111 ريض - حساب التكامل
الفصل الدراسي الثالث 1444 هـ
حل الاختبار الفصلي
د. طارق عبدالرحمن مُحَمَّد الفاضل

السؤال الأول (12 درجة) :

(1) استخدم مجموع ريمان لحساب التكامل المحدد $\int_{-1}^{4} (2x+3)\, dx$.

الحل : $[a,b]=[-1,4]$ و $f(x)=2x+3$.

$$\Delta_x = \frac{b-a}{n} = \frac{4-(-1)}{n} = \frac{5}{n}$$

$$x_k = a + k\,\Delta_x = -1 + k\left(\frac{5}{n}\right) = -1 + \frac{5k}{n}$$

$$f(x_k) = 2\left(-1+\frac{5k}{n}\right) + 3 = -2 + \frac{10k}{n} + 3 = \frac{10k}{n} + 1$$

$$R_n = \sum_{k=1}^{n} f(x_k)\ \Delta_x = \sum_{k=1}^{n}\left(\frac{10k}{n}+1\right)\ \left(\frac{5}{n}\right) = \sum_{k=1}^{n}\left(\frac{50k}{n^2}+\frac{5}{n}\right) = \sum_{k=1}^{n}\frac{50k}{n^2} + \sum_{k=1}^{n}\frac{5}{n}$$

$$= \frac{50}{n^2}\sum_{k=1}^{n} k + \frac{5}{n}\sum_{k=1}^{n} 1 = \frac{50}{n^2}\ \frac{n(n+1)}{2} + \frac{5}{n}\,(n) = 25\left(\frac{n+1}{n}\right) + 5$$

$$\int_{-1}^{4} (2x+3)\, dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[25\left(\frac{n+1}{n}\right)+5\right] = 25(1)+5 = 30$$

(2) جد $F'(x)$ إذا كانت $F(x) = \int_{\frac{1}{x}}^{5^{-2x}} \frac{t+1}{\sqrt{t^2+2}}\, dt$

الحل :

$$F'(x) = \frac{d}{dx}\int_{\frac{1}{x}}^{5^{-2x}} \frac{t+1}{\sqrt{t^2+2}}\, dt$$

$$= \frac{5^{-2x}+1}{\sqrt{\left(5^{-2x}\right)^2+2}}\ \left[5^{-2x}\,(-2)\ \ln 5\right] - \frac{\frac{1}{x}+1}{\sqrt{\left(\frac{1}{x}\right)^2+2}}\ \left(\frac{-1}{x^2}\right)$$

$$= \frac{-2\,5^{-2x}\ \ln 5\left(5^{-2x}+1\right)}{\sqrt{5^{-4x}+2}} + \frac{\frac{1}{x}+1}{x^2\sqrt{\frac{1}{x^2}+2}}$$

احسب $\frac{dy}{dx}$ فيمايلي :

$$y = 2^{\sqrt[3]{x}}\ \tanh^{-1}(x) \quad (3)$$

الحل :

$$\frac{dy}{dx} = \left(2^{\sqrt[3]{x}}\left(\frac{1}{3}\,x^{-\frac{2}{3}}\right)\ln 2\right)\tanh^{-1}(x) + 2^{\sqrt[3]{x}}\ \frac{1}{1-x^2}$$

$$= \frac{2^{\sqrt[3]{x}}\tanh^{-1}(x)\ln 2}{3\,x^{\frac{2}{3}}} + \frac{2^{\sqrt[3]{x}}}{1-x^2}$$

$$y = \sin^{-1}(2x)\ \log_7|1-\ln|3x|| \quad (4)$$

الحل :

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1-(2x)^2}}\,(2)\right)\log_7|1-\ln|3x|| + \sin^{-1}(2x)\left(\frac{0-\frac{3}{3x}}{1-\ln|3x|}\ \frac{1}{\ln 7}\right)$$

$$= \frac{2\ \log_7|1-\ln|3x||}{\sqrt{1-4x^2}} - \frac{\sin^{-1}(2x)}{x\,(1-\ln|3x|)\ln 7}$$

$$y = (\tan x)^{\cos(e^x)} \quad (5)$$

الحل :

$$y = (\tan x)^{\cos(e^x)} \implies \ln|y| = \ln\left|(\tan x)^{\cos(e^x)}\right| = \cos(e^x)\ \ln|\tan x|$$

