

**MATH 111 - Integral Calculus**  
**Second Semester - 1446 H**  
**Solution of the First Exam**  
*Dr Tariq A. Alfadhel*

**Question (1):** [9 marks]

1. Use Riemann Sum to evaluate the definite integral  $\int_0^4 (2x^2 + 1) dx$  . [3]

**Solution :**  $[a, b] = [0, 4]$  ,  $f(x) = 2x^2 + 1$  .

$$\Delta_x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}$$

$$x_k = a + k \Delta_x = 0 + k \left( \frac{4}{n} \right) = \frac{4k}{n}$$

$$f(x_k) = 2 \left( \frac{4k}{n} \right)^2 + 1 = 2 \left( \frac{16k^2}{n^2} \right) + 1 = \frac{32k^2}{n^2} + 1$$

$$R_n = \sum_{k=1}^n f(x_k) \Delta_x = \sum_{k=1}^n \left( \frac{32k^2}{n^2} + 1 \right) \left( \frac{4}{n} \right)$$

$$= \sum_{k=1}^n \left( \frac{128k^2}{n^3} + \frac{4}{n} \right) = \sum_{k=1}^n \frac{128k^2}{n^3} + \sum_{k=1}^n \frac{4}{n}$$

$$= \frac{128}{n^3} \sum_{k=1}^n k^2 + \frac{4}{n} \sum_{k=1}^n 1 = \frac{128}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{4}{n} (n)$$

$$= \frac{64}{3} \left( \frac{(n+1)(2n+1)}{n^2} \right) + 4$$

$$\int_0^4 (2x^2 + 1) dx = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \left[ \frac{64}{3} \left( \frac{(n+1)(2n+1)}{n^2} \right) + 4 \right]$$

$$= \frac{64}{3} (2) + 4 = \frac{128}{3} + 4 = \frac{140}{3} .$$

2. Find  $F'(x)$ , if  $F(x) = \int_{\tan(\sqrt[3]{x})}^{3^{5x}} \sqrt{t^2 + 1} dt$  . [2]

**Solution :**

$$F'(x) = \frac{d}{dx} \int_{\tan(\sqrt[3]{x})}^{3^{5x}} \sqrt{t^2 + 1} dt$$

$$= \sqrt{(3^{5x})^2 + 1} (3^{5x} (5) \ln 3) - \sqrt{(\tan(\sqrt[3]{x}))^2 + 1} \left( \sec^2(\sqrt[3]{x}) \left( \frac{1}{3} x^{-\frac{2}{3}} \right) \right)$$

$$= 5 \cdot 3^{5x} \ln 3 \sqrt{3^{10x} + 1} - \frac{\sec^2(\sqrt[3]{x}) \sqrt{\tan^2(\sqrt[3]{x}) + 1}}{3 x^{\frac{2}{3}}} .$$

Find  $\frac{dy}{dx}$  of the following :

3.  $y = \left[ \csc^{-1} \left( \frac{3}{x} \right) \right] \log |\sec(2x) + \tan(2x)|$  . [2]

**Solution :**

$$\begin{aligned} \frac{dy}{dx} &= \left( \frac{-1}{\left(\frac{3}{x}\right) \sqrt{\left(\frac{3}{x}\right)^2 - 1}} \left( \frac{-3}{x^2} \right) \right) \log |\sec(2x) + \tan(2x)| \\ &+ \left[ \csc^{-1} \left( \frac{3}{x} \right) \right] \left( \frac{\sec(2x) \tan(2x) (2) + \sec^2(2x) (2)}{\sec(2x) + \tan(2x)} \frac{1}{\ln 10} \right) \\ &= \frac{\log |\sec(2x) + \tan(2x)|}{x \sqrt{\frac{9}{x^2} - 1}} + \frac{2 \sec(2x) \csc^{-1} \left( \frac{3}{x} \right)}{\ln 10} . \end{aligned}$$

4.  $y = (\cos x)^{\csc x} + 3^{2x^2}$  . [2]

**Solution :**

Let  $y = f(x) + g(x)$ , where  $f(x) = (\cos x)^{\csc x}$  and  $g(x) = 3^{2x^2}$  .

Then  $\frac{dy}{dx} = y' = f'(x) + g'(x)$

First :  $g'(x) = 3^{2x^2} (4x) \ln 3 = 4x 3^{x^2} \ln 3$

Second : Finding  $f'(x)$

$$f(x) = (\cos x)^{\csc x} \implies \ln |f(x)| = \ln |(\cos x)^{\csc x}| = \csc x \ln |\cos x|$$

Differentiate both sides.

$$\frac{f'(x)}{f(x)} = (-\csc x \cot x) \ln |\cos x| + \csc x \left( \frac{-\sin x}{\cos x} \right)$$

$$f'(x) = f(x) \left[ -\csc x \cot x \ln |\cos x| - \frac{\csc x \sin x}{\cos x} \right]$$

$$= (\cos x)^{\csc x} \left[ -\csc x \cos x \ln |\cos x| - \frac{1}{\cos x} \right]$$

$$\text{Therefore, } \frac{dy}{dx} = (\cos x)^{\csc x} [-\csc x \cos x \ln |\cos x| - \sec x] + 4x 3^{x^2} \ln 3$$

**Question (2):** [16 marks]

Evaluate the following integrals :

1.  $\int \left( x^{\frac{2}{3}} \sqrt[3]{\sec(x^3)} \right)^3 dx$  . [2]

