MATH 111 - Integral Calculus
First Semester - 1447 H
Solution of the Final Exam
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Question (1): [$3+2+2=7$ marks]

1. Find the value of c that satisfies the mean value theorem of the definite integral for the function $f(x)=\dfrac{1}{\sqrt{x+1}}$ on the interval $[3,8]$.

   Solution: Using the formula $(b-a)\ f(c)=\displaystyle\int_a^b f(x)\ dx$.

   $$(8-3)\ \frac{1}{\sqrt{c+1}}=\int_3^8 \frac{1}{\sqrt{x+1}}\ dx=\int_3^8 (x+1)^{-\frac{1}{2}}\ dx$$

   $$\frac{5}{\sqrt{c+1}}=\left[\frac{(x+1)^{\frac{1}{2}}}{\frac{1}{2}}\right]_3^8=2\sqrt{8+1}-2\sqrt{3+1}=2(3)-2(2)=6-4=2$$

   $$\frac{5}{\sqrt{c+1}}=2 \implies \sqrt{c+1}=\frac{5}{2} \implies c+1=\frac{25}{4} \implies c=\frac{25}{4}-1=\frac{21}{4}$$

   The desired value is $c=\dfrac{21}{4}\in(3,8)$.

2. Find $F'(x)$, if $F(x)=\displaystyle\int_{\sinh x}^{2^{x+1}}(1+t^2)\ dt$.

   Solution:

   $$F'(x)=\frac{d}{dx}\int_{\sinh x}^{2^{x+1}}(1+t^2)\ dt$$

   $$=\left[1+\left(2^{x+1}\right)^2\right]\ \left(2^{x+1}\ln 2\right)-\left[1+(\sinh x)^2\right]\ (\cosh x)$$

   $$=2^{x+1}\ln 2\ \left(1+2^{2x+2}\right)\ -\cosh x\left(1+\sinh^2 x\right)\ .$$

   $$=2^{x+1}\ln 2\ \left(1+2^{2x+2}\right)\ -\cosh^3 x\ .$$

3. Find y' if $y(x)=(\cos x)^x+e^{-5x}$.

   Solution:

   $$y(x)=(\cos x)^x+e^{-5x}=e^{\ln|\cos x|^x}+e^{-5x}=e^{x\ln|\cos x|}+e^{-5x}$$

   $$y'(x)=e^{x\ln|\cos x|}\ \left[(1)\ln|\cos x|+x\left(\frac{-\sin x}{\cos x}\right)\right]+e^{-5x}\ (-5)\ .$$

   $$=(\cos x)^x\left[\ln|\cos x|-x\tan x\right]-5e^{-5x}$$

Question (2): [2 + 3 + 2 + 3 + 3 + 2 = 15 marks]
Evaluate the following integrals :

1. $\displaystyle\int \frac{1}{\sqrt{x^2-4x}}\, dx$.

   Solution: By completing the square.

   $x^2 - 4x = \left(x^2 - 4x + 4\right) - 4 = (x-2)^2 - (2)^2$

   $$\int \frac{1}{\sqrt{x^2-4x}}\, dx = \int \frac{1}{\sqrt{(x-2)^2-(2)^2}}\, dx = \cosh^{-1}\left(\frac{x-2}{2}\right) + c\ .$$

2. $\displaystyle\int x^2 \cos(3x)\, dx$.

   Solution: Using integration by parts .

   $u = x^2 \qquad dv = \cos(3x)\, dx$

   $du = 2x\, dx \qquad v = \frac{1}{3}\, \sin(3x)$

   $$\int x^2 \cos(3x)\, dx = x^2\left(\frac{1}{3}\sin(3x)\right) - \int \frac{1}{3}\sin(3x)\ 2x\ dx$$

   $$= \frac{x^2}{3}\sin(3x) - \frac{2}{3}\int x\sin(3x)\, dx$$

   Using integration by parts again .

