

MATH 111 - Integral Calculus
First Semester - 1447 H
Solution of the First Exam
Dr Tariq A. Alfadhel

Question (1): [9 marks]

1. Use Riemann Sum to evaluate the definite integral $\int_0^1 (3x^2 - x) dx$. [3]

Solution : $[a, b] = [0, 1]$, $f(x) = 3x^2 - x$.

$$\Delta_x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$x_k = a + k \Delta_x = 0 + k \left(\frac{1}{n} \right) = \frac{k}{n}$$

$$f(x_k) = 3 \left(\frac{k}{n} \right)^2 - \frac{k}{n} = \frac{3k^2}{n^2} - \frac{k}{n}$$

$$R_n = \sum_{k=1}^n f(x_k) \Delta_x = \sum_{k=1}^n \left(\frac{3k^2}{n^2} - \frac{k}{n} \right) \left(\frac{1}{n} \right)$$

$$= \sum_{k=1}^n \left(\frac{3k^2}{n^3} - \frac{k}{n^2} \right) = \sum_{k=1}^n \frac{3k^2}{n^3} - \sum_{k=1}^n \frac{k}{n^2}$$

$$= \frac{3}{n^3} \sum_{k=1}^n k^2 - \frac{1}{n^2} \sum_{k=1}^n k = \frac{3}{n^3} \frac{n(n+1)(2n+1)}{6} - \frac{1}{n^2} \frac{n(n+1)}{2}$$

$$= \frac{(n+1)(2n+1)}{2n^2} - \frac{n+1}{2n}$$

$$\int_0^1 (3x^2 - x) dx = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \left[\frac{(n+1)(2n+1)}{2n^2} - \frac{n+1}{2n} \right]$$

$$= \frac{2}{2} - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2} .$$

2. Find $F'(x)$, if $F(x) = \int_{\cos(\sqrt{x+1})}^{e^{x^2}} \frac{1}{1-t^2} dt$. [2]

Solution :

$$F'(x) = \frac{d}{dx} \int_{\cos(\sqrt{x+1})}^{e^{x^2}} \frac{1}{1-t^2} dt$$

$$= \frac{1}{1-(e^{x^2})^2} (e^{x^2} (2x)) - \frac{1}{1-(\cos(\sqrt{x+1}))^2} \left(-\sin(\sqrt{x+1}) \frac{1}{2\sqrt{x+1}} \right)$$

$$= \frac{2x e^{x^2}}{1-e^{2x^2}} + \frac{1}{2\sqrt{x+1} \sin(\sqrt{x+1})} .$$

Find $\frac{dy}{dx}$ of the following :

3. $y = \sin^{-1}(5^{3x+1}) + \log_2(\sqrt[5]{x^4 + 2})$. [2]

Solution :

$$y = \sin^{-1}(5^{3x+1}) + \log_2(\sqrt[5]{x^4 + 2}) = \sin^{-1}(5^{3x+1}) + \log_2(x^4 + 2)^{\frac{1}{5}}$$

$$= \sin^{-1}(5^{3x+1}) + \frac{1}{5} \log_2(x^4 + 2)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (5^{3x+1})^2}} (5^{3x+1} (3 \ln 5)) + \frac{1}{5} \frac{4x^3}{x^4 + 2} \frac{1}{\ln 2}$$

$$= \frac{3 \ln 5 \cdot 5^{3x+1}}{\sqrt{1 - 5^{6x+2}}} + \frac{4x^3}{5 \ln 2 (x^4 + 2)} .$$

4. $y = (\sin x)^{\sqrt{x}}$. [2]

Solution :

$$y = (\sin x)^{\sqrt{x}} \implies \ln |y| = \ln |(\sin x)^{\sqrt{x}}| = \sqrt{x} \ln |\sin x|$$

Differentiate both sides.

$$\frac{y'}{y} = \frac{1}{2\sqrt{x}} \ln |\sin x| + \sqrt{x} \left(\frac{\cos x}{\sin x} \right)$$

$$y' = y \left[\frac{\ln |\sin x|}{2\sqrt{x}} + \sqrt{x} \cot x \right]$$

$$= (\sin x)^{\sqrt{x}} \left[\frac{\ln |\sin x|}{2\sqrt{x}} + \sqrt{x} \cot x \right] .$$

Question (2): [16 marks]

Evaluate the following integrals :

1. $\int \frac{\cos x}{(1 + \sin x)^4} dx$. [2]

Solution :

$$\int \frac{\cos x}{(1 + \sin x)^4} dx = \int (1 + \sin x)^{-4} \cos x dx$$

$$= \frac{(1 + \sin x)^{-3}}{-3} + c = \frac{1}{-3(1 + \sin x)^3} + c .$$

2. $\int \frac{\sec(x^{\frac{2}{5}}) \tan(x^{\frac{2}{5}})}{x^{\frac{3}{5}}} dx$. [2]

Solution : .

$$\begin{aligned} \int \frac{\sec(x^{\frac{2}{5}}) \tan(x^{\frac{2}{5}})}{x^{\frac{3}{5}}} dx &= \int \sec\left(x^{\frac{2}{5}}\right) \tan\left(x^{\frac{2}{5}}\right) x^{-\frac{3}{5}} dx \\ &= \frac{5}{2} \int \sec\left(x^{\frac{2}{5}}\right) \tan\left(x^{\frac{2}{5}}\right) \left(\frac{2}{5} x^{-\frac{3}{5}}\right) dx = \frac{5}{2} \sec\left(x^{\frac{2}{5}}\right) + c . \end{aligned}$$

3. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})} dx$. [2]

Solution :

$$\begin{aligned} \int \frac{1}{\sqrt{x}(1+\sqrt{x})} dx &= 2 \int \frac{1}{2\sqrt{x}(1+\sqrt{x})} dx \\ &= 2 \int \frac{\left(\frac{1}{2\sqrt{x}}\right)}{1+\sqrt{x}} dx = 2 \ln|1+\sqrt{x}| + c . \end{aligned}$$

4. $\int \frac{7^{\cot(3x)}}{\sin^2(3x)} dx$. [2]

Solution :

$$\begin{aligned} \int \frac{7^{\cot(3x)}}{\sin^2(3x)} dx &= \int 7^{\cot(3x)} \frac{1}{\sin^2(3x)} dx \\ &= \frac{-1}{3} \int 7^{\cot(3x)} (-\csc^2(3x) (3)) dx = -\frac{1}{3} \frac{7^{\cot(3x)}}{\ln 7} + c . \end{aligned}$$

5. $\int_0^1 \frac{x}{x+1} dx$. [2]

Solution :

$$\begin{aligned} \int_0^1 \frac{x}{x+1} dx &= \int_0^1 \frac{(x+1)-1}{x+1} dx = \int_0^1 \left(\frac{x+1}{x+1} - \frac{1}{x+1}\right) dx \\ &= \int_0^1 \left(1 - \frac{1}{x+1}\right) dx = [x - \ln|x+1|]_0^1 \\ &= (1 - \ln 2) - (0 - \ln(1)) = 1 - \ln 2 . \end{aligned}$$

6. $\int \frac{x-5}{\sqrt{1-x^2}} dx$. [2]

Solution :

$$\begin{aligned} \int \frac{x-5}{\sqrt{1-x^2}} dx &= \int \left(\frac{x}{\sqrt{1-x^2}} - \frac{5}{\sqrt{1-x^2}}\right) dx \\ &= \frac{1}{-2} \int (1-x^2)^{-\frac{1}{2}} (-2x) dx - 5 \int \frac{1}{\sqrt{1-x^2}} dx \end{aligned}$$

$$= \frac{1}{-2} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} - 5 \sin^{-1} x + c = -\sqrt{1-x^2} - 5 \sin^{-1} x + c .$$

7. $\int \frac{1}{x\sqrt{x^6-9}} dx$. [2]

Solution :

$$\begin{aligned} \int \frac{1}{x\sqrt{x^6-9}} dx &= \int \frac{1}{x\sqrt{(x^3)^2-(3)^2}} dx = \int \frac{x^2}{x^3\sqrt{(x^3)^2-(3)^2}} dx \\ &= \frac{1}{3} \int \frac{3x^2}{x^3\sqrt{(x^3)^2-(3)^2}} dx = \frac{1}{3} \frac{1}{3} \sec^{-1} \left(\frac{x^3}{3} \right) + c . \end{aligned}$$

8. $\int \frac{1}{e^{-x}+e^x} dx$. [2]

Solution :

$$\begin{aligned} \int \frac{1}{e^{-x}+e^x} dx &= \int \frac{e^x}{e^x(e^{-x}+e^x)} dx = \int \frac{e^x}{1+e^{2x}} dx \\ &= \int \frac{e^x}{(1)^2+(e^x)^2} dx = \tan^{-1}(e^x) + c . \end{aligned}$$

MATH 111 - Integral Calculus
First Semester - 1447 H
Solution of the Second Exam
Dr Tariq A. Alfadhel

Question (1): [2 + 2 marks]

Find $\frac{dy}{dx}$ of the following :

1. $y = \sinh(\sqrt{x^2 + 1}) + \cosh^{-1}(e^x) .$

Solution :

$$\begin{aligned} \frac{dy}{dx} &= \cosh(\sqrt{x^2 + 1}) \left(\frac{2x}{2\sqrt{x^2 + 1}} \right) + \frac{1}{\sqrt{(e^x)^2 - 1}} (e^x) \\ &= \frac{x \cosh(\sqrt{x^2 + 1})}{\sqrt{x^2 + 1}} + \frac{e^x}{\sqrt{e^{2x} - 1}} . \end{aligned}$$