**Solution :**

$$\begin{aligned} \int \left( x^{\frac{2}{3}} \sqrt[3]{\sec(x^3)} \right)^3 dx &= \int \left( x^{\frac{2}{3}} \right)^3 \left( \sqrt[3]{\sec(x^3)} \right)^3 dx = \int x^2 \sec(x^3) dx \\ &= \frac{1}{3} \int \sec(x^3) (3x^2) dx = \frac{1}{3} \ln |\sec(x^3) + \tan(x^3)| + c . \end{aligned}$$

2.  $\int \frac{2}{\sqrt{e^{4x}-4}} dx$  . [2]

**Solution :**

$$\begin{aligned} \int \frac{2}{\sqrt{e^{4x}-4}} dx &= \int \frac{2}{\sqrt{(e^{2x})^2 - (2)^2}} dx = \int \frac{e^{2x} (2)}{e^{2x} \sqrt{(e^{2x})^2 - (2)^2}} dx \\ &= \frac{1}{2} \sec^{-1} \left( \frac{e^{2x}}{2} \right) + c . \end{aligned}$$

3.  $\int_0^1 10^{2 \log x} 3^{3x^3} dx$  . [2]

**Solution :**

$$\begin{aligned} \int_0^1 10^{2 \log x} 3^{3x^3} dx &= \int_0^1 10^{\log x^2} 3^{3x^3} dx = \int_0^1 x^2 3^{3x^3} dx \\ &= \frac{1}{9} \int_0^1 3^{3x^3} (9x^2) dx = \frac{1}{9} \left[ \frac{3^{3x^3}}{\ln 3} \right]_0^1 = \frac{1}{9} \left[ \frac{3^{3(1)^3}}{\ln 3} - \frac{3^{3(0)^3}}{\ln 3} \right] \\ &= \frac{1}{9} \left[ \frac{3^3}{\ln 3} - \frac{3^0}{\ln 3} \right] = \frac{1}{9} \left( \frac{27-1}{\ln 3} \right) = \frac{26}{9 \ln 3} . \end{aligned}$$

4.  $\int \frac{x^2}{4+x^2} dx$  . [2]

**Solution :**

$$\begin{aligned} \int \frac{x^2}{4+x^2} dx &= \int \frac{(x^2+4)-4}{x^2+4} dx = \int \left[ \frac{x^2+4}{x^2+4} - \frac{4}{x^2+4} \right] dx \\ &= \int \left[ 1 - \frac{4}{x^2+4} \right] dx = \int 1 dx - 4 \int \frac{1}{(x)^2 + (2)^2} dx \\ &= x - 4 \left( \frac{1}{2} \tan^{-1} \left( \frac{x}{2} \right) \right) + c = x - 2 \tan^{-1} \left( \frac{x}{2} \right) + c \end{aligned}$$

$$5. \int \frac{\cos(\ln(x^2))}{x} dx . \text{ [2]}$$

**Solution :**

$$\begin{aligned} \int \frac{\cos(\ln(x^2))}{x} dx &= \int \cos(2 \ln |x|) \frac{1}{x} dx \\ &= \frac{1}{2} \int \cos(2 \ln |x|) \frac{2}{x} dx = \frac{1}{2} \sin(2 \ln |x|) + c \end{aligned}$$

$$6. \int \frac{\cot(\sqrt[3]{x})}{x^{\frac{2}{3}}} dx . \text{ [2]}$$

**Solution :**

$$\begin{aligned} \int \frac{\cot(\sqrt[3]{x})}{x^{\frac{2}{3}}} dx &= \int \cot(x^{\frac{1}{3}}) x^{-\frac{2}{3}} dx \\ &= 3 \int \cot(x^{\frac{1}{3}}) \left(\frac{1}{3} x^{-\frac{2}{3}}\right) dx = 3 \ln |\sin(\sqrt[3]{x})| + c \end{aligned}$$

$$7. \int \frac{(1 + \sec^{-1} x)^5}{\sqrt{x^4 - x^2}} dx . \text{ [2]}$$

**Solution :**

$$\begin{aligned} \int \frac{(1 + \sec^{-1} x)^5}{\sqrt{x^4 - x^2}} dx &= \int \frac{(1 + \sec^{-1} x)^5}{\sqrt{x^2(x^2 - 1)}} dx \\ &= \int \frac{(1 + \sec^{-1} x)^5}{|x|\sqrt{x^2 - 1}} dx = \int (1 + \sec^{-1} x)^5 \left(\frac{1}{x\sqrt{x^2 - 1}}\right) dx \\ &= \frac{(1 + \sec^{-1} x)^6}{6} + c , \text{ where } x > 0 . \end{aligned}$$

$$8. \int (\sec x \tan x) (\sec^2 x)^{-1} dx . \text{ [2]}$$

**First solution :**

$$\begin{aligned} \int (\sec x \tan x) (\sec^2 x)^{-1} dx &= \int (\sec x)^{-2} (\sec x \tan x) dx \\ &= \frac{(\sec x)^{-1}}{-1} + c = -\cos x + c \end{aligned}$$

**Second solution :**

$$\begin{aligned} \int (\sec x \tan x) (\sec^2 x)^{-1} dx &= \int \frac{\sec x \tan x}{\sec^2 x} dx = \int \frac{\tan x}{\sec x} dx \\ &= \int \sin x dx = -\cos x + c . \end{aligned}$$