MATH 111 - Integral Calculus
First Semester - 1446 H
Solution of the First Exam
Dr Tariq A. Alfadhel

Question (1): [9 marks]

1. Use Riemann Sum to evaluate the definite integral $\int_0^2 (x^2+3)\ dx$. [3]

   Solution : $[a,b]=[0,2]$, $f(x)=x^2+3$.

   $$\Delta_x=\frac{b-a}{n}=\frac{2-0}{n}=\frac{2}{n}$$

   $$x_k=a+k\ \Delta_x=0+k\left(\frac{2}{n}\right)=\frac{2k}{n}$$

   $$f(x_k)=\left(\frac{2k}{n}\right)^2+3=\frac{4k^2}{n^2}+3$$

   $$R_n=\sum_{k=1}^{n} f(x_k)\ \Delta_x=\sum_{k=1}^{n}\left(\frac{4k^2}{n^2}+3\right)\ \left(\frac{2}{n}\right)$$

   $$=\sum_{k=1}^{n}\left(\frac{8k^2}{n^3}+\frac{6}{n}\right)=\sum_{k=1}^{n}\frac{8k^2}{n^3}+\sum_{k=1}^{n}\frac{6}{n}$$

   $$=\frac{8}{n^3}\sum_{k=1}^{n}k^2+\frac{6}{n}\sum_{k=1}^{n}1=\frac{8}{n^3}\ \frac{n(n+1)(2n+1)}{6}+\frac{6}{n}\ (n)$$

   $$=\frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)+6$$

   $$\int_0^2 (x^2+3)\ dx=\lim_{n\to\infty} R_n=\lim_{n\to\infty}\left[\frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)+6\right]$$

   $$=\frac{4}{3}\ (2)+6=\frac{8}{3}+6=\frac{26}{3}\ .$$

2. Find $F'(x)$, if $F(x)=\int_{\tan\left(\frac{x}{2}\right)}^{3^{5x}}\sqrt{t^2+1}\ dt$. [2]

   Solution :

   $$F'(x)=\frac{d}{dx}\int_{\tan\left(\frac{x}{2}\right)}^{3^{5x}}\sqrt{t^2+1}\ dt$$

   $$=\sqrt{(3^{5x})^2+1}\ \left(3^{5x}(5)\ \ln 3\right)-\sqrt{\left(\tan\left(\frac{x}{2}\right)\right)^2+1}\ \left(\sec^2\left(\frac{x}{2}\right)\ \left(\frac{1}{2}\right)\right)$$

   $$=5\ 3^{5x}\ \ln 3\ \sqrt{3^{10x}+1}-\frac{1}{2}\ \sec^2\left(\frac{x}{2}\right)\ \sqrt{\tan^2\left(\frac{x}{2}\right)+1}\ .$$

Find $\frac{dy}{dx}$ of the following :

3. $y = \left[\cos^{-1}(3x)\right] \ \log|\csc x - \cot x|$. [2]

Solution :

$$\frac{dy}{dx} = \left(\frac{-1}{\sqrt{1-(3x)^2}}\ (3)\right) \ \log|\csc x - \cot x|$$

$$+ \left[\cos^{-1}(3x)\right] \ \left(\frac{-\csc x \cot x - (-\csc^2)x)}{\csc x - \cot x}\ \frac{1}{\ln 10}\right)$$

$$= \frac{-3 \ \log|\csc x - \cot x|}{\sqrt{1-9x^2}} + \frac{\csc x \ \cos^{-1}(3x)}{\ln 10} \ .$$

4. $y = (\cot x)^{\csc x} + 3^{x^2}$. [2]

Solution :

Let $y = f(x) + g(x)$, where $f(x) = (\cot x)^{\csc x}$ and $g(x) = 3^{x^2}$.

Then $\frac{dy}{dx} = y' = f'(x) + g'(x)$

First - $g'(x) = 3^{x^2} \ (2x)\ln 3 = 2x \ 3^{x^2} \ \ln 3$

Second - Finding $f'(x)$

$f(x) = (\cot x)^{\csc x} \implies \ln|f(x)| = \ln|(\cot x)^{\csc x}| = \csc x \ \ln|\cot x|$

Differentiate both sides.

$$\frac{f'(x)}{f(x)} = (-\csc x \ \cot x) \ \ln|\cot x| + \csc x \left(\frac{-\csc^2 x}{\cot x}\right)$$

$$f'(x) = f(x)\left[-\csc x \ \cot x \ \ln|\cot x| - \frac{\csc^3 x}{\cot x}\right]$$

$$= (\cot x)^{\csc x} \ \left[-\csc x \ \cot x \ \ln|\cot x| - \frac{\csc^3 x}{\cot x}\right]$$

