**MATH 111 - Integral Calculus**
**First Semester - 1445 H**
**Solution of the First Exam**
*Dr Tariq A. Alfadhel*

**Question (1):** [9 marks]

1. Use Riemann Sum to evaluate the definite integral $\int_0^4 (x^2+1)\, dx$ . [3]

   **Solution :** $[a,b]=[0,4]$ $f(x)=x^2+1$ .

   $\Delta_x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}$

   $x_k = a + k\ \Delta_x = 0 + k\left(\frac{4}{n}\right) = \frac{4k}{n}$

   $f(x_k) = \left(\frac{4k}{n}\right)^2 + 1 = \frac{16k^2}{n^2} + 1$

   $R_n = \sum_{k=1}^{n} f(x_k)\ \Delta_x = \sum_{k=1}^{n}\left(\frac{16k^2}{n^2}+1\right)\ \left(\frac{4}{n}\right)$

   $= \sum_{k=1}^{n}\left(\frac{64k^2}{n^3}+\frac{4}{n}\right) = \sum_{k=1}^{n}\frac{64k^2}{n^3} + \sum_{k=1}^{n}\frac{4}{n}$

   $= \frac{64}{n^3}\sum_{k=1}^{n}k^2 + \frac{4}{n}\sum_{k=1}^{n}1 = \frac{64}{n^3}\ \frac{n(n+1)(2n+1)}{6} + \frac{4}{n}\ (n)$

   $= \frac{32}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right) + 4$

   $\int_0^4 (x^2+1)\ dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{32}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)+4\right]$

   $= \frac{32}{3}\ (2) + 4 = \frac{64}{3} + 4 = \frac{76}{3}$ .

2. Find $F'(x)$, if $F(x) = \int_{\sin(x^2)}^{\pi^{2x}} \sqrt{2t^3+2}\ dt$ . [2]

   **Solution :**

   $F'(x) = \frac{d}{dx}\int_{\sin(x^2)}^{\pi^{2x}} \sqrt{2t^3+2}\ dt$

   $= \sqrt{2\left(\pi^{2x}\right)^3+2}\ \left(\pi^{2x}\ (2)\ln\pi\right) - \sqrt{2\left(\sin(x^2)\right)^3+2}\ \left(\cos(x^2)\ (2x)\right)$

   $= 2\pi^{2x}\ln\pi\sqrt{2\pi^{6x}+2} - 2x\ \cos(x^2)\sqrt{2\sin^3(x^2)+2}$ .

Find $\frac{dy}{dx}$ of the following :

3. $y = \tan^{-1}(3x)\ \log_5|1 - sec(3x)|$ . [2]

**Solution :**

$$\frac{dy}{dx} = \left(\frac{1}{1+(3x)^2}\ (3)\right)\ \log_5|1 - sec(3x)| + \tan^{-1}(3x)\ \left(\frac{-\sec(3x)\tan(3x)\ (3)}{1-\sec(3x)}\frac{1}{\ln 5}\right)$$

$$= \frac{3\log_5|1-\sec(3x)|}{1+9x^2} - \frac{3\sec(3x)\tan(3x)\ \tan^{-1}(3x)}{\ln 5\,(1-\sec(3x))} .$$

4. $y = (\cot x)^{\sin x} + 4^x$ . [2]

**Solution :**

Let $y = f(x) + g(x)$, where $f(x) = (\cot x)^{\sin x}$ and $g(x) = 4^x$ .

Then $\frac{dy}{dx} = y' = f'(x) + g'(x)$

First - $g'(x) = 4^x\ (1)\ln 4 = 4^x \ln 4$

Second - Finding $f'(x)$

$$f(x) = (\cot x)^{\sin x} \implies \ln|f(x)| = \ln\left|(\cot x)^{\sin x}\right| = \sin x\ \ln|\cot x|$$

Differentiate both sides.

$$\frac{f'(x)}{f(x)} = \cos x\ \ln|\cot x| + \sin x\left(\frac{-\csc^2 x}{\cot x}\right)$$

$$f'(x) = f(x)\left[\cos x\ \ln|\cot x| - \frac{\sin x\csc^2 x}{\cot x}\right] = (\cot x)^{\sin x}\ \ [\cos x\ \ln|\cot x| - \sec x]$$

Therefore, $\frac{dy}{dx} = (\cot x)^{\sin x}\ \ [\cos x\ \ln|\cot x| - \sec x] + 4^x \ln 4$

**Question (2):** [16 marks]
Evaluate the following integrals :

1. $\int \left(\sqrt{x}\ e^{x^2}\right)^2\ dx$ . [2]

**Solution :**

$$\int \left(\sqrt{x}\ e^{x^2}\right)^2\ dx = \int \left(\sqrt{x}\right)^2\left(e^{x^2}\right)^2\ dx = \int x\ e^{2x^2}\ dx$$

$$= \frac{1}{4}\int e^{2x^2}(4x)\ dx = \frac{1}{4}e^{2x^2} + c$$

Using The formula :

$$\int e^{f(x)}\ f'(x)\ dx = e^{f(x)} + c .$$

2. $\int x\ \sqrt{x+1}\ dx$ . [2]

**Solution :** Put $u = x+1 \implies x = u-1$, hence $dx = du$ .

$$\int x\ \sqrt{x+1}\ dx = \int (u-1)\sqrt{u}\ du = \int (u-1)u^{\frac{1}{2}}\ du$$

$$= \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)\ du = \frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{5}\ (x+1)^{\frac{5}{2}} - \frac{2}{3}\ (x+1)^{\frac{3}{2}} + c\ .$$

3. $\int_0^1 x\ 5^{2-x^2}\ dx$ . [2]

**Solution :**

$$\int_0^1 x\ 5^{2-x^2}\ dx = \frac{1}{-2}\int_0^1 5^{2-x^2}(-2x)\ dx = \frac{1}{-2}\left[\frac{5^{2-x^2}}{\ln 5}\right]_0^1$$

$$= \frac{1}{-2}\left[\frac{5^{2-(1)^2}}{\ln 5} - \frac{5^{2-(0)^2}}{\ln 5}\right] = \frac{1}{-2}\left[\frac{5^1}{\ln 5} - \frac{5^2}{\ln 5}\right] = \frac{1}{-2}\left(\frac{-20}{\ln 5}\right) = \frac{10}{\ln 5}$$

Using The formula : $\int a^{f(x)}\ f'(x)\ dx = \frac{a^{f(x)}}{\ln a} + c$ .

4. $\int \frac{2-x}{\sqrt{1-x^2}}\ dx$ . [2]

**Solution :**

$$\int \frac{2-x}{\sqrt{1-x^2}}\ dx = \int \left(\frac{2}{\sqrt{1-x^2}} + \frac{-x}{\sqrt{1-x^2}}\right)\ dx$$

$$= \int \frac{2}{\sqrt{1-x^2}}\ dx + \int \left(1-x^2\right)^{-\frac{1}{2}}(-x)\ dx$$

$$= 2\int \frac{1}{\sqrt{1-x^2}}\ dx + \frac{1}{2}\int \left(1-x^2\right)^{-\frac{1}{2}}(-2x)\ dx$$

$$= 2\sin^{-1} x + \frac{1}{2}\ \frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sin^{-1} x + \sqrt{1-x^2} + c$$

5. $\int \frac{\tan\left(\ln(x^2)\right)}{x}\ dx$ . [2]

**Solution :**

$$\int \frac{\tan\left(\ln(x^2)\right)}{x}\ dx = \int \tan\left(2\ln|x|\right)\ \frac{1}{x}\ dx$$

$$= \frac{1}{2} \int \tan\left(2\ln|x|\right) \ \frac{2}{x} \ dx = \frac{1}{2} \ln\left|\sec\left(2\ln|x|\right)\right| + c$$