بإشتقاق الطرفين

$$\frac{y'}{y} = -\sin(e^x)\ e^x\ \ln|\tan x| + \cos(e^x)\ \frac{\sec^2 x}{\tan x}$$

$$y' = y\ \left[-e^x\sin(e^x)\ln|\tan x| + \frac{\sec^2 x\ \cos(e^x)}{\tan x}\right]$$

$$y' = (\tan x)^{\cos(e^x)}\ \left[\frac{\sec^2 x\ \cos(e^x)}{\tan x} - e^x\sin(e^x)\ln|\tan x|\right]$$

$$y = \coth^{-1}\left(x^2\right) \quad (6)$$

الحل :

$$\frac{dy}{dx} = \frac{-1}{1-\left(x^2\right)^2}\ (2x) = \frac{-2x}{1-x^4}$$

السؤال الثاني (18 درجة) : أحسب التكاملات التالية

$$\int e^{3x} \ \tanh\left(e^{3x}\right) \ dx \quad (1)$$

الحل :

$$\int e^{3x} \ \tanh\left(e^{3x}\right) \ dx = \frac{1}{3}\int \tanh\left(e^{3x}\right) \ \left(3e^{3x}\right) \ dx = \frac{1}{3}\ln\left|\cosh\left(e^{3x}\right)\right| + c$$

باستخدام القانون

$$\int \tanh\left(f(x)\right) \ f'(x)\,dx = \ln\left|\cosh\left(f(x)\right)\right| + c$$

$$\int \sqrt{1-x^3}\,x^2\,dx \quad (2)$$

الحل :

$$\int \sqrt{1-x^3}\,x^2\,dx = -\frac{1}{3}\int \left(1-x^3\right)^{\frac{1}{2}} \ \left(-3x^2\right)dx$$

$$= -\frac{1}{3}\,\frac{\left(1-x^3\right)^{\frac{3}{2}}}{\frac{3}{2}} + c = -\frac{2}{9}\ \left(1-x^3\right)^{\frac{3}{2}} + c$$

باستخدام القانون $\int [f(x)]^n \ f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$ حيث $n \neq -1$.

$$\int_0^1 x\,4^{2-x^2}\,dx \quad (3)$$

الحل :

$$\int_0^1 x\,4^{2-x^2}\,dx = \frac{1}{-2}\int_0^1 4^{2-x^2}(-2x)\,dx = \frac{1}{-2}\left[\frac{4^{2-x^2}}{\ln 4}\right]_0^1$$

$$= \frac{1}{-2}\left[\frac{4^{2-(1)^2}}{\ln 4} - \frac{4^{2-(0)^2}}{\ln 4}\right] = \frac{1}{-2}\left[\frac{4^1}{\ln 4} - \frac{4^2}{\ln 4}\right] = \frac{1}{-2}\left(\frac{-12}{\ln 4}\right) = \frac{6}{\ln 4}$$

باستخدام القانون $\int a^{f(x)} \ f'(x)\,dx = \frac{a^{f(x)}}{\ln a} + c$.

$$\int \frac{x-1}{\sqrt{4-x^2}}\,dx \quad (4)$$

الحل :

$$\int \frac{x-1}{\sqrt{4-x^2}}\,dx = \int \left(\frac{x}{\sqrt{4-x^2}} - \frac{1}{\sqrt{4-x^2}}\right)\,dx$$

$$= \int \frac{x}{\sqrt{4-x^2}}\,dx - \int \frac{1}{\sqrt{4-x^2}}\,dx = -\frac{1}{2}\int \left(4-x^2\right)^{-\frac{1}{2}}\ (-2x)\,dx - \int \frac{1}{\sqrt{(2)^2-x^2}}\,dx$$

