111 ريض - حساب التكامل
الفصل الدراسي الثاني 1445 هـ
حل الاختبار الفصلي الأول
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السؤال الأول (9 درجات) :

(1) استخدم مجموع ريمان لحساب التكامل المحدد $\int_0^2 (x^2-1)\,dx$. [3]

الحل : $[a,b]=[0,2]$ و $f(x)=x^2-1$.

$$\Delta_x=\frac{b-a}{n}=\frac{2-0}{n}=\frac{2}{n}$$

$$x_k=a+k\,\Delta_x=0+k\left(\frac{2}{n}\right)=\frac{2k}{n}$$

$$f(x_k)=\left(\frac{2k}{n}\right)^2-1=\frac{4k^2}{n^2}-1$$

$$R_n=\sum_{k=1}^{n} f(x_k)\ \Delta_x=\sum_{k=1}^{n}\left(\frac{4k^2}{n^2}-1\right)\ \left(\frac{2}{n}\right)=\sum_{k=1}^{n}\left(\frac{8k^2}{n^3}-\frac{2}{n}\right)=\sum_{k=1}^{n}\frac{8k^2}{n^3}-\sum_{k=1}^{n}\frac{2}{n}$$

$$=\frac{8}{n^3}\sum_{k=1}^{n}k^2-\frac{2}{n}\sum_{k=1}^{n}1=\frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6}-\frac{2}{n}\,(n)=\frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)-2$$

$$\int_0^2 (x^2-1)\,dx=\lim_{n\to\infty}R_n=\lim_{n\to\infty}\left[\frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)-2\right]$$

$$=\frac{4}{3}\,(2)-2=\frac{8}{3}-2=\frac{2}{3}$$

(2) جد $F'(x)$ إذا كانت $F(x)=\int_{\sin\left(\frac{x}{2}\right)}^{5^{3x}}\sqrt{t^2+1}\,dt$. [2]

الحل :

$$F'(x)=\frac{d}{dx}\int_{\sin\left(\frac{x}{2}\right)}^{5^{3x}}\sqrt{t^2+1}\,dt$$

$$=\sqrt{\left(5^{3x}\right)^2+1}\ \left(5^{3x}\,(3)\ln 5\right)-\sqrt{\left(\sin\left(\frac{x}{2}\right)\right)^2+1}\ \left(\cos\left(\frac{x}{2}\right)\ \left(\frac{1}{2}\right)\right)$$

$$=3\,5^{2x}\ln 5\sqrt{5^{6x}+1}-\frac{1}{2}\ \cos\left(\frac{x}{2}\right)\sqrt{\sin^2\left(\frac{x}{2}\right)+1}$$

احسب $\frac{dy}{dx}$ فيمايلي :

(3) $y = \tan^{-1}(2x)\ \log|\sec x + \tan x|$. [2]

الحل :

$$\frac{dy}{dx} = \left(\frac{1}{1+(2x)^2}\,(2)\right)\ \log|\sec x + \tan x| + \tan^{-1}(2x)\ \left(\frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\,\frac{1}{\ln 10}\right)$$

$$= \frac{2\log|\sec x + \tan x|}{1+4x^2} + \frac{\sec x\ \tan^{-1}(2x)}{\ln 10}$$

(4) $y = (\tan x)^{\sec x} + 5^x$. [2]

الحل :

لتكن $y = f(x) + g(x)$ حيث $f(x) = (\tan x)^{\sec x}$ و $g(x) = 5^x$.

عندئذ $\frac{dy}{dx} = y' = f'(x) + g'(x)$

أولاً - $g'(x) = 5^x\ (1)\ln 5 = 5^x \ln 5$

ثانياً - حساب $f'(x)$

$$f(x) = (\tan x)^{\sec x} \implies \ln|f(x)| = \ln\left|(\tan x)^{\sec x}\right| = \sec x\ \ln|\tan x|$$

بإشتقاق الطرفين

$$\frac{f'(x)}{f(x)} = \sec x\ \tan x\ \ln|\tan x| + \sec x\left(\frac{\sec^2 x}{\tan x}\right)$$

$$f'(x) = f(x)\left[\sec x\ \tan x\ \ln|\tan x| + \frac{\sec^3 x}{\tan x}\right]$$

$$= (\tan x)^{\sec x}\left[\sec x\ \tan x\ \ln|\tan x| + \frac{\sec^3 x}{\tan x}\right]$$

أي أن $\frac{dy}{dx} = (\tan x)^{\sec x}\left[\sec x\ \tan x\ \ln|\tan x| + \frac{\sec^3 x}{\tan x}\right] + 5^x \ln 5$

السؤال الثاني (16 درجة) : أحسب التكاملات التالية

(1) $\int \left(\sqrt[3]{x}\, e^{\frac{x^2}{3}}\right)^3\, dx$. [2]

الحل :

$$\int \left(\sqrt[3]{x}\, e^{\frac{x^2}{3}}\right)^3 dx = \int \left(\sqrt[3]{x}\right)^3 \left(e^{\frac{x^2}{3}}\right)^3 dx = \int x\, e^{x^2}\, dx$$

$$= \frac{1}{2} \int e^{x^2} (2x)\, dx = \frac{1}{2} e^{x^2} + c$$

باستخدام القانون

$$\int e^{f(x)}\, f'(x)\, dx = e^{f(x)} + c$$

(2) $\int \frac{1}{\sqrt{e^{4x} - 9}}\, dx$. [2]