   $u = x \qquad dv = \sin(3x)\, dx$

   $du = dx \qquad v = -\frac{1}{3}\, \cos(3x)$

   $$\int x^2 \cos(3x)\, dx = \frac{x^2}{3}\sin(3x) - \frac{2}{3}\left[x\ \left(-\frac{1}{3}\cos(3x)\right) - \int -\frac{1}{3}\cos(3x)\ dx\right]$$

   $$= \frac{x^2}{3}\sin(3x) + \frac{2x}{9}\cos(3x) - \frac{2}{9}\int \cos(3x)\, dx$$

   $$= \frac{x^2}{3}\sin(3x) + \frac{2x}{9}\cos(3x) - \frac{2}{27}\sin(3x) + c$$

3. $\displaystyle\int \frac{(\sqrt{x}-1)\sec\left(\sqrt{x}-\ln\sqrt{x}\right)}{x}\, dx$

   Solution:

   $$\int \frac{(\sqrt{x}-1)\sec\left(\sqrt{x}-\ln\sqrt{x}\right)}{x}\, dx = \int \sec\left(\sqrt{x} - \ln\left|x^{\frac{1}{2}}\right|\right)\ \left(\frac{\sqrt{x}-1}{x}\right)\ dx$$

   $$= \int \sec\left(\sqrt{x} - \frac{1}{2}\ln|x|\right)\ \left(\frac{\sqrt{x}}{x} - \frac{1}{x}\right)\ dx = \int \sec\left(\sqrt{x} - \frac{1}{2}\ln|x|\right)\ \left(\frac{1}{\sqrt{x}} - \frac{1}{x}\right)\ dx$$

   $$= 2\int \sec\left(\sqrt{x} - \frac{1}{2}\ln|x|\right)\ \left[\frac{1}{2}\left(\frac{1}{\sqrt{x}} - \frac{1}{x}\right)\right]\ dx$$

$$= 2\int \sec\left(\sqrt{x} - \frac{1}{2}\ln|x|\right)\ \left(\frac{1}{2\sqrt{x}} - \frac{1}{2x}\right)\ dx$$

$$= 2\ln\left|\sec\left(\sqrt{x} - \frac{1}{2}\ln|x|\right) + \tan\left(\sqrt{x} - \frac{1}{2}\ln|x|\right)\right| + c\ .$$

4. $\displaystyle\int \frac{5x^2+6x+4}{(x+1)(x^2+x+1)}\ dx\ .$

Solution : Using the method of partial fractions.

$$\frac{5x^2+6x+4}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$

$$\frac{5x^2+6x+4}{(x+1)(x^2+x+1)} = \frac{A\ (x^2+x+1)}{(x+1)\ (x^2+x+1)} + \frac{(Bx+C)\ (x+1)}{(x^2+x+1)\ (x+1)}$$

$$5x^2+6x+4 = Ax^2+Ax+A+Bx^2+Bx+Cx+C$$

$$5x^2+6x+4 = (A+B)x^2+(A+B+C)x+(A+C)$$

By comparing the coefficients of the two polynomials in each side :

$$\begin{array}{lll} A+B=5 & \longrightarrow & (1) \\ A+B+C=6 & \longrightarrow & (2) \\ A+C=4 & \longrightarrow & (3) \end{array}$$

Equation (2) - Equation (1) : $C = 1$.

From Equation (3) : $A+1=4 \implies A=3$.

From Equation (1) : $3+B=5 \implies B=2$.

$$\int \frac{5x^2+6x+4}{(x+1)(x^2+x+1)}\ dx = \int \left(\frac{3}{x+1} + \frac{2x+1}{x^2+x+1}\right)\ dx$$

$$= 3\int \frac{1}{x+1}\ dx + \int \frac{2x+1}{x^2+x+1}\ dx = 3\ln|x+1| + \ln\left|x^2+x+1\right| + c$$

5. $\displaystyle\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\ dx\ .$

Solution : Using trigonometric substitutions.