Therefore, $\frac{dy}{dx} = (\cot x)^{\csc x} \ \left[-\csc x \ \cot x \ \ln|\cot x| - \frac{\csc^3 x}{\cot x}\right] + 2x \ 3^{x^2} \ \ln 3$

Question (2): [16 marks]
Evaluate the following integrals :

1. $\int \left(x^{\frac{2}{5}} \ \sqrt[5]{\sin(x^3)}\right)^5 \ dx$. [2]

Solution :

$$\int \left(x^{\frac{2}{5}} \ \sqrt[5]{\sin(x^3)}\right)^5 \ dx = \int \left(x^{\frac{2}{5}}\right)^5 \ \left(\sqrt[5]{\sin(x^3)}\right)^5 \ dx = \int x^2 \ \sin(x^3) \ dx$$

$$= \frac{1}{3}\int \sin(x^3)\ (3x^2)\ dx = \frac{1}{3}\ \left(-\cos(x^3)\right) + c = -\frac{1}{3}\ \cos(x^3) + c\ .$$

2. $\displaystyle\int \frac{-1}{\sqrt{e^{6x}-4}}\ dx$. [2]

Solution : .

$$\int \frac{-1}{\sqrt{e^{6x}-4}}\ dx = \int \frac{-1}{\sqrt{\left(e^{3x}\right)^2-(2)^2}}\ dx = -\int \frac{e^{3x}}{e^{3x}\ \sqrt{\left(e^{3x}\right)^2-(2)^2}}\ dx$$

$$= -\frac{1}{3}\int \frac{e^{3x}\ (3)}{e^{3x}\ \sqrt{\left(e^{3x}\right)^2-(2)^2}}\ dx = -\frac{1}{3}\left(\frac{1}{2}\ \sec^{-1}\left(\frac{e^{3x}}{2}\right)\right) + c\ .$$

$$= -\frac{1}{6}\ \sec^{-1}\left(\frac{e^{3x}}{2}\right) + c\ .$$

3. $\displaystyle\int_0^1 e^{2\ln x}\ 3^{3x^3}\ dx$. [2]

Solution :

$$\int_0^1 e^{2\ln x}\ 3^{3x^3}\ dx = \int_0^1 e^{\ln x^2}\ 3^{3x^3}\ dx = \int_0^1 x^2\ 3^{3x^3}\ dx$$

$$= \frac{1}{9}\int_0^1 3^{3x^3}(9x^2)\ dx = \frac{1}{9}\left[\frac{3^{3x^3}}{\ln 3}\right]_0^1 = \frac{1}{9}\left[\frac{3^{3(1)^3}}{\ln 3} - \frac{3^{3(0)^3}}{\ln 3}\right]$$

$$= \frac{1}{9}\left[\frac{3^3}{\ln 3} - \frac{3^0}{\ln 3}\right] = \frac{1}{9}\ \left(\frac{27-1}{\ln 3}\right) = \frac{26}{9\ln 3}\ .$$

4. $\displaystyle\int \frac{x+1}{\sqrt{1-x^2}}\ dx$. [2]

Solution :

$$\int \frac{x+1}{\sqrt{1-x^2}}\ dx = \int \left[\frac{x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\right]\ dx$$

$$= \int \frac{x}{\sqrt{1-x^2}}\ dx + \int \frac{1}{\sqrt{1-x^2}}\ dx$$

$$= -\frac{1}{2}\int \left(1-x^2\right)^{-\frac{1}{2}}(-2x)\ dx + \int \frac{1}{\sqrt{1-x^2}}\ dx$$

$$= -\frac{1}{2}\ \frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + \sin^{-1}x + c = -\sqrt{1-x^2} + \sin^{-1}x + c\ .$$

5. $\displaystyle\int \frac{\sec\left(\ln(x^2)\right)}{x}\ dx$. [2]

Solution :

$$\int \frac{\sec\left(\ln(x^2)\right)}{x}\ dx = \int \sec\left(2\ln|x|\right)\ \frac{1}{x}\ dx$$

$$= \frac{1}{2}\int \sec\left(2\ln|x|\right)\ \frac{2}{x}\ dx = \frac{1}{2}\ln\left|\sec\left(2\ln|x|\right) + \tan\left(2\ln|x|\right)\right| + c$$

6. $\displaystyle\int \frac{\tan\left(\sqrt[3]{x}\right)}{x^{\frac{2}{3}}}\ dx$. [2]

