MATH 111 - Integral Calculus
First Semester - 1447 H
Solution of the First Exam

Dr Tariq A. Alfadhel

Question (1): [9 marks]

1. Use Riemann Sum to evaluate the definite integral $\displaystyle\int_0^1 (3x^2 - x)\, dx$. [3]

   Solution : $[a,b] = [0,1]$, $f(x) = 3x^2 - x$.

   $$\Delta_x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

   $$x_k = a + k\ \Delta_x = 0 + k\left(\frac{1}{n}\right) = \frac{k}{n}$$

   $$f(x_k) = 3\left(\frac{k}{n}\right)^2 - \frac{k}{n} = \frac{3k^2}{n^2} - \frac{k}{n}$$

   $$R_n = \sum_{k=1}^{n} f(x_k)\ \Delta_x = \sum_{k=1}^{n}\left(\frac{3k^2}{n^2} - \frac{k}{n}\right)\ \left(\frac{1}{n}\right)$$

   $$= \sum_{k=1}^{n}\left(\frac{3k^2}{n^3} - \frac{k}{n^2}\right) = \sum_{k=1}^{n}\frac{3k^2}{n^3} - \sum_{k=1}^{n}\frac{k}{n^2}$$

   $$= \frac{3}{n^3}\sum_{k=1}^{n} k^2 - \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{3}{n^3}\ \frac{n(n+1)(2n+1)}{6} - \frac{1}{n^2}\ \frac{n(n+1)}{2}$$

   $$= \frac{(n+1)(2n+1)}{2n^2} - \frac{n+1}{2n}$$

   $$\int_0^1 (3x^2 - x)\ dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{(n+1)(2n+1)}{2n^2} - \frac{n+1}{2n}\right]$$

   $$= \frac{2}{2} - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}\ .$$

2. Find $F'(x)$, if $F(x) = \displaystyle\int_{\cos(\sqrt{x+1})}^{e^{x^2}} \frac{1}{1-t^2}dt$. [2]

   Solution :

   $$F'(x) = \frac{d}{dx}\int_{\cos(\sqrt{x+1})}^{e^{x^2}} \frac{1}{1-t^2}dt$$

   $$= \frac{1}{1-\left(e^{x^2}\right)^2}\ \left(e^{x^2}\ (2x)\right) - \frac{1}{1-\left(\cos\left(\sqrt{x+1}\right)\right)^2}\ \left(-\sin\left(\sqrt{x+1}\right)\ \frac{1}{2\sqrt{x+1}}\right)$$

   $$= \frac{2x\ e^{x^2}}{1-e^{2x^2}} + \frac{1}{2\sqrt{x+1}\ \sin\left(\sqrt{x+1}\right)}\ .$$

Find $\dfrac{dy}{dx}$ of the following :

3. $y = \sin^{-1}\left(5^{3x+1}\right) + \log_2(\sqrt[5]{x^4+2})$. [2]

Solution :

$y = \sin^{-1}\left(5^{3x+1}\right) + \log_2\left(\sqrt[5]{x^4+2}\right) = \sin^{-1}\left(5^{3x+1}\right) + \log_2\left(x^4+2\right)^{\frac{1}{5}}$

$= \sin^{-1}\left(5^{3x+1}\right) + \dfrac{1}{5}\ \log_2\left(x^4+2\right)$

$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\left(5^{3x+1}\right)^2}}\ \left(5^{3x+1}\ (3)\ \ln 5\right) + \dfrac{1}{5}\ \dfrac{4x^3}{x^4+2}\ \dfrac{1}{\ln 2}$

$= \dfrac{3\ \ln 5\ 5^{3x+1}}{\sqrt{1-5^{6x+2}}} + \dfrac{4x^3}{5\ \ln 2\ (x^4+2)}$.

4. $y = (\sin x)^{\sqrt{x}}$. [2]

Solution :

$y = (\sin x)^{\sqrt{x}} \implies \ln|y| = \ln\left|(\sin x)^{\sqrt{x}}\right| = \sqrt{x}\ \ln|\sin x|$

Differentiate both sides.

$\dfrac{y'}{y} = \dfrac{1}{2\sqrt{x}} \ln|\sin x| + \sqrt{x}\ \left(\dfrac{\cos x}{\sin x}\right)$

$y' = y\ \left[\dfrac{\ln|\sin x|}{2\sqrt{x}} + \sqrt{x}\ \cot x\right]$

$= (\sin x)^{\sqrt{x}} \left[\dfrac{\ln|\sin x|}{2\sqrt{x}} + \sqrt{x}\ \cot x\right]$.