Using The formula :

$$\int \tan\left(f(x)\right) \ f'(x) \ dx = \ln\left|\sec\left(f(x)\right)\right| + c \ .$$

6. $\displaystyle\int \frac{\sec\left(\sqrt{x}\right)\tan\left(\sqrt{x}\right)}{\sqrt{x}} \ dx$ . [2]

**Solution :**

$$\int \frac{\sec\left(\sqrt{x}\right)\tan\left(\sqrt{x}\right)}{\sqrt{x}} \ dx = \int \sec\left(\sqrt{x}\right)\tan\left(\sqrt{x}\right) \ \frac{1}{\sqrt{x}} \ dx$$

$$= 2\int \sec\left(\sqrt{x}\right)\tan\left(\sqrt{x}\right) \ \frac{1}{2\sqrt{x}} \ dx = 2\sec\left(\sqrt{x}\right) + c$$

Using The formula :

$$\int \sec\left(f(x)\right)\tan\left(f(x)\right) \ f'(x) \ dx = \sec\left(f(x)\right) + c \ .$$

7. $\displaystyle\int \frac{\left(\tan^{-1}x\right)^2}{x^2+1} \ dx$ . [2]

**Solution :**

$$\int \frac{\left(\tan^{-1}x\right)^2}{x^2+1} \ dx = \int \left(\tan^{-1}x\right)^2 \ \frac{1}{x^2+1} \ dx = \frac{\left(\tan^{-1}x\right)^3}{3} + c$$

Using The formula :

$$\int \left[f(x)\right]^n \ f'(x) \ dx = \frac{\left[f(x)\right]^{n+1}}{n+1} + c \quad n \neq -1 \ .$$

8. $\displaystyle\int \frac{\sin(2x) \ \cos(2x)}{\sin^2(2x)} \ dx$ . [2]

**Solution :**

$$\int \frac{\sin(2x) \ \cos(2x)}{\sin^2(2x)} \ dx = \int \frac{\sin(2x) \ \cos(2x)}{\sin(2x) \ \sin(2x)} \ dx = \int \frac{\cos(2x)}{\sin(2x)} \ dx$$

$$= \frac{1}{2} \int \frac{\cos(2x) \ (2)}{\sin(2x)} \ dx = \ln\left|\sin(2x)\right| + c$$

Using The formula :

$$\int \frac{f'(x)}{f(x)} \ dx = \ln\left|f(x)\right| + c \ .$$

**MATH 111 - Integral Calculus**
**First Semester - 1445 H**
**Solution of the Second Exam**
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**Question (1):** [4 marks]

Find $\dfrac{dy}{dx}$ of the following :

1. $y = \cosh(3x^2) + sech^{-1}(2x)$ . [2]

   **Solution :**

   $$\frac{dy}{dx} = \sinh(3x^2)\ (6x) + \frac{-1}{2x\ \sqrt{1-(2x)^2}}\ (2)$$

   $$= 6x\sinh(3x^2) - \frac{1}{x\ \sqrt{1-4x^2}}\ .$$

2. $y = \coth^{-1}(3x) + \tanh^2\sqrt{2x}$ . [2]

   **Solution :**

   $$\frac{dy}{dx} = \frac{-1}{1-(3x)^2}(3) + 2\left(\tanh\sqrt{2x}\right)^1\ sech^2\sqrt{2x}\ \frac{1}{2\sqrt{2x}}(2)$$

   $$= \frac{-3}{1-9x^2} + \frac{2\tanh\sqrt{2x}\ sech^2\sqrt{2x}}{\sqrt{2x}}\ .$$

**Question (2):** [21 marks]

Evaluate the following integrals :

1. $\displaystyle\int e^{-x}\cosh x\ dx$ . [2]

   **Solution :**

   $$\int e^{-x}\cosh x\ dx = \int e^{-x}\left(\frac{e^x+e^{-x}}{2}\right)\ dx = \int\left(\frac{e^0+e^{-2x}}{2}\right)\ dx$$

   $$= \int\frac{1}{2}\ dx + \int\frac{e^{-2x}}{2}\ dx = \frac{1}{2}\int 1\ dx + \frac{1}{2}\ \frac{1}{-2}\int e^{-2x}\ (-2)\ dx$$

   $$= \frac{1}{2}\ x - \frac{1}{4}\ e^{-2x} + c = \frac{x}{2} - \frac{e^{-2x}}{4} + c\ .$$

   Using The formula :

   $$\int e^{f(x)}\ f'(x)\ dx = e^{f(x)} + c\ .$$

2. $\displaystyle\int \frac{dx}{\sqrt{x}\,\sqrt{1+x}}$ . [2]

**Solution (1) :**

$$\int \frac{dx}{\sqrt{x}\,\sqrt{1+x}} = \int \frac{1}{\sqrt{x}\,\sqrt{(1)^2+(\sqrt{x})^2}}\,dx$$

$$= 2\int \frac{\frac{1}{2\sqrt{x}}}{\sqrt{(1)^2+(\sqrt{x})^2}}\,dx = 2\sinh^{-1}\left(\sqrt{x}\right) + c\ .$$

Using The formula :

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2+a^2}}\,dx = \sinh^{-1}\left(\frac{f(x)}{a}\right) + c \quad a>0\ .$$

**Solution (2) :** Using the substitution $u = \sqrt{x+1} \implies u^2 = x + 1 \implies u^2 - 1 = x$ .

Hence $2u\,du = dx$ and $\sqrt{x} = \sqrt{u^2-1}$ .

$$\int \frac{dx}{\sqrt{x}\,\sqrt{1+x}} = \int \frac{2u}{\sqrt{u^2-1}\,u}\,du = 2\int \frac{1}{\sqrt{u^2-1}}\,du$$

$$= 2\cosh^{-1}(u) + c = 2\cosh^{-1}\left(\sqrt{x+1}\right) + c\ .$$

Using The formula :

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2-a^2}}\,dx = \cosh^{-1}\left(\frac{f(x)}{a}\right) + c \quad |f(x)|>a \quad a>0\ .$$

3. $\displaystyle\int_1^e x^3 \ln x\,dx$ . [2]

**Solution :** Using integration by parts.

$$u = \ln x \qquad dv = x^3\,dx$$
$$du = \frac{1}{x}\,dx \qquad v = \frac{x^4}{4}$$

$$\int_1^e x^3\ln x\,dx = \left[\frac{x^4}{4}\,\ln x\right]_1^e - \int_1^e \frac{x^4}{4}\,\frac{1}{x}\,dx = \left[\frac{x^4}{4}\,\ln x\right]_1^e - \frac{1}{4}\int_1^e x^3\,dx$$

$$= \left[\frac{x^4}{4}\,\ln x\right]_1^e - \frac{1}{4}\left[\frac{x^4}{4}\right]_1^e = \left[\frac{e^4}{4}\ln(e) - \frac{1}{4}\ln(1)\right] - \frac{1}{4}\left[\frac{e^4}{4} - \frac{1}{4}\right]$$

$$= \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16} = \frac{3e^4}{16} + \frac{1}{16}\ .$$

4. $\int \sin^5 x \ \cos^3 x \ dx$ . [2]

**Solution :**

Using the substitution $u = \sin x$ .