Solution :

$$\int \frac{\tan\left(\sqrt[3]{x}\right)}{x^{\frac{2}{3}}}\ dx = \int \tan\left(x^{\frac{1}{3}}\right)\ x^{-\frac{2}{3}}\ dx$$

$$= 3\int \tan\left(x^{\frac{1}{3}}\right)\left(\frac{1}{3}\ x^{-\frac{2}{3}}\right)\ dx = 3\ln\left|\sec\left(\sqrt[3]{x}\right)\right| + c$$

7. $\displaystyle\int \frac{\left(1+\sec^{-1}x\right)^5}{\sqrt{x^4-x^2}}\ dx$. [2]

Solution :

$$\int \frac{\left(1+\sec^{-1}x\right)^5}{\sqrt{x^4-x^2}}\ dx = \int \frac{\left(1+\sec^{-1}x\right)^5}{\sqrt{x^2\left(x^2-1\right)}}\ dx$$

$$= \int \frac{\left(1+\sec^{-1}x\right)^5}{|x|\sqrt{x^2-1}}\ dx = \int \left(1+\sec^{-1}x\right)^5\left(\frac{1}{x\sqrt{x^2-1}}\right)\ dx$$

$$= \frac{\left(1+\sec^{-1}x\right)^6}{6} + c\ ,\ \text{where } x > 0\ .$$

8. $\displaystyle\int \cos x\ \left(\sin^2 x\right)^{-1}\ dx$. [2]

First solution :

$$\int \cos x\ \left(\sin^2 x\right)^{-1}\ dx = \int (\sin x)^{-2}\cos x\ dx = \frac{(\sin x)^{-1}}{-1} + c = -\csc x + c$$

Second solution :

$$\int \cos x\ \left(\sin^2 x\right)^{-1}\ dx = \int \frac{\cos x}{\sin^2 x}\ dx = \int \frac{1}{\sin x}\frac{\cos x}{\sin x}\ dx$$

$$= \int \csc x\ \cot x\ dx = -\csc x + c\ .$$

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Question (1): [4 marks]
Find $\frac{dy}{dx}$ of the following :

1. $y = csch\left(\frac{2}{x}\right) + \tanh^{-1}\left(e^{5x}\right)$. [2]

   Solution :
   $$\frac{dy}{dx} = -csch\left(\frac{2}{x}\right)\coth\left(\frac{2}{x}\right)\;\left(\frac{-2}{x^2}\right) + \frac{1}{1-\left(e^{5x}\right)^2}\;\left(e^{5x}\;(5)\right)$$
   $$= \frac{2}{x^2}\;csch\left(\frac{2}{x}\right)\coth\left(\frac{2}{x}\right) + \frac{5e^{5x}}{1-e^{10x}} .$$

2. $y = \sinh^{-1}\left(\sqrt{x}\,\right) + \coth(4x^2)$. [2]

   Solution :
   $$\frac{dy}{dx} = \frac{1}{\sqrt{1+\left(\sqrt{x}\right)^2}}\;\left(\frac{1}{2\sqrt{x}}\right) + \left(-csch^2(4x^2)\;(8x)\right)$$
   $$= \frac{1}{2\sqrt{x}\sqrt{1+x}} - 8x\;csch^2(4x^2) .$$

Question (2): [21 marks]
Evaluate the following integrals :

1. $\int \frac{sech^2(x^{-2})}{x^3}\,dx$. [2]

   Solution :
   $$\int \frac{sech^2(x^{-2})}{x^3}\,dx = \int sech^2(x^{-2})\;x^{-3}\,dx$$
   $$= \frac{1}{-2}\int sech^2(x^{-2})\;\left(-2x^{-3}\right)\,dx = -\frac{1}{2}\tanh(x^{-2}) + c$$

2. $\int \frac{3}{\sqrt{x^2+6x-16}}\,dx$. [3]

   Solution : By completing the square.

   $x^2 + 6x - 16 = (x^2 + 6x + 9) - 16 - 9 = (x+3)^2 - (5)^2$.

$$\int \frac{3}{\sqrt{x^2+6x-16}}\ dx = 3\int \frac{1}{\sqrt{(x+3)^2-(5)^2}}\ dx$$

$$= 3\cosh^{-1}\left(\frac{x+3}{5}\right) + c\ .$$

3. $\int e^x\ \cosh 2x\ dx\ .$ [2]

Solution :

$$\int e^x\ \cosh 2x\ dx = \int e^x\left(\frac{e^{2x}+e^{-2x}}{2}\right)\ dx = \int \left(\frac{e^{3x}+e^{-x}}{2}\right)\ dx$$

$$\int \left(\frac{e^{3x}}{2}+\frac{e^{-x}}{2}\right)\ dx = \frac{1}{2}\ \frac{1}{3}\int e^{3x}(3)\ dx + \frac{1}{2}\ \frac{1}{-1}\int e^{-x}(-1)\ dx$$

$$= \frac{1}{6}\ e^{3x} - \frac{1}{2}\ e^{-x} + c = \frac{e^{3x}}{6} - \frac{e^{-x}}{2} + c\ .$$

4. $\int (3x-2)\ \sinh x\ dx\ .$ [2]

Solution : Using integration by parts.