Put $x = 2\sin\theta \implies \sin\theta = \dfrac{x}{2}$.

$dx = 2\cos\theta\ d\theta$.

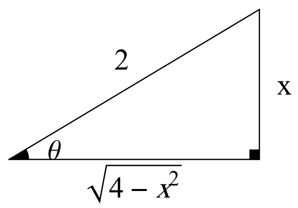
$$\left(4-x^2\right)^{\frac{3}{2}} = \left(4-4\sin^2\theta\right)^{\frac{3}{2}} = \left[4\left(1-\sin^2\theta\right)\right]^{\frac{3}{2}} = \left[2^2\ \cos^2\theta\right]^{\frac{3}{2}}$$

$$= \left(2^2\right)^{\frac{3}{2}}\ \left(\cos^2\theta\right)^{\frac{3}{2}} = 2^3\ \cos^3\theta$$

$$\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\ dx = \int \frac{4\sin^2\theta\ 2\cos\theta}{2^3\ \cos^3\theta}\ d\theta = \int \frac{\sin^2\theta}{\cos^2\theta}\ d\theta = \int \tan^2\theta\ d\theta$$

$$= \int \left(\sec^2\theta - 1\right)\ d\theta = \tan\theta - \theta + c$$

$\sin\theta = \dfrac{x}{2} \implies \theta = \sin^{-1}\left(\dfrac{x}{2}\right)$.



From the triangle :
$\tan\theta = \dfrac{x}{\sqrt{4-x^2}}$

$$\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\ dx = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + c\ .$$

6. $\int \tan^3 x\ \sec^2 x\ dx$.

   Solution : .

   $$\int \tan^3 x\ \sec^2 x\ dx = \int (\tan x)^3 \sec^2 x\ dx = \frac{(\tan x)^4}{4} + c = \frac{\tan^4 x}{4} + c\ .$$

Question (3): [$2+2+3+3+2+2+4 = 18$ marks]

1. Calculate $\lim\limits_{x\to\infty} x^{\frac{1}{x}}$.

   Solution:

   $$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln|x|^{\frac{1}{x}}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln|x|} = \lim_{x\to\infty} e^{\frac{\ln|x|}{x}}$$

   $$\lim_{x\to\infty} \frac{\ln|x|}{x} \qquad \left(\frac{\infty}{\infty}\right)$$

   Using L'Hôpital's rule.

   $$\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0\ .$$

   Therefore, $\lim\limits_{x\to\infty} \dfrac{\ln|x|}{x} = 0$.

   So, $\lim\limits_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1$.

2. Determine whether the improper integral $\displaystyle\int_0^\infty \frac{x}{(1+x^2)^2}\ dx$ converges or diverges.

   Solution:

   $$\int_0^\infty \frac{x}{(1+x^2)^2}\ dx = \lim_{t\to\infty} \int_0^t \frac{x}{(1+x^2)^2}\ dx = \lim_{t\to\infty} \left(\frac{1}{2}\int_0^t (1+x^2)^{-2}\ (2x)\ dx\right)$$

$$= \lim_{t\to\infty} \left(\frac{1}{2} \left[\frac{(1+x^2)^{-1}}{-1} \right]_0^t \right) = \lim_{t\to\infty} \left(\frac{1}{2} \left[\frac{-1}{1+x^2} \right]_0^t \right)$$

$$= \lim_{t\to\infty} \left(\frac{1}{2} \left[\frac{-1}{1+t^2} - \frac{-1}{1+(0)^2} \right] \right) = \frac{1}{2}[0-(-1)] = \frac{1}{2} \ .$$

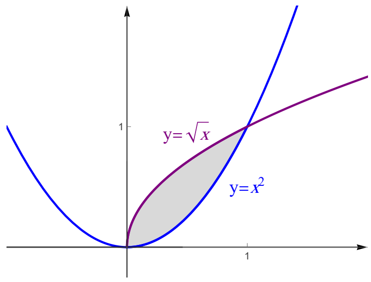
Hence, the improper integral converges.