Hence $du = \cos x \ dx$ .

$$\int \sin^5 x \ \cos^3 x \ dx = \int \sin^5 x \ \cos^2 x \ \cos x \ dx \int \sin^5 x \ \left(1 - \sin^2 x\right) \ \cos x \ dx$$

$$= \int u^5 \ \left(1 - u^2\right) \ du = \int \left(u^5 - u^7\right) \ du$$

$$= \frac{u^6}{6} - \frac{u^8}{8} + c = \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + c \ .$$

5. $\int \cosh^{-1} x \ dx$ . [2]

**Solution :** Using integration by parts.

$$u = \cosh^{-1} x \qquad dv = 1 \ dx$$

$$du = \frac{1}{\sqrt{x^2 - 1}} \ dx \qquad v = x$$

$$\int \cosh^{-1} x \ dx = x \cosh^{-1} x - \int \frac{1}{\sqrt{x^2 - 1}} \ x \ dx$$

$$= x \cosh^{-1} x - \frac{1}{2} \int \left(x^2 - 1\right)^{-\frac{1}{2}} (2x) \ dx$$

$$= x \cosh^{-1} x - \frac{1}{2} \frac{\left(x^2 - 1\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = x \cosh^{-1} x - \sqrt{x^2 - 1} + c \ .$$

6. $\int \frac{dx}{\left(4 + x^2\right)^{\frac{3}{2}}}$ . [3]

**Solution :** Using trigonometric substitutions.

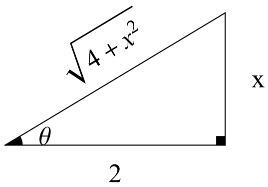
Put $x = 2 \tan \theta \implies \tan \theta = \frac{x}{2}$ .

$dx = 2 \sec^2 \theta \ d\theta$ .

$$\int \frac{dx}{\left(4 + x^2\right)^{\frac{3}{2}}} = \int \frac{2 \sec^2 \theta}{\left(4 + 4 \tan^2 \theta\right)^{\frac{3}{2}}} \ d\theta = \int \frac{2 \sec^2 \theta}{\left(4 \left(1 + \tan^2 \theta\right)\right)^{\frac{3}{2}}} \ d\theta = \int \frac{2 \sec^2 \theta}{\left(4 \sec^2 \theta\right)^{\frac{3}{2}}} \ d\theta$$

$$= \int \frac{2 \sec^2 \theta}{(4)^{\frac{3}{2}} \sec^3 \theta} \ d\theta = \frac{2}{8} \int \frac{1}{\sec \theta} \ d\theta = \frac{1}{4} \int \cos \theta \ d\theta = \frac{1}{4} \sin \theta + c$$

$\tan\theta = \dfrac{x}{2}$ .




From the triangle :
$\sin\theta = \dfrac{x}{\sqrt{4+x^2}}$

$$\int \frac{dx}{(4+x^2)^{\frac{3}{2}}} = \frac{1}{4}\,\frac{x}{\sqrt{4+x^2}} + c \ .$$

7. $\displaystyle\int \frac{\sin x}{4-\cos^2 x}\,dx$ . [3]

**Solution :**

$$\int \frac{\sin x}{4-\cos^2 x}\,dx = -\int \frac{-\sin x}{(2)^2-(\cos x)^2}\,dx = -\frac{1}{2}\tanh^{-1}\left(\frac{\cos x}{2}\right) + c$$

Using The formula :

$$\int \frac{f'(x)}{a^2-[f(x)]^2}\,dx = \frac{1}{a}\tanh^{-1}\left(\frac{f(x)}{a}\right) + c, \text{ where } |f(x)| < a \text{ and } a > 0 \ .$$

8. $\displaystyle\int \frac{x+3}{x^3+9x}\,dx$ . [3]

**Solution :** Using the method of partial fractions.

$$\frac{x+3}{x^3+9x} = \frac{x+3}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$

$$\frac{x+3}{x(x^2+9)} = \frac{A(x^2+9)}{x(x^2+9)} + \frac{x\,(Bx+C)}{x(x^2+9)}$$

$$x+3 = A(x^2+9) + x(Bx+C)$$

$$x+3 = Ax^2 + 9A + Bx^2 + Cx = (A+B)x^2 + Cx + 9A$$

By comparing the coefficients of the two polynomials in each side :

$$\begin{array}{lll} A+B=0 & \longrightarrow & (1) \\ C=1 & \longrightarrow & (2) \\ 9A=3 & \longrightarrow & (3) \end{array}$$

From equation (3) : $9A = 3 \implies A = \dfrac{1}{3}$ .

From equation (1)$\dfrac{1}{3} + B = 0 \implies B = -\dfrac{1}{3}$ .

$$\int \frac{x+3}{x^3+9x}\,dx = \int \left(\frac{\frac{1}{3}}{x} + \frac{-\frac{1}{3}x+1}{x^2+9}\right)\,dx$$

$$= \int \frac{\frac{1}{3}}{x}\,dx + \int \frac{-\frac{1}{3}x}{x^2+9}\,dx + \int \frac{1}{x^2+9}\,dx$$

$$= \frac{1}{3}\int \frac{1}{x}\ dx - \frac{1}{3}\ \frac{1}{2}\int \frac{2x}{x^2+9}\ dx + \int \frac{1}{x^2+(3)^2}\ dx$$

$$= \frac{1}{3}\ln|x| - \frac{1}{6}\ln(x^2+9) + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c\ .$$

9. $\displaystyle\int \frac{dx}{\sqrt[4]{x}+\sqrt{x}}$ . [2]

**Solution :** $\displaystyle\int \frac{dx}{\sqrt[4]{x}+\sqrt{x}} = \int \frac{1}{x^{\frac{1}{4}}+x^{\frac{1}{2}}}\ dx$

Using the substitution $x = u^4$, then $u = x^{\frac{1}{4}}$ .

$dx = 4u^3\ du$ .

$$\int \frac{dx}{x^{\frac{1}{4}}+x^{\frac{1}{2}}} = \int \frac{4u^3}{(u^4)^{\frac{1}{4}}+(u^4)^{\frac{1}{2}}}\ du = \int \frac{4u^3}{u+u^2}\ du$$

$$= \int \frac{4u^3}{u(1+u)}\ du = 4\int \frac{u^2}{u+1}\ du$$

Using long division of polynomials :

$$4\int \frac{u^2}{u+1}\ du = 4\int \left(u - 1 + \frac{1}{u+1}\right)\ du$$

$$= 4\left(\frac{u^2}{2} - u + \ln|u+1|\right) + c = 2u^2 - 4u + 4\ln|u-1| + c$$

$$\int \frac{dx}{x^{\frac{1}{4}}+x^{\frac{1}{2}}} = 2\left(x^{\frac{1}{4}}\right)^2 - 4x^{\frac{1}{4}} + 4\ln\left|x^{\frac{1}{4}}+1\right| + c$$

$$= 2x^{\frac{1}{2}} - 4x^{\frac{1}{4}} + 4\ln\left|x^{\frac{1}{4}}+1\right| + c\ .$$

**MATH 111 - Integral Calculus**
**First Semester - 1445 H**
**Solution of the Final Exam**
*Dr Tariq A. Alfadhel*

**First Part** [7 marks] .

1. Find the value of $c$ that satisfies the mean value theorem of integration for the function $f(x) = (x-1)^2$ on the interval $[1,4]$ . [3]

   **Solution :** using the formula $(b-a)f(c) = \int_a^b f(x)\ dx$ ,

   where $f(x) = (x-1)^2$ and $[a,b] = [1,4]$ .

   $$(4-1)\ (c-1)^2 = \int_1^4 (x-1)^2\ dx = \left[\frac{(x-1)^3}{3}\right]_1^4$$

   $$3(c-1)^2 = \frac{(4-1)^3}{3} - \frac{(1-1)^3}{3} = \frac{3^3}{3} - 0 = 9$$

   $$(c-1)^2 = 3 \implies c-1 = \pm\sqrt{3} \implies c = 1 \pm \sqrt{3}\ .$$

   Note that $c = 1+\sqrt{3} \in (1,4)$, while $c = 1-\sqrt{3} \notin (1,4)$

   The desired value of $c$ is $1+\sqrt{3}$ .