$u = 3x-2 \qquad dv = \sinh x\ dx$

$du = 3\ dx \qquad v = \cosh x$

$$\int (3x-2)\ \sinh x\ dx = (3x-2)\ \cosh x - \int 3\cosh x\ dx$$

$$= (3x-2)\ \cosh x - 3\int \cosh x\ dx = (3x-2)\ \cosh x - 3\sinh x + c\ .$$

5. $\int x^{-2}\ \ln x\ dx\ .$ [2]

Solution : Using integration by parts.

$u = \ln x \qquad dv = x^{-2}\ dx$

$du = \frac{1}{x}\ dx \qquad v = \frac{x^{-1}}{-1} = \frac{-1}{x}$

$$\int x^{-2}\ \ln x\ dx = \frac{-1}{x}\ln x - \int \frac{-1}{x}\ \frac{1}{x}\ dx$$

$$= \frac{-\ln x}{x} - \int \frac{-1}{x^2}\ dx = \frac{-\ln x}{x} - \frac{1}{x} + c\ .$$

6. $\displaystyle\int \sin^4 x \ \cos^5 x \ dx$. [2]

Solution :

Using the substitution $u = \sin x$.

Hence $du = \cos x \ dx$.

$$\int \sin^4 x \ \cos^5 x \ dx = \int \sin^4 x \ \cos^4 x \ \cos x \ dx = \int \sin^4 x \ \left(\cos^2 x\right)^2 \cos x \ dx$$

$$= \int \sin^4 x \ \left(1 - \sin^2 x\right)^2 \cos x \ dx = \int u^4 \left(1 - u^2\right)^2 \ du = \int u^4 \left(1 - 2u^2 + u^4\right) \ du$$

$$= \int \left(u^4 - 2u^6 + u^8\right) \ du = \frac{u^5}{5} - 2\,\frac{u^7}{7} + \frac{u^9}{9} + c$$

$$= \frac{\sin^5 x}{5} - 2\,\frac{\sin^7 x}{7} + \frac{\sin^9 x}{9} + c$$

7. $\displaystyle\int \frac{\sqrt{x^2 - 9}}{x} \ dx$. [3]

Solution : Using trigonometric substitutions.

Put $x = 3\sec\theta \implies \sec\theta = \dfrac{x}{3} \implies \cos\theta = \dfrac{3}{x}$.

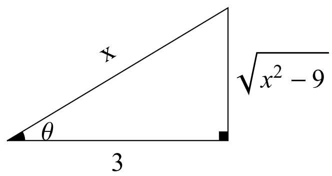
$dx = 3\sec\theta \ \tan\theta \ d\theta$.

$$\sqrt{x^2 - 9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9\left(\sec^2\theta - 1\right)} = \sqrt{9\tan^2\theta} = 3\tan\theta$$

$$= \int \frac{\sqrt{x^2 - 9}}{x} \ dx = \int \frac{(3\tan\theta) \ (3\sec\theta \ \tan\theta)}{3\sec\theta} \ d\theta = 3\int \tan^2\theta \ d\theta$$

$$= 3\int \left(\sec^2\theta - 1\right) \ d\theta = 3\left(\tan\theta - \theta\right) + c = 3\tan\theta - 3\theta + c$$

$\cos\theta = \dfrac{3}{x}$.



From the triangle :

$\tan\theta = \dfrac{\sqrt{x^2 - 9}}{3}$

$\theta = \sec^{-1}\left(\dfrac{x}{3}\right)$

$$\int \frac{\sqrt{x^2 - 9}}{x} \ dx = 3\left(\frac{\sqrt{x^2 - 9}}{3}\right) - 3\sec^{-1}\left(\frac{x}{3}\right) + c$$

$$= \sqrt{x^2 - 9} - 3\sec^{-1}\left(\frac{x}{3}\right) + c \ .$$

8. $\displaystyle\int \frac{5x^2+x+8}{x^3+4x}\,dx$. [3]

Solution : Using the method of partial fractions.

$$\frac{5x^2+x+8}{x^3+4x} = \frac{5x^2+x+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\frac{5x^2+x+8}{x(x^2+4)} = \frac{A(x^2+4)}{x(x^2+4)} + \frac{x\,(Bx+C)}{x(x^2+4)}$$

$$5x^2+x+8 = A(x^2+4) + x(Bx+C)$$

$$5x^2+x+8 = Ax^2+4A+Bx^2+Cx = (A+B)x^2+Cx+4A$$

By comparing the coefficients of the two polynomials in each side :

$$\begin{array}{lcl} A+B=5 & \longrightarrow & (1) \\ C=1 & \longrightarrow & (2) \\ 4A=8 & \longrightarrow & (3) \end{array}$$

From equation (3) : $4A = 8 \implies A = \frac{8}{4} = 2$.

