MATH 111 - Integral Calculus
First Semester - 1447 H
Solution of the First Exam
Dr Tariq A. Alfadhel

Question (1): [9 marks]

1. Use Riemann Sum to evaluate the definite integral $\int_0^1 (3x^2 - x)\, dx$. [3]

   Solution : $[a, b] = [0, 1]$, $f(x) = 3x^2 - x$.

   $\Delta_x = \dfrac{b-a}{n} = \dfrac{1-0}{n} = \dfrac{1}{n}$

   $x_k = a + k\ \Delta_x = 0 + k\left(\dfrac{1}{n}\right) = \dfrac{k}{n}$

   $f(x_k) = 3\left(\dfrac{k}{n}\right)^2 - \dfrac{k}{n} = \dfrac{3k^2}{n^2} - \dfrac{k}{n}$

   $R_n = \displaystyle\sum_{k=1}^{n} f(x_k)\ \Delta_x = \sum_{k=1}^{n}\left(\frac{3k^2}{n^2} - \frac{k}{n}\right)\ \left(\frac{1}{n}\right)$

   $= \displaystyle\sum_{k=1}^{n}\left(\frac{3k^2}{n^3} - \frac{k}{n^2}\right) = \sum_{k=1}^{n}\frac{3k^2}{n^3} - \sum_{k=1}^{n}\frac{k}{n^2}$

   $= \displaystyle\frac{3}{n^3}\sum_{k=1}^{n}k^2 - \frac{1}{n^2}\sum_{k=1}^{n}k = \frac{3}{n^3}\ \frac{n(n+1)(2n+1)}{6} - \frac{1}{n^2}\ \frac{n(n+1)}{2}$

   $= \dfrac{(n+1)(2n+1)}{2n^2} - \dfrac{n+1}{2n}$

   $\displaystyle\int_0^1 (3x^2 - x)\ dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{(n+1)(2n+1)}{2n^2} - \frac{n+1}{2n}\right]$

   $= \dfrac{2}{2} - \dfrac{1}{2} = 1 - \dfrac{1}{2} = \dfrac{1}{2}$.

2. Find $F'(x)$, if $F(x) = \displaystyle\int_{\cos(\sqrt{x+1})}^{e^{x^2}} \frac{1}{1-t^2}dt$. [2]

   Solution :

   $F'(x) = \displaystyle\frac{d}{dx}\int_{\cos(\sqrt{x+1})}^{e^{x^2}} \frac{1}{1-t^2}dt$

   $= \dfrac{1}{1-\left(e^{x^2}\right)^2}\ \left(e^{x^2}\ (2x)\right) - \dfrac{1}{1-\left(\cos\left(\sqrt{x+1}\right)\right)^2}\ \left(-\sin\left(\sqrt{x+1}\right)\ \dfrac{1}{2\sqrt{x+1}}\right)$

   $= \dfrac{2x\ e^{x^2}}{1-e^{2x^2}} + \dfrac{1}{2\sqrt{x+1}\ \sin\left(\sqrt{x+1}\right)}$.

Find $\dfrac{dy}{dx}$ of the following :

3. $y = \sin^{-1}\left(5^{3x+1}\right) + \log_2(\sqrt[5]{x^4+2})$. [2]

   Solution :

   $y = \sin^{-1}\left(5^{3x+1}\right) + \log_2\left(\sqrt[5]{x^4+2}\right) = \sin^{-1}\left(5^{3x+1}\right) + \log_2\left(x^4+2\right)^{\frac{1}{5}}$

   $= \sin^{-1}\left(5^{3x+1}\right) + \dfrac{1}{5}\ \log_2\left(x^4+2\right)$

   $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\left(5^{3x+1}\right)^2}}\ \left(5^{3x+1}\ (3)\ \ln 5\right) + \dfrac{1}{5}\ \dfrac{4x^3}{x^4+2}\ \dfrac{1}{\ln 2}$

   $= \dfrac{3\ \ln 5\ 5^{3x+1}}{\sqrt{1-5^{6x+2}}} + \dfrac{4x^3}{5\ \ln 2\ \left(x^4+2\right)}$.

4. $y = (\sin x)^{\sqrt{x}}$. [2]

   Solution :

   $y = (\sin x)^{\sqrt{x}} \implies \ln|y| = \ln\left|(\sin x)^{\sqrt{x}}\right| = \sqrt{x}\ \ln|\sin x|$

   Differentiate both sides.

   $\dfrac{y'}{y} = \dfrac{1}{2\sqrt{x}} \ln|\sin x| + \sqrt{x}\ \left(\dfrac{\cos x}{\sin x}\right)$

   $y' = y\ \left[\dfrac{\ln|\sin x|}{2\sqrt{x}} + \sqrt{x}\ \cot x\right]$

   $= (\sin x)^{\sqrt{x}} \left[\dfrac{\ln|\sin x|}{2\sqrt{x}} + \sqrt{x}\ \cot x\right]$.

Question (2): [16 marks]
Evaluate the following integrals :

1. $\displaystyle\int \frac{\cos x}{(1+\sin x)^4}\ dx$. [2]

   Solution :

   $\displaystyle\int \frac{\cos x}{(1+\sin x)^4}\ dx = \int (1+\sin x)^{-4}\ \cos x\ dx$

   $= \dfrac{(1+\sin x)^{-3}}{-3} + c = \dfrac{1}{-3\,(1+\sin x)^3} + c$.