2. $y = \operatorname{sech}(5x^2) + \tanh^{-1}(\sin x) .$

Solution :

$$\begin{aligned} \frac{dy}{dx} &= -\operatorname{sech}(5x^2) \tanh(5x^2) (10x) + \frac{1}{1 - (\sin x)^2} (\cos x) \\ &= -10x \operatorname{sech}(5x^2) \tanh(5x^2) + \frac{\cos x}{1 - \sin^2 x} . \end{aligned}$$

Question (2): [2 + 2 + 2 + 3 + 2 + 3 + 2 + 3 + 2 marks]

Evaluate the following integrals :

1. $\int x \tanh(4x^2 - 1) dx .$

Solution :

$$\begin{aligned} \int x \tanh(4x^2 - 1) dx &= \frac{1}{8} \int \tanh(4x^2 - 1) (8x) dx \\ &= \frac{1}{8} \ln |\cosh(4x^2 - 1)| + c \end{aligned}$$

2. $\int \frac{1}{\sqrt{5^{2x} + 1}} dx$

Solution :

$$\int \frac{1}{\sqrt{5^{2x} + 1}} dx = \int \frac{1}{\sqrt{(5^x)^2 + (1)^2}} dx$$

$$= \frac{1}{\ln 5} \int \frac{5^x \ln 5}{5^x \sqrt{(5^x)^2 + (1)^2}} dx = -\frac{1}{\ln 5} \operatorname{csch}^{-1}(5^x) + c$$

$$3. \int \sin^2 x \cos^5 x dx$$

Solution :

Using the substitution $u = \sin x$.

Hence $du = \cos x dx$.

$$\begin{aligned} \int \sin^2 x \cos^5 x dx &= \int \sin^2 x \cos^4 x \cos x dx = \int \sin^2 x (\cos^2 x)^2 \cos x dx \\ &= \int \sin^2 x (1 - \sin^2 x)^2 \cos x dx = \int u^2 (1 - u^2)^2 du \\ &= \int u^2 (1 - 2u^2 + u^4) du = \int (u^2 - 2u^4 + u^6) du \\ &= \frac{u^3}{3} - 2 \frac{u^5}{5} + \frac{u^7}{7} + c = \frac{\sin^3 x}{3} - 2 \frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + c \end{aligned}$$

$$4. \int \frac{1}{(x-3)\sqrt{7-x^2+6x}} dx.$$

Solution : By completing the square.

$$\begin{aligned} 7 - x^2 + 6x &= 7 - (x^2 - 6x) = 7 - (x^2 - 6x + 9) + 9 \\ &= 16 - (x^2 - 6x + 9) = (4)^2 - (x-3)^2. \end{aligned}$$

$$\begin{aligned} \int \frac{1}{(x-3)\sqrt{7-x^2+6x}} dx &= \int \frac{1}{(x-3)\sqrt{(4)^2 - (x-3)^2}} dx \\ &= -\frac{1}{4} \operatorname{sech}^{-1}\left(\frac{x-3}{4}\right) + c. \end{aligned}$$

$$5. \int \sin^{-1} x dx.$$

Solution : Using integration by parts.

$$\begin{aligned} u &= \sin^{-1} x & dv &= dx \\ du &= \frac{1}{\sqrt{1-x^2}} dx & v &= x \\ \int \sin^{-1} x dx &= x \sin^{-1} x - \int x \frac{1}{\sqrt{1-x^2}} dx \\ &= x \sin^{-1} x - \int x (1-x^2)^{-\frac{1}{2}} dx \end{aligned}$$

$$\begin{aligned}
&= x \sin^{-1} x - \frac{1}{-2} \int (1-x^2)^{-\frac{1}{2}} (-2x) dx \\
&= x \sin^{-1} x + \frac{1}{2} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c = x \sin^{-1} x + \sqrt{1-x^2} + c.
\end{aligned}$$

6. $\int \frac{1}{(9+x^2)^{\frac{3}{2}}} dx$.

Solution : Using trigonometric substitutions.

Put $x = 3 \tan \theta \implies \tan \theta = \frac{x}{3}$.

$$dx = 3 \sec^2 \theta d\theta.$$

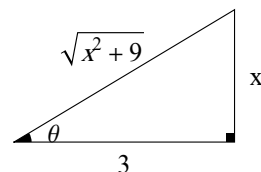
$$(9+x^2)^{\frac{3}{2}} = (9+9\tan^2 \theta)^{\frac{3}{2}} = [9(1+\tan^2 \theta)]^{\frac{3}{2}} = [(3)^2 \sec^2 \theta]^{\frac{3}{2}} = 3^3 \sec^3 \theta.$$

$$\begin{aligned}
\int \frac{1}{(9+x^2)^{\frac{3}{2}}} dx &= \int \frac{3 \sec^2 \theta}{3^3 \sec^3 \theta} d\theta = \frac{1}{3^2} \int \frac{1}{\sec \theta} d\theta \\
&= \frac{1}{9} \int \cos \theta d\theta = \frac{1}{9} \sin \theta + c
\end{aligned}$$

$$\tan \theta = \frac{x}{3} .$$

From the triangle :

$$\sin \theta = \frac{x}{\sqrt{x^2+9}}$$



$$\int \frac{1}{(9+x^2)^{\frac{3}{2}}} dx = \frac{1}{9} \frac{x}{\sqrt{x^2+9}} + c$$

7. $\int x^3 \ln |2x| dx$

Solution : Using integration by parts.

$$u = \ln |2x| \quad dv = x^3 dx$$

$$du = \frac{1}{x} dx \quad v = \frac{x^4}{4}$$

$$\begin{aligned}
\int x^3 \ln |2x| dx &= \frac{x^4}{4} \ln |2x| - \int \frac{x^4}{4} \frac{1}{x} dx \\
&= \frac{x^4}{4} \ln |2x| - \frac{1}{4} \int x^3 dx = \frac{x^4}{4} \ln |2x| - \frac{1}{4} \frac{x^4}{4} + c
\end{aligned}$$

8. $\int \frac{2x+1}{x^2(1+x^2)} dx$

Solution : Using the method of partial fractions.

$$\begin{aligned}\frac{2x+1}{x^2(1+x^2)} &= \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{Bx+C}{1+x^2} \\ \frac{2x+1}{x^2(1+x^2)} &= \frac{A_1 x(1+x^2)}{x^2(1+x^2)} + \frac{A_2(1+x^2)}{x^2(1+x^2)} + \frac{(Bx+C)x^2}{x^2(1+x^2)} \\ 2x+1 &= A_1 x(1+x^2) + A_2(1+x^2) + (Bx+C)x^2 \\ 2x+1 &= A_1 x + A_1 x^3 + A_2 + A_2 x^2 + Bx^3 + Cx^2 \\ 2x+1 &= (A_1+B)x^3 + (A_2+C)x^2 + A_1 x + A_2\end{aligned}$$

By comparing the coefficients of the two polynomials in each side :

$$\begin{aligned}A_1 + B &= 0 & \longrightarrow (1) \\ A_2 + C &= 0 & \longrightarrow (2) \\ A_1 &= 2 & \longrightarrow (3) \\ A_2 &= 1 & \longrightarrow (4)\end{aligned}$$

From equations (1) and (3) : $2 + B = 0 \implies B = -2$.

From equations (2) and (4) : $1 + C = 0 \implies C = -1$.

$$\begin{aligned}\int \frac{2x+1}{x^2(1+x^2)} dx &= \int \left(\frac{2}{x} + \frac{1}{x^2} + \frac{-2x-1}{1+x^2} \right) dx \\ &= \int \frac{2}{x} dx + \int \frac{1}{x^2} dx + \int \frac{-2x-1}{1+x^2} dx \\ &= 2 \int \frac{1}{x} dx + \int x^{-2} dx - \int \frac{2x}{1+x^2} dx - \int \frac{1}{1+x^2} dx \\ &= 2 \ln|x| + \frac{x^{-1}}{-1} - \ln(1+x^2) - \tan^{-1} x + c .\end{aligned}$$

9. $\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{2}{3}}} dx$.

Solution :

Using the substitution $x = u^6$, then $u = x^{\frac{1}{6}}$.

$$dx = 6u^5 du .$$

$$\begin{aligned}\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{2}{3}}} dx &= \int \frac{6u^5}{(u^6)^{\frac{1}{2}} + (u^6)^{\frac{2}{3}}} du = 6 \int \frac{u^5}{u^3 + u^4} du \\ &= 6 \int \frac{u^5}{u^3(u+1)} du = 6 \int \frac{u^2}{u+1} du\end{aligned}$$

Using long division of polynomials :

$$\begin{aligned}6 \int \frac{u^2}{u+1} du &= 6 \int \left(u - 1 + \frac{1}{u+1} \right) du = 6 \left(\frac{u^2}{2} - u + \ln|u+1| \right) + c \\ &= 3u^2 - 6u + 6 \ln|u+1| + c = 3x^{\frac{1}{3}} - 6x^{\frac{1}{6}} + 6 \ln \left| x^{\frac{1}{6}} + 1 \right| + c\end{aligned}$$

MATH 111 - Integral Calculus
First Semester - 1447 H
Solution of the Final Exam
Dr Tariq A. Alfadhel

Question (1): [3 + 2 + 2 = 7 marks]

- Find the value of c that satisfies the mean value theorem of the definite integral for the function $f(x) = \frac{1}{\sqrt{x+1}}$ on the interval $[3, 8]$.

