

**MATH 111 - Integral Calculus**  
**First Semester - 1448 H**  
**Solution of the First Exam**  
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**Question (1):** [2 + 1 = 3 marks]

1. Use Riemann Sum to evaluate the definite integral  $\int_0^1 (6x^2 + 2x) dx$ .

**Solution :**  $[a, b] = [0, 1]$ ,  $f(x) = 6x^2 + 2x$ .

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}.$$

$$x_k = a + k \Delta x = 0 + k \left( \frac{1}{n} \right) = \frac{k}{n}.$$

$$f(x_k) = 6 \left( \frac{k}{n} \right)^2 + 2 \left( \frac{k}{n} \right) = \frac{6k^2}{n^2} + \frac{2k}{n}.$$

$$R_n = \sum_{k=1}^n f(x_k) \Delta x = \sum_{k=1}^n \left( \frac{6k^2}{n^2} + \frac{2k}{n} \right) \left( \frac{1}{n} \right) = \sum_{k=1}^n \left( \frac{6k^2}{n^3} + \frac{2k}{n^2} \right)$$

$$= \sum_{k=1}^n \frac{6k^2}{n^3} + \sum_{k=1}^n \frac{2k}{n^2} = \frac{6}{n^3} \sum_{k=1}^n k^2 + \frac{2}{n^2} \sum_{k=1}^n k$$

$$= \frac{6}{n^3} \left( \frac{n(n+1)(2n+1)}{6} \right) + \frac{2}{n^2} \left( \frac{n(n+1)}{2} \right)$$

$$= \left( \frac{n+1}{n} \cdot \frac{2n+1}{n} \right) + \frac{n+1}{n}.$$

$$\int_0^1 (6x^2 + 2x) dx = \lim_{n \rightarrow \infty} R_n$$

$$= \lim_{n \rightarrow \infty} \left[ \left( \frac{n+1}{n} \cdot \frac{2n+1}{n} \right) + \frac{n+1}{n} \right] = (1)(2) + 1 = 3.$$

2. Find  $F'(x)$ , if  $F(x) = \int_{\cosh \sqrt{x}}^{\sec(x^2+1)} (1+t) dt$ .

**Solution :**

$$F'(x) = \frac{d}{dx} \int_{\cosh \sqrt{x}}^{\sec(x^2+1)} (1+t) dt$$

$$= (1 + \sec(x^2 + 1)) [\sec(x^2 + 1) \tan(x^2 + 1) (2x)] - (1 + \cosh \sqrt{x}) \left[ \sinh \sqrt{x} \left( \frac{1}{2\sqrt{x}} \right) \right]$$

$$= 2x \sec(x^2 + 1) \tan(x^2 + 1) (1 + \sec(x^2 + 1)) - \frac{\sinh \sqrt{x} (1 + \cosh \sqrt{x})}{2\sqrt{x}}.$$

**Question (2):** Find  $\frac{dy}{dx}$  of the following : [1 + 1 = 2 marks]

1.  $y(x) = [\ln(e^x + 1) + \cos^{-1}(x^2)]^3$  .

**Solution :**

$$\begin{aligned} y'(x) &= 3 [\ln(e^x + 1) + \cos^{-1}(x^2)]^2 \left( \frac{e^x}{e^x + 1} + \frac{-1}{\sqrt{1 - (x^2)^2}} (2x) \right) \\ &= 3 [\ln(e^x + 1) + \cos^{-1}(x^2)]^2 \left( \frac{e^x}{e^x + 1} - \frac{2x}{\sqrt{1 - x^4}} \right) . \end{aligned}$$

2.  $y(x) = (\csc x)^x$

**Solution :**

$$y(x) = (\csc x)^x \implies \ln |y(x)| = \ln |(\csc x)^x| = x \ln |\csc x|$$

Differentiate both sides.

$$\frac{y'(x)}{y(x)} = (1) \ln |\csc x| + x \left( \frac{-\csc x \cot x}{\csc x} \right)$$

$$y'(x) = y(x) [\ln |\csc x| - x \cot x] = (\csc x)^x [\ln |\csc x| - x \cot x] .$$

**Question (3):** [1 + 1 + 1 + 1 + 1 + 1 + 1 + 2 + 2 = 10 marks]

Evaluate the following integrals :

1.  $\int x^{\frac{1}{3}} \tanh \left( x^{\frac{4}{3}} \right) dx$  .