2. $\displaystyle\int \frac{\sec(x^{\frac{2}{5}})\tan(x^{\frac{2}{5}})}{x^{\frac{3}{5}}}\ dx$. [2]

   Solution : .

$$\int \frac{\sec(x^{\frac{2}{5}})\tan(x^{\frac{2}{5}})}{x^{\frac{3}{5}}}\ dx = \int \sec\left(x^{\frac{2}{5}}\right)\tan\left(x^{\frac{2}{5}}\right)\ x^{-\frac{3}{5}}\ dx$$

$$= \frac{5}{2}\int \sec\left(x^{\frac{2}{5}}\right)\tan\left(x^{\frac{2}{5}}\right)\ \left(\frac{2}{5}\ x^{-\frac{3}{5}}\right)\ dx = \frac{5}{2}\ \sec\left(x^{\frac{2}{5}}\right) + c\ .$$

3. $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\ dx$. [2]

Solution :

$$\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\ dx = 2\int \frac{1}{2\sqrt{x}(1+\sqrt{x})}\ dx$$

$$= 2\int \frac{\left(\frac{1}{2\sqrt{x}}\right)}{1+\sqrt{x}}\ dx = 2\ln\left|1+\sqrt{x}\right| + c\ .$$

4. $\displaystyle\int \frac{7^{\cot(3x)}}{\sin^2(3x)}\ dx$. [2]

Solution :

$$\int \frac{7^{\cot(3x)}}{\sin^2(3x)}\ dx = \int 7^{\cot(3x)}\ \frac{1}{\sin^2(3x)}\ dx$$

$$= \frac{-1}{3}\int 7^{\cot(3x)}\ \left(-\csc^2(3x)\ (3)\right) dx = -\frac{1}{3}\ \frac{7^{\cot(3x)}}{\ln 7} + c\ .$$

5. $\displaystyle\int_0^1 \frac{x}{x+1}\ dx$. [2]

Solution :

$$\int_0^1 \frac{x}{x+1}\ dx = \int_0^1 \frac{(x+1)-1}{x+1}\ dx = \int_0^1 \left(\frac{x+1}{x+1} - \frac{1}{x+1}\right)\ dx$$

$$= \int_0^1 \left(1 - \frac{1}{x+1}\right)\ dx = \left[x - \ln|x+1|\right]_0^1$$

$$= (1-\ln 2) - (0 - \ln(1)) = 1 - \ln 2\ .$$

6. $\displaystyle\int \frac{x-5}{\sqrt{1-x^2}}\ dx$. [2]

Solution :

$$\int \frac{x-5}{\sqrt{1-x^2}}\ dx = \int \left(\frac{x}{\sqrt{1-x^2}} - \frac{5}{\sqrt{1-x^2}}\right)\ dx$$

$$= \frac{1}{-2}\int \left(1-x^2\right)^{-\frac{1}{2}}\ (-2x)\ dx - 5\int \frac{1}{\sqrt{1-x^2}}\ dx$$

$$= \frac{1}{-2}\,\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} - 5\sin^{-1}x + c = -\sqrt{1-x^2} - 5\sin^{-1}x + c\ .$$

7. $\displaystyle\int \frac{1}{x\sqrt{x^6-9}}\,dx\ .$ [2]

Solution :

$$\int \frac{1}{x\sqrt{x^6-9}}\,dx = \int \frac{1}{x\sqrt{\left(x^3\right)^2-(3)^2}}\,dx = \int \frac{x^2}{x^3\sqrt{\left(x^3\right)^2-(3)^2}}\,dx$$

$$= \frac{1}{3}\int \frac{3x^2}{x^3\sqrt{\left(x^3\right)^2-(3)^2}}\,dx = \frac{1}{3}\,\frac{1}{3}\ \sec^{-1}\left(\frac{x^3}{3}\right) + c\ .$$

8. $\displaystyle\int \frac{1}{e^{-x}+e^{x}}\,dx\ .$ [2]

Solution :

$$\int \frac{1}{e^{-x}+e^{x}}\,dx = \int \frac{e^x}{e^x\ \left(e^{-x}+e^{x}\right)}\,dx = \int \frac{e^x}{1+e^{2x}}\,dx$$

$$= \int \frac{e^x}{(1)^2+\left(e^x\right)^2}\,dx = \tan^{-1}\left(e^x\right) + c\ .$$

MATH 111 - Integral Calculus
First Semester - 1447 H
Solution of the Second Exam
Dr Tariq A. Alfadhel

Question (1): [2 + 2 marks]

Find $\dfrac{dy}{dx}$ of the following :

1. $y = \sinh\left(\sqrt{x^2+1}\right) + \cosh^{-1}\left(e^x\right)$.

   Solution :

   $$\frac{dy}{dx} = \cosh\left(\sqrt{x^2+1}\right)\ \left(\frac{2x}{2\sqrt{x^2+1}}\right) + \frac{1}{\sqrt{\left(e^x\right)^2-1}}\ \left(e^x\right)$$

   $$= \frac{x\ \cosh\left(\sqrt{x^2+1}\right)}{\sqrt{x^2+1}} + \frac{e^x}{\sqrt{e^{2x}-1}} \ .$$

2. $y = \operatorname{sech}\left(5x^2\right) + \tanh^{-1}\left(\sin x\right)$.

   Solution :

   $$\frac{dy}{dx} = -\operatorname{sech}\left(5x^2\right)\ \tanh\left(5x^2\right)\ (10x) + \frac{1}{1-\left(\sin x\right)^2}\ (\cos x)$$

   $$= -10x\ \operatorname{sech}\left(5x^2\right)\ \tanh\left(5x^2\right) + \frac{\cos x}{1-\sin^2 x} \ .$$

Question (2): [2 + 2 + 2 + 3 + 2 + 3 + 2 + 3 + 2 marks]

Evaluate the following integrals :

1. $\displaystyle\int x\ \tanh(4x^2-1)\ dx$.

   Solution :

   $$\int x\ \tanh(4x^2-1)\ dx = \frac{1}{8}\int \tanh(4x^2-1)\ (8x)\ dx$$

   $$= \frac{1}{8}\ \ln\left|\cosh(4x^2-1)\right| + c$$

2. $\displaystyle\int \frac{1}{\sqrt{5^{2x}+1}}\ dx$

   Solution :

   $$\int \frac{1}{\sqrt{5^{2x}+1}}\ dx = \int \frac{1}{\sqrt{\left(5^x\right)^2+(1)^2}}\ dx$$

$$= \frac{1}{\ln 5} \int \frac{5^x \ \ln 5}{5^x \ \sqrt{\left(5^x\right)^2 + (1)^2}} \ dx = -\frac{1}{\ln 5} \ \text{csch}^{-1}\left(5^x\right) + c$$

3. $\displaystyle\int \sin^2 x \ \cos^5 x \ dx$

Solution :

Using the substitution $u = \sin x$.

