

MATH 111 - Integral Calculus
Second Semester - 1446 H
Solution of the Second Exam
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Question (1): [4 marks]

Find $\frac{dy}{dx}$ of the following :

1. $y = \coth(\sqrt[3]{x}) + \operatorname{csch}^{-1}\left(\frac{2}{x}\right)$. [2]

Solution :

$$\begin{aligned}\frac{dy}{dx} &= -\operatorname{csch}^2(\sqrt[3]{x}) \left(\frac{1}{3}x^{-\frac{2}{3}}\right) + \frac{-1}{\frac{2}{x}\sqrt{\left(\frac{2}{x}\right)^2 + 1}} \left(\frac{-2}{x^2}\right) \\ &= \frac{-\operatorname{csch}^2(\sqrt[3]{x})}{3x^{\frac{2}{3}}} + \frac{1}{x\sqrt{\frac{4}{x^2} + 1}} .\end{aligned}$$

2. $y = \coth^{-1}\left(\frac{x^3}{3}\right) + \cosh^{-1}(3x)$. [2]

Solution :

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1 - \left(\frac{x^3}{3}\right)^2} (x^2) + \frac{1}{\sqrt{(3x)^2 - 1}} (3) \\ &= \frac{x^2}{1 - \frac{x^6}{9}} + \frac{3}{\sqrt{9x^2 - 1}} .\end{aligned}$$

Question (2): [21 marks]

Evaluate the following integrals :

1. $\int x^{-2} \operatorname{csch}^2\left(\frac{1}{x}\right) dx$. [2]

Solution :

$$\begin{aligned}\int x^{-2} \operatorname{csch}^2\left(\frac{1}{x}\right) dx &= \int \operatorname{csch}^2\left(\frac{1}{x}\right) \left(\frac{1}{x^2}\right) dx \\ &= \int -\operatorname{csch}^2\left(\frac{1}{x}\right) \left(\frac{-1}{x^2}\right) dx = \coth\left(\frac{1}{x}\right) + c .\end{aligned}$$

2. $\int \frac{1}{\sqrt{x^2 + 2x + 10}} dx$. [2]

Solution : By completing the square.

$$x^2 + 2x + 10 = (x^2 + 2x + 1) + 10 - 1 = (x + 1)^2 + (3)^2 .$$

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 + 2x + 10}} dx &= \int \frac{1}{\sqrt{(x + 1)^2 + (3)^2}} dx \\ &= \sinh^{-1} \left(\frac{x + 1}{3} \right) + c . \end{aligned}$$

3. $\int (2x - 1) \cosh x \, dx$. [2]

Solution : Using integration by parts.

$$\begin{aligned} u &= 2x - 1 & dv &= \cosh x \, dx \\ du &= 2 \, dx & v &= \sinh x \end{aligned}$$

$$\begin{aligned} \int (2x - 1) \cosh x \, dx &= (2x - 1) \sinh x - \int 2 \sinh x \, dx \\ &= (2x - 1) \sinh x - 2 \int \sinh x \, dx = (2x - 1) \sinh x - 2 \cosh x + c . \end{aligned}$$

4. $\int \cot^4 x \csc^4 x \, dx$. [2]

Solution :

Using the substitution $u = \cot x$.

Hence $du = -\csc^2 x \, dx \implies (-1) du = \csc^2 x \, dx$.

$$\begin{aligned} \int \cot^4 x \csc^4 x \, dx &= \int \cot^4 x \csc^2 x \csc^2 x \, dx \\ &= \int \cot^4 x (1 + \cot^2 x) \csc^2 x \, dx = \int u^4 (1 + u^2) (-1) \, du \\ &= - \int (u^4 + u^6) \, du = - \left[\frac{u^5}{5} + \frac{u^7}{7} \right] + c = - \frac{\cot^5 x}{5} - \frac{\cot^7 x}{7} + c \end{aligned}$$

5. $\int e^{\frac{x}{2}} \sinh(3x) \, dx$. [2]

Solution :

$$\begin{aligned} \int e^{\frac{x}{2}} \sinh(3x) \, dx &= \int e^{\frac{x}{2}} \left(\frac{e^{3x} - e^{-3x}}{2} \right) dx = \int \left(\frac{e^{\frac{7x}{2}} + e^{-\frac{5x}{2}}}{2} \right) dx \\ &= \int \left(\frac{e^{\frac{7x}{2}}}{2} + \frac{e^{-\frac{5x}{2}}}{2} \right) dx = \frac{1}{2} \cdot \frac{2}{7} \int e^{\frac{7x}{2}} \left(\frac{7}{2} \right) dx + \frac{1}{2} \cdot \frac{2}{-5} \int e^{-\frac{5x}{2}} \left(-\frac{5}{2} \right) dx \\ &= \frac{1}{7} e^{\frac{7x}{2}} - \frac{1}{5} e^{-\frac{5x}{2}} + c = \frac{e^{\frac{7x}{2}}}{7} - \frac{e^{-\frac{5x}{2}}}{5} + c . \end{aligned}$$

6. $\int \frac{1}{(9-x^2)^{\frac{3}{2}}} dx$. [3]

Solution : Using trigonometric substitutions.