Solution: Using the formula $(b-a) f(c) = \int_a^b f(x) dx$.

$$(8-3) \frac{1}{\sqrt{c+1}} = \int_3^8 \frac{1}{\sqrt{x+1}} dx = \int_3^8 (x+1)^{-\frac{1}{2}} dx$$

$$\frac{5}{\sqrt{c+1}} = \left[\frac{(x+1)^{\frac{1}{2}}}{\frac{1}{2}} \right]_3^8 = 2\sqrt{8+1} - 2\sqrt{3+1} = 2(3) - 2(2) = 6 - 4 = 2$$

$$\frac{5}{\sqrt{c+1}} = 2 \implies \sqrt{c+1} = \frac{5}{2} \implies c+1 = \frac{25}{4} \implies c = \frac{25}{4} - 1 = \frac{21}{4}$$

The desired value is $c = \frac{21}{4} \in (3, 8)$.

- Find $F'(x)$, if $F(x) = \int_{\sinh x}^{2^{x+1}} (1+t^2) dt$.

Solution:

$$\begin{aligned} F'(x) &= \frac{d}{dx} \int_{\sinh x}^{2^{x+1}} (1+t^2) dt \\ &= \left[1 + (2^{x+1})^2 \right] (2^{x+1} \ln 2) - \left[1 + (\sinh x)^2 \right] (\cosh x) \\ &= 2^{x+1} \ln 2 (1 + 2^{2x+2}) - \cosh x (1 + \sinh^2 x) \\ &= 2^{x+1} \ln 2 (1 + 2^{2x+2}) - \cosh^3 x \end{aligned}$$

- Find y' if $y(x) = (\cos x)^x + e^{-5x}$.

Solution:

$$\begin{aligned} y(x) &= (\cos x)^x + e^{-5x} = e^{\ln|\cos x|^x} + e^{-5x} = e^{x \ln|\cos x|} + e^{-5x} \\ y'(x) &= e^{x \ln|\cos x|} \left[(1) \ln|\cos x| + x \left(\frac{-\sin x}{\cos x} \right) \right] + e^{-5x} (-5) \\ &= (\cos x)^x [\ln|\cos x| - x \tan x] - 5e^{-5x} \end{aligned}$$

Question (2): [2 + 3 + 2 + 3 + 3 + 2 = 15 marks]

Evaluate the following integrals :

1. $\int \frac{1}{\sqrt{x^2 - 4x}} dx$.

Solution: By completing the square.

$$x^2 - 4x = (x^2 - 4x + 4) - 4 = (x - 2)^2 - (2)^2$$

$$\int \frac{1}{\sqrt{x^2 - 4x}} dx = \int \frac{1}{\sqrt{(x - 2)^2 - (2)^2}} dx = \cosh^{-1} \left(\frac{x - 2}{2} \right) + c .$$

2. $\int x^2 \cos(3x) dx$.

Solution: Using integration by parts .

$$\begin{aligned} u &= x^2 & dv &= \cos(3x) dx \\ du &= 2x dx & v &= \frac{1}{3} \sin(3x) \end{aligned}$$

$$\begin{aligned} \int x^2 \cos(3x) dx &= x^2 \left(\frac{1}{3} \sin(3x) \right) - \int \frac{1}{3} \sin(3x) 2x dx \\ &= \frac{x^2}{3} \sin(3x) - \frac{2}{3} \int x \sin(3x) dx \end{aligned}$$

Using integration by parts again .

$$\begin{aligned} u &= x & dv &= \sin(3x) dx \\ du &= dx & v &= -\frac{1}{3} \cos(3x) \end{aligned}$$

$$\begin{aligned} \int x^2 \cos(3x) dx &= \frac{x^2}{3} \sin(3x) - \frac{2}{3} \left[x \left(-\frac{1}{3} \cos(3x) \right) - \int -\frac{1}{3} \cos(3x) dx \right] \\ &= \frac{x^2}{3} \sin(3x) + \frac{2x}{9} \cos(3x) - \frac{2}{9} \int \cos(3x) dx \\ &= \frac{x^2}{3} \sin(3x) + \frac{2x}{9} \cos(3x) - \frac{2}{27} \sin(3x) + c \end{aligned}$$

3. $\int \frac{(\sqrt{x} - 1) \sec(\sqrt{x} - \ln \sqrt{x})}{x} dx$

Solution:

$$\begin{aligned} \int \frac{(\sqrt{x} - 1) \sec(\sqrt{x} - \ln \sqrt{x})}{x} dx &= \int \sec \left(\sqrt{x} - \ln \left| x^{\frac{1}{2}} \right| \right) \left(\frac{\sqrt{x} - 1}{x} \right) dx \\ &= \int \sec \left(\sqrt{x} - \frac{1}{2} \ln |x| \right) \left(\frac{\sqrt{x}}{x} - \frac{1}{x} \right) dx = \int \sec \left(\sqrt{x} - \frac{1}{2} \ln |x| \right) \left(\frac{1}{\sqrt{x}} - \frac{1}{x} \right) dx \\ &= 2 \int \sec \left(\sqrt{x} - \frac{1}{2} \ln |x| \right) \left[\frac{1}{2} \left(\frac{1}{\sqrt{x}} - \frac{1}{x} \right) \right] dx \end{aligned}$$

$$\begin{aligned}
&= 2 \int \sec \left(\sqrt{x} - \frac{1}{2} \ln |x| \right) \left(\frac{1}{2\sqrt{x}} - \frac{1}{2x} \right) dx \\
&= 2 \ln \left| \sec \left(\sqrt{x} - \frac{1}{2} \ln |x| \right) + \tan \left(\sqrt{x} - \frac{1}{2} \ln |x| \right) \right| + c .
\end{aligned}$$

$$4. \int \frac{5x^2 + 6x + 4}{(x+1)(x^2 + x + 1)} dx .$$

Solution : Using the method of partial fractions.

$$\begin{aligned}
\frac{5x^2 + 6x + 4}{(x+1)(x^2 + x + 1)} &= \frac{A}{x+1} + \frac{Bx + C}{x^2 + x + 1} \\
\frac{5x^2 + 6x + 4}{(x+1)(x^2 + x + 1)} &= \frac{A(x^2 + x + 1)}{(x+1)(x^2 + x + 1)} + \frac{(Bx + C)(x+1)}{(x^2 + x + 1)(x+1)} \\
5x^2 + 6x + 4 &= Ax^2 + Ax + A + Bx^2 + Bx + Cx + C \\
5x^2 + 6x + 4 &= (A+B)x^2 + (A+B+C)x + (A+C)
\end{aligned}$$

By comparing the coefficients of the two polynomials in each side :

$$\begin{aligned}
A + B &= 5 & \longrightarrow (1) \\
A + B + C &= 6 & \longrightarrow (2) \\
A + C &= 4 & \longrightarrow (3)
\end{aligned}$$

Equation (2) - Equation (1) : $C = 1$.

From Equation (3) : $A + 1 = 4 \implies A = 3$.

From Equation (1) : $3 + B = 5 \implies B = 2$.

$$\begin{aligned}
\int \frac{5x^2 + 6x + 4}{(x+1)(x^2 + x + 1)} dx &= \int \left(\frac{3}{x+1} + \frac{2x+1}{x^2 + x + 1} \right) dx \\
&= 3 \int \frac{1}{x+1} dx + \int \frac{2x+1}{x^2 + x + 1} dx = 3 \ln |x+1| + \ln |x^2 + x + 1| + c
\end{aligned}$$

$$5. \int \frac{x^2}{(4-x^2)^{\frac{3}{2}}} dx .$$

Solution : Using trigonometric substitutions.

$$\text{Put } x = 2 \sin \theta \implies \sin \theta = \frac{x}{2} .$$

$$dx = 2 \cos \theta d\theta .$$

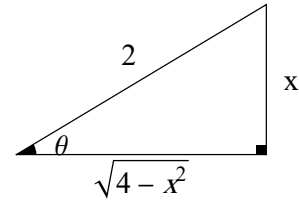
$$\begin{aligned}
(4-x^2)^{\frac{3}{2}} &= (4-4\sin^2 \theta)^{\frac{3}{2}} = [4(1-\sin^2 \theta)]^{\frac{3}{2}} = [2^2 \cos^2 \theta]^{\frac{3}{2}} \\
&= (2^2)^{\frac{3}{2}} (\cos^2 \theta)^{\frac{3}{2}} = 2^3 \cos^3 \theta \\
\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}} dx &= \int \frac{4 \sin^2 \theta \cdot 2 \cos \theta}{2^3 \cos^3 \theta} d\theta = \int \frac{\sin^2 \theta}{\cos^2 \theta} d\theta = \int \tan^2 \theta d\theta
\end{aligned}$$

$$= \int (\sec^2 \theta - 1) d\theta = \tan \theta - \theta + c$$

$$\sin \theta = \frac{x}{2} \implies \theta = \sin^{-1} \left(\frac{x}{2} \right).$$

From the triangle :

$$\tan \theta = \frac{x}{\sqrt{4-x^2}}$$



$$\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}} dx = \frac{x}{\sqrt{4-x^2}} - \sin^{-1} \left(\frac{x}{2} \right) + c.$$

6. $\int \tan^3 x \sec^2 x dx$.

Solution : .

$$\int \tan^3 x \sec^2 x dx = \int (\tan x)^3 \sec^2 x dx = \frac{(\tan x)^4}{4} + c = \frac{\tan^4 x}{4} + c .$$

Question (3): [2 + 2 + 3 + 3 + 2 + 2 + 4 = 18 marks]

1. Calculate $\lim_{x \rightarrow \infty} x^{\frac{1}{x}}$.

Solution:

$$\lim_{x \rightarrow \infty} x^{\frac{1}{x}} = \lim_{x \rightarrow \infty} e^{\ln|x|^{\frac{1}{x}}} = \lim_{x \rightarrow \infty} e^{\frac{1}{x} \ln|x|} = \lim_{x \rightarrow \infty} e^{\frac{\ln|x|}{x}}$$

$$\lim_{x \rightarrow \infty} \frac{\ln|x|}{x} \quad \left(\frac{\infty}{\infty} \right)$$

Using L'Hôpital's rule.