Hence $du = \cos x \ dx$.

$$\int \sin^2 x \ \cos^5 x \ dx = \int \sin^2 x \ \cos^4 x \ \cos x \ dx = \int \sin^2 x \ \left(\cos^2 x\right)^2 \ \cos x \ dx$$

$$= \int \sin^2 x \ \left(1 - \sin^2 x\right)^2 \ \cos x \ dx = \int u^2 \left(1 - u^2\right)^2 \ du$$

$$= \int u^2 \left(1 - 2u^2 + u^4\right) \ du = \int \left(u^2 - 2u^4 + u^6\right) \ du$$

$$= \frac{u^3}{3} - 2 \ \frac{u^5}{5} + \frac{u^7}{7} + c = \frac{\sin^3 x}{3} - 2 \ \frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + c$$

4. $\displaystyle\int \frac{1}{(x-3)\sqrt{7 - x^2 + 6x}} \ dx$.

Solution : By completing the square.

$7 - x^2 + 6x = 7 - \left(x^2 - 6x\right) = 7 - \left(x^2 - 6x + 9\right) + 9$

$= 16 - \left(x^2 - 6x + 9\right) = (4)^2 - (x-3)^2$.

$$\int \frac{1}{(x-3)\sqrt{7 - x^2 + 6x}} \ dx = \int \frac{1}{(x-3)\sqrt{(4)^2 - (x-3)^2}} \ dx$$

$$= -\frac{1}{4} \ \text{sech}^{-1}\left(\frac{x-3}{4}\right) + c \ .$$

5. $\displaystyle\int \sin^{-1} x \ dx$.

Solution : Using integration by parts.

$u = \sin^{-1} x$	$dv = dx$
$du = \dfrac{1}{\sqrt{1-x^2}} \ dx$	$v = x$

$$\int \sin^{-1} x \ dx = x \ \sin^1 x - \int x \ \frac{1}{\sqrt{1-x^2}} \ dx$$

$$= x \ \sin^1 x - \int x \ \left(1 - x^2\right)^{-\frac{1}{2}} \ dx$$

$$= x\ \sin^1 x - \frac{1}{-2}\int \left(1-x^2\right)^{-\frac{1}{2}}\ (-2x)\ dx$$

$$= x\ \sin^1 x + \frac{1}{2}\ \frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = x\sin^{-1} x + \sqrt{1-x^2} + c.$$

6. $\displaystyle\int \frac{1}{(9+x^2)^{\frac{3}{2}}}\ dx$.

Solution : Using trigonometric substitutions.

Put $x = 3\tan\theta \implies \tan\theta = \dfrac{x}{3}$.

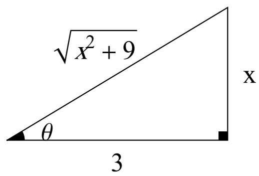
$dx = 3\sec^2\theta\ d\theta.$

$\left(9+x^2\right)^{\frac{3}{2}} = \left(9+9\tan^2\theta\right)^{\frac{3}{2}} = \left[9\left(1+\tan^2\theta\right)\right]^{\frac{3}{2}} = \left[(3)^2\ \sec^2\theta\right]^{\frac{3}{2}} = 3^3\ \sec^3\theta.$

$$\int \frac{1}{(9+x^2)^{\frac{3}{2}}}\ dx = \int \frac{3\sec^2\theta}{3^3\ \sec^3\theta}\ d\theta = \frac{1}{3^2}\int \frac{1}{\sec\theta}\ d\theta$$

$$= \frac{1}{9}\int \cos\theta\ d\theta = \frac{1}{9}\sin\theta + c$$

$\tan\theta = \dfrac{x}{3}$.



From the triangle :
$\sin\theta = \dfrac{x}{\sqrt{x^2+9}}$

$$\int \frac{1}{(9+x^2)^{\frac{3}{2}}}\ dx = \frac{1}{9}\ \frac{x}{\sqrt{x^2+9}} + c$$

7. $\displaystyle\int x^3\ \ln|2x|\ dx$

Solution : Using integration by parts.

$u = \ln|2x| \qquad dv = x^3\ dx$

$du = \dfrac{2}{2x}\ dx = \dfrac{1}{x}\ dx \qquad v = \dfrac{x^4}{4}$

$$\int x^3\ \ln|2x|\ dx = \frac{x^4}{4}\ \ln|2x| - \int \frac{x^4}{4}\ \frac{1}{x}\ dx$$

$$= \frac{x^4}{4}\ \ln|2x| - \frac{1}{4}\int x^3\ dx = \frac{x^4}{4}\ \ln|2x| - \frac{1}{4}\ \frac{x^4}{4} + c$$

8. $\displaystyle\int \frac{2x+1}{x^2(1+x^2)}\ dx$

Solution : Using the method of partial fractions.

$$\frac{2x+1}{x^2(1+x^2)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{Bx+C}{1+x^2}$$

$$\frac{2x+1}{x^2(1+x^2)} = \frac{A_1\ x(1+x^2)}{x^2(1+x^2)} + \frac{A_2(1+x^2)}{x^2(1+x^2)} + \frac{(Bx+C)x^2}{x^2(1+x^2)}$$

$$2x+1 = A_1\ x(1+x^2) + A_2(1+x^2) + (Bx+C)x^2$$

$$2x+1 = A_1x + A_1x^3 + A_2 + A_2x^2 + Bx^3 + Cx^2$$

$$2x+1 = (A_1+B)\,x^3 + (A_2+C)\,x^2 + A_1x + A_2$$

By comparing the coefficients of the two polynomials in each side :

$$\begin{array}{lll} A_1+B=0 & \longrightarrow & (1) \\ A_2+C=0 & \longrightarrow & (2) \\ A_1=2 & \longrightarrow & (3) \\ A_2=1 & \longrightarrow & (4) \end{array}$$

From equations (1) and (3) : $2+B=0 \implies B=-2$.