Put $x = 3 \sin \theta \implies \sin \theta = \frac{x}{3}$

$dx = 3 \cos \theta d\theta$

$(9-x^2)^{\frac{3}{2}} = (9-9\sin^2 \theta)^{\frac{3}{2}} = [9(1-\sin^2 \theta)]^{\frac{3}{2}}$

$= (9\cos^2 \theta)^{\frac{3}{2}} = (9)^{\frac{3}{2}} (\cos^2 \theta)^{\frac{3}{2}} = (3^2)^{\frac{3}{2}} (\cos^2 \theta)^{\frac{3}{2}} = 3^3 \cos^3 \theta$

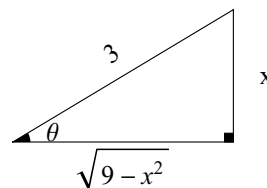
$\int \frac{1}{(9-x^2)^{\frac{3}{2}}} dx = \int \frac{3 \cos \theta}{3^3 \cos^3 \theta} d\theta = \frac{1}{3^2} \int \frac{1}{\cos^2 \theta} d\theta$

$= \frac{1}{9} \int \sec^2 \theta d\theta = \frac{1}{9} \tan \theta + c$

$\sin \theta = \frac{x}{3}$.

From the triangle :

$\tan \theta = \frac{x}{\sqrt{9-x^2}}$



$\int \frac{1}{(9-x^2)^{\frac{3}{2}}} dx = \frac{1}{9} \frac{x}{\sqrt{9-x^2}} + c$

7. $\int \frac{1}{\sqrt{e^{6x}+1}} dx$. [2]

Solution :

$\int \frac{1}{\sqrt{e^{6x}+1}} dx = \int \frac{1}{\sqrt{(e^{3x})^2 + (1)^2}} dx$

$= \int \frac{e^{3x}}{e^{3x} \sqrt{(e^{3x})^2 + (1)^2}} dx = \frac{1}{3} \int \frac{e^{3x} (3)}{e^{3x} \sqrt{(e^{3x})^2 + (1)^2}} dx$

$= \frac{1}{3} (-\operatorname{csch}^{-1}(e^{3x})) + c = -\frac{1}{3} \operatorname{csch}^{-1}(e^{3x}) + c$

8. $\int \tan^{-1} x dx$. [3]

Solution : Using integration by parts.

$u = \tan^{-1} x \quad dv = 1 dx$

$du = \frac{1}{x^2+1} dx \quad v = x$

$$\begin{aligned}\int \tan^{-1} x \, dx &= x \tan^{-1} x - \int x \frac{1}{x^2 + 1} \, dx \\ &= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{x^2 + 1} \, dx = x \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) + c .\end{aligned}$$

9. $\int \frac{x^2 + x + 9}{x^3 + 9x} \, dx$. [3]

Solution : Using the method of partial fractions.

$$\frac{x^2 + x + 9}{x^3 + 9x} = \frac{x^2 + x + 9}{x(x^2 + 9)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 9}$$

$$\frac{x^2 + x + 9}{x(x^2 + 9)} = \frac{A(x^2 + 9)}{x(x^2 + 9)} + \frac{x(Bx + C)}{x(x^2 + 9)}$$

$$x^2 + x + 9 = A(x^2 + 9) + x(Bx + C)$$

$$x^2 + x + 9 = Ax^2 + 9A + Bx^2 + Cx = (A + B)x^2 + Cx + 9A$$

By comparing the coefficients of the two polynomials in each side :

$$A + B = 1 \quad \longrightarrow \quad (1)$$

$$C = 1 \quad \longrightarrow \quad (2)$$

$$9A = 9 \quad \longrightarrow \quad (3)$$

From equation (3) : $9A = 9 \implies A = \frac{9}{9} = 1$.

From equation (1) : $1 + B = 0 \implies B = -1$.

$$\begin{aligned}\int \frac{x^2 + x + 9}{x^3 + 9x} \, dx &= \int \left(\frac{1}{x} + \frac{-x + 1}{x^2 + 9} \right) \, dx \\ &= \int \frac{1}{x} \, dx + \int \frac{-x}{x^2 + 9} \, dx + \int \frac{1}{x^2 + 9} \, dx \\ &= \int \frac{1}{x} \, dx - \frac{1}{2} \int \frac{2x}{x^2 + 9} \, dx + \int \frac{1}{x^2 + (3)^2} \, dx \\ &= \ln|x| - \frac{1}{2} \ln(x^2 + 9) + \frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right) + c .\end{aligned}$$