

MATH 111 - Integral Calculus
First Semester - 1447 H
Solution of the Second Exam
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Question (1): [2 + 2 marks]

Find $\frac{dy}{dx}$ of the following :

1. $y = \sinh(\sqrt{x^2 + 1}) + \cosh^{-1}(e^x)$.

Solution :

$$\begin{aligned}\frac{dy}{dx} &= \cosh(\sqrt{x^2 + 1}) \left(\frac{2x}{2\sqrt{x^2 + 1}} \right) + \frac{1}{\sqrt{(e^x)^2 - 1}} (e^x) \\ &= \frac{x \cosh(\sqrt{x^2 + 1})}{\sqrt{x^2 + 1}} + \frac{e^x}{\sqrt{e^{2x} - 1}} .\end{aligned}$$

2. $y = \operatorname{sech}(5x^2) + \tanh^{-1}(\sin x)$.

Solution :

$$\begin{aligned}\frac{dy}{dx} &= -\operatorname{sech}(5x^2) \tanh(5x^2) (10x) + \frac{1}{1 - (\sin x)^2} (\cos x) \\ &= -10x \operatorname{sech}(5x^2) \tanh(5x^2) + \frac{\cos x}{1 - \sin^2 x} .\end{aligned}$$

Question (2): [2 + 2 + 2 + 3 + 2 + 3 + 2 + 3 + 2 marks]

Evaluate the following integrals :

1. $\int x \tanh(4x^2 - 1) dx$.

Solution :

$$\begin{aligned}\int x \tanh(4x^2 - 1) dx &= \frac{1}{8} \int \tanh(4x^2 - 1) (8x) dx \\ &= \frac{1}{8} \ln |\cosh(4x^2 - 1)| + c\end{aligned}$$

2. $\int \frac{1}{\sqrt{5^{2x} + 1}} dx$

Solution :

$$\int \frac{1}{\sqrt{5^{2x} + 1}} dx = \int \frac{1}{\sqrt{(5^x)^2 + (1)^2}} dx$$

$$= \frac{1}{\ln 5} \int \frac{5^x \ln 5}{5^x \sqrt{(5^x)^2 + (1)^2}} dx = -\frac{1}{\ln 5} \operatorname{csch}^{-1}(5^x) + c$$

$$3. \int \sin^2 x \cos^5 x dx$$

Solution :

Using the substitution $u = \sin x$.

Hence $du = \cos x dx$.

$$\begin{aligned} \int \sin^2 x \cos^5 x dx &= \int \sin^2 x \cos^4 x \cos x dx = \int \sin^2 x (\cos^2 x)^2 \cos x dx \\ &= \int \sin^2 x (1 - \sin^2 x)^2 \cos x dx = \int u^2 (1 - u^2)^2 du \\ &= \int u^2 (1 - 2u^2 + u^4) du = \int (u^2 - 2u^4 + u^6) du \\ &= \frac{u^3}{3} - 2 \frac{u^5}{5} + \frac{u^7}{7} + c = \frac{\sin^3 x}{3} - 2 \frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + c \end{aligned}$$

$$4. \int \frac{1}{(x-3)\sqrt{7-x^2+6x}} dx.$$

Solution : By completing the square.

$$\begin{aligned} 7 - x^2 + 6x &= 7 - (x^2 - 6x) = 7 - (x^2 - 6x + 9) + 9 \\ &= 16 - (x^2 - 6x + 9) = (4)^2 - (x-3)^2. \end{aligned}$$

$$\begin{aligned} \int \frac{1}{(x-3)\sqrt{7-x^2+6x}} dx &= \int \frac{1}{(x-3)\sqrt{(4)^2 - (x-3)^2}} dx \\ &= -\frac{1}{4} \operatorname{sech}^{-1}\left(\frac{x-3}{4}\right) + c. \end{aligned}$$

$$5. \int \sin^{-1} x dx.$$

Solution : Using integration by parts.

$$\begin{aligned} u &= \sin^{-1} x & dv &= dx \\ du &= \frac{1}{\sqrt{1-x^2}} dx & v &= x \end{aligned}$$

$$\begin{aligned} \int \sin^{-1} x dx &= x \sin^{-1} x - \int x \frac{1}{\sqrt{1-x^2}} dx \\ &= x \sin^{-1} x - \int x (1-x^2)^{-\frac{1}{2}} dx \end{aligned}$$

$$\begin{aligned}
&= x \sin^{-1} x - \frac{1}{-2} \int (1-x^2)^{-\frac{1}{2}} (-2x) dx \\
&= x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c = x \sin^{-1} x + \sqrt{1-x^2} + c.
\end{aligned}$$

6. $\int \frac{1}{(9+x^2)^{\frac{3}{2}}} dx$.

Solution : Using trigonometric substitutions.