$$= -\frac{1}{2}\,\frac{\left(4-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} - \sin^{-1}\left(\frac{x}{2}\right) + c = -\left(4-x^2\right)^{\frac{1}{2}} - \sin^{-1}\left(\frac{x}{2}\right) + c$$

$$\int \frac{e^{2x}}{4+e^{4x}}\,dx \quad (5)$$

الحل :

$$\int \frac{e^{2x}}{4+e^{4x}}\,dx = \frac{1}{2}\int \frac{e^{2x}\ (2)}{(2)^2+\left(e^{2x}\right)^2}\,dx$$

$$= \frac{1}{2}\ \left(\frac{1}{2}\ \tan^{-1}\left(\frac{e^{2x}}{2}\right)\right) + c = \frac{1}{4}\ \tan^{-1}\left(\frac{e^{2x}}{2}\right) + c$$

باستخدام القانون

$$\int \frac{f'(x)}{a^2+\left[f(x)\right]^2}\,dx = \frac{1}{a}\ \tan^{-1}\left(\frac{f(x)}{a}\right) + c$$

$$\int \frac{3\,csch\,(\ln|x|)\coth(\ln|x|)}{x}\,dx \quad (6)$$

الحل :

$$\int \frac{3\,csch\,(\ln|x|)\coth(\ln|x|)}{x}\,dx = 3\int csch\,(\ln|x|)\coth(\ln|x|)\ \frac{1}{x}\,dx$$

$$= 3\,(-csch\,(\ln|x|)) + c = -3\,csch\,(\ln|x|) + c$$

باستخدام القانون

$$\int csch\,(f(x))\coth(f(x))\ f'(x)\,dx = -csch\,(f(x)) + c$$

$$\int \frac{2}{\sqrt{16-e^{2x}}}\,dx \quad (7)$$

الحل :

$$\int \frac{2}{\sqrt{16-e^{2x}}}\,dx = 2\int \frac{1}{\sqrt{(4)^2-\left(e^x\right)^2}}\,dx = 2\int \frac{e^x}{e^x\,\sqrt{(4)^2-\left(e^x\right)^2}}\,dx$$

$$= 2\left[-\frac{1}{4}sech^{-1}\left(\frac{e^x}{4}\right)\right] + c = -\frac{1}{2}\ sech^{-1}\left(\frac{e^x}{4}\right) + c$$

باستخدام القانون

$$.\int \frac{f'(x)}{f(x)\sqrt{a^2 - [f(x)]^2}}\,dx = -\frac{1}{a}\ sech^{-1}\left(\frac{f(x)}{a}\right) + c$$

$$\int (2x-1)\sqrt{x+2}\,dx \quad (8)$$

الحل : بوضع $x = u - 2 \implies u = x + 2$

$dx = du$ و $2x - 1 = 2(u-2) - 1 = 2u - 5$

$$\int (2x-1)\sqrt{x+2}\,dx = \int (2u-5)\sqrt{u}\,du = \int (2u-5)u^{\frac{1}{2}}\,du$$

$$= \int \left(2u^{\frac{3}{2}} - 5u^{\frac{1}{2}}\right)\,du = 2\,\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - 5\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{4}{5}\,(x+2)^{\frac{5}{2}} - \frac{10}{3}\,(x+2)^{\frac{3}{2}} + c$$

$$\int e^{-2x}\cosh 2x\,dx \quad (9)$$

الحل :

$$\int e^{-2x}\cosh 2x\,dx = \int e^{-2x}\left(\frac{e^{2x}+e^{-2x}}{2}\right)\,dx = \int \left(\frac{e^0+e^{-4x}}{2}\right)\,dx$$

$$= \int \left(\frac{1}{2} + \frac{e^{-4x}}{2}\right)\,dx = \frac{1}{2}\int 1\,dx + \frac{1}{2}\,\frac{1}{-4}\,\int e^{-4x}\,(-4)\,dx$$

$$= \frac{1}{2}\,x - \frac{1}{8}\,e^{-4x} + c = \frac{x}{2} - \frac{e^{-4x}}{8} + c$$

باستخدام القانون $\int e^{f(x)}\,f'(x)\,dx = e^{f(x)} + c$