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0 .$$

$$\text{Therefore, } \lim_{x \rightarrow \infty} \frac{\ln|x|}{x} = 0 .$$

$$\text{So, } \lim_{x \rightarrow \infty} x^{\frac{1}{x}} = e^0 = 1 .$$

2. Determine whether the improper integral $\int_0^{\infty} \frac{x}{(1+x^2)^2} dx$ converges or diverges.

Solution:

$$\int_0^{\infty} \frac{x}{(1+x^2)^2} dx = \lim_{t \rightarrow \infty} \int_0^t \frac{x}{(1+x^2)^2} dx = \lim_{t \rightarrow \infty} \left(\frac{1}{2} \int_0^t (1+x^2)^{-2} (2x) dx \right)$$

$$\begin{aligned}
&= \lim_{t \rightarrow \infty} \left(\frac{1}{2} \left[\frac{(1+x^2)^{-1}}{-1} \right]_0^t \right) = \lim_{t \rightarrow \infty} \left(\frac{1}{2} \left[\frac{-1}{1+x^2} \right]_0^t \right) \\
&= \lim_{t \rightarrow \infty} \left(\frac{1}{2} \left[\frac{-1}{1+t^2} - \frac{-1}{1+(0)^2} \right] \right) = \frac{1}{2} [0 - (-1)] = \frac{1}{2} .
\end{aligned}$$

Hence, the improper integral converges.

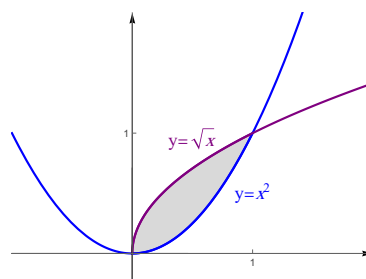
3. Sketch the region bounded by the graphs of the curves $y = x^2$ and $y = \sqrt{x}$, then find its area.

Solution:

$y = x^2$ represents a parabola opens upwards, and its vertex is $(0, 0)$.

$y = \sqrt{x}$ represents the upper half of a parabola opens to the right and its vertex is $(0, 0)$.

Points of intersection of $y = x^2$ and $y = \sqrt{x}$:
 $x^2 = \sqrt{x} \implies x^4 = x$
 $\implies x^4 - x = 0$
 $\implies x(x^3 - 1) = 0$
 $\implies x = 0, x = 1.$



$$\begin{aligned}
A &= \int_0^1 (\sqrt{x} - x^2) dx = \left[2\frac{x^{\frac{3}{2}}}{3} - \frac{x^3}{3} \right]_0^1 \\
&= \left(\frac{2}{3} - \frac{1}{3} \right) - (0 - 0) = \frac{1}{3} .
\end{aligned}$$

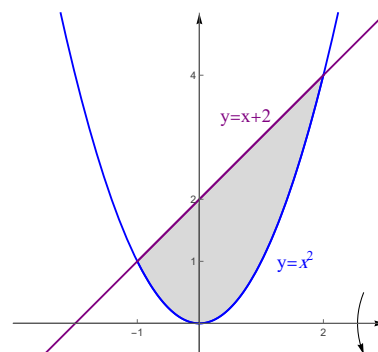
4. Sketch the region bounded by the graphs of the curves $y = x^2$ and $y = x + 2$, then find the volume of the solid generated by revolving this region about the x -axis.

Solution :

$y = x^2$ represents a parabola opens upwards, and its vertex is $(0, 0)$.

$y = x + 2$ represents a straight line passing through $(0, 2)$ with slope 1.

Points of intersection of $y = x^2$ and $y = x + 2$:
 $x^2 = x + 2 \implies x^2 - x - 2 = 0$
 $\implies (x + 1)(x - 2) = 0$
 $\implies x = -1, x = 2$



Using Washer Method :

$$\begin{aligned}
 V &= \pi \int_{-1}^2 \left[(x+2)^2 - (x^2)^2 \right] dx \\
 &= \pi \int_{-1}^2 [x^2 + 4x + 4 - x^4] dx = \pi \left[-\frac{x^5}{5} + \frac{x^3}{3} + 2x^2 + 4x \right]_{-1}^2 \\
 &= \pi \left[\left(-\frac{2^5}{5} + \frac{2^3}{3} + 2(2^2) + 4(2) \right) - \left(-\frac{(-1)^5}{5} + \frac{(-1)^3}{3} + 2((-1)^2) + 4(-1) \right) \right] \\
 &= \pi \left[\left(-\frac{32}{5} + \frac{8}{3} + 8 + 8 \right) - \left(\frac{1}{5} - \frac{1}{3} + 2 - 4 \right) \right] \\
 &= \pi \left(-\frac{33}{5} + \frac{9}{3} + 18 \right) = \pi \left(-\frac{33}{5} + 21 \right) = \frac{72\pi}{5} .
 \end{aligned}$$

5. Find the arc length of $y = \frac{2}{3}x\sqrt{x}$, from $x = 0$ to $x = 1$.

Solution :

$$\begin{aligned}
 y &= \frac{2}{3}x\sqrt{x} = \frac{2}{3} x^{\frac{3}{2}} . \\
 y' &= \frac{2}{3} \left(\frac{3}{2} x^{\frac{1}{2}} \right) = x^{\frac{1}{2}} . \\
 L &= \int_0^1 \sqrt{1 + \left(x^{\frac{1}{2}} \right)^2} dx = \int_0^1 \sqrt{1 + x} dx = \int_0^1 (1 + x)^{\frac{1}{2}} dx \\
 &= \left[\frac{2}{3} (1 + x)^{\frac{3}{2}} \right]_0^1 = \frac{2}{3} (1 + 1)^{\frac{3}{2}} - \frac{2}{3} (1 + 0)^{\frac{3}{2}} = \frac{2}{3} (\sqrt{8} - 1) .
 \end{aligned}$$

6. Convert the polar equation $r = \cos \theta + \sec \theta$ into a Cartesian equation.

Solution:

$$\begin{aligned}
 r &= \cos \theta + \sec \theta \implies r^2 = r \cos \theta + r \sec \theta = r \cos \theta + \frac{r}{\cos \theta} \\
 \implies x^2 + y^2 &= x + \frac{r}{\frac{x}{r}} = x + \frac{r^2}{x} = x + \frac{x^2 + y^2}{x} \\
 \implies x^3 + xy^2 &= x^2 + x^2 + y^2 \implies x^3 + xy^2 - 2x^2 - y^2 = 0
 \end{aligned}$$

7. Sketch the common region between the curves $r = 2$ and $r = 2 + 2 \cos \theta$, then find its area.

Solution:

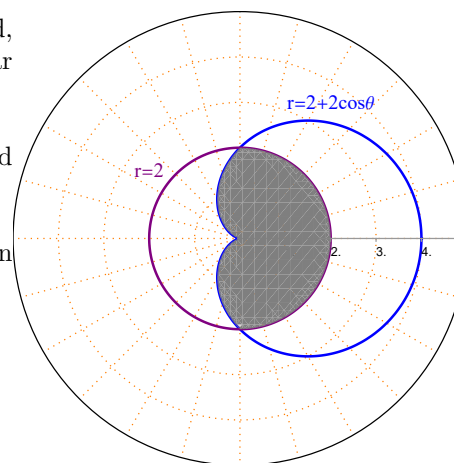
$r = 2 + 2 \cos \theta$ represents a cardioid, symmetric with respect to the polar axis.

$r = 2$ represents a circle centered at the pole, and its radius is 2.

Points of intersection between $r = 2 + 2 \cos \theta$ and $r = 2$:

$$2 + 2 \cos \theta = 2 \implies 2 \cos \theta = 0$$

$$\implies \cos \theta = 0 \implies \theta = \frac{\pi}{2}, \theta = \frac{3\pi}{2}$$



Note that the shaded region is symmetric with respect to the polar axis.

$$\begin{aligned} A &= 2 \left(\frac{1}{2} \int_0^{\frac{\pi}{2}} (2)^2 d\theta + \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} (2 + 2 \cos \theta)^2 d\theta \right) \\ &= \int_0^{\frac{\pi}{2}} 4 d\theta + \int_{\frac{\pi}{2}}^{\pi} (4 + 8 \cos \theta + 4 \cos^2 \theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} 4 d\theta + \int_{\frac{\pi}{2}}^{\pi} \left(4 + 8 \cos \theta + 4 \left(\frac{1 + \cos 2\theta}{2} \right) \right) d\theta \\ &= \int_0^{\frac{\pi}{2}} 4 d\theta + \int_{\frac{\pi}{2}}^{\pi} (4 + 8 \cos \theta + (2 + 2 \cos 2\theta)) d\theta \\ &= \int_0^{\frac{\pi}{2}} 4 d\theta + \int_{\frac{\pi}{2}}^{\pi} (6 + 8 \cos \theta + 2 \cos 2\theta) d\theta \\ &= [4\theta]_0^{\frac{\pi}{2}} + [6\theta + 8 \sin \theta + \sin 2\theta]_{\frac{\pi}{2}}^{\pi} \\ &= \left[4 \left(\frac{\pi}{2} \right) - 4(0) \right] + \left[(6\pi + 8 \sin(\pi) + \sin(2\pi)) - \left(6 \left(\frac{\pi}{2} \right) + 8 \sin \left(\frac{\pi}{2} \right) + \sin(\pi) \right) \right] \\ &= [2\pi - 0] + [(6\pi + 0 + 0) - (3\pi + 8 + 0)] = 2\pi + (3\pi - 8) = 5\pi - 8 . \end{aligned}$$

MATH 111 - Integral Calculus
Second Semester - 1447 H
Solution of the First Exam
Dr Tariq A. Alfadhel

Question (1): [2 + 1 = 3 marks]

1. Use Riemann Sum to evaluate the definite integral $\int_0^1 x^3 dx$.

Solution : $[a, b] = [0, 1]$, $f(x) = x^3$.