From equations (2) and (4) : $1+C=0 \implies C=-1$.

$$\int \frac{2x+1}{x^2(1+x^2)}\ dx = \int \left(\frac{2}{x} + \frac{1}{x^2} + \frac{-2x-1}{1+x^2}\right)\ dx$$

$$= \int \frac{2}{x}\ dx + \int \frac{1}{x^2}\ dx + \int \frac{-2x-1}{1+x^2}\ dx$$

$$= 2\int \frac{1}{x}\ dx + \int x^{-2}\ dx - \int \frac{2x}{1+x^2}\ dx - \int \frac{1}{1+x^2}\ dx$$

$$= 2\ln|x| + \frac{x^{-1}}{-1} - \ln(1+x^2) - \tan^{-1}x + c\ .$$

9. $\displaystyle\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{2}{3}}}\ dx$.

Solution :

Using the substitution $x = u^6$, then $u = x^{\frac{1}{6}}$.

$dx = 6u^5\ du$.

$$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{2}{3}}}\ dx = \int \frac{6u^5}{(u^6)^{\frac{1}{2}} + (u^6)^{\frac{2}{3}}}\ du = 6\int \frac{u^5}{u^3+u^4}\ du$$

$$= 6\int \frac{u^5}{u^3\ (u+1)}\ du = 6\int \frac{u^2}{u+1}\ du$$

Using long division of polynomials :

$$6\int \frac{u^2}{u+1}\ du = 6\int \left(u - 1 + \frac{1}{u+1}\right)\ du = 6\left(\frac{u^2}{2} - u + \ln|u+1|\right) + c$$

$$= 3u^2 - 6u + 6\ln|u+1| + c = 3x^{\frac{1}{3}} - 6x^{\frac{1}{6}} + 6\ln\left|x^{\frac{1}{6}} + 1\right| + c$$

MATH 111 - Integral Calculus
First Semester - 1447 H
Solution of the Final Exam
Dr Tariq A. Alfadhel

Question (1): [$3+2+2=7$ marks]

1. Find the value of c that satisfies the mean value theorem of the definite integral for the function $f(x)=\dfrac{1}{\sqrt{x+1}}$ on the interval $[3,8]$.

   Solution: Using the formula $(b-a)\ f(c)=\displaystyle\int_a^b f(x)\ dx$.

   $$(8-3)\ \frac{1}{\sqrt{c+1}}=\int_3^8 \frac{1}{\sqrt{x+1}}\ dx=\int_3^8 (x+1)^{-\frac{1}{2}}\ dx$$

   $$\frac{5}{\sqrt{c+1}}=\left[\frac{(x+1)^{\frac{1}{2}}}{\frac{1}{2}}\right]_3^8=2\sqrt{8+1}-2\sqrt{3+1}=2(3)-2(2)=6-4=2$$

   $$\frac{5}{\sqrt{c+1}}=2 \implies \sqrt{c+1}=\frac{5}{2} \implies c+1=\frac{25}{4} \implies c=\frac{25}{4}-1=\frac{21}{4}$$

   The desired value is $c=\dfrac{21}{4}\in(3,8)$.

2. Find $F'(x)$, if $F(x)=\displaystyle\int_{\sinh x}^{2^{x+1}} (1+t^2)\ dt$.

   Solution:

   $$F'(x)=\frac{d}{dx}\int_{\sinh x}^{2^{x+1}} (1+t^2)\ dt$$

   $$=\left[1+\left(2^{x+1}\right)^2\right]\ \left(2^{x+1}\ln 2\right)-\left[1+(\sinh x)^2\right]\ (\cosh x)$$

   $$=2^{x+1}\ln 2\ \left(1+2^{2x+2}\right)\ -\cosh x\left(1+\sinh^2 x\right)\ .$$

   $$=2^{x+1}\ln 2\ \left(1+2^{2x+2}\right)\ -\cosh^3 x\ .$$

3. Find y' if $y(x)=(\cos x)^x+e^{-5x}$.

   Solution:

   $$y(x)=(\cos x)^x+e^{-5x}=e^{\ln|\cos x|^x}+e^{-5x}=e^{x\ln|\cos x|}+e^{-5x}$$

   $$y'(x)=e^{x\ln|\cos x|}\ \left[(1)\ln|\cos x|+x\left(\frac{-\sin x}{\cos x}\right)\right]+e^{-5x}\ (-5)\ .$$

   $$=(\cos x)^x\left[\ln|\cos x|-x\tan x\right]-5e^{-5x}$$

Question (2): [2 + 3 + 2 + 3 + 3 + 2 = 15 marks]
Evaluate the following integrals :

1. $\displaystyle\int \frac{1}{\sqrt{x^2-4x}}\ dx$.

   Solution: By completing the square.

   $x^2-4x = \left(x^2-4x+4\right)-4 = (x-2)^2-(2)^2$

   $$\int \frac{1}{\sqrt{x^2-4x}}\ dx = \int \frac{1}{\sqrt{(x-2)^2-(2)^2}}\ dx = \cosh^{-1}\left(\frac{x-2}{2}\right)+c\ .$$

2. $\displaystyle\int x^2\cos(3x)\ dx$.

   Solution: Using integration by parts .

   $u = x^2 \qquad dv = \cos(3x)\ dx$

   $du = 2x\ dx \qquad v = \frac{1}{3}\ \sin(3x)$

   $$\int x^2\cos(3x)\ dx = x^2\left(\frac{1}{3}\sin(3x)\right) - \int \frac{1}{3}\sin(3x)\ 2x\ dx$$

   $$= \frac{x^2}{3}\sin(3x) - \frac{2}{3}\int x\sin(3x)\ dx$$

   Using integration by parts again .