From equation (1) : $2 + B = 0 \implies B = 3$.

$$\int \frac{5x^2+x+8}{x^3+4x}\,dx = \int \left(\frac{2}{x} + \frac{3x+1}{x^2+4}\right)\,dx$$

$$= \int \frac{2}{x}\,dx + \int \frac{3x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$$

$$= 2\int \frac{1}{x}\,dx + \frac{3}{2}\int \frac{2x}{x^2+4}\,dx + \int \frac{1}{x^2+(2)^2}\,dx$$

$$= 2\ln|x| + \frac{3}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c\ .$$

9. $\displaystyle\int \frac{dx}{\sqrt[4]{x}+\sqrt{x}}$. [2]

Solution : $\displaystyle\int \frac{dx}{\sqrt[4]{x}+\sqrt{x}} = \int \frac{1}{x^{\frac{1}{4}}+x^{\frac{1}{2}}}\,dx$

Using the substitution $x = u^4$, then $u = x^{\frac{1}{4}}$.

$dx = 4u^3\,du$.

$$\int \frac{dx}{x^{\frac{1}{4}}+x^{\frac{1}{2}}} = \int \frac{4u^3}{(u^4)^{\frac{1}{4}}+(u^4)^{\frac{1}{2}}}\,du = \int \frac{4u^3}{u+u^2}\,du$$

$$= \int \frac{4u^3}{u(1+u)}\,du = 4\int \frac{u^2}{u+1}\,du$$

Using long division of polynomials :

$$4\int \frac{u^2}{u+1}\ du = 4\int \left(u - 1 + \frac{1}{u+1}\right)\ du$$

$$= 4\left(\frac{u^2}{2} - u + \ln|u+1|\right) + c = 2u^2 - 4u + 4\ln|u-1| + c$$

$$\int \frac{dx}{x^{\frac{1}{4}} + x^{\frac{1}{2}}} = 2\left(x^{\frac{1}{4}}\right)^2 - 4x^{\frac{1}{4}} + 4\ln\left|x^{\frac{1}{4}} + 1\right| + c$$

$$= 2x^{\frac{1}{2}} - 4x^{\frac{1}{4}} + 4\ln\left|x^{\frac{1}{4}} + 1\right| + c\ .$$

MATH 111 - Integral Calculus
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Solution of the Final Exam
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Question (1): [7 marks]

1. Find the value of c that satisfies the mean value theorem of the definite integral for the function $f(x) = 4x - x^2$ on the interval $[0, 3]$. [3]

   Solution : Using the formula $(b-a)\ f(c) = \int_a^b f(x)\ dx$.

   $$(3-0)\ (4c - c^2) = \int_0^3 (4x - x^2)\ dx = \left[2x^2 - \frac{x^3}{3}\right]_0^3$$

   $$3(4c - c^2) = \left(2(3)^2 - \frac{(3)^3}{3}\right) - \left(2(0)^2 - \frac{(0)^3}{3}\right) = 18 - 9 = 9$$

   $$3(4c - c^2) = 9 \implies 4c - c^2 = 3 \implies c^2 - 4c + 3 = 0$$

   $$\implies (c-3)(c-1) = 0 \implies c = 3\ ,\ c = 1.$$

   Note that $c = 1 \in (0,3)$ while $c = 3 \notin (0,3)$.

   The desired value is $c = 1$.

2. Find $F'(x)$, if $F(x) = \int_{\sec x}^{3^{2x}} \sqrt{2t^2 - 1}\ dt$. [2]

   Solution :

   $$F'(x) = \frac{d}{dx} \int_{\sec x}^{3^{2x}} \sqrt{2t^2 - 1}\ dt$$

   $$= \sqrt{2\left(3^{2x}\right)^2 - 1}\ \left(3^{2x}(2)\ \ln 3\right) - \sqrt{2\left(\sec x\right)^2 - 1}\ \left(\sec x\ \tan x\right)$$

   $$= 2 \ln 3 \left(3^{2x}\right) \sqrt{2\ \left(3^{4x}\right) - 1} - \sec x \tan x \sqrt{2 \sec^2 x - 1}\ .$$

3. Find y' if $y = \cosh^{-1}\left(3^{x^2-1}\right) + \log|\tanh(2x)|$. [2]

   Solution :

   $$y' = \frac{1}{\sqrt{\left(3^{x^2-1}\right)^2 - 1}} \left(3^{x^2-1}\ (2x)\ \ln 3\right) + \frac{sech^2(2x)\ (2)}{\tanh(2x)} \frac{1}{\ln 10}$$

   $$= \frac{2x \ln 3\ \left(3^{x^2-1}\right)}{\sqrt{3^{2x^2-2} - 1}} + \frac{2\ sech^2(2x)}{\tanh(2x)\ \ln 10}\ .$$

Question (2): [14 marks]
Evaluate the following integrals :

1. $\displaystyle\int \frac{2}{\sqrt{-x^2+6x-8}}\, dx$. [3]

   Solution : By completing the square.

