111 ريض - حساب التكامل
الفصل الدراسي الثاني 1445 هـ
حل الاختبار الفصلي الثاني
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السؤال الأول (4 درجات) : أحسب $\frac{dy}{dx}$ فيما يلي :

(1) $y = \tanh(2x^3) + sech^{-1}(3x)$. [2]

الحل :

$$\frac{dy}{dx} = sech^2(2x^3)\,(6x^2) + \frac{-1}{3x\,\sqrt{1-(3x)^2}}\,(3)$$

$$= 6x^2\, sech^2(2x^3) - \frac{1}{x\,\sqrt{1-9x^2}}$$

(2) $y = \cosh^{-1}\left(\sqrt{x}\right) + \sinh^3\left(\frac{1}{x}\right)$. [2]

الحل :

$$\frac{dy}{dx} = \frac{1}{\sqrt{\left(\sqrt{x}\right)^2 - 1}}\left(\frac{1}{2\sqrt{x}}\right) + 3\left(\sinh\left(\frac{1}{x}\right)\right)^2 \cosh\left(\frac{1}{x}\right)\ \left(\frac{-1}{x^2}\right)$$

$$= \frac{1}{2\sqrt{x}\,\sqrt{x-1}} - \left(\frac{3}{x^2}\right)\sinh^2\left(\frac{1}{x}\right)\ \cosh\left(\frac{1}{x}\right)$$

السؤال الثاني (21 درجة): أحسب التكاملات التالية

(1) $\int \frac{\coth\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$. [2]

الحل :

$$\int \frac{\coth\left(\sqrt{x}\right)}{\sqrt{x}}\,dx = \int \coth\left(\sqrt{x}\right)\ \frac{1}{\sqrt{x}}\,dx$$

$$= 2\int \coth\left(\sqrt{x}\right)\ \frac{1}{2\sqrt{x}}\,dx = 2\ln\left|\sinh\left(\sqrt{x}\right)\right| + c$$

باستخدام القانون

$$\int \coth\left(f(x)\right)\ f'(x)\,dx = \ln\left|\sinh\left(f(x)\right)\right| + c$$

(2) $\int (2x+1)\ \cos x\, dx$. [2]

الحل : باستخدام طريقة التكامل بالتجزئ

$$u = 2x+1 \qquad dv = \cos x\, dx$$
$$du = 2\, dx \qquad v = \sin x$$

$$\int (2x+1)\ \cos x\, dx = (2x+1)\ \sin x - \int 2\sin x\, dx$$

$$= (2x+1)\ \sin x - 2\int \sin x\, dx = (2x+1)\ \sin x - 2(-\cos x) + c$$

$$= (2x+1)\ \sin x + 2\cos x + c$$

(3) $\int \tan^2 x\ \sec^4 x\, dx$. [2]

الحل :

باستخدام التعويض $u = \tan x$

عندئذ $du = \sec^2 x\, dx$

$$\int \tan^2 x\ \sec^4 x\, dx = \int \tan^2 x\ \sec^2 x\ \sec^2 x\, dx$$

$$\int \tan^2 x\ \left(1+\tan^2 x\right)\ \sec^2 x\, dx = \int u^2\ \left(1+u^2\right)\ du$$

$$= \int \left(u^2+u^4\right)\ du = \frac{u^3}{3} + \frac{u^5}{5} + c = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + c$$

(4) $\int \frac{1}{x\,\sqrt{x^6+25}}\, dx$. [3]

الحل :

$$\int \frac{1}{x\,\sqrt{x^6+25}}\, dx = \int \frac{1}{x\,\sqrt{\left(x^3\right)^2+(5)^2}}\, dx = \int \frac{x^2}{x^2\ x\,\sqrt{\left(x^3\right)^2+(5)^2}}\, dx$$

$$= \frac{1}{3}\int \frac{3x^2}{x^3\,\sqrt{\left(x^3\right)^2+(5)^2}}\, dx = \frac{1}{3}\left[-\frac{1}{5}\, csch^{-1}\left(\frac{x^3}{5}\right)\right] + c = -\frac{1}{15}\, csch^{-1}\left(\frac{x^3}{5}\right) + c$$

باستخدام القانون

$$\int \frac{f'(x)}{f(x)\sqrt{\left[f(x)\right]^2+a^2}}\, dx = -\frac{1}{a}\, csch^{-1}\left(\frac{f(x)}{a}\right) + c$$ حيث $a > 0$.

