin^3 \theta) = 0 \end{aligned}$$

(Note that $\lim_{r \rightarrow 0^+} r^2 = 0$ and $\cos \theta \sin^3 \theta$ is bounded)

Another solution : Using Squeeze theorem,

$$0 \leq \left| \frac{xy^3}{x^2 + y^2} \right| = \left| (xy) \frac{y^2}{x^2 + y^2} \right| \leq |x| |y| = |x| |y|$$

$$\text{Note that } \lim_{(x,y) \rightarrow (0,0)} 0 = \lim_{(x,y) \rightarrow (0,0)} |x| |y| = 0.$$

So, $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{x^2 + y^2} = 0$.

Therefore, $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = f(0, 0)$. Hence f is continuous at $(0, 0)$.

2. Study the differentiability of f at $(0, 0)$.

Solution :

$$f(0, 0) = 0.$$

$$f(0 + \Delta x, 0 + \Delta y) = f(\Delta x, \Delta y) = \frac{\Delta x (\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2}$$

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\left(\frac{h(0)^3}{h^2 + 0}\right)}{h} = 0.$$

$$f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\left(\frac{(0)h^3}{0 + h^2}\right)}{h} = 0.$$

$$\begin{aligned} & \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \frac{f(\Delta x, \Delta y) - f(0, 0) - f_x(0, 0) \Delta x - f_y(0, 0) \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \\ &= \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \frac{\left(\frac{\Delta x (\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2}\right)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \frac{\Delta x (\Delta y)^3}{\left[(\Delta x)^2 + (\Delta y)^2\right]^{\frac{3}{2}}} \end{aligned}$$

Using polar coordinates :

$$\begin{aligned} & \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \frac{\Delta x (\Delta y)^3}{\left[(\Delta x)^2 + (\Delta y)^2\right]^{\frac{3}{2}}} = \lim_{r \rightarrow 0^+} \frac{r \cos \theta (r \sin \theta)^3}{(r^2)^{\frac{3}{2}}} \\ &= \lim_{r \rightarrow 0^+} \frac{r^4 (\cos \theta \sin^3 \theta)}{r^3} = \lim_{r \rightarrow 0^+} r (\cos \theta \sin^3 \theta) = 0 \end{aligned}$$

(Note that $\lim_{r \rightarrow 0^+} r = 0$ and $(\cos \theta \sin^3 \theta)$ is bounded)

Therefore, f is differentiable at $(0, 0)$.

Question (3): [4 points]

Let $w = \ln(xy + z^2)$, where $x = u^2 + v$, $y = u + v^2$ and $z = uv$.

Use the chain rule to find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.

Solution :

$$(1). \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u}$$

$$\begin{aligned}
&= \left(\frac{y}{xy + z^2} \right) (2u) + \left(\frac{x}{xy + z^2} \right) (1) + \left(\frac{2z}{xy + z^2} \right) (v) \\
&= \frac{2u(u + v^2) + u^2 + v + 2v(uv)}{(u^2 + v)(u + v^2) + u^2v^2}.
\end{aligned}$$

$$\begin{aligned}
(2). \quad \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v} \\
&= \left(\frac{y}{xy + z^2} \right) (1) + \left(\frac{x}{xy + z^2} \right) (2v) + \left(\frac{2z}{xy + z^2} \right) (u) \\
&= \frac{u + v^2 + 2v(u^2 + v) + 2u(uv)}{(u^2 + v)(u + v^2) + u^2v^2}.
\end{aligned}$$

Question (4): [4 points]

Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$x^2 + y^2 + z^2 = xyz + 1$$

Solution :

$$x^2 + y^2 + z^2 = xyz + 1 \implies x^2 + y^2 + z^2 - xyz - 1 = 0.$$

Let $F(x, y, z) = x^2 + y^2 + z^2 - xyz - 1$, then $F(x, y, z) = 0$.

If z is differentiable in x and y , then :

$$(1). \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x - yz}{2z - xy} .$$

$$(2). \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2y - xz}{2z - xy} .$$

Question (5): [3 + 3 = 6 points]

Consider the function $f(x, y) = 3x - x^3 + 3y - y^3$,

1. Find the critical points of f .

Solution :

$$f_x(x, y) = 0 \implies 3 - 3x^2 = 0 \implies x^2 = 1 \implies x = \pm 1 .$$

$$f_y(x, y) = 0 \implies 3 - 3y^2 = 0 \implies y^2 = 1 \implies y = \pm 1 .$$

Therefore, the critical points of f are $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$.

2. Classify each critical point as a local minimum, maximum or saddle point.

Solution :

$$f_{xx}(x, y) = -6x , f_{yy}(x, y) = -6y \text{ and } f_{xy}(x, y) = 0 .$$

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 = 36xy .$$

(a). The critical point $(1, 1)$:

$$D(1, 1) = 36(1)(1) = 36 > 0 \text{ and } f_{xx}(1, 1) = -6 < 0 .$$

$$f(1, 1) = 3 - 1 + 3 - 1 = 4.$$

Therefore, $(1, 1, 4)$ is a local maximum.

(b). The critical point $(1, -1)$:

$$D(1, -1) = 36(1)(-1) = -36 < 0$$

$$f(1, -1) = 3 - 1 - 3 + 1 = 0.$$

Therefore, $(1, -1, 0)$ is a saddle point.

(c). The critical point $(-1, 1)$:

$$D(-1, 1) = 36(-1)(1) = -36 < 0 .$$

$$f(-1, 1) = -3 + 1 + 3 - 1 = 0.$$

Therefore, $(-1, 1, 0)$ is a saddle point.

(d). The critical point $(-1, -1)$:

$$D(-1, -1) = 36(-1)(-1) = 36 > 0 \text{ and } f_{xx}(-1, -1) = 6 > 0 .$$

$$f(-1, -1) = -3 + 1 - 3 + 1 = -4.$$

Therefore, $(-1, -1, -4)$ is a local minimum.