3. Sketch the region bounded by the graphs of the curves $y = x^2$ and $y = \sqrt{x}$, then find its area.

Solution:

$y = x^2$ represents a parabola opens upwards, and its vertex is $(0, 0)$.

$y = \sqrt{x}$ represents the upper half of a parabola opens to to right and its vertex is $(0, 0)$.



Points of intersection of $y = x^2$ and $y = \sqrt{x}$:
$x^2 = \sqrt{x} \implies x^4 = x$
$\implies x^4 - x = 0$
$\implies x(x^3 - 1) = 0$
$\implies x = 0 \ , \ x = 1.$

$$\text{A} = \int_0^1 \left(\sqrt{x} - x^2\right) \, dx = \left[2\frac{x^{\frac{3}{2}}}{3} - \frac{x^3}{3} \right]_0^1$$

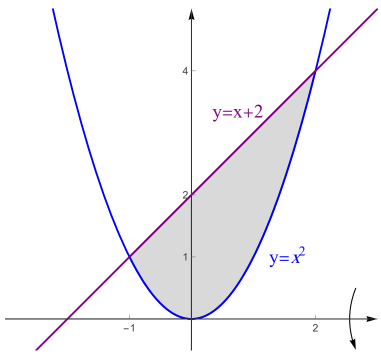
$$= \left(\frac{2}{3} - \frac{1}{3} \right) - (0-0) = \frac{1}{3} \ .$$

4. Sketch the region bounded by the graphs of the curves $y = x^2$ and $y = x + 2$, then find the volume of the solid generated by revolving this region about the $x-$axis.

Solution :

$y = x^2$ represents a parabola opens upwards, and its vertex is $(0, 0)$.

$y = x + 2$ represents a straight line passing through $(0, 2)$ with slope 1.



Points of intersection of $y = x^2$ and $y = x + 2$:
$x^2 = x + 2 \implies x^2 - x - 2 = 0$
$\implies (x + 1)(x - 2) = 0$
$\implies x = -1 \ , \ x = 2$

Using Washer Method :

$$V = \pi \int_{-1}^{2} \left[(x+2)^2 - \left(x^2\right)^2 \right] \ dx$$

$$= \pi \int_{-1}^{2} \left[x^2 + 4x + 4 - x^4 \right] \ dx = \pi \left[-\frac{x^5}{5} + \frac{x^3}{3} + 2x^2 + 4x \right]_{-1}^{2}$$

$$= \pi \left[\left(-\frac{2^5}{5} + \frac{2^3}{3} + 2(2^2) + 4(2) \right) - \left(-\frac{(-1)^5}{5} + \frac{(-1)^3}{3} + 2((-1)^2) + 4(-1) \right) \right]$$

$$= \pi \left[\left(-\frac{32}{5} + \frac{8}{3} + 8 + 8 \right) - \left(\frac{1}{5} - \frac{1}{3} + 2 - 4 \right) \right]$$

$$= \pi \left(-\frac{33}{5} + \frac{9}{3} + 18 \right) = \pi \left(-\frac{33}{5} + 21 \right) = \frac{72\pi}{5} \ .$$

5. Find the arc length of $y = \frac{2}{3} x \sqrt{x}$, from $x = 0$ to $x = 1$.

Solution :

$$y = \frac{2}{3} x \sqrt{x} = \frac{2}{3} \ x^{\frac{3}{2}} \ .$$

$$y' = \frac{2}{3} \left(\frac{3}{2} \ x^{\frac{1}{2}} \right) = x^{\frac{1}{2}} \ .$$

$$L = \int_0^1 \sqrt{1 + \left(x^{\frac{1}{2}} \right)^2} \ dx = \int_0^1 \sqrt{1+x} \ dx = \int_0^1 (1+x)^{\frac{1}{2}} \ dx$$

$$= \left[\frac{2}{3} \ (1+x)^{\frac{3}{2}} \right]_0^1 = \frac{2}{3} \ (1+1)^{\frac{3}{2}} - \frac{2}{3} \ (1+0)^{\frac{3}{2}} = \frac{2}{3} \left(\sqrt{8} - 1 \right) \ .$$