   $-x^2+6x-8 = -(x^2-6x)-8 = -(x^2-6x+9)-8+9$

   $= -(x-3)^2+1 = 1-(x-3)^2$.

   $$\int \frac{2}{\sqrt{-x^2+6x-8}}\, dx = 2\int \frac{1}{\sqrt{1-(x-3)^2}}\, dx = 2\sin^{-1}(x-3)+c \ .$$

2. $\displaystyle\int x^{-3}\ \ln|x|\ dx$. [3]

   Solution : Using integration by parts .

   $u = \ln x \qquad dv = x^{-3}\ dx$

   $du = \dfrac{1}{x}\ dx \qquad v = \dfrac{x^{-2}}{-2} = \dfrac{1}{-2x^2}$

   $$\int x^{-3}\ \ln x\ dx = \frac{1}{-2x^2}\ln x - \int \frac{1}{-2x^2}\ \frac{1}{x}\ dx$$

   $$= \frac{\ln x}{-2x^2} + \frac{1}{2}\int x^{-3}\ dx = \frac{\ln x}{-2x^2} + \frac{1}{2}\left(\frac{x^{-2}}{-2}\right) + c = \frac{\ln x}{-2x^2} - \frac{1}{4x^2} + c \ .$$

3. $\displaystyle\int \frac{\sqrt{x^2+9}}{x^4}\, dx$. [3]

   Solution : Using trigonometric substitutions.

   Put $x = 3\tan\theta \implies \tan\theta = \dfrac{x}{3}$.

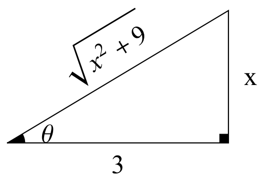
   $dx = 3\sec^2\theta\ d\theta$.

   $\sqrt{x^2+9} = \sqrt{9\tan^2\theta+9} = \sqrt{9\left(\tan^2\theta+1\right)} = \sqrt{9\sec^2\theta} = 3\sec\theta$.

   $$\int \frac{\sqrt{x^2+9}}{x^4}\, dx = \int \frac{3\sec\theta\ 3\sec^2\theta}{(3\tan\theta)^4}\, d\theta = \frac{3^2}{3^4}\int \frac{\sec^3\theta}{\tan^4\theta}\, d\theta$$

   $$= \frac{1}{9}\int \frac{\cos^4\theta}{\sin^4\theta\ \cos^3\theta}\, d\theta = \frac{1}{9}\int (\sin\theta)^{-4}\ \cos\theta\ d\theta = \frac{1}{9}\ \frac{(\sin\theta)^{-3}}{-3} + c$$

   $\tan\theta = \dfrac{x}{3}$.

   From the triangle :
   $\sin\theta = \dfrac{x}{\sqrt{x^2+9}}$

$$\int \frac{\sqrt{x^2+9}}{x^4}\, dx = \frac{1}{-27}\, \left(\frac{x}{\sqrt{x^2+9}}\right)^{-3} + c\ .$$

4. $\int \frac{3x+1}{x^3+x}\, dx$. [3]

Solution : Using the method of partial fractions.

$$\frac{3x+1}{x^3+x} = \frac{3x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\frac{3x+1}{x(x^2+1)} = \frac{A(x^2+1)}{x(x^2+1)} + \frac{x\,(Bx+C)}{x(x^2+1)}$$

$$3x+1 = A(x^2+1) + x(Bx+C)$$

$$3x+1 = Ax^2 + A + Bx^2 + Cx = (A+B)x^2 + Cx + A$$

By comparing the coefficients of the two polynomials in each side :

$$\begin{array}{lll} A+B=0 & \longrightarrow & (1) \\ C=3 & \longrightarrow & (2) \\ A=1 & \longrightarrow & (3) \end{array}$$

From equation (1) : $1+B=0 \implies B=-1$.

$$\int \frac{3x+1}{x^3+x}\, dx = \int \left(\frac{1}{x} + \frac{-x+3}{x^2+1}\right)\, dx$$

$$= \int \frac{1}{x}\, dx + \int \frac{-x}{x^2+1}\, dx + \int \frac{3}{x^2+1}\, dx$$

$$= \int \frac{1}{x}\, dx - \frac{1}{2}\int \frac{2x}{x^2+1}\, dx + 3\int \frac{1}{x^2+(1)^2}\, dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + 3\tan^{-1}(x) + c\ .$$

5. $\int \frac{1}{\sqrt{x}\,(1+x)}\, dx$. [2]

Solution : Using the substitution $u = \sqrt{x} \implies u^2 = x$.