(5) $\int x \ \ln|x| \, dx$. [2]

الحل :

الحل : باستخدام طريقة التكامل بالتجزئ

$$\begin{aligned} u &= \ln|x| \qquad & dv &= x\,dx \\ du &= \frac{1}{x}\,dx \qquad & v &= \frac{x^2}{2} \end{aligned}$$

$$\int x \ \ln|x| \, dx = \frac{x^2}{2}\ \ln|x| - \int \frac{x^2}{2}\,\frac{1}{x}\,dx$$

$$= \frac{x^2}{2}\ \ln|x| - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ \ln|x| - \frac{1}{2}\,\frac{x^2}{2} + c = \frac{x^2}{2}\ \ln|x| - \frac{x^2}{4} + c$$

(6) $\int \frac{1}{\sqrt{4+\sqrt{x}}}\,dx$. [2]

الحل :

ضع $u = \sqrt{4+\sqrt{x}} \implies u^2 = 4+\sqrt{x}$

$$\implies u^2 - 4 = \sqrt{x} \implies (u^2-4)^2 = x$$

عندئذ $2(u^2-4)\,2u\,du = dx \implies 4u(u^2-4)\,du = dx$

$$\int \frac{1}{\sqrt{4+\sqrt{x}}}\,dx = \int \frac{4u(u^2-4)}{u}\,du = 4\int (u^2-4)\,du$$

$$= 4\left[\frac{u^3}{3} - 4u\right] + c = 4\left[\frac{\left(\sqrt{4+\sqrt{x}}\right)^3}{3} - 4\left(\sqrt{4+\sqrt{x}}\right)\right] + c$$

(7) $\int \frac{1}{\sqrt{x^2+6x+13}}\,dx$. [2]

الحل :باستخدام الإكمال إلى مربع كامل

$$x^2+6x+13 = \left(x^2+6x+9\right) + 4 = (x+3)^2 + (2)^2$$

$$\int \frac{1}{\sqrt{x^2+6x+13}}\,dx = \int \frac{1}{\sqrt{(x+3)^2+(2)^2}}\,dx = \sinh^{-1}\left(\frac{x+3}{2}\right) + c$$

باستخدام القانون

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2 + a^2}}\,dx = \sinh^{-1}\left(\frac{f(x)}{a}\right) + c$$ حيث $a > 0$.

(8) $\int \frac{3x^2+3x+8}{x^3+4x}\,dx$. [3]

الحل : باستخدام طريقة الكسور الجزئية

$$\frac{3x^2+3x+8}{x^3+4x} = \frac{3x^2+3x+8}{x\,(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\frac{3x^2+3x+8}{x^3+4x} = \frac{A(x^2+4)}{x(x^2+4)} + \frac{(Bx+C)\,x}{x(x^2+4)}$$

$$3x^2+3x+8 = A(x^2+4) + x(Bx+C)$$

$$3x^2+3x+8 = Ax^2+4A+Bx^2+Cx = (A+B)x^2+Cx+4A$$

بمقارنة معاملات كثيرتي الحدود قي الطرفين نحصل على :

$$\begin{array}{ll} A+B=3 & \longrightarrow (1) \\ C=3 & \longrightarrow (2) \\ 4A=8 & \longrightarrow (3) \end{array}$$

من المعادلة (3) نحصل على : $4A = 8 \implies A = 2$.

من المعادلة (1) نحصل على : $2 + B = 3 \implies B = 1$.

$$\int \frac{3x^2+3x+8}{x^3+4x}\,dx = \int \left(\frac{2}{x} + \frac{x+3}{x^2+4}\right)\,dx$$

$$= \int \frac{2}{x}\,dx + \int \frac{x}{x^2+4}\,dx + \int \frac{3}{x^2+4}\,dx$$

$$= 2\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{2x}{x^2+4}\,dx + 3\int \frac{1}{x^2+2^2}\,dx$$

$$= 2\ln|x| + \frac{1}{2}\,\ln\left(x^2+4\right) + 3\left[\frac{1}{2}\,\tan^{-1}\left(\frac{x}{2}\right)\right] + c$$

(9) $\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx$. [3]

الحل : باستخدام التعويضات المثلثية

ضع $\sin\theta = \frac{x}{2} \implies x = 2\sin\theta$

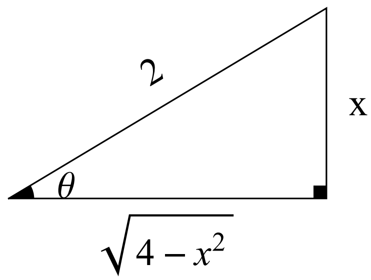
$$dx = 2\cos\theta\,d\theta$$

$$\left(4-x^2\right)^{\frac{3}{2}}=\left(4-4\sin^2\theta\right)^{\frac{3}{2}}=\left[4\left(1-\sin^2\theta\right)\right]^{\frac{3}{2}}$$

$$=\left(4\cos^2\theta\right)^{\frac{3}{2}}=(4)^{\frac{3}{2}}\ \left(\cos^2\theta\right)^{\frac{3}{2}}=\left(2^2\right)^{\frac{3}{2}}\ \left(\cos^2\theta\right)^{\frac{3}{2}}=2^3\ \cos^3\theta$$

$$\int\frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx=\int\frac{2\cos\theta}{2^3\ \cos^3\theta}\,d\theta=\frac{1}{2^2}\ \int\frac{1}{\cos^2\theta}\,d\theta$$

$$=\frac{1}{4}\int\sec^2\theta\,d\theta=\frac{1}{4}\ \tan\theta+c$$



من المثلث : نلاحظ أن $\tan\theta=\frac{x}{\sqrt{4-x^2}}$

$$\int\frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx=\frac{1}{4}\ \frac{x}{\sqrt{4-x^2}}+c$$