الحل :

$$\int \frac{1}{\sqrt{e^{4x} - 9}}\, dx = \int \frac{1}{\sqrt{(e^{2x})^2 - (3)^2}}\, dx = \frac{1}{2} \int \frac{e^{2x}\,(2)}{e^{2x}\sqrt{(e^{2x})^2 - (3)^2}}\, dx$$

$$= \frac{1}{2} \left(\frac{1}{3} \sec^{-1} \left(\frac{e^{2x}}{3} \right) \right) + c = \frac{1}{6} \sec^{-1} \left(\frac{e^{2x}}{3} \right) + c$$

باستخدام القانون

$\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2 - a^2}}\, dx = \frac{1}{a} \sec^{-1} \left(\frac{f(x)}{a} \right) + c$ ، حيث $|f(x)| > a$ و $a > 0$.

(3) $\int_0^1 x\, 5^{2x^2}\, dx$. [2]

الحل :

$$\int_0^1 x\, 5^{2x^2}\, dx = \frac{1}{4} \int_0^1 5^{2x^2} (4x)\, dx = \frac{1}{4} \left[\frac{5^{2x^2}}{\ln 5} \right]_0^1$$

$$= \frac{1}{4} \left[\frac{5^{2(1)^2}}{\ln 5} - \frac{5^{2(0)^2}}{\ln 5} \right] = \frac{1}{4} \left[\frac{5^2}{\ln 5} - \frac{5^0}{\ln 5} \right] = \frac{1}{4} \left(\frac{25 - 1}{\ln 5} \right) = \frac{6}{\ln 5}$$

باستخدام القانون

$$\int a^{f(x)}\, f'(x)\, dx = \frac{a^{f(x)}}{\ln a} + c \,.$$

(4) $\int \frac{x + 2}{x^2 + 1}\, dx$. [2]

الحل :

$$\int \frac{x + 2}{x^2 + 1}\, dx = \int \left[\frac{x}{x^2 + 1} + \frac{2}{x^2 + 1} \right] dx$$

$$= \int \frac{x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx$$

$$= \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln(x^2+1) + 2\tan^{-1}(x) + c$$

(5) $\int x^{-1}\ \sin\left(\ln(x^2)\right)\ dx$. [2]

الحل :

$$\int x^{-1}\ \sin\left(\ln(x^2)\right)\ dx = \int \sin\left(2\ln|x|\right)\ \frac{1}{x}\,dx$$

$$= \frac{1}{2}\int \sin\left(2\ln|x|\right)\ \frac{2}{x}\,dx = \frac{1}{2}\left(-\cos\left(2\ln|x|\right)\right) + c = -\frac{\cos\left(2\ln|x|\right)}{2} + c$$

باستخدام القانون

$$\int \sin\left(f(x)\right)\ f'(x)\,dx = -\cos\left(f(x)\right) + c$$

(6) $\int x^{-2}\ \csc\left(\frac{1}{x}\right)\cot\left(\frac{1}{x}\right)\ dx$. [2]

الحل :

$$\int x^{-2}\ \csc\left(\frac{1}{x}\right)\cot\left(\frac{1}{x}\right)\ dx = \int \csc\left(\frac{1}{x}\right)\cot\left(\frac{1}{x}\right)\ \left(\frac{1}{x^2}\right)\ dx$$

$$= \int -\csc\left(\frac{1}{x}\right)\cot\left(\frac{1}{x}\right)\ \left(-\frac{1}{x^2}\right)\ dx = \csc\left(\frac{1}{x}\right) + c$$

باستخدام القانون

$$\int \csc\left(f(x)\right)\cot\left(f(x)\right)\ f'(x)\,dx = -\csc\left(f(x)\right) + c$$

(7) $\int \frac{\left(3+\sin^{-1}x\right)^3}{\sqrt{1-x^2}}\ dx$. [2]

الحل :

$$\int \frac{\left(3+\sin^{-1}x\right)^3}{\sqrt{1-x^2}}\ dx = \int \left(3+\sin^{-1}x\right)^3\ \frac{1}{\sqrt{1-x^2}}\ dx = \frac{\left(3+\sin^{-1}x\right)^4}{4} + c$$

باستخدام القانون

$$\int [f(x)]^n \; f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c$$ ، حيث $n \neq -1$.

(8) $\int \frac{\sin x}{\cos^2 x}\, dx$. [2]

الحل الأول :

$$\int \frac{\sin x}{\cos^2 x}\, dx = \int \frac{1}{\cos x} \, \frac{\sin x}{\cos x}\, dx$$

$$= \int \sec x \; \tan x \, dx = \sec x + c$$

باستخدام القانون

$$\int \sec x \; \tan x \, dx = \sec x + c$$

الحل الثاني :

$$\int \frac{\sin x}{\cos^2 x}\, dx = \int (\cos x)^{-2} \; \sin x \, dx = -\int (\cos x)^{-2} \; (-\sin x)\, dx$$

$$= -\frac{(\cos x)^{-1}}{-1} + c = \frac{1}{\cos x} + c = \sec x + c$$

$$\int [f(x)]^n \; f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c$$ ، حيث $n \neq -1$.