Put $x = 3 \tan \theta \implies \tan \theta = \frac{x}{3}$.

$$dx = 3 \sec^2 \theta d\theta.$$

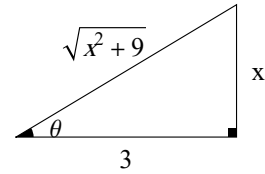
$$(9+x^2)^{\frac{3}{2}} = (9+9\tan^2 \theta)^{\frac{3}{2}} = [9(1+\tan^2 \theta)]^{\frac{3}{2}} = [(3)^2 \sec^2 \theta]^{\frac{3}{2}} = 3^3 \sec^3 \theta.$$

$$\begin{aligned}
\int \frac{1}{(9+x^2)^{\frac{3}{2}}} dx &= \int \frac{3 \sec^2 \theta}{3^3 \sec^3 \theta} d\theta = \frac{1}{3^2} \int \frac{1}{\sec \theta} d\theta \\
&= \frac{1}{9} \int \cos \theta d\theta = \frac{1}{9} \sin \theta + c
\end{aligned}$$

$$\tan \theta = \frac{x}{3} .$$

From the triangle :

$$\sin \theta = \frac{x}{\sqrt{x^2+9}}$$



$$\int \frac{1}{(9+x^2)^{\frac{3}{2}}} dx = \frac{1}{9} \frac{x}{\sqrt{x^2+9}} + c$$

7. $\int x^3 \ln |2x| dx$

Solution : Using integration by parts.

$$u = \ln |2x| \quad dv = x^3 dx$$

$$du = \frac{1}{x} dx \quad v = \frac{x^4}{4}$$

$$\begin{aligned}
\int x^3 \ln |2x| dx &= \frac{x^4}{4} \ln |2x| - \int \frac{x^4}{4} \frac{1}{x} dx \\
&= \frac{x^4}{4} \ln |2x| - \frac{1}{4} \int x^3 dx = \frac{x^4}{4} \ln |2x| - \frac{1}{4} \frac{x^4}{4} + c
\end{aligned}$$

8. $\int \frac{2x+1}{x^2(1+x^2)} dx$

Solution : Using the method of partial fractions.

$$\begin{aligned}\frac{2x+1}{x^2(1+x^2)} &= \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{Bx+C}{1+x^2} \\ \frac{2x+1}{x^2(1+x^2)} &= \frac{A_1 x(1+x^2)}{x^2(1+x^2)} + \frac{A_2(1+x^2)}{x^2(1+x^2)} + \frac{(Bx+C)x^2}{x^2(1+x^2)} \\ 2x+1 &= A_1 x(1+x^2) + A_2(1+x^2) + (Bx+C)x^2 \\ 2x+1 &= A_1 x + A_1 x^3 + A_2 + A_2 x^2 + Bx^3 + Cx^2 \\ 2x+1 &= (A_1+B)x^3 + (A_2+C)x^2 + A_1 x + A_2\end{aligned}$$

By comparing the coefficients of the two polynomials in each side :

$$\begin{aligned}A_1 + B &= 0 & \longrightarrow (1) \\ A_2 + C &= 0 & \longrightarrow (2) \\ A_1 &= 2 & \longrightarrow (3) \\ A_2 &= 1 & \longrightarrow (4)\end{aligned}$$

From equations (1) and (3) : $2 + B = 0 \implies B = -2$.

From equations (2) and (4) : $1 + C = 0 \implies C = -1$.

$$\begin{aligned}\int \frac{2x+1}{x^2(1+x^2)} dx &= \int \left(\frac{2}{x} + \frac{1}{x^2} + \frac{-2x-1}{1+x^2} \right) dx \\ &= \int \frac{2}{x} dx + \int \frac{1}{x^2} dx + \int \frac{-2x-1}{1+x^2} dx \\ &= 2 \int \frac{1}{x} dx + \int x^{-2} dx - \int \frac{2x}{1+x^2} dx - \int \frac{1}{1+x^2} dx \\ &= 2 \ln |x| + \frac{x^{-1}}{-1} - \ln(1+x^2) - \tan^{-1} x + c .\end{aligned}$$

9. $\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{2}{3}}} dx$.

Solution :

Using the substitution $x = u^6$, then $u = x^{\frac{1}{6}}$.

$$dx = 6u^5 du .$$

$$\begin{aligned}\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{2}{3}}} dx &= \int \frac{6u^5}{(u^6)^{\frac{1}{2}} + (u^6)^{\frac{2}{3}}} du = 6 \int \frac{u^5}{u^3 + u^4} du \\ &= 6 \int \frac{u^5}{u^3(u+1)} du = 6 \int \frac{u^2}{u+1} du\end{aligned}$$

Using long division of polynomials :

$$\begin{aligned}6 \int \frac{u^2}{u+1} du &= 6 \int \left(u - 1 + \frac{1}{u+1} \right) du = 6 \left(\frac{u^2}{2} - u + \ln |u+1| \right) + c \\ &= 3u^2 - 6u + 6 \ln |u+1| + c = 3x^{\frac{1}{3}} - 6x^{\frac{1}{6}} + 6 \ln \left| x^{\frac{1}{6}} + 1 \right| + c\end{aligned}$$