   $u = x \qquad dv = \sin(3x)\ dx$

   $du = dx \qquad v = -\frac{1}{3}\ \cos(3x)$

   $$\int x^2\cos(3x)\ dx = \frac{x^2}{3}\sin(3x) - \frac{2}{3}\left[x\ \left(-\frac{1}{3}\cos(3x)\right) - \int -\frac{1}{3}\cos(3x)\ dx\right]$$

   $$= \frac{x^2}{3}\sin(3x) + \frac{2x}{9}\cos(3x) - \frac{2}{9}\int \cos(3x)\ dx$$

   $$= \frac{x^2}{3}\sin(3x) + \frac{2x}{9}\cos(3x) - \frac{2}{27}\sin(3x) + c$$

3. $\displaystyle\int \frac{(\sqrt{x}-1)\sec\left(\sqrt{x}-\ln\sqrt{x}\right)}{x}\ dx$

   Solution:

   $$\int \frac{(\sqrt{x}-1)\sec\left(\sqrt{x}-\ln\sqrt{x}\right)}{x}\ dx = \int \sec\left(\sqrt{x}-\ln\left|x^{\frac{1}{2}}\right|\right)\ \left(\frac{\sqrt{x}-1}{x}\right)\ dx$$

   $$= \int \sec\left(\sqrt{x}-\frac{1}{2}\ln|x|\right)\ \left(\frac{\sqrt{x}}{x}-\frac{1}{x}\right)\ dx = \int \sec\left(\sqrt{x}-\frac{1}{2}\ln|x|\right)\ \left(\frac{1}{\sqrt{x}}-\frac{1}{x}\right)\ dx$$

   $$= 2\int \sec\left(\sqrt{x}-\frac{1}{2}\ln|x|\right)\ \left[\frac{1}{2}\left(\frac{1}{\sqrt{x}}-\frac{1}{x}\right)\right]\ dx$$

$$= 2\int \sec\left(\sqrt{x} - \frac{1}{2}\ln|x|\right)\ \left(\frac{1}{2\sqrt{x}} - \frac{1}{2x}\right)\ dx$$

$$= 2\ln\left|\sec\left(\sqrt{x} - \frac{1}{2}\ln|x|\right) + \tan\left(\sqrt{x} - \frac{1}{2}\ln|x|\right)\right| + c\ .$$

4. $\displaystyle\int \frac{5x^2 + 6x + 4}{(x+1)(x^2+x+1)}\ dx\ .$

   Solution : Using the method of partial fractions.

   $$\frac{5x^2 + 6x + 4}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx + C}{x^2+x+1}$$

   $$\frac{5x^2 + 6x + 4}{(x+1)(x^2+x+1)} = \frac{A\ (x^2+x+1)}{(x+1)\ (x^2+x+1)} + \frac{(Bx+C)\ (x+1)}{(x^2+x+1)\ (x+1)}$$

   $$5x^2 + 6x + 4 = Ax^2 + Ax + A + Bx^2 + Bx + Cx + C$$

   $$5x^2 + 6x + 4 = (A+B)x^2 + (A+B+C)x + (A+C)$$

   By comparing the coefficients of the two polynomials in each side :

   $$\begin{array}{lcl} A + B = 5 & \longrightarrow & (1) \\ A + B + C = 6 & \longrightarrow & (2) \\ A + C = 4 & \longrightarrow & (3) \end{array}$$

   Equation (2) - Equation (1) : $C = 1$.

   From Equation (3) : $A + 1 = 4 \implies A = 3$.

   From Equation (1) : $3 + B = 5 \implies B = 2$.

   $$\int \frac{5x^2 + 6x + 4}{(x+1)(x^2+x+1)}\ dx = \int \left(\frac{3}{x+1} + \frac{2x+1}{x^2+x+1}\right)\ dx$$

   $$= 3\int \frac{1}{x+1}\ dx + \int \frac{2x+1}{x^2+x+1}\ dx = 3\ln|x+1| + \ln\left|x^2+x+1\right| + c$$

5. $\displaystyle\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\ dx\ .$

   Solution : Using trigonometric substitutions.

   Put $x = 2\sin\theta \implies \sin\theta = \dfrac{x}{2}$.

   $dx = 2\cos\theta\ d\theta$.

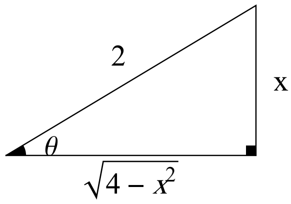
   $$\left(4 - x^2\right)^{\frac{3}{2}} = \left(4 - 4\sin^2\theta\right)^{\frac{3}{2}} = \left[4\left(1 - \sin^2\theta\right)\right]^{\frac{3}{2}} = \left[2^2\ \cos^2\theta\right]^{\frac{3}{2}}$$

   $$= \left(2^2\right)^{\frac{3}{2}}\ \left(\cos^2\theta\right)^{\frac{3}{2}} = 2^3\ \cos^3\theta$$

   $$\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\ dx = \int \frac{4\sin^2\theta\ 2\cos\theta}{2^3\ \cos^3\theta}\ d\theta = \int \frac{\sin^2\theta}{\cos^2\theta}\ d\theta = \int \tan^2\theta\ d\theta$$

$$= \int \left(\sec^2\theta - 1\right)\ d\theta = \tan\theta - \theta + c$$

$\sin\theta = \dfrac{x}{2} \implies \theta = \sin^{-1}\left(\dfrac{x}{2}\right)$.



From the triangle :
$\tan\theta = \dfrac{x}{\sqrt{4-x^2}}$

$$\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\ dx = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + c\ .$$

6. $\displaystyle\int \tan^3 x\ \sec^2 x\ dx$.