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n} .$$

$$x_k = a + k \Delta x = 0 + k \left(\frac{1}{n} \right) = \frac{k}{n} .$$

$$f(x_k) = \left(\frac{k}{n} \right)^3 = \frac{k^3}{n^3} .$$

$$R_n = \sum_{k=1}^n f(x_k) \Delta x = \sum_{k=1}^n \left(\frac{k^3}{n^3} \right) \left(\frac{1}{n} \right) = \sum_{k=1}^n \frac{k^3}{n^4} = \frac{1}{n^4} \sum_{k=1}^n k^3$$

$$= \frac{1}{n^4} \left[\frac{n(n+1)}{2} \right]^2 = \frac{1}{n^4} \left[\frac{n^2(n+1)^2}{4} \right] = \frac{1}{4} \left[\frac{n^2(n+1)^2}{n^4} \right]$$

$$= \frac{1}{4} \frac{(n+1)^2}{n^2} = \frac{1}{4} \left(\frac{n+1}{n} \right)^2 .$$

$$\int_0^1 x^3 dx = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \left[\frac{1}{4} \left(\frac{n+1}{n} \right)^2 \right] = \frac{1}{4} (1)^2 = \frac{1}{4} .$$

2. Find $F'(x)$, if $F(x) = \int_{\sqrt{3x^2+4}}^{\sec(x^3)} \sqrt{4+t^2} dt$.

Solution :

$$F'(x) = \frac{d}{dx} \int_{\sqrt{3x^2+4}}^{\sec(x^3)} \sqrt{4+t^2} dt$$

$$= \sqrt{4 + [\sec(x^3)]^2} (\sec(x^3) \tan(x^3)(3x^2)) - \sqrt{4 + (\sqrt{3x^2+4})^2} \left(\frac{1}{2\sqrt{3x^2+4}} (6x) \right)$$

$$= 3x^2 \sec(x^3) \tan(x^3) \sqrt{4 + \sec^2(x^3)} - \frac{3x\sqrt{3x^2+8}}{\sqrt{3x^2+4}} .$$

Question (2): Find $\frac{dy}{dx}$ of the following :[1 + 1 = 2 marks]

1. $y(x) = \left[\sec^{-1}(\ln x) + e^{x^4} \right]^5$.

Solution :

$$\begin{aligned} y'(x) &= 5 \left[\sec^{-1}(\ln x) + e^{x^4} \right]^4 \left(\frac{1}{\ln x \sqrt{(\ln x)^2 - 1}} \frac{1}{x} + e^{x^4} (4x^3) \right) \\ &= 5 \left[\sec^{-1}(\ln x) + e^{x^4} \right]^4 \left(\frac{1}{x \ln x \sqrt{(\ln x)^2 - 1}} + 4x^3 e^{x^4} \right) . \end{aligned}$$

2. $y(x) = (\tan x)^x$

Solution :

$$y(x) = (\tan x)^x \implies \ln |y(x)| = \ln |(\tan x)^x| = x \ln |\tan x|$$

Differentiate both sides.

$$\frac{y'(x)}{y(x)} = (1) \ln |\tan x| + x \left(\frac{\sec^2 x}{\tan x} \right)$$

$$y'(x) = y(x) \left[\ln |\tan x| + \frac{x \sec^2 x}{\tan x} \right] = (\tan x)^x \left[\ln |\tan x| + \frac{x \sec^2 x}{\tan x} \right] .$$

Question (3): [1 + 1 + 1 + 1 + 1 + 1 + 1 + 2 + 2 = 10 marks]

Evaluate the following integrals :

1. $\int (x^3 + 1) \sec(x^4 + 4x) dx$.

Solution :

$$\begin{aligned} \int (x^3 + 1) \sec(x^4 + 4x) dx &= \frac{1}{4} \int \sec(x^4 + 4x) [4(x^3 + 1)] dx \\ &= \frac{1}{4} \int \sec(x^4 + 4x) (4x^3 + 4) dx = \frac{1}{4} \ln |\sec(x^4 + 4x) + \tan(x^4 + 4x)| + c. \end{aligned}$$

2. $\int \frac{x^5 + x}{\sqrt{x^6 + 3x^2 + 8}} dx$.

Solution :

$$\begin{aligned} \int \frac{x^5 + x}{\sqrt{x^6 + 3x^2 + 8}} dx &= \int (x^6 + 3x^2 + 8)^{-\frac{1}{2}} (x^5 + x) dx \\ &= \frac{1}{6} \int (x^6 + 3x^2 + 8)^{-\frac{1}{2}} [6(x^5 + x)] dx = \frac{1}{6} \int (x^6 + 3x^2 + 8)^{-\frac{1}{2}} (6x^5 + 6x) dx \end{aligned}$$

$$= \frac{1}{6} \frac{(x^6 + 3x^2 + 8)^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{1}{3} (x^6 + 3x^2 + 8)^{\frac{1}{2}} + c$$

$$3. \int \frac{1}{\operatorname{csch} x (1 + \cosh x)} dx .$$

Solution :

$$\int \frac{1}{\operatorname{csch} x (1 + \cosh x)} dx = \int \frac{\sinh x}{1 + \cosh x} dx = \ln |1 + \cosh x| + c .$$

$$4. \int \frac{5^{\tanh x}}{\cosh^2 x} dx .$$

Solution :

$$\int \frac{5^{\tanh x}}{\cosh^2 x} dx = \int 5^{\tanh x} \operatorname{sech}^2 x dx = \frac{5^{\tanh x}}{\ln 5} + c .$$

$$5. \int \frac{2^x}{\sqrt{9 - 4^x}} dx$$

Solution :

$$\begin{aligned} \int \frac{2^x}{\sqrt{9 - 4^x}} dx &= \int \frac{2^x}{\sqrt{9 - (2^2)^x}} dx = \int \frac{2^x}{\sqrt{9 - (2^x)^2}} dx \\ &= \frac{1}{\ln 2} \int \frac{2^x \ln 2}{\sqrt{(3)^2 - (2^x)^2}} dx = \frac{1}{\ln 2} \sin^{-1} \left(\frac{2^x}{3} \right) + c \end{aligned}$$

$$6. \int \frac{(e^x - 1) \cos(x + e^{-x})}{e^x} dx$$

Solution :

$$\begin{aligned} \int \frac{(e^x - 1) \cos(x + e^{-x})}{e^x} dx &= \int \cos(x + e^{-x}) \frac{(e^x - 1)}{e^x} dx \\ &= \int \cos(x + e^{-x}) \left(\frac{e^x}{e^x} - \frac{1}{e^x} \right) dx = \int \cos(x + e^{-x}) (1 - e^{-x}) dx \\ &= \int \cos(x + e^{-x}) (1 + e^{-x}(-1)) dx = \sin(x + e^{-x}) + c . \end{aligned}$$

$$7. \int_0^1 \frac{x}{x+1} dx .$$

Solution :

$$\int_0^1 \frac{x}{x+1} dx = \int_0^1 \frac{(x+1) - 1}{x+1} dx = \int_0^1 \left(\frac{x+1}{x+1} - \frac{1}{x+1} \right) dx$$

$$\begin{aligned}
&= \int_0^1 \left(1 - \frac{1}{x+1}\right) dx = [x - \ln|x+1|]_0^1 \\
&= (1 - \ln 2) - (0 - \ln(1)) = 1 - \ln 2 .
\end{aligned}$$

$$8. \int \frac{1}{x^2 + 8x + 25} dx .$$

Solution : By completing the square.

$$\begin{aligned}
x^2 + 8x + 25 &= (x^2 + 8x + 16) + 9 = (x + 4)^2 + (3)^2 \\
&= \int \frac{1}{x^2 + 8x + 25} dx = \int \frac{1}{(x + 4)^2 + (3)^2} dx \\
&= \frac{1}{3} \tan^{-1} \left(\frac{x + 4}{3} \right) + c .
\end{aligned}$$

MATH 111 - Integral Calculus
Second Semester - 1447 H
Solution of the Second Exam
Dr Tariq A. Alfadhel

Question (1): [1 + 1 = 2 marks]

Find the derivative of the following functions :

1. $f(x) = \sinh^{-1}(\ln|x+1|)$.

Solution :

$$f'(x) = \frac{1}{\sqrt{1 + (\ln|x+1|)^2}} \cdot \frac{1}{x+1} = \frac{1}{(x+1)\sqrt{1 + (\ln|x+1|)^2}}.$$

2. $g(x) = \operatorname{csch}^{-1}(\sqrt{x+2})$.

Solution :

$$g'(x) = \frac{-1}{\sqrt{x+2} \sqrt{(\sqrt{x+2})^2 + 1}} \cdot \frac{1}{2\sqrt{x+2}} = \frac{-1}{2(x+2) \sqrt{x+3}}.$$

Question (2): [2 marks]

Calculate $\lim_{x \rightarrow 0^+} \left(\frac{1}{\ln(x+1)} - \frac{1}{x} \right)$.