2. Find $F'(x)$ , if $F(x) = \int_{\ln|x|}^{e^x} \sqrt{t^2+5}\ dt$ . [2]

   **Solution :** $F'(x) = \frac{d}{dx}\int_{\ln|x|}^{e^x} \sqrt{t^2+5}\ dt$

   $$F'(x) = \sqrt{(e^x)^2+5}\ (e^x) - \sqrt{(\ln|x|)^2+5}\ \left(\frac{1}{x}\right)$$

   $$= e^x\ \sqrt{e^{2x}+5} - \frac{\sqrt{\ln^2|x|+5}}{x}\ .$$

3. Find $f'(x)$ , if $f(x) = \sinh^{-1}(3^x) + \ln|\tanh(4x)|$ . [2]

   **Solution :**

   $$f'(x) = \frac{1}{\sqrt{1+(3^x)^2}}\ (3^x \ln 3) + \frac{sech^2(4x)\ (4)}{\tanh(4x)}$$

   $$= \frac{3^x \ln 3}{\sqrt{1+3^{2x}}} + \frac{4\ sech^2(4x)}{\tanh(4x)}.$$

**Second Part** [14 marks] .
Evaluate the following integrals :

1. $\displaystyle\int \frac{2}{\sqrt{-x^2-6x}}\,dx$ . [3]

   **Solution :** Using completing the square method.

   $$\int \frac{2}{\sqrt{-x^2-6x}}\,dx = \int \frac{2}{\sqrt{-(x^2+6x+9)+9}}\,dx$$

   $$= 2\int \frac{1}{\sqrt{9-(x^2+6x+9)}}\,dx = 2\int \frac{1}{\sqrt{(3)^2-(x+3)^2}}\,dx$$

   $$= \ 2\ \sin^{-1}\left(\frac{x+3}{3}\right) + c\ .$$

   Using the formula :

   $$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\left(\frac{f(x)}{a}\right) + c\ ,\ \text{where } a>0\ .$$

2. $\displaystyle\int x^2\ \cosh x\ dx$ . [3]

   **Solution :** using integration by parts.

   $u = x^2 \qquad dv = \cosh x\ dx$
   $du = 2x\ dx \qquad v = \sinh x$

   $$\int x^2\ \cosh x\ dx = x^2\ \sinh x - \int 2x\ \sinh x\ dx$$

   using integration by parts again .

   $u = 2x \qquad dv = \sinh x\ dx$
   $du = 2\ dx \qquad v = \cosh x$

   $$\int x^2\ \cosh x\ dx = x^2\ \sinh x - \left[2x\ \cosh x - \int 2\cosh x\ dx\right]$$

   $$= x^2\ \sinh x - 2x\ \cosh x + 2\int \cosh x\ dx$$

   $$= x^2\ \sinh x - 2x\ \cosh x + 2\sinh x + c\ .$$

3. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$ . [3]

   **Solution :** using trigonometric substitutions.

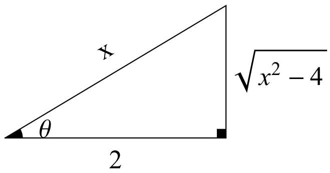
   Put $x = 2\sec\theta \implies \sec\theta = \dfrac{x}{2} \implies \cos\theta = \dfrac{2}{x}$ .

   $dx = 2\sec\theta\ \tan\theta\ d\theta$ .

$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = \sqrt{4\,(\sec^2\theta - 1)} = \sqrt{4\tan^2\theta} = 2\tan\theta$ .

$$\int \frac{1}{x^2\sqrt{x^2-4}}\,dx = \int \frac{2\sec\theta\ \tan\theta}{(2\sec\theta)^2\ 2\tan\theta}\,d\theta = \int \frac{\sec\theta}{4\sec^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{4}\int \cos\theta\ d\theta = \frac{1}{4}\ \sin\theta + c\ .$$




$\cos\theta = \dfrac{2}{x}$ .

From the triangle :

$\sin\theta = \dfrac{\sqrt{x^2-4}}{x}$ .

$$\int \frac{1}{x^2\sqrt{x^2-4}}\,dx = \frac{1}{4}\ \frac{\sqrt{x^2-4}}{x} + c$$

4. $\displaystyle\int \frac{4x^2 - x + 12}{x^3 + 4x}\,dx$ . [3]

**Solution :** using the method of partial fractions.

$$\frac{4x^2-x+12}{x^3+4x} = \frac{4x^2-x+12}{x\ (x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\frac{4x^2-x+12}{x\ (x^2+4)} = \frac{A\ (x^2+4)}{x\ (x^2+4)} + \frac{(Bx+C)\ x}{(x^2+4)\ x}$$

$$4x^2 - x + 12 = A(x^2+4) + (Bx+C)x$$

$$4x^2 - x + 12 = Ax^2 + 4A + Bx^2 + Cx$$

$$4x^2 - x + 12 = (A+B)x^2 + Cx + 4A$$

By comparing the coefficients of the two polynomials in each side :

$A + B = 4 \quad \longrightarrow \quad (1)$
$C = -1 \quad \longrightarrow \quad (2)$
$4A = 12 \quad \longrightarrow \quad (3)$

From equation (2) : $C = -1$ .

From equation (3) : $A = 3$ .

From equation (1) : : $3 + B = 4 \implies B = 1$ .

$$\int \frac{4x^2-x+12}{x^3+4x}\,dx = \int \left(\frac{3}{x} + \frac{x-1}{x^2+4}\right)\,dx$$

$$= 3\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{2x}{x^2+4}\,dx - \int \frac{1}{(x)^2+(2)^2}\,dx$$

$$= 3\ln|x| + \frac{1}{2}\ \ln(x^2+4) - \frac{1}{2}\ \tan^{-1}\left(\frac{x}{2}\right) + c\ .$$

5. $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\, dx$ . [2]

**Solution :** $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = \int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}\, dx$

Using the substitution $x = u^6$ , hence $u = x^{\frac{1}{6}}$ .

$dx = 6u^5\ du$

$$\int \frac{dx}{x^{\frac{1}{2}}+x^{\frac{1}{3}}} = \int \frac{6u^5}{(u^6)^{\frac{1}{2}}+(u^6)^{\frac{1}{3}}}\, du = \int \frac{6u^5}{u^3+u^2}\, du$$

$$= \int \frac{6u^5}{u^2(1+u)}\, du = 6\int \frac{u^3}{u+1}\, du$$

Using long division of polynomials.

$$6\int \frac{u^3}{u+1}\, du = 6\int \left(u^2 - u + 1 + \frac{-1}{u+1}\right)\ du$$

$$= 6\left(\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1|\right) + c = 2u^3 - 3u^2 + 6u - 6\ln|u-1| + c$$

$$\int \frac{dx}{x^{\frac{1}{2}}+x^{\frac{1}{3}}} = 2\left(x^{\frac{1}{6}}\right)^3 - 3\left(x^{\frac{1}{6}}\right)^2 + 6x^{\frac{1}{6}} - 6\ln|x^{\frac{1}{6}}-1| + c$$

$$= 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\ln|x^{\frac{1}{6}}-1| + c\ .$$

**Third Part** [19 marks] .

1. Calculate $\displaystyle\lim_{x\to\infty} \frac{e^x+5x}{e^{2x}+2x+1}$ . [2]

**Solution :** $\displaystyle\lim_{x\to\infty} \frac{e^x+5x}{e^{2x}+2x+1} \qquad \left(\frac{\infty}{\infty}\right)$

Using L'Hôpital's rule .

$$\lim_{x\to\infty} \frac{e^x+5}{e^{2x}\ (2)+2} = \lim_{x\to\infty} \frac{e^x+5}{2e^{2x}+2} \qquad \left(\frac{\infty}{\infty}\right)$$

Using L'Hôpital's rule again .

$$\lim_{x\to\infty} \frac{e^x}{2e^{2x}\ (2)} = \lim_{x\to\infty} \frac{e^x}{4e^{2x}} = \lim_{x\to\infty} \frac{1}{4e^x} = 0$$

Therefore, $\displaystyle\lim_{x\to\infty} \frac{e^x+5x}{e^{2x}+2x+1} = 0$ .