   Solution : .

   $$\int \tan^3 x\ \sec^2 x\ dx = \int (\tan x)^3 \sec^2 x\ dx = \frac{(\tan x)^4}{4} + c = \frac{\tan^4 x}{4} + c\ .$$

Question (3): [$2+2+3+3+2+2+4 = 18$ marks]

1. Calculate $\lim_{x\to\infty} x^{\frac{1}{x}}$.

   Solution:

   $$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln|x|^{\frac{1}{x}}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln|x|} = \lim_{x\to\infty} e^{\frac{\ln|x|}{x}}$$

   $$\lim_{x\to\infty} \frac{\ln|x|}{x} \qquad \left(\frac{\infty}{\infty}\right)$$

   Using L'Hôpital's rule.

   $$\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0\ .$$

   Therefore, $\displaystyle\lim_{x\to\infty} \frac{\ln|x|}{x} = 0$.

   So, $\displaystyle\lim_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1$.

2. Determine whether the improper integral $\displaystyle\int_0^\infty \frac{x}{(1+x^2)^2}\ dx$ converges or diverges.

   Solution:

   $$\int_0^\infty \frac{x}{(1+x^2)^2}\ dx = \lim_{t\to\infty} \int_0^t \frac{x}{(1+x^2)^2}\ dx = \lim_{t\to\infty} \left(\frac{1}{2}\int_0^t (1+x^2)^{-2}\ (2x)\ dx\right)$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\left[\frac{(1+x^2)^{-1}}{-1}\right]_0^t\right) = \lim_{t\to\infty}\left(\frac{1}{2}\left[\frac{-1}{1+x^2}\right]_0^t\right)$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\left[\frac{-1}{1+t^2} - \frac{-1}{1+(0)^2}\right]\right) = \frac{1}{2}[0-(-1)] = \frac{1}{2} \ .$$

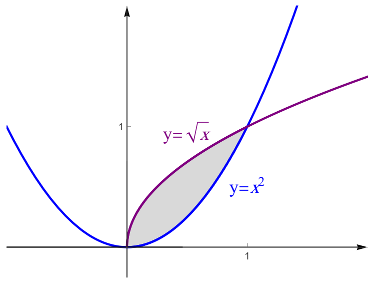
Hence, the improper integral converges.

3. Sketch the region bounded by the graphs of the curves $y = x^2$ and $y = \sqrt{x}$, then find its area.

Solution:

$y = x^2$ represents a parabola opens upwards, and its vertex is $(0,0)$.

$y = \sqrt{x}$ represents the upper half of a parabola opens to to right and its vertex is $(0,0)$.



Points of intersection of $y = x^2$ and $y = \sqrt{x}$:
$x^2 = \sqrt{x} \implies x^4 = x$
$\implies x^4 - x = 0$
$\implies x(x^3-1) = 0$
$\implies x = 0 \ , \ x = 1.$

$$\text{A} = \int_0^1 \left(\sqrt{x} - x^2\right) \, dx = \left[2\frac{x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\right]_0^1$$

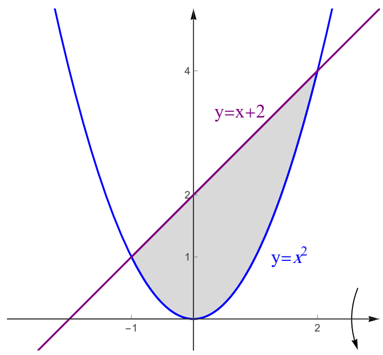
$$= \left(\frac{2}{3} - \frac{1}{3}\right) - (0-0) = \frac{1}{3} \ .$$

4. Sketch the region bounded by the graphs of the curves $y = x^2$ and $y = x + 2$, then find the volume of the solid generated by revolving this region about the $x-$axis.

Solution :

$y = x^2$ represents a parabola opens upwards, and its vertex is $(0,0)$.

$y = x + 2$ represents a straight line passing through $(0,2)$ with slope 1.



Points of intersection of $y = x^2$ and $y = x + 2$:
$x^2 = x + 2 \implies x^2 - x - 2 = 0$
$\implies (x+1)(x-2) = 0$
$\implies x = -1 \ , \ x = 2$

Using Washer Method :

$$V = \pi \int_{-1}^{2} \left[(x+2)^2 - \left(x^2\right)^2 \right] \ dx$$

$$= \pi \int_{-1}^{2} \left[x^2 + 4x + 4 - x^4 \right] \ dx = \pi \left[-\frac{x^5}{5} + \frac{x^3}{3} + 2x^2 + 4x \right]_{-1}^{2}$$

$$= \pi \left[\left(-\frac{2^5}{5} + \frac{2^3}{3} + 2(2^2) + 4(2) \right) - \left(-\frac{(-1)^5}{5} + \frac{(-1)^3}{3} + 2((-1)^2) + 4(-1) \right) \right]$$

$$= \pi \left[\left(-\frac{32}{5} + \frac{8}{3} + 8 + 8 \right) - \left(\frac{1}{5} - \frac{1}{3} + 2 - 4 \right) \right]$$

$$= \pi \left(-\frac{33}{5} + \frac{9}{3} + 18 \right) = \pi \left(-\frac{33}{5} + 21 \right) = \frac{72\pi}{5} \ .$$

5. Find the arc length of $y = \frac{2}{3} x\sqrt{x}$, from $x = 0$ to $x = 1$.

Solution :

$$y = \frac{2}{3} x\sqrt{x} = \frac{2}{3} \ x^{\frac{3}{2}} \ .$$

$$y' = \frac{2}{3} \left(\frac{3}{2} \ x^{\frac{1}{2}} \right) = x^{\frac{1}{2}} \ .$$

$$L = \int_{0}^{1} \sqrt{1 + \left(x^{\frac{1}{2}}\right)^2} \ dx = \int_{0}^{1} \sqrt{1+x} \ dx = \int_{0}^{1} (1+x)^{\frac{1}{2}} \ dx$$

$$= \left[\frac{2}{3} \ (1+x)^{\frac{3}{2}} \right]_{0}^{1} = \frac{2}{3} \ (1+1)^{\frac{3}{2}} - \frac{2}{3} \ (1+0)^{\frac{3}{2}} = \frac{2}{3} \left(\sqrt{8} - 1 \right) \ .$$