Question (2): [16 marks]
Evaluate the following integrals :

1. $\displaystyle\int \frac{\cos x}{(1+\sin x)^4}\ dx$. [2]

Solution :

$\displaystyle\int \frac{\cos x}{(1+\sin x)^4}\ dx = \int (1+\sin x)^{-4}\ \cos x\ dx$

$= \dfrac{(1+\sin x)^{-3}}{-3} + c = \dfrac{1}{-3\,(1+\sin x)^3} + c$.

2. $\displaystyle\int \frac{\sec(x^{\frac{2}{5}})\tan(x^{\frac{2}{5}})}{x^{\frac{3}{5}}}\ dx$. [2]

Solution : .

$$\int \frac{\sec(x^{\frac{2}{5}})\tan(x^{\frac{2}{5}})}{x^{\frac{3}{5}}}\ dx = \int \sec\left(x^{\frac{2}{5}}\right)\tan\left(x^{\frac{2}{5}}\right)\ x^{-\frac{3}{5}}\ dx$$

$$= \frac{5}{2}\int \sec\left(x^{\frac{2}{5}}\right)\tan\left(x^{\frac{2}{5}}\right)\ \left(\frac{2}{5}\ x^{-\frac{3}{5}}\right)\ dx = \frac{5}{2}\ \sec\left(x^{\frac{2}{5}}\right) + c\ .$$

3. $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\ dx\ .$ [2]

Solution :

$$\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\ dx = 2\int \frac{1}{2\sqrt{x}(1+\sqrt{x})}\ dx$$

$$= 2\int \frac{\left(\frac{1}{2\sqrt{x}}\right)}{1+\sqrt{x}}\ dx = 2\ln\left|1+\sqrt{x}\right| + c\ .$$

4. $\displaystyle\int \frac{7^{\cot(3x)}}{\sin^2(3x)}\ dx\ .$ [2]

Solution :

$$\int \frac{7^{\cot(3x)}}{\sin^2(3x)}\ dx = \int 7^{\cot(3x)}\ \frac{1}{\sin^2(3x)}\ dx$$

$$= \frac{-1}{3}\int 7^{\cot(3x)}\ \left(-\csc^2(3x)\ (3)\right)dx = -\frac{1}{3}\ \frac{7^{\cot(3x)}}{\ln 7} + c\ .$$

5. $\displaystyle\int_0^1 \frac{x}{x+1}\ dx\ .$ [2]

Solution :

$$\int_0^1 \frac{x}{x+1}\ dx = \int_0^1 \frac{(x+1)-1}{x+1}\ dx = \int_0^1 \left(\frac{x+1}{x+1} - \frac{1}{x+1}\right)\ dx$$

$$= \int_0^1 \left(1 - \frac{1}{x+1}\right)\ dx = \left[x - \ln|x+1|\right]_0^1$$

$$= (1-\ln 2) - (0 - \ln(1)) = 1 - \ln 2\ .$$

6. $\displaystyle\int \frac{x-5}{\sqrt{1-x^2}}\ dx\ .$ [2]

Solution :

$$\int \frac{x-5}{\sqrt{1-x^2}}\ dx = \int \left(\frac{x}{\sqrt{1-x^2}} - \frac{5}{\sqrt{1-x^2}}\right)\ dx$$

$$= \frac{1}{-2}\int \left(1-x^2\right)^{-\frac{1}{2}}\ (-2x)\ dx - 5\int \frac{1}{\sqrt{1-x^2}}\ dx$$

$$= \frac{1}{-2} \frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} - 5\sin^{-1} x + c = -\sqrt{1-x^2} - 5\sin^{-1} x + c \ .$$

7. $\displaystyle\int \frac{1}{x\sqrt{x^6-9}}\, dx$. [2]

Solution :

$$\int \frac{1}{x\sqrt{x^6-9}}\, dx = \int \frac{1}{x\sqrt{\left(x^3\right)^2-(3)^2}}\, dx = \int \frac{x^2}{x^3\sqrt{\left(x^3\right)^2-(3)^2}}\, dx$$

$$= \frac{1}{3} \int \frac{3x^2}{x^3\sqrt{\left(x^3\right)^2-(3)^2}}\, dx = \frac{1}{3}\,\frac{1}{3}\ \sec^{-1}\left(\frac{x^3}{3}\right) + c \ .$$

8. $\displaystyle\int \frac{1}{e^{-x}+e^{x}}\, dx$. [2]

Solution :

$$\int \frac{1}{e^{-x}+e^{x}}\, dx = \int \frac{e^x}{e^x\ \left(e^{-x}+e^{x}\right)}\, dx = \int \frac{e^x}{1+e^{2x}}\, dx$$

$$= \int \frac{e^x}{(1)^2+\left(e^x\right)^2}\, dx = \tan^{-1}\left(e^x\right) + c \ .$$