$2u\ du = dx$.

$$\int \frac{1}{\sqrt{x}\,(1+x)}\, dx = \int \frac{2u}{u(1+u^2)}\, du = 2\int \frac{1}{1+u^2}\, du$$

$$= 2\tan^{-1}(u) + c = 2\tan^{-1}\left(\sqrt{x}\right) + c\ .$$

Question (3): [19 marks]

1. Evaluate the limit $\lim_{x\to\infty} e^{-x^3}\ (x^4+1)$. [2]

   Solution :

   $\lim_{x\to\infty} e^{-x^3}\ (x^4+1) \qquad (0.\infty)$

   $\lim_{x\to\infty} e^{-x^3}\ (x^4+1) = \lim_{x\to\infty} \frac{x^4+1}{e^{x^3}} \qquad \left(\frac{\infty}{\infty}\right)$

   Using L'Hôpital's rule.

   $\lim_{x\to\infty} \frac{4\ x^3}{e^{x^3}\ (3x^2)} = \lim_{x\to\infty} \frac{4x}{3e^{x^3}} \qquad \left(\frac{\infty}{\infty}\right)$

   Using L'Hôpital's rule.

   $\lim_{x\to\infty} \frac{4}{3e^{x^3}(3x^2)} = \lim_{x\to\infty} \frac{4}{9x^2\ e^{x^3}} = 0$.

   Therefore, $\lim_{x\to\infty} e^{-x^3}\ (x^4+1) = 0$.

   Note that $\lim_{x\to\infty} e^{x^3} = \infty$ and $\lim_{x\to\infty} 9x^2 = \infty$.

2. Discus whether the improper integral $\int_2^\infty \frac{1}{x\,(\ln x)^2}\,dx$ converges or diverges. [3]

   Solution :

   $\int_2^\infty \frac{1}{x\,(\ln x)^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\,(\ln x)^2}\,dx = \lim_{t\to\infty} \int_2^t (\ln x)^{-2}\,\frac{1}{x}\,dx$

   $= \lim_{t\to\infty} \left[\frac{(\ln x)^{-1}}{-1}\right]_2^t = \lim_{t\to\infty} \left[\frac{-1}{\ln x}\right]_2^t = \lim_{t\to\infty} \left[\frac{-1}{\ln t} - \frac{-1}{\ln 2}\right]$

   $= 0 - \frac{-1}{\ln 2} = \frac{1}{\ln 2}$.

   Hence, the improper integral converges.

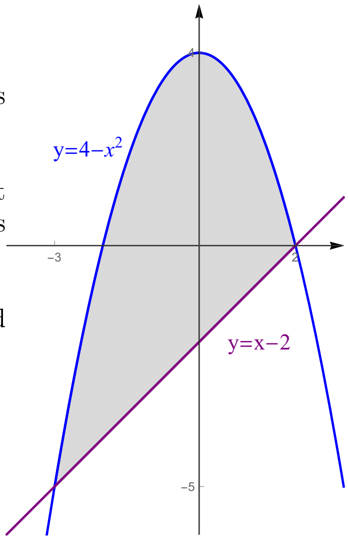
3. Sketch the region region bounded by the curves $y = 4 - x^2$ and $y = x - 2$, and find its area. [3]

   Solution :

$y = 4 - x^2$ represents a parabola opens downwards, and its vertex is $(0, 4)$.

$y = x - 2$ represents a straight line passing through $(0, -2)$, and its slope equals 1.