6. Convert the polar equation $r = \cos\theta + \sec\theta$ into a Cartesian equation.

Solution:

$$r = \cos\theta + \sec\theta \implies r^2 = r\cos\theta + r\sec\theta = r\cos\theta + \frac{r}{\cos\theta}$$

$$\implies x^2 + y^2 = x + \frac{r}{\frac{x}{r}} = x + \frac{r^2}{x} = x + \frac{x^2+y^2}{x}$$

$$\implies x^3 + xy^2 = x^2 + x^2 + y^2 \implies x^3 + xy^2 - 2x^2 - y^2 = 0$$

7. Sketch the common region between the curves $r = 2$ and $r = 2 + 2\cos\theta$, then find its area.

Solution:

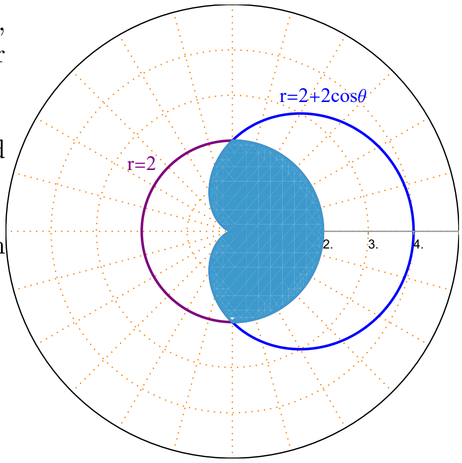
$r = 2 + 2\cos\theta$ represents a cardioid, symmetric with respect to the polar axis.

$r = 2$ represents a circle centered at the pole, and its radius is 2.

Points of intersection between $r = 2 + 2\cos\theta$ and $r = 2$:

$2 + 2\cos\theta = 2 \implies 2\cos\theta = 0$

$\implies \cos\theta = 0 \implies \theta = \frac{\pi}{2}\ ,\ \theta = \frac{3\pi}{2}$



Note that the shaded region is symmetric with respect to the polar axis.

$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (2)^2\ d\theta + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (2 + 2\cos\theta)^2\ d\theta\right)$$

$$= \int_0^{\frac{\pi}{2}} 4\ d\theta + \int_{\frac{\pi}{2}}^{\pi} \left(4 + 8\cos\theta + 4\cos^2\theta\right)\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4\ d\theta + \int_{\frac{\pi}{2}}^{\pi} \left(4 + 8\cos\theta + 4\left(\frac{1 + \cos 2\theta}{2}\right)\right)\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4\ d\theta + \int_{\frac{\pi}{2}}^{\pi} (4 + 8\cos\theta + (2 + 2\cos 2\theta))\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4\ d\theta + \int_{\frac{\pi}{2}}^{\pi} (6 + 8\cos\theta + 2\cos 2\theta)\ d\theta$$

$$= [4\theta]_0^{\frac{\pi}{2}} + [6\theta + 8\sin\theta + \sin 2\theta]_{\frac{\pi}{2}}^{\pi}$$

$$= \left[4\left(\frac{\pi}{2}\right) - 4(0)\right] + \left[(6\pi + 8\sin(\pi) + \sin(2\pi)) - \left(6\left(\frac{\pi}{2}\right) + 8\sin\left(\frac{\pi}{2}\right) + \sin(\pi)\right)\right]$$

$$= [2\pi - 0] + [(6\pi + 0 + 0) - (3\pi + 8 + 0)] = 2\pi + (3\pi - 8) = 5\pi - 8\ .$$

MATH 111 - Integral Calculus
Second Semester - 1447 H
Solution of the First Exam
Dr Tariq A. Alfadhel

Question (1): [2 + 1 = 3 marks]

1. Use Riemann Sum to evaluate the definite integral $\displaystyle\int_0^1 x^3\ dx$.

   Solution : $[a,b] = [0,1]$, $f(x) = x^3$.

   $$\Delta_x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n} .$$

   $$x_k = a + k\ \Delta_x = 0 + k\left(\frac{1}{n}\right) = \frac{k}{n} .$$

   $$f\left(x_k\right) = \left(\frac{k}{n}\right)^3 = \frac{k^3}{n^3} .$$

   $$R_n = \sum_{k=1}^{n} f\left(x_k\right)\ \Delta_x = \sum_{k=1}^{n}\left(\frac{k^3}{n^3}\right)\left(\frac{1}{n}\right) = \sum_{k=1}^{n}\frac{k^3}{n^4} = \frac{1}{n^4}\sum_{k=1}^{n}k^3$$

   $$= \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2 = \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] = \frac{1}{4}\left[\frac{n^2(n+1)^2}{n^4}\right]$$

   $$= \frac{1}{4}\frac{(n+1)^2}{n^2} = \frac{1}{4}\left(\frac{n+1}{n}\right)^2 .$$

   $$\int_0^1 x^3\ dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{1}{4}\left(\frac{n+1}{n}\right)^2\right] = \frac{1}{4}(1)^2 = \frac{1}{4} .$$

2. Find $F'(x)$, if $F(x) = \displaystyle\int_{\sqrt{3x^2+4}}^{\sec(x^3)}\sqrt{4+t^2}dt$.

   Solution :

   $$F'(x) = \frac{d}{dx}\int_{\sqrt{3x^2+4}}^{\sec(x^3)}\sqrt{4+t^2}dt$$

   $$= \sqrt{4+[\sec(x^3)]^2}\left(\sec(x^3)\tan(x^3)(3x^2)\right) - \sqrt{4+(\sqrt{3x^2+4})^2}\left(\frac{1}{2\sqrt{3x^2+4}}\ (6x)\right)$$

   $$= 3x^2\sec(x^3)\tan(x^3)\sqrt{4+\sec^2(x^3)} - \frac{3x\sqrt{3x^2+8}}{\sqrt{3x^2+4}} .$$

Question (2): Find $\frac{dy}{dx}$ of the following :[$1 + 1 = 2$ marks]

1. $y(x) = \left[\sec^{-1}(\ln x) + e^{x^4}\right]^5$.