2. Determine whether the improper integral $\displaystyle\int_1^\infty \frac{1}{(2x-1)^3}\, dx$ converges or diverges . [2]

**Solution :**

$$\int_1^\infty \frac{1}{(2x-1)^3}\ dx = \lim_{t\to\infty}\int_1^t \frac{1}{(2x-1)^3}\ dx = \lim_{t\to\infty}\left(\frac{1}{2}\ \int_1^t (2x-1)^{-3}(2)\ dx\right)$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\left[\frac{(2x-1)^{-2}}{-2}\right]_1^t\right) = \lim_{t\to\infty}\left(\frac{1}{2}\left[\frac{1}{-2(2x-1)^2}\right]_1^t\right)$$

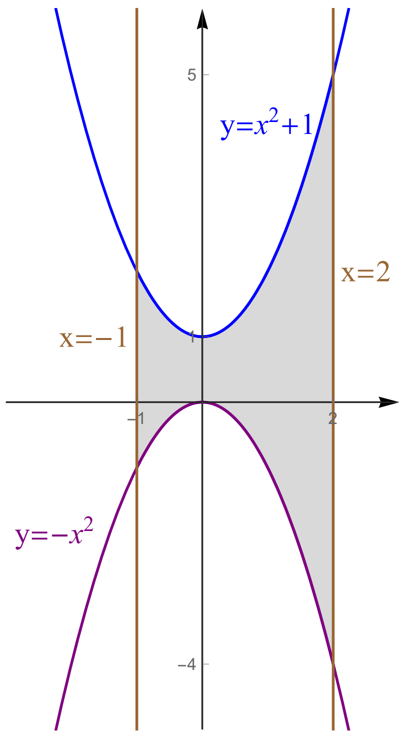
$$= \lim_{t\to\infty}\left(\frac{1}{2}\left[\frac{1}{-2(2t-1)^2} - \frac{1}{-2(2(1)-1)^2}\right]\right) = \lim_{t\to\infty}\left(\frac{1}{2}\left[\frac{1}{-2(2t-1)^2} + \frac{1}{2}\right]\right)$$

$$= \frac{1}{2}\left[0+\frac{1}{2}\right] = \frac{1}{4}\ .$$

The improper integral converges.

3. Sketch the region bounded by the graphs of the curves $y=-x^2$, $y=x^2+1$, $x=-1$ and $x=2$, then find its area . [3]

**Solution :**




$y = x^2+1$ represents a parabola opens upwards, with vertex $(0,1)$ .
$y=-x^2$ represents a parabola opens downwards, with vertex $(0,0)$ .
$x=-1$ represents a straight line parallel to the $y$-axis and passes through $(-1,0)$ .
$x=2$ represents a straight line parallel to the $y$-axis and passes through $(2,0)$ .

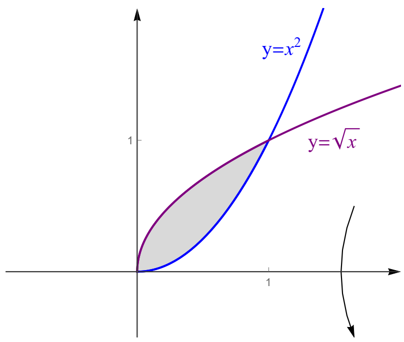
$$\mathbf{A} = \int_{-1}^2 \left[(x^2+1)-(-x^2)\right]\ dx = \int_{-1}^2 (2x^2+1)\ dx = \left[2\ \frac{x^3}{3}+x\right]_{-1}^2$$

$$= \left(2\ \left(\frac{8}{3}\right)+2\right) - \left(2\ \left(\frac{-1}{3}\right)-1\right) = \frac{16}{3}+2+\frac{2}{3}+1 = \frac{18}{3}+3 = 6+3 = 9$$

4. Sketch the region bounded by the graphs of the curves $y = x^2$ and $y = \sqrt{x}$, then find the volume of the solid generated by revolving this region about the $x$-axis. [3]

**Solution :**

$y = x^2$ represents a parabola opens upwards, with vertex $(0, 0)$ .
$y = \sqrt{x}$ represents the upper half of the parabola $x = y^2$ which opens to the right, with vertex $(0, 0)$ .




Points of intersection between $y = x^2$ and $y = \sqrt{x}$ :
$x^2 = \sqrt{x} \implies x^4 = x$
$\implies x^4 - x = 0 \implies x(x^3 - 1) = 0$
$\implies x = 0 \ , \ x = 1$ .

Using Washer method :

$$\mathbf{V} = \pi \int_0^1 \left[ \left(\sqrt{x}\right)^2 - (x^2)^2 \right] \, dx = \pi \int_0^1 (x - x^4)] \, dx = \pi \left[ \frac{x^2}{2} - \frac{x^5}{5} \right]_0^1$$

$$= \pi \left[ \left( \frac{1}{2} - \frac{1}{5} \right) - \left( \frac{0}{2} - \frac{0}{5} \right) \right] = \pi \left( \frac{5}{10} - \frac{2}{10} \right) = \frac{3\pi}{10} \ .$$

5. Find the arc length of $y = 2 + \cosh x$ , from $x = 0$ to $x = \ln 2$ . [3]

**Solution :**

$y' = 0 + \sinh x = \sinh x$ .

$$\mathbf{L} = \int_0^{\ln 2} \sqrt{1 + (\sinh x)^2} \, dx = \int_0^{\ln 2} \sqrt{1 + \sinh^2 x} \, dx = \int_0^{\ln 2} \sqrt{\cosh^2 x} \, dx$$

$$= \int_0^{\ln 2} |\cosh x| \, dx = \int_0^{\ln 2} \cosh x \, dx = [\sinh x]_0^{\ln 2} = \sinh(\ln 2) - \sinh(0)$$

$$= \sinh(\ln 2) - 0 = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4} \ .$$

6. Convert the polar equation $r = 8\cos\theta + 6\sin\theta$ into a Cartesian equation. [2]

**Solution :**

$$r = 8\cos\theta + 6\sin\theta \implies r^2 = 8(r\cos\theta) + 6(r\sin\theta) \implies x^2 + y^2 = 8x + 6y$$

$$\implies x^2 - 8x + y^2 - 6y = 0 \implies (x^2 - 8x + 16) + (y^2 - 6y + 9) = 16 + 9$$

$$\implies (x - 4)^2 + (y - 3)^2 = 5^2$$

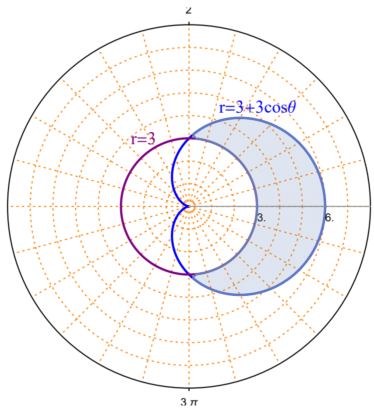
It represents a circle centered at$(4, 3)$ , with radius equals 5 .

7. Sketch the region inside the curve $r = 3 + 3\cos\theta$ and outside the curve $r = 3$ , then find its area . [3]

**Solution :**

$r = 3+3\cos\theta$ represents a cardioid symmetric with respect to the polar axis.
$r = 3$ represents a circle centered at the pole with radius equals 3 .