Solution :

$$\begin{aligned} & \lim_{x \rightarrow 0^+} \left(\frac{1}{\ln(x+1)} - \frac{1}{x} \right) \quad (\infty - \infty) \\ & \lim_{x \rightarrow 0^+} \left(\frac{1}{\ln(x+1)} - \frac{1}{x} \right) = \lim_{x \rightarrow 0^+} \frac{x - \ln(x+1)}{x \ln(x+1)} \quad \left(\frac{0}{0} \right) \end{aligned}$$

Using L'Hôpital's rule.

$$\begin{aligned} & \lim_{x \rightarrow 0^+} \frac{1 - \frac{1}{x+1}}{\ln(x+1) + x \frac{1}{x+1}} = \lim_{x \rightarrow 0^+} \frac{\left(\frac{(x+1)-1}{x+1} \right)}{\left(\frac{(x+1)\ln(x+1)+x}{x+1} \right)} \\ & = \lim_{x \rightarrow 0^+} \frac{x}{(x+1)\ln(x+1) + x} \quad \left(\frac{0}{0} \right) \end{aligned}$$

Using L'Hôpital's rule.

$$\lim_{x \rightarrow 0^+} \frac{1}{\ln(x+1) + (x+1) \frac{1}{x+1} + 1} = \lim_{x \rightarrow 0^+} \frac{1}{\ln(x+1) + 2} = \frac{1}{0+2} = \frac{1}{2}.$$

$$\text{Therefore, } \lim_{x \rightarrow 0^+} \left(\frac{1}{\ln(x+1)} - \frac{1}{x} \right) = \frac{1}{2}.$$

Question (3): [1.5 + 1 + 1 + 2 + 2 + 2 + 1.5 marks]

Evaluate the following integrals :

1. $\int x \operatorname{sech}^{-1} x \, dx$.

Solution : Using integration by parts.

$$\begin{aligned} u &= \operatorname{sech}^{-1} x & dv &= x \, dx \\ du &= \frac{-1}{x\sqrt{1-x^2}} \, dx & v &= \frac{x^2}{2} \\ \int x \operatorname{sech}^{-1} x \, dx &= \frac{x^2}{2} \operatorname{sech}^{-1} x - \int \frac{x^2}{2} \frac{-1}{x\sqrt{1-x^2}} \, dx \\ &= \frac{x^2}{2} \operatorname{sech}^{-1} x + \frac{1}{2} \int \frac{x}{\sqrt{1-x^2}} \, dx \\ &= \frac{x^2}{2} \operatorname{sech}^{-1} x + \frac{1}{2} \frac{1}{-2} \int (1-x^2)^{-\frac{1}{2}} (-2x) \, dx \\ &= \frac{x^2}{2} \operatorname{sech}^{-1} x - \frac{1}{4} \frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{x^2}{2} \operatorname{sech}^{-1} x - \frac{\sqrt{1-x^2}}{2} + c \end{aligned}$$

2. $\int \frac{1}{x\sqrt{(\ln|x|)^2 - 1}} \, dx$

Solution :

$$\int \frac{1}{x\sqrt{(\ln|x|)^2 - 1}} \, dx = \int \frac{\left(\frac{1}{x}\right)}{\sqrt{(\ln|x|)^2 - 1}} \, dx = \cosh^{-1}(\ln|x|) + c .$$

3. $\int \sec^3 x \tan^3 x \, dx$

Solution :

Using the substitution $u = \sec x$.

Hence $du = \sec x \tan x \, dx$.

$$\begin{aligned} \int \sec^3 x \tan^3 x \, dx &= \int \sec^2 x \tan^2 x \sec x \tan x \, dx \\ &= \int \sec^2 x (\sec^2 x - 1) \sec x \tan x \, dx = \int u^2 (u^2 - 1) \, du \\ &= \int (u^4 - u^2) \, du = \frac{u^5}{5} - \frac{u^3}{3} + c = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + c \end{aligned}$$

$$4. \int \frac{1}{(x-2)\sqrt{-x^2+4x-2}} dx .$$

Solution : By completing the square.

$$-x^2 + 4x - 2 = -(x^2 - 4x) - 2 = -(x^2 - 4x + 4) - 2 + 4$$

$$= 2 - (x^2 - 4x + 4) = (\sqrt{2})^2 - (x-2)^2 .$$

$$\int \frac{1}{(x-2)\sqrt{-x^2+4x-2}} dx = \int \frac{1}{(x-2)\sqrt{(\sqrt{2})^2 - (x-2)^2}} dx$$

$$= -\frac{1}{\sqrt{2}} \operatorname{sech}^{-1} \left(\frac{x-2}{\sqrt{2}} \right) + c .$$

$$5. \int \frac{1}{(4-x^2)^{\frac{3}{2}}} dx .$$

Solution : Using trigonometric substitutions.

$$\text{Put } x = 2 \sin \theta \implies \sin \theta = \frac{x}{2} .$$

$$dx = 2 \cos \theta d\theta .$$

$$(4-x^2)^{\frac{3}{2}} = (4-4\sin^2 \theta)^{\frac{3}{2}} = [4(1-\sin^2 \theta)]^{\frac{3}{2}} = [(2)^2 \cos^2 \theta]^{\frac{3}{2}} = 2^3 \cos^3 \theta .$$

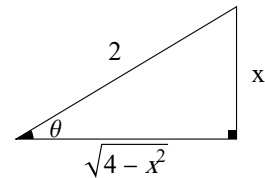
$$\int \frac{1}{(4-x^2)^{\frac{3}{2}}} dx = \int \frac{2 \cos \theta}{2^3 \cos^3 \theta} d\theta = \frac{1}{2^2} \int \frac{1}{\cos^2 \theta} d\theta$$

$$= \frac{1}{4} \int \sec^2 \theta d\theta = \frac{1}{4} \tan \theta + c$$

$$\sin \theta = \frac{x}{2} .$$

From the triangle :

$$\tan \theta = \frac{x}{\sqrt{4-x^2}}$$



$$\int \frac{1}{(4-x^2)^{\frac{3}{2}}} dx = \frac{1}{4} \frac{x}{\sqrt{4-x^2}} + c$$

$$6. \int \frac{3x^2 + x + 4}{x(x^2 + 4)} dx$$

Solution : Using the method of partial fractions.

$$\frac{3x^2 + x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$\frac{3x^2 + x + 4}{x(x^2 + 4)} = \frac{A(x^2 + 4)}{x(x^2 + 4)} + \frac{(Bx + C)x}{x(x^2 + 4)}$$

$$3x^2 + x + 4 = Ax^2 + 4A + Bx^2 + Cx$$

$$3x^2 + x + 4 = (A + B)x^2 + Cx + 4A$$

By comparing the coefficients of the two polynomials in each side :

$$A + B = 3 \quad \longrightarrow \quad (1)$$

$$C = 1 \quad \longrightarrow \quad (2)$$

$$4A = 4 \quad \longrightarrow \quad (3)$$

$$\text{From equations (3) : } 4A = 4 \implies A = \frac{4}{4} = 1 .$$

$$\text{From equations (1) : } A + B = 3 \implies 1 + B = 3 \implies B = 2 .$$

$$\begin{aligned} \int \frac{3x^2 + x + 4}{x(x^2 + 4)} dx &= \int \left(\frac{1}{x} + \frac{2x + 1}{x^2 + 4} \right) dx \\ &= \int \frac{1}{x} dx + \int \frac{2x + 1}{x^2 + 4} dx \\ &= \int \frac{1}{x} dx + \int \frac{2x}{x^2 + 4} dx + \int \frac{1}{(x)^2 + (2)^2} dx \\ &= \ln |x| + \ln(x^2 + 4) + \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + c . \end{aligned}$$

$$7. \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{3}{4}}} dx .$$

Solution :

Using the substitution $x = u^4$, then $u = x^{\frac{1}{4}}$.

$$dx = 4u^3 du .$$

$$\begin{aligned} \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{3}{4}}} dx &= \int \frac{4u^3}{(u^4)^{\frac{1}{2}} + (u^4)^{\frac{3}{4}}} du = 4 \int \frac{u^3}{u^2 + u^3} du \\ &= 4 \int \frac{u^3}{u^2(1 + u)} du = 4 \int \frac{u}{u + 1} du = 4 \int \frac{(u + 1) - 1}{u + 1} du \\ &= 4 \int \left[\frac{u + 1}{u + 1} - \frac{1}{u + 1} \right] du = 4 \int \left(1 - \frac{1}{u + 1} \right) du \\ &= 4(u - \ln |u + 1|) + c = 4u - 4 \ln |u + 1| + c = 4x^{\frac{1}{4}} - 4 \ln \left| x^{\frac{1}{4}} + 1 \right| + c . \end{aligned}$$

MATH 111 - Integral Calculus
Second Semester - 1447 H
Solution of the Final Exam
Dr Tariq A. Alfadhel

Question (1): [2 + 2 + 2 = 6 marks]

1. Find the value of c that satisfies the mean value theorem of the definite integral for the function $f(x) = (x+1)^2$ on the interval $[-1, 2]$.

Solution: Using the formula $(b-a) f(c) = \int_a^b f(x) dx$.

$$(2 - (-1)) (c+1)^2 = \int_{-1}^2 (x+1)^2 dx = \left[\frac{(x+1)^3}{3} \right]_{-1}^2$$

$$3(c+1)^2 = \frac{(2+1)^3}{3} - \frac{(-1+1)^3}{3} = \frac{27}{3} - \frac{0}{3} = 9$$

$$(c+1)^2 = 3 \implies c+1 = \pm\sqrt{3} \implies c = -1 \pm \sqrt{3}.$$

Note that $c = -1 - \sqrt{3} \notin (-1, 2)$ and $c = -1 + \sqrt{3} \in (-1, 2)$.