   Solution :

   $$y'(x) = 5\left[\sec^{-1}(\ln x) + e^{x^4}\right]^4 \left(\frac{1}{\ln x\sqrt{(\ln x)^2 - 1}}\ \frac{1}{x} + e^{x^4}(4x^3)\right)$$

   $$= 5\left[\sec^{-1}(\ln x) + e^{x^4}\right]^4 \left(\frac{1}{x\ln x\sqrt{(\ln x)^2 - 1}} + 4x^3\ e^{x^4}\right) .$$

2. $y(x) = (\tan x)^x$

   Solution :

   $$y(x) = (\tan x)^x \implies \ln|y(x)| = \ln|(\tan x)^x| = x\ \ln|\tan x|$$

   Differentiate both sides.

   $$\frac{y'(x)}{y(x)} = (1)\ln|\tan x| + x\left(\frac{\sec^2 x}{\tan x}\right)$$

   $$y'(x) = y(x)\ \left[\ln|\tan x| + \frac{x\sec^2 x}{\tan x}\right] = (\tan x)^x\left[\ln|\tan x| + \frac{x\sec^2 x}{\tan x}\right] .$$

Question (3): [$1 + 1 + 1 + 1 + 1 + 1 + 2 + 2 = 10$ marks]
Evaluate the following integrals :

1. $\int (x^3 + 1)\sec(x^4 + 4x)\ dx$.

   Solution :

   $$\int (x^3 + 1)\sec(x^4 + 4x)\ dx = \frac{1}{4}\int \sec(x^4 + 4x)\ \left[4(x^3 + 1)\right]\ dx$$

   $$= \frac{1}{4}\int \sec(x^4 + 4x)(4x^3 + 4)\ dx = \frac{1}{4}\ln\left|\sec(x^4 + 4x) + \tan(x^4 + 4x)\right| + c.$$

2. $\int \frac{x^5 + x}{\sqrt{x^6 + 3x^2 + 8}}\ dx$.

   Solution : .

   $$\int \frac{x^5 + x}{\sqrt{x^6 + 3x^2 + 8}}\ dx = \int \left(x^6 + 3x^2 + 8\right)^{-\frac{1}{2}}(x^5 + x)\ dx$$

   $$= \frac{1}{6}\int \left(x^6 + 3x^2 + 8\right)^{-\frac{1}{2}}\left[6(x^5 + x)\right]\ dx = \frac{1}{6}\int \left(x^6 + 3x^2 + 8\right)^{-\frac{1}{2}}\left(6x^5 + 6x\right)\ dx$$

$$= \frac{1}{6}\frac{\left(x^6+3x^2+8\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{1}{3}\left(x^6+3x^2+8\right)^{\frac{1}{2}} + c$$

3. $\displaystyle\int \frac{1}{\operatorname{csch}x(1+\cosh x)}\ dx$.

Solution :

$$\int \frac{1}{\operatorname{csch}x(1+\cosh x)}\ dx = \int \frac{\sinh x}{1+\cosh x}\ dx = \ln|1+\cosh x| + c\ .$$

4. $\displaystyle\int \frac{5^{\tanh x}}{\cosh^2 x}\ dx$.

Solution :

$$\int \frac{5^{\tanh x}}{\cosh^2 x}\ dx = \int 5^{\tanh x}\operatorname{sech}^2 x\ dx = \frac{5^{\tanh x}}{\ln 5} + c\ .$$

5. $\displaystyle\int \frac{2^x}{\sqrt{9-4^x}}\ dx$

Solution :

$$\int \frac{2^x}{\sqrt{9-4^x}}\ dx = \int \frac{2^x}{\sqrt{9-\left(2^2\right)^x}}\ dx = \int \frac{2^x}{\sqrt{9-\left(2^x\right)^2}}\ dx$$

$$= \frac{1}{\ln 2}\int \frac{2^x\ \ln 2}{\sqrt{(3)^2-\left(2^x\right)^2}}\ dx = \frac{1}{\ln 2}\sin^{-1}\left(\frac{2^x}{3}\right) + c$$

6. $\displaystyle\int \frac{(e^x-1)\cos\left(x+e^{-x}\right)}{e^x}\ dx$

Solution :

$$\int \frac{(e^x-1)\cos\left(x+e^{-x}\right)}{e^x}\ dx = \int \cos\left(x+e^{-x}\right)\frac{(e^x-1)}{e^x}\ dx$$

$$= \int \cos\left(x+e^{-x}\right)\left(\frac{e^x}{e^x}-\frac{1}{e^x}\right)\ dx = \int \cos\left(x+e^{-x}\right)\left(1-e^{-x}\right)\ dx$$

$$= \int \cos\left(x+e^{-x}\right)\left(1+e^{-x}(-1)\right)\ dx = \sin\left(x+e^{-x}\right) + c\ .$$

7. $\displaystyle\int_0^1 \frac{x}{x+1}\ dx$.

Solution :

$$\int_0^1 \frac{x}{x+1}\ dx = \int_0^1 \frac{(x+1)-1}{x+1}\ dx = \int_0^1 \left(\frac{x+1}{x+1}-\frac{1}{x+1}\right)\ dx$$

$$= \int_0^1 \left(1 - \frac{1}{x+1}\right) \, dx = [x - \ln|x+1|]_0^1$$

$$= (1 - \ln 2) - (0 - \ln(1)) = 1 - \ln 2 \ .$$

8. $\displaystyle\int \frac{1}{x^2 + 8x + 25} \, dx \ .$

Solution : By completing the square.

$x^2 + 8x + 25 = (x^2 + 8x + 16) + 9 = (x+4)^2 + (3)^2$

$$= \int \frac{1}{x^2 + 8x + 25} \, dx = \int \frac{1}{(x+4)^2 + (3)^2} \, dx$$

$$= \frac{1}{3} \tan^{-1} \left(\frac{x+4}{3}\right) + c \ .$$