Points of intersection between $r = 3 + 3\cos\theta$ and $r = 3$ :
$3 + 3\cos\theta = 3 \implies 3\cos\theta = 0$
$\implies \cos\theta = 0 \implies \theta = \frac{\pi}{2}$ , $\theta = \frac{3\pi}{2}$




Note that the shaded region is symmetric with respect to the polar axis.

$$\mathbf{A} = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \left[(3+3\cos\theta)^2 - (3)^2\right]\ d\theta\right) = \int_0^{\frac{\pi}{2}} \left[9 + 18\cos\theta + 9\cos^2\theta - 9\right]\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[18\cos\theta + 9\left(\frac{1+\cos 2\theta}{2}\right)\right]\ d\theta = \int_0^{\frac{\pi}{2}} \left(18\cos\theta + \frac{9}{2} + \frac{9}{2}\cos 2\theta\right)\ d\theta$$

$$= \left[18\sin\theta + \frac{9}{2}\theta + \frac{9}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \left(18\sin\left(\frac{\pi}{2}\right) + \frac{9}{2}\frac{\pi}{2} + \frac{9}{4}\sin(\pi)\right) - \left(18\sin(0) + \frac{9}{2}(0) + \frac{9}{4}\sin(0)\right)$$

$$= \left(18(1) + \frac{9\pi}{4} + \frac{9}{4}(0)\right) - (18(0) + 0 + \frac{9}{4}(0)) = 18 + \frac{9\pi}{4}\ .$$

**MATH 111 - Integral Calculus**
**Second Semester - 1445 H**
**Solution of the First Exam**
*Dr Tariq A. Alfadhel*

**Question (1):** [9 marks]

1. Use Riemann Sum to evaluate the definite integral $\int_0^2 (x^2-1)\, dx$ . [3]

   **Solution :** $[a,b]=[0,2]$ and $f(x)=x^2-1$ .

   $$\Delta_x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n} .$$

   $$x_k = a + k\ \Delta_x = 0 + k\left(\frac{2}{n}\right) = \frac{2k}{n} .$$

   $$f(x_k) = \left(\frac{2k}{n}\right)^2 - 1 = \frac{4k^2}{n^2} - 1 .$$

   $$R_n = \sum_{k=1}^{n} f(x_k)\ \Delta_x = \sum_{k=1}^{n}\left(\frac{4k^2}{n^2}-1\right)\ \left(\frac{2}{n}\right) = \sum_{k=1}^{n}\left(\frac{8k^2}{n^3}-\frac{2}{n}\right)$$

   $$= \sum_{k=1}^{n}\frac{8k^2}{n^3} - \sum_{k=1}^{n}\frac{2}{n}$$

   $$= \frac{8}{n^3}\sum_{k=1}^{n}k^2 - \frac{2}{n}\sum_{k=1}^{n}1 = \frac{8}{n^3}\ \frac{n(n+1)(2n+1)}{6} - \frac{2}{n}\ (n)$$

   $$= \frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right) - 2 .$$

   $$\int_0^2 (x^2-1)\ dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right) - 2\right]$$

   $$= \frac{4}{3}\ (2) - 2 = \frac{8}{3} - 2 = \frac{2}{3} .$$

2. Find $F'(x)$ , if $F(x) = \int_{\sin\left(\frac{x}{2}\right)}^{5^{3x}} \sqrt{t^2+1}\ dt$ . [2]

   **Solution :**

   $$F'(x) = \frac{d}{dx}\int_{\sin\left(\frac{x}{2}\right)}^{5^{3x}} \sqrt{t^2+1}\ dt$$

   $$= \sqrt{\left(5^{3x}\right)^2+1}\ \left(5^{3x}\ (3)\ln 5\right) - \sqrt{\left(\sin\left(\frac{x}{2}\right)\right)^2+1}\ \left(\cos\left(\frac{x}{2}\right)\ \left(\frac{1}{2}\right)\right)$$

   $$= 3\ 5^{2x}\ln 5\sqrt{5^{6x}+1} - \frac{1}{2}\ \cos\left(\frac{x}{2}\right)\sqrt{\sin^2\left(\frac{x}{2}\right)+1} .$$

Find $\frac{dy}{dx}$ of the following :

3. $y = \tan^{-1}(2x)\ \log|\sec x + \tan x|$ . [2]

**Solution :**

$$\frac{dy}{dx} = \left(\frac{1}{1+(2x)^2}\ (2)\right)\ \log|\sec x + \tan x| + \tan^{-1}(2x)\ \left(\frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\,\frac{1}{\ln 10}\right)$$

$$= \frac{2\log|\sec x + \tan x|}{1+4x^2} + \frac{\sec x\ \tan^{-1}(2x)}{\ln 10} .$$

4. $y = (\tan x)^{\sec x} + 5^x$ . [2]

**Solution :**

let $y = f(x) + g(x)$ , where $f(x) = (\tan x)^{\sec x}$ and $g(x) = 5^x$ .

Then $\frac{dy}{dx} = y' = f'(x) + g'(x)$

First - $g'(x) = 5^x\ (1)\ln 5 = 5^x \ln 5$

Second - Finding $f'(x)$

$f(x) = (\tan x)^{\sec x} \implies \ln|f(x)| = \ln|(\tan x)^{\sec x}| = \sec x\ \ln|\tan x|$ .

Differentiate both sides :

$$\frac{f'(x)}{f(x)} = \sec x\ \tan x\ \ln|\tan x| + \sec x\left(\frac{\sec^2 x}{\tan x}\right)$$

$$f'(x) = f(x)\left[\sec x\ \tan x\ \ln|\tan x| + \frac{\sec^3 x}{\tan x}\right]$$

$$= (\tan x)^{\sec x}\left[\sec x\ \tan x\ \ln|\tan x| + \frac{\sec^3 x}{\tan x}\right] .$$

Hence, $\frac{dy}{dx} = (\tan x)^{\sec x}\left[\sec x\ \tan x\ \ln|\tan x| + \frac{\sec^3 x}{\tan x}\right] + 5^x \ln 5$ .

**Question (2):** [16 marks]
Evaluate the following integrals :

1. $\int \left(\sqrt[3]{x}\ e^{\frac{x^2}{3}}\right)^3\ dx$ . [2]

**Solution :**

$$\int \left(\sqrt[3]{x}\ e^{\frac{x^2}{3}}\right)^3\ dx = \int \left(\sqrt[3]{x}\right)^3 \left(e^{\frac{x^2}{3}}\right)^3\ dx = \int x\ e^{x^2}\ dx$$

$$= \frac{1}{2}\int e^{x^2}(2x)\ dx = \frac{1}{2}e^{x^2} + c$$

Using the formula :

$$\int e^{f(x)}\ f'(x)\ dx = e^{f(x)} + c\ .$$

2. $\displaystyle\int \frac{1}{\sqrt{e^{4x}-9}}\ dx\ .$ [2]

**Solution :**

$$\int \frac{1}{\sqrt{e^{4x}-9}}\ dx = \int \frac{1}{\sqrt{\left(e^{2x}\right)^2-(3)^2}}\ dx = \frac{1}{2}\int \frac{e^{2x}\ (2)}{e^{2x}\sqrt{\left(e^{2x}\right)^2-(3)^2}}\ dx$$

$$= \frac{1}{2}\left(\frac{1}{3}\sec^{-1}\left(\frac{e^{2x}}{3}\right)\right) + c = \frac{1}{6}\sec^{-1}\left(\frac{e^{2x}}{3}\right) + c\ .$$

Using the formula :

$\displaystyle\int \frac{f'(x)}{f(x)\sqrt{\left[f(x)\right]^2-a^2}}\ dx = \frac{1}{a}\ \sec^{-1}\left(\frac{f(x)}{a}\right) + c$ , where $|f(x)| > a$ and $a > 0$.