The desired value is $c = -1 + \sqrt{3}$.

2. Find $F'(x)$, if $F(x) = \int_{\sec x}^{\log_2 x} \frac{1}{\sqrt{1+t}} dt$.

Solution:

$$\begin{aligned} F'(x) &= \frac{d}{dx} \int_{\sec x}^{\log_2 x} \frac{1}{\sqrt{1+t}} dt \\ &= \frac{1}{\sqrt{1+\log_2 x}} \left(\frac{1}{x} \frac{1}{\ln 2} \right) - \frac{1}{\sqrt{1+\sec x}} (\sec x \tan x) \\ &= \frac{1}{x \ln 2 \sqrt{1+\log_2 x}} - \frac{\sec x \tan x}{\sqrt{1+\sec x}}. \end{aligned}$$

3. Find y' if $y(x) = \ln \left| \frac{(x^2+1)^{\frac{2}{3}} \cos^7(5x)}{\sqrt[3]{x^4+1}} \right|$.

Solution:

$$\begin{aligned} y(x) &= \ln \left| (x^2+1)^{\frac{2}{3}} \cos^7(5x) \right| - \ln \left| \sqrt[3]{x^4+1} \right| \\ y(x) &= \ln \left| (x^2+1)^{\frac{2}{3}} \right| + \ln |\cos^7(5x)| - \ln \left| (x^4+1)^{\frac{1}{3}} \right| \\ y(x) &= \frac{2}{3} \ln |x^2+1| + 7 \ln |\cos(5x)| - \frac{1}{3} \ln |x^4+1| \end{aligned}$$

$$y'(x) = \frac{2}{3} \left(\frac{2x}{x^2 + 1} \right) + 7 \left(\frac{-\sin(5x) (5)}{\cos(5x)} \right) - \frac{1}{3} \left(\frac{4x^3}{x^4 + 1} \right)$$

Question (2): [2 + 2 + 3 + 3 + 3 + 3 = 16 marks]

Evaluate the following integrals :

$$1. \int \frac{1}{x + x \ln |x|} dx .$$

Solution:

$$\begin{aligned} \int \frac{1}{x + x \ln |x|} dx &= \int \frac{1}{x (1 + \ln |x|)} dx \\ &= \int \frac{\left(\frac{1}{x}\right)}{1 + \ln |x|} dx = \ln |1 + \ln |x|| + c . \end{aligned}$$

$$2. \int \frac{\sec x \tan x}{1 + \sec^2 x} dx$$

Solution:

$$\int \frac{\sec x \tan x}{1 + \sec^2 x} dx = \int \frac{\sec x \tan x}{(1)^2 + (\sec x)^2} dx = \tan^{-1}(\sec x) + c .$$

$$3. \int e^{2x} \sin x dx .$$

Solution: Using integration by parts .

$$\begin{aligned} u &= \sin x & dv &= e^{2x} dx \\ du &= \cos x dx & v &= \frac{1}{2} e^{2x} \end{aligned}$$

$$\begin{aligned} \int e^{2x} \sin x dx &= \sin x \left(\frac{1}{2} e^{2x} \right) - \int \frac{1}{2} e^{2x} \cos x dx \\ &= \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x dx \end{aligned}$$

Using integration by parts again .

$$\begin{aligned} u &= \cos x & dv &= e^{2x} dx \\ du &= -\sin x dx & v &= \frac{1}{2} e^{2x} \end{aligned}$$

$$\begin{aligned} \int e^{2x} \sin x dx &= \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \left[\frac{1}{2} e^{2x} \cos x - \int \frac{1}{2} e^{2x} (-\sin x) dx \right] \\ &= \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x - \frac{1}{4} \int e^{2x} \sin x dx \\ \int e^{2x} \sin x dx + \frac{1}{4} \int e^{2x} \sin x dx &= \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x + c \end{aligned}$$

$$\frac{5}{4} \int e^{2x} \sin x \, dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x + c$$

$$\int e^{2x} \sin x \, dx = \frac{4}{5} \left[\frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x + c \right].$$

4. $\int \frac{3x^2 + 1}{(x^2 - 1)(x^2 + 1)} \, dx$.

Solution : Using the method of partial fractions.

$$\frac{3x^2 + 1}{(x^2 - 1)(x^2 + 1)} = \frac{3x^2 + 1}{(x - 1)(x + 1)(x^2 + 1)} = \frac{A_1}{x - 1} + \frac{A_2}{x + 1} + \frac{Bx + C}{x^2 + 1}$$

$$3x^2 + 1 = A_1(x + 1)(x^2 + 1) + A_2(x - 1)(x^2 + 1) + (Bx + C)(x^2 - 1)$$

$$3x^2 + 1 = A_1x^3 + A_1x + A_1x^2 + A_1 + A_2x^3 + A_2x - A_2x^2 - A_2$$

$$+ Bx^3 - Bx + Cx^2 - C$$

$$3x^2 + 1 = (A_1 + A_2 + B)x^3 + (A_1 - A_2 + C)x^2 + (A_1 + A_2 - B)x$$

$$+ (A_1 - A_2 - C)$$

By comparing the coefficients of the two polynomials in each side :

$$\begin{aligned} A_1 + A_2 + B &= 0 & \longrightarrow (1) \\ A_1 - A_2 + C &= 3 & \longrightarrow (2) \\ A_1 + A_2 - B &= 0 & \longrightarrow (3) \\ A_1 - A_2 - C &= 1 & \longrightarrow (4) \end{aligned}$$

Equation (1) - Equation (3) : $2B = 0 \implies B = 0$.

Equation (2) - Equation (4) : $2C = 2 \implies C = 1$.

Equation (1) becomes : $A_1 + A_2 = 0 \longrightarrow (5)$.

Equation (2) becomes : $A_1 - A_2 = 2 \longrightarrow (6)$.

Equation (5) + Equation (6) : $2A_1 = 2 \implies A_1 = 1$.

From Equation (5) : $1 + A_2 = 0 \implies A_2 = -1$

$$\int \frac{3x^2 + 1}{(x^2 - 1)(x^2 + 1)} \, dx = \int \left(\frac{1}{x - 1} + \frac{-1}{x + 1} + \frac{1}{x^2 + 1} \right) \, dx$$

$$= \int \frac{1}{x - 1} \, dx - \int \frac{1}{x + 1} \, dx + \int \frac{1}{x^2 + 1} \, dx$$

$$= \ln |x - 1| - \ln |x + 1| + \tan^{-1} x + c.$$

5. $\int \frac{1}{x^2 \sqrt{9 - x^2}} \, dx$.

Solution : Using trigonometric substitutions.

Put $x = 3 \sin \theta \implies \sin \theta = \frac{x}{3}$.

$dx = 3 \cos \theta \, d\theta$.

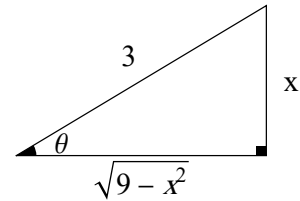
$\sqrt{9 - x^2} = \sqrt{9 - 9 \sin^2 \theta} = \sqrt{9(1 - \sin^2 \theta)} = \sqrt{9 \cos^2 \theta} = 3 \cos \theta$

$$\begin{aligned} \int \frac{1}{x^2 \sqrt{9 - x^2}} dx &= \int \frac{3 \cos \theta}{(3 \sin \theta)^2 3 \cos \theta} d\theta = \int \frac{1}{9 \sin^2 \theta} d\theta \\ &= \frac{1}{9} \int \frac{1}{\sin^2 \theta} d\theta = \frac{1}{9} \int \csc^2 \theta \, d\theta = -\frac{1}{9} \cot \theta + c \end{aligned}$$

$\sin \theta = \frac{x}{3}$

From the triangle :

$\cot \theta = \frac{\sqrt{9 - x^2}}{x}$



$\int \frac{1}{x^2 \sqrt{9 - x^2}} dx = -\frac{1}{9} \frac{\sqrt{9 - x^2}}{x} + c$.

6. $\int \frac{1}{(x + 3)\sqrt{x^2 + 6x + 13}} dx$.

Solution : By completing the square.