**Solution :**

$$\begin{aligned} \int x^{\frac{1}{3}} \tanh \left( x^{\frac{4}{3}} \right) dx &= \frac{3}{4} \int \tanh \left( x^{\frac{4}{3}} \right) \left( \frac{4}{3} x^{\frac{1}{3}} \right) dx \\ &= \frac{3}{4} \ln \left| \cosh \left( x^{\frac{4}{3}} \right) \right| + c . \end{aligned}$$

2.  $\int \frac{x-1}{\sqrt{(x-1)^2 + 1}} dx$  .

**Solution : .**

$$\begin{aligned} \int \frac{x-1}{\sqrt{(x-1)^2 + 1}} dx &= \int [(x-1)^2 + 1]^{-\frac{1}{2}} (x-1) dx \\ &= \frac{1}{2} \int [(x-1)^2 + 1]^{-\frac{1}{2}} (2(x-1)) dx = \frac{1}{2} \frac{[(x-1)^2 + 1]^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= [(x-1)^2 + 1]^{\frac{1}{2}} + c = \sqrt{(x-1)^2 + 1} + c \end{aligned}$$

$$3. \int \frac{\log(x^3)}{x} dx$$

**Solution :**

$$\begin{aligned} \int \frac{\log(x^3)}{x} dx &= \int \frac{3 \log(x)}{x} dx = 3 \int \log(x) \frac{1}{x} dx \\ &= 3 \ln 10 \int (\log(x))^1 \left( \frac{1}{x} \frac{1}{\ln 10} \right) dx = 3 \ln 10 \frac{(\log(x))^2}{2} + c \end{aligned}$$

$$4. \int \frac{1}{\operatorname{sech} x (1 + \sinh x)} dx .$$

**Solution :**

$$\int \frac{1}{\operatorname{sech} x (1 + \sinh x)} dx = \int \frac{\cosh x}{1 + \sinh x} dx = \ln |1 + \sinh x| + c .$$

$$5. \int 3^x 7^{3^x} dx .$$

**Solution :**

$$\int 3^x 7^{3^x} dx = \frac{1}{\ln 3} \int 7^{3^x} (3^x \ln 3) dx = \frac{1}{\ln 3} \frac{7^{3^x}}{\ln 7} + c .$$

$$6. \int \frac{1}{\sqrt{e^{6x} - 4}} dx$$

**Solution :**

$$\begin{aligned} \int \frac{1}{\sqrt{e^{6x} - 4}} dx &= \int \frac{1}{\sqrt{(e^{3x})^2 - (2)^2}} dx = \int \frac{e^{3x}}{e^{3x} \sqrt{(e^{3x})^2 - (2)^2}} dx \\ &= \frac{1}{3} \int \frac{e^{3x} (3)}{e^{3x} \sqrt{(e^{3x})^2 - (2)^2}} dx = \frac{1}{3} \left( \frac{1}{2} \sec^{-1} \left( \frac{e^{3x}}{2} \right) \right) + c \end{aligned}$$

$$7. \int_0^1 \frac{x}{2x+1} dx .$$

**Solution :**

$$\begin{aligned} \int_0^1 \frac{x}{2x+1} dx &= \frac{1}{2} \int_0^1 \frac{2x}{2x+1} dx = \frac{1}{2} \int_0^1 \frac{(2x+1) - 1}{2x+1} dx \\ &= \frac{1}{2} \int_0^1 \left( \frac{2x+1}{2x+1} - \frac{1}{2x+1} \right) dx = \frac{1}{2} \int_0^1 \left( 1 - \frac{1}{2x+1} \right) dx \\ &= \frac{1}{2} \left( \int_0^1 1 dx - \int_0^1 \frac{1}{2x+1} dx \right) = \frac{1}{2} \left( \int_0^1 1 dx - \frac{1}{2} \int_0^1 \frac{2}{2x+1} dx \right) . \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left( [x]_0^1 - \frac{1}{2} [\ln |2x+1|]_0^1 \right) = \frac{1}{2} \left( [1-0] - \frac{1}{2} [\ln(3) - \ln(1)] \right) \\
&= \frac{1}{2} \left( 1 - \frac{1}{2} \ln(3) \right) = \frac{1}{2} - \frac{\ln(3)}{4} .
\end{aligned}$$

8.  $\int \frac{1}{x^2+6x+18} dx$  .

**Solution :** By completing the square.

$$\begin{aligned}
x^2+6x+18 &= (x^2+6x+9)+9 = (x+3)^2+(3)^2 \\
&= \int \frac{1}{x^2+6x+18} dx = \int \frac{1}{(x+3)^2+(3)^2} dx \\
&= \frac{1}{3} \tan^{-1} \left( \frac{x+3}{3} \right) + c .
\end{aligned}$$