3. $\displaystyle\int_0^1 x\ 5^{2x^2}\ dx\ .$ [2]

**Solution :**

$$\int_0^1 x\ 5^{2x^2}\ dx = \frac{1}{4}\int_0^1 5^{2x^2}(4x)\ dx = \frac{1}{4}\left[\frac{5^{2x^2}}{\ln 5}\right]_0^1$$

$$= \frac{1}{4}\left[\frac{5^{2(1)^2}}{\ln 5} - \frac{5^{2(0)^2}}{\ln 5}\right] = \frac{1}{4}\left[\frac{5^2}{\ln 5} - \frac{5^0}{\ln 5}\right] = \frac{1}{4}\left(\frac{25-1}{\ln 5}\right) = \frac{6}{\ln 5}\ .$$

Using the formula :

$$\int a^{f(x)}\ f'(x)\ dx = \frac{a^{f(x)}}{\ln a} + c\ .$$

4. $\displaystyle\int \frac{x+2}{x^2+1}\ dx\ .$ [2]

**Solution :**

$$\int \frac{x+2}{x^2+1}\ dx = \int \left[\frac{x}{x^2+1} + \frac{2}{x^2+1}\right]\ dx$$

$$= \int \frac{x}{x^2+1}\ dx + \int \frac{2}{x^2+1}\ dx$$

$$= \frac{1}{2}\int \frac{2x}{x^2+1}\ dx + 2\int \frac{1}{x^2+1}\ dx$$

$$= \frac{1}{2} \ln(x^2+1) + 2\tan^{-1}(x) + c \ .$$

5. $\displaystyle\int x^{-1} \ \sin\left(\ln(x^2)\right) \ dx \ .$ [2]

**Solution :**

$$\int x^{-1} \ \sin\left(\ln(x^2)\right) \ dx = \int \sin\left(2\ln|x|\right) \ \frac{1}{x} \ dx$$

$$= \frac{1}{2}\int \sin\left(2\ln|x|\right) \ \frac{2}{x} \ dx = \frac{1}{2}\left(-\cos\left(2\ln|x|\right)\right) + c = -\frac{\cos\left(2\ln|x|\right)}{2} + c \ .$$

Using the formula :

$$\int \sin\left(f(x)\right) \ f'(x) \ dx = -\cos\left(f(x)\right) + c \ .$$

6. $\displaystyle\int x^{-2} \ \csc\left(\frac{1}{x}\right)\cot\left(\frac{1}{x}\right) \ dx \ .$ [2]

**Solution :**

$$\int x^{-2} \ \csc\left(\frac{1}{x}\right)\cot\left(\frac{1}{x}\right) \ dx = \int \csc\left(\frac{1}{x}\right)\cot\left(\frac{1}{x}\right) \ \left(\frac{1}{x^2}\right) \ dx$$

$$= \int -\csc\left(\frac{1}{x}\right)\cot\left(\frac{1}{x}\right) \ \left(-\frac{1}{x^2}\right) \ dx = \csc\left(\frac{1}{x}\right) + c \ .$$

Using the formula :

$$\int \csc\left(f(x)\right)\cot\left(f(x)\right) \ f'(x) \ dx = -\csc\left(f(x)\right) + c$$

7. $\displaystyle\int \frac{\left(3+\sin^{-1}x\right)^3}{\sqrt{1-x^2}} \ dx \ .$ [2]

**Solution :**

$$\int \frac{\left(3+\sin^{-1}x\right)^3}{\sqrt{1-x^2}} \ dx = \int \left(3+\sin^{-1}x\right)^3 \ \frac{1}{\sqrt{1-x^2}} \ dx$$

$$= \frac{\left(3+\sin^{-1}x\right)^4}{4} + c \ .$$

Using the formula :

$$\int [f(x)]^n \ f'(x) \ dx = \frac{[f(x)]^{n+1}}{n+1} + c \quad n \neq -1 \ .$$

8. $\int \frac{\sin x}{\cos^2 x}\ dx$ . [2]

**First Solution :**

$$\int \frac{\sin x}{\cos^2 x}\ dx = \int \frac{1}{\cos x}\ \frac{\sin x}{\cos x}\ dx$$

$$= \int \sec x\ \tan x\ dx = \sec x + c\ .$$

Using the formula :

$$\int \sec x\ \tan x\ dx = \sec x + c\ .$$

**Second Solution :**

$$\int \frac{\sin x}{\cos^2 x}\ dx = \int (\cos x)^{-2}\ \sin x\ dx = -\int (\cos x)^{-2}\ (-\sin x)\ dx$$

$$= -\frac{(\cos x)^{-1}}{-1} + c = \frac{1}{\cos x} + c = \sec x + c\ .$$

Using the formula :

$$\int [f(x)]^n\ f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1} + c \quad n \neq -1\ .$$

**MATH 111 - Integral Calculus**
**Second Semester - 1445 H**
**Solution of the Second Exam**
*Dr Tariq A. Alfadhel*

**Question (1):** [4 marks]

Find $\frac{dy}{dx}$ of the following :

1. $y = \tanh(2x^3) + sech^{-1}(3x)$ . [2]

   **Solution :**

   $$\frac{dy}{dx} = sech^2(2x^3)\ (6x^2) + \frac{-1}{3x\ \sqrt{1-(3x)^2}}\ (3)$$

   $$= 6x^2\ sech^2(2x^3) - \frac{1}{x\ \sqrt{1-9x^2}}\ .$$

2. $y = \cosh^{-1}\left(\sqrt{x}\right) + \sinh^3\left(\frac{1}{x}\right)$ . [2]

   **Solution :**

   $$\frac{dy}{dx} = \frac{1}{\sqrt{\left(\sqrt{x}\right)^2 - 1}}\left(\frac{1}{2\sqrt{x}}\right) + 3\left(\sinh\left(\frac{1}{x}\right)\right)^2\ \cosh\left(\frac{1}{x}\right)\ \left(\frac{-1}{x^2}\right)$$

   $$= \frac{1}{2\sqrt{x}\ \sqrt{x-1}} - \left(\frac{3}{x^2}\right)\sinh^2\left(\frac{1}{x}\right)\ \cosh\left(\frac{1}{x}\right)\ .$$

**Question (2):** [21 marks]

Evaluate the following integrals :

1. $\int \frac{\coth\left(\sqrt{x}\right)}{\sqrt{x}}\ dx$ . [2]

   **Solution :**

   $$\int \frac{\coth\left(\sqrt{x}\right)}{\sqrt{x}}\ dx = \int \coth\left(\sqrt{x}\right)\ \frac{1}{\sqrt{x}}\ dx$$

   $$= 2\int \coth\left(\sqrt{x}\right)\ \frac{1}{2\sqrt{x}}\ dx = 2\ln\left|\sinh\left(\sqrt{x}\right)\right| + c\ .$$

   Using the formula :

   $$\int \coth\left(f(x)\right)\ f'(x)\ dx = \ln\left|\sinh\left(f(x)\right)\right| + c\ .$$

2. $\int (2x+1)\ \cos x\ dx$ . [2]

**Solution :** Using integration by parts.

$$u = 2x+1 \qquad dv = \cos x \ dx$$
$$du = 2 \ dx \qquad v = \sin x$$

$$\int (2x+1) \ \cos x \ dx = (2x+1) \ \sin x - \int 2\sin x \ dx$$

$$= (2x+1) \ \sin x - 2\int \sin x \ dx = (2x+1) \ \sin x - 2(-\cos x) + c$$

$$= (2x+1) \ \sin x + 2\cos x + c \ .$$

3. $\int \tan^2 x \ \sec^4 x \ dx$ . [2]