$x^2 + 6x + 13 = (x^2 + 6x + 9) + 4 = (x + 3)^2 + (2)^2$

$$\begin{aligned} \int \frac{1}{(x + 3)\sqrt{x^2 + 6x + 13}} dx &= \int \frac{1}{(x + 3)\sqrt{(x + 3)^2 + (2)^2}} dx \\ &= -\frac{1}{2} \operatorname{csch}^{-1} \left(\frac{x + 3}{2} \right) + c. \end{aligned}$$

Question (3): [2.5 + 2 + 3 + 3 + 2.5 + 2 + 3 = 18 marks]

1. Calculate $\lim_{x \rightarrow \infty} \left(1 + \frac{4}{x}\right)^{2x}$.

Solution:

$\lim_{x \rightarrow \infty} \left(1 + \frac{4}{x}\right)^{2x} \quad (1^\infty).$

$\lim_{x \rightarrow \infty} \left(1 + \frac{4}{x}\right)^{2x} = \lim_{x \rightarrow \infty} e^{\ln \left|1 + \frac{4}{x}\right|^{2x}} = \lim_{x \rightarrow \infty} e^{2x \ln \left|1 + \frac{4}{x}\right|}$

$\lim_{x \rightarrow \infty} 2x \ln \left|1 + \frac{4}{x}\right| \quad (\infty \cdot 0).$

$$= \lim_{x \rightarrow \infty} \frac{\ln \left| 1 + \frac{4}{x} \right|}{\left(\frac{1}{2x} \right)} \quad \left(\frac{0}{0} \right).$$

Using L'Hôpital's rule.

$$\lim_{x \rightarrow \infty} \frac{\left(\frac{4 \left(\frac{-1}{x^2} \right)}{\left(1 + \frac{4}{x} \right)} \right)}{\left(\frac{1}{2} \cdot \frac{-1}{x^2} \right)} = \lim_{x \rightarrow \infty} \frac{8}{\left(1 + \frac{4}{x} \right)} = \frac{8}{1} = 8.$$

Therefore, $\lim_{x \rightarrow \infty} 2x \ln \left| 1 + \frac{4}{x} \right| = 8.$

So, $\lim_{x \rightarrow \infty} \left(1 + \frac{4}{x} \right)^{2x} = e^8.$

2. Determine whether the improper integral $\int_0^2 \frac{1}{\sqrt{2-x}} dx$ converges or diverges.

Solution:

$$\begin{aligned} \int_0^2 \frac{1}{\sqrt{2-x}} dx &= \lim_{t \rightarrow 2^-} \int_0^t (2-x)^{-\frac{1}{2}} dx = \lim_{t \rightarrow 2^-} \left((-1) \int_0^t (2-x)^{-\frac{1}{2}} (-1) dx \right) \\ &= \lim_{t \rightarrow 2^-} \left((-1) \left[\frac{(2-x)^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^t \right) = \lim_{t \rightarrow 2^-} ((-1) [2\sqrt{2-t} - 2\sqrt{2-0}]) \\ &= (-1) [0 - 2\sqrt{2}] = 2\sqrt{2}. \end{aligned}$$

Hence, the improper integral converges.

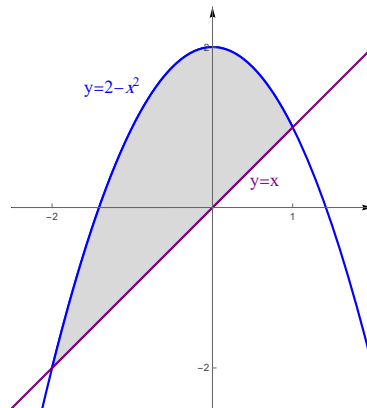
3. Sketch the region bounded by the graphs of the curves $y = 2 - x^2$ and $y = x$, then find its area.

Solution:

$y = 2 - x^2$ represents a parabola opens downwards, and its vertex is $(0, 2)$.

$y = x$ represents a straight line passing through $(0, 0)$ and its slope is 1.

Points of intersection of $y = 2 - x^2$ and $y = x$:
 $x = 2 - x^2 \implies x^2 + x - 2 = 0$
 $\implies (x+2)(x-1) = 0$
 $\implies x = -2, x = 1.$



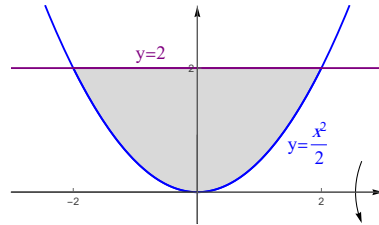
$$\begin{aligned}
A &= \int_{-2}^1 [(2 - x^2) - x] \, dx = \int_{-2}^1 (-x^2 - x + 2) \, dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^1 \\
&= \left(-\frac{1^3}{3} - \frac{1^2}{2} + 2(1) \right) - \left(-\frac{(-2)^3}{3} - \frac{(-2)^2}{2} + 2(-2) \right) . \\
&= -\frac{1}{3} - \frac{1}{2} + 2 - \left(\frac{8}{3} - 2 - 4 \right) = -\frac{9}{3} - \frac{1}{2} + 8 = 5 - \frac{1}{2} = \frac{9}{2} .
\end{aligned}$$

4. Sketch the region bounded by the graphs of the curves $y = \frac{x^2}{2}$ and $y = 2$, then find the volume of the solid generated by revolving this region about the x -axis.

Solution :

$y = \frac{x^2}{2}$ represents a parabola opens upwards, and its vertex is $(0, 0)$.

$y = 2$ represents a straight line parallel to the x -axis and passing through $(0, 2)$.



Points of intersection of $y = \frac{x^2}{2}$ and $y = 2$:

$$\frac{x^2}{2} = 2 \implies x^2 = 4 \implies x = \pm 2.$$

Using Washer Method :

$$\begin{aligned}
V &= \pi \int_{-2}^2 \left[(2)^2 - \left(\frac{x^2}{2} \right)^2 \right] dx = \pi \int_{-2}^2 \left(4 - \frac{x^4}{4} \right) dx \\
&= \pi \left[4x - \frac{x^5}{20} \right]_{-2}^2 = \pi \left[\left(4(2) - \frac{(2)^5}{20} \right) - \left(4(-2) - \frac{(-2)^5}{20} \right) \right] \\
&= \pi \left[8 - \frac{32}{20} - \left(-8 + \frac{32}{20} \right) \right] = \pi \left(8 - \frac{8}{5} + 8 - \frac{8}{5} \right) \\
&= \pi \left(16 - \frac{16}{5} \right) = \frac{64\pi}{5} .
\end{aligned}$$

5. Find the arc length of $y = \ln |\sec x|$, from $x = 0$ to $x = \frac{\pi}{4}$.

Solution :

$$y' = \frac{\sec x \tan x}{\sec x} = \tan x .$$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + (\tan x)^2} \, dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x} \, dx$$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x} \, dx = \int_0^{\frac{\pi}{4}} |\sec x| \, dx = \int_0^{\frac{\pi}{4}} \sec x \, dx \\
&= [\ln |\sec x + \tan x|]_0^{\frac{\pi}{4}} = \ln \left| \sec \left(\frac{\pi}{4} \right) + \tan \left(\frac{\pi}{4} \right) \right| - \ln |\sec(0) + \tan(0)| \\
&= \ln(\sqrt{2} + 1) - \ln(1 + 0) = \ln(1 + \sqrt{2})
\end{aligned}$$

6. Convert the polar equation $\frac{3}{r} = \sin \theta - \sec \theta$ into a Cartesian equation.

Solution:

$$\begin{aligned}
\frac{3}{r} = \sin \theta - \sec \theta &\implies 3 = r \sin \theta - r \sec \theta = r \sin \theta - \frac{r}{\cos \theta} \\
\implies 3 = r \sin \theta - \frac{r^2}{r \cos \theta} &\implies 3 = y - \frac{x^2 + y^2}{x} \\
\implies 3x = xy - (x^2 + y^2) &\implies x^2 + y^2 + 3x - xy = 0 .
\end{aligned}$$

7. Sketch the region in the first quadrant, inside the curve $r = 1 + \sin \theta$ and outside the curve $r = 1$, then find its area.

Solution:

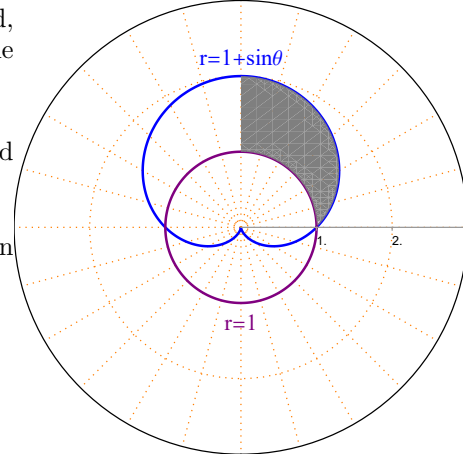
$r = 1 + \sin \theta$ represents a cardioid, symmetric with respect to the line $\theta = \frac{\pi}{2}$.

$r = 1$ represents a circle centered at the pole, and its radius is 1.

Points of intersection between $r = 1 + \sin \theta$ and $r = 1$:

$$1 + \sin \theta = 1 \implies \sin \theta = 0$$

$$\implies \theta = 0, \theta = \pi$$



Since the region lies in the first quadrant then $0 \leq \theta \leq \frac{\pi}{2}$.

$$\begin{aligned}
A &= \frac{1}{2} \int_0^{\frac{\pi}{2}} [(1 + \sin \theta)^2 - (1)^2] \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} [1 + 2 \sin \theta + \sin^2 \theta - 1] \, d\theta \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} [2 \sin \theta + \sin^2 \theta] \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \left[2 \sin \theta + \left(\frac{1 - \cos 2\theta}{2} \right) \right] \, d\theta \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \left[\frac{1}{2} + 2 \sin \theta - \frac{\cos 2\theta}{2} \right] \, d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \left[\frac{1}{2} + 2 \sin \theta - \frac{1}{4} \cos 2\theta \right] \, d\theta
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left[\frac{1}{2} \theta - 2 \cos \theta - \frac{1}{4} \sin 2\theta \right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{2} \left[\left(\frac{1}{2} \left(\frac{\pi}{2} \right) - 2 \cos \left(\frac{\pi}{2} \right) - \frac{1}{4} \sin(\pi) \right) - \left(\frac{1}{2} (0) - 2 \cos(0) - \frac{1}{4} \sin(0) \right) \right] \\
&= \frac{1}{2} \left[\left(\frac{\pi}{4} - 0 - 0 \right) - (0 - 2 - 0) \right] = \frac{1}{2} \left(2 + \frac{\pi}{4} \right) = 1 + \frac{\pi}{8} .
\end{aligned}$$