6. Convert the polar equation $r = \cos\theta + \sec\theta$ into a Cartesian equation.

Solution:

$$r = \cos\theta + \sec\theta \implies r^2 = r\cos\theta + r\sec\theta = r\cos\theta + \frac{r}{\cos\theta}$$

$$\implies x^2 + y^2 = x + \frac{r}{\frac{x}{r}} = x + \frac{r^2}{x} = x + \frac{x^2 + y^2}{x}$$

$$\implies x^3 + xy^2 = x^2 + x^2 + y^2 \implies x^3 + xy^2 - 2x^2 - y^2 = 0$$

7. Sketch the common region between the curves $r = 2$ and $r = 2 + 2\cos\theta$, then find its area.

Solution:

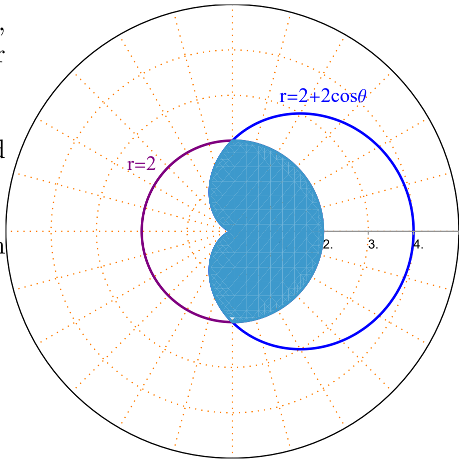
$r = 2 + 2\cos\theta$ represents a cardioid, symmetric with respect to the polar axis.

$r = 2$ represents a circle centered at the pole, and its radius is 2.

Points of intersection between $r = 2 + 2\cos\theta$ and $r = 2$:

$2 + 2\cos\theta = 2 \implies 2\cos\theta = 0$

$\implies \cos\theta = 0 \implies \theta = \dfrac{\pi}{2}\ ,\ \theta = \dfrac{3\pi}{2}$



Note that the shaded region is symmetric with respect to the polar axis.

$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} (2)^2\ d\theta + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (2 + 2\cos\theta)^2\ d\theta\right)$$

$$= \int_0^{\frac{\pi}{2}} 4\ d\theta + \int_{\frac{\pi}{2}}^{\pi} \left(4 + 8\cos\theta + 4\cos^2\theta\right)\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4\ d\theta + \int_{\frac{\pi}{2}}^{\pi} \left(4 + 8\cos\theta + 4\left(\frac{1 + \cos 2\theta}{2}\right)\right)\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4\ d\theta + \int_{\frac{\pi}{2}}^{\pi} (4 + 8\cos\theta + (2 + 2\cos 2\theta))\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4\ d\theta + \int_{\frac{\pi}{2}}^{\pi} (6 + 8\cos\theta + 2\cos 2\theta)\ d\theta$$

$$= [4\theta]_0^{\frac{\pi}{2}} + [6\theta + 8\sin\theta + \sin 2\theta]_{\frac{\pi}{2}}^{\pi}$$

$$= \left[4\left(\frac{\pi}{2}\right) - 4(0)\right] + \left[(6\pi + 8\sin(\pi) + \sin(2\pi)) - \left(6\left(\frac{\pi}{2}\right) + 8\sin\left(\frac{\pi}{2}\right) + \sin(\pi)\right)\right]$$

$$= [2\pi - 0 + [(6\pi + 0 + 0) - (3\pi + 8 + 0)] = 2\pi + (3\pi - 8) = 5\pi - 8\ .$$