Points of intersection of $y = 4 - x^2$ and $y = x - 2$:
$x - 2 = 4 - x^2 \implies x^2 + x - 6 = 0$
$\implies (x + 3)(x - 2) = 0$
$\implies x = -3$, $x = 2$.



$$\mathbf{A} = \int_{-3}^{2} \left[(4 - x^2) - (x - 2)\right] \; dx = \int_{-3}^{2} (4 - x^2 - x + 2) \; dx$$

$$= \int_{-3}^{2} (-x^2 - x + 6) \; dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 6x\right]_{-3}^{2}$$

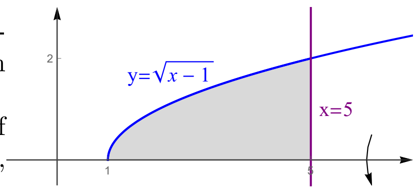
$$= \left(-\frac{(2)^3}{3} - \frac{(2)^2}{2} + 6(2)\right) - \left(-\frac{(-3)^3}{3} - \frac{(-3)^2}{2} + 6(-3)\right)$$

$$= \left(\frac{-8}{3} - 2 + 12\right) - \left(9 - \frac{9}{2} - 18\right) = 19 - \frac{8}{3} + \frac{9}{2} = \frac{125}{6} \; .$$

4. Sketch the region bounded bu the curves $y = \sqrt{x - 1}$, $y = 0$ and $x = 5$, and find the volume of the solid generated by revolving this region about the $x-$axis. [3]

Solution :

$y = 0$ represents the x-axis.
$x = 5$ represents a straight line parallel to the y-axis, and passing through $(5, 0)$.
$y = \sqrt{x - 1}$ represents the upper half of the parabola $x = y^2$ opens to the right, and its vertex is $(0, 1)$.



Using the disk method.

$$\mathbf{V} = \pi \int_{1}^{5} \left(\sqrt{x - 1}\right)^2 \; dx = \pi \int_{1}^{5} (x - 1) \; dx = \left[\frac{x^2}{2} - x\right]_{1}^{5}$$

$$= \left(\frac{25}{2} - 5\right) - \left(\frac{1}{2} - 1\right) = \frac{25}{2} - 5 - \frac{1}{2} + 1 = 12 - 4 = 8 \; .$$

5. Find the arc length of the function $y = \ln|\sec x|$ from $x = 0$ to $x = \frac{\pi}{4}$. [3]

Solution :

$y' = \dfrac{\sec x \tan x}{\sec x} = \tan x$.

$\mathbf{L} = \displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1 + (\tan x)^2}\ dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x}\ dx = \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\ dx$

$= \displaystyle\int_0^{\frac{\pi}{4}} |\sec x|\ dx = \int_0^{\frac{\pi}{4}} \sec x\ dx = [\ln|\sec x + \tan x|]_0^{\frac{\pi}{4}}$

$= \ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right| - \ln|\sec(0) + \tan(0)|$

$= \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0| = \ln\left|1 + \sqrt{2}\right|$.

6. Convert the polar equation $r = \dfrac{1}{\sin\theta - 2\cos\theta}$ into a Cartesian equation.[1]

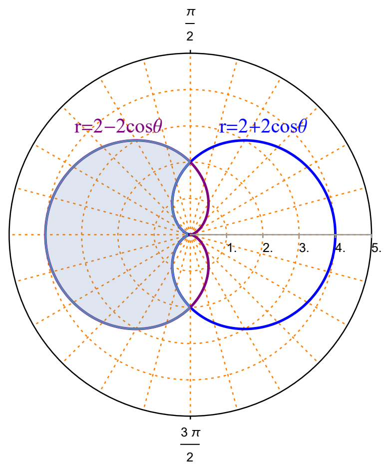
Solution :

$r = \dfrac{1}{\sin\theta - 2\cos\theta} \implies r(\sin\theta - 2\cos\theta) = 1$

$\implies r\sin\theta - 2(r\cos\theta) = 1 \implies y - 2x = 1 \implies y = 2x + 1$.

7. Sketch the region inside the graph of the polar equation $r = 2 - 2\cos\theta$ and outside the graph of $r = 2 + 2\cos\theta$. Then compute its area. [4]

Solution :



Points of intersection of $r = 2 - 2\cos\theta$ and $r = 2 + 2\cos\theta$:
$2 + 2\cos\theta = 2 - 2\cos\theta$
$\implies 4\cos\theta = 0$
$\implies \cos\theta = 0$
$\implies \theta = \frac{\pi}{2}$, $\theta = \frac{3\pi}{2}$.

Note that the shaded region is symmetric with respect to the polar axis.

$$\mathbf{A} = 2\left(\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \left[(2 - 2\cos\theta)^2 - (2 + 2\cos\theta)^2\right]\ d\theta\right)$$

$$= \int_{\frac{\pi}{2}}^{\pi} \left[4 - 8\cos\theta + 4\cos^2\theta - \left(4 + 8\cos\theta + 4\cos^2\theta\right)\right] \; d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} \left(4 - 8\cos\theta + 4\cos^2\theta - 4 - 8\cos\theta - 4\cos^2\theta\right) \; d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} -16\cos\theta \; d\theta = -16\left[\sin\theta\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -16\left[\sin(\pi) - \sin\left(\frac{\pi}{2}\right)\right] = -16(0-1) = 16 \; .$$