**Solution :** Using the substitution $u = \tan x$

Then $du = \sec^2 x \ dx$

$$\int \tan^2 x \ \sec^4 x \ dx = \int \tan^2 x \ \sec^2 x \ \sec^2 x \ dx$$

$$\int \tan^2 x \ \left(1+\tan^2 x\right) \ \sec^2 x \ dx = \int u^2 \ \left(1+u^2\right) \ du$$

$$= \int \left(u^2+u^4\right) \ du = \frac{u^3}{3} + \frac{u^5}{5} + c = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + c \ .$$

4. $\int \frac{1}{x \ \sqrt{x^6+25}} \ dx$ . [3]

**Solution :**

$$\int \frac{1}{x \ \sqrt{x^6+25}} \ dx = \int \frac{1}{x \ \sqrt{(x^3)^2+(5)^2}} \ dx = \int \frac{x^2}{x^2 \ x \ \sqrt{(x^3)^2+(5)^2}} \ dx$$

$$= \frac{1}{3}\int \frac{3x^2}{x^3 \ \sqrt{(x^3)^2+(5)^2}} \ dx = \frac{1}{3}\left[-\frac{1}{5} \ csch^{-1}\left(\frac{x^3}{5}\right)\right] + c = -\frac{1}{15} \ csch^{-1}\left(\frac{x^3}{5}\right) + c$$

Using the formula :

$$\int \frac{f'(x)}{f(x)\sqrt{[f(x)]^2+a^2}} \ dx = -\frac{1}{a} \ csch^{-1}\left(\frac{f(x)}{a}\right) + c \ , \text{ where } a > 0 \ .$$

5. $\int x \ \ln|x| \ dx$ . [2]

**Solution :** Using integration by parts.

$$u = \ln|x| \qquad dv = x \ dx$$
$$du = \frac{1}{x} \ dx \qquad v = \frac{x^2}{2}$$

$$\int x\ \ln|x|\ dx = \frac{x^2}{2}\ \ln|x| - \int \frac{x^2}{2}\ \frac{1}{x}\ dx$$

$$= \frac{x^2}{2}\ \ln|x| - \frac{1}{2}\int x\ dx = \frac{x^2}{2}\ \ln|x| - \frac{1}{2}\ \frac{x^2}{2} + c = \frac{x^2}{2}\ \ln|x| - \frac{x^2}{4} + c\ .$$

6. $\displaystyle\int \frac{1}{\sqrt{4+\sqrt{x}}}\ dx$ . [2]

**Solution :**

Put $u = \sqrt{4+\sqrt{x}} \implies u^2 = 4+\sqrt{x}$

$\implies u^2 - 4 = \sqrt{x} \implies (u^2-4)^2 = x$ .

Then $2(u^2-4)\ 2u\ du = dx \implies 4u(u^2-4)\ du = dx$ .

$$\int \frac{1}{\sqrt{4+\sqrt{x}}}\ dx = \int \frac{4u(u^2-4)}{u}\ du = 4\int (u^2-4)\ du$$

$$= 4\left[\frac{u^3}{3} - 4u\right] + c = 4\left[\frac{\left(\sqrt{4+\sqrt{x}}\right)^3}{3} - 4\left(\sqrt{4+\sqrt{x}}\right)\right] + c\ .$$

7. $\displaystyle\int \frac{1}{\sqrt{x^2+6x+13}}\ dx$ . [2]

**Solution :** Using completing the square.

$x^2+6x+13 = \left(x^2+6x+9\right) + 4 = (x+3)^2 + (2)^2$

$$\int \frac{1}{\sqrt{x^2+6x+13}}\ dx = \int \frac{1}{\sqrt{(x+3)^2+(2)^2}}\ dx = \sinh^{-1}\left(\frac{x+3}{2}\right) + c\ .$$

using the formula :

$$\int \frac{f'(x)}{\sqrt{[f(x)]^2+a^2}}\ dx = \sinh^{-1}\left(\frac{f(x)}{a}\right) + c\ , \text{ where } a > 0\ .$$

8. $\displaystyle\int \frac{3x^2+3x+8}{x^3+4x}\ dx$ . [3]

**Solution :** Using the method of partial fractions.

$$\frac{3x^2+3x+8}{x^3+4x} = \frac{3x^2+3x+8}{x\ (x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\frac{3x^2+3x+8}{x^3+4x} = \frac{A(x^2+4)}{x(x^2+4)} + \frac{(Bx+C)\,x}{x(x^2+4)}$$

$3x^2+3x+8 = A(x^2+4) + x(Bx+C)$

$$3x^2 + 3x + 8 = Ax^2 + 4A + Bx^2 + Cx = (A+B)x^2 + Cx + 4A$$

By comparing the coefficients of the two polynomials in each side :

$A + B = 3 \longrightarrow (1)$
$C = 3 \longrightarrow (2)$
$4A = 8 \longrightarrow (3)$

From equation (3) : $4A = 8 \implies A = 2$ .

From equation (1) : $2 + B = 3 \implies B = 1$ .

$$\int \frac{3x^2+3x+8}{x^3+4x}\, dx = \int \left(\frac{2}{x} + \frac{x+3}{x^2+4}\right) dx$$

$$= \int \frac{2}{x}\, dx + \int \frac{x}{x^2+4}\, dx + \int \frac{3}{x^2+4}\, dx$$

$$= 2\int \frac{1}{x}\, dx + \frac{1}{2} \int \frac{2x}{x^2+4}\, dx + 3\int \frac{1}{x^2+2^2}\, dx$$

$$= 2\ln|x| + \frac{1}{2} \ln\left(x^2+4\right) + 3\left[\frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right)\right] + c \ .$$

9. $\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\, dx$ . [3]

**Solution :** Using trigonometric substitutions.

Put $x = 2\sin\theta \implies \sin\theta = \frac{x}{2}$ .

$dx = 2\cos\theta\, d\theta$ .

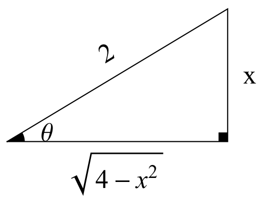
$$\left(4-x^2\right)^{\frac{3}{2}} = \left(4 - 4\sin^2\theta\right)^{\frac{3}{2}} = \left[4\left(1-\sin^2\theta\right)\right]^{\frac{3}{2}}$$

$$= \left(4\cos^2\theta\right)^{\frac{3}{2}} = (4)^{\frac{3}{2}} \left(\cos^2\theta\right)^{\frac{3}{2}} = \left(2^2\right)^{\frac{3}{2}} \left(\cos^2\theta\right)^{\frac{3}{2}} = 2^3 \cos^3\theta \ .$$

$$\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\, dx = \int \frac{2\cos\theta}{2^3 \cos^3\theta}\, d\theta = \frac{1}{2^2} \int \frac{1}{\cos^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \sec^2\theta\, d\theta = \frac{1}{4} \tan\theta + c$$

$\sin\theta = \frac{x}{2}$ .




From the triangle :
$\tan\theta = \frac{x}{\sqrt{4-x^2}}$ .

$$\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\, dx = \frac{1}{4} \frac{x}{\sqrt{4-x^2}} + c \ .$$

**MATH 111 - Integral Calculus**
**Second Semester - 1445 H**
**Solution of the Final Exam**
*Dr Tariq A. Alfadhel*

**First Part** [7 marks] .

1. Find the value of $c$ that satisfies the mean value theorem of integration for the function $f(x) = \sqrt[3]{x+1}$ on the interval $[-2, 0]$ . [3]

   **Solution :** Using the formula $(b-a)f(c) = \int_a^b f(x)\ dx$ ,

   where $f(x) = \sqrt[3]{x+1}$ and $[a, b] = [-2, 0]$ .

   $$(0-(-2))\ \sqrt[3]{c+1} = \int_{-2}^{0} \sqrt[3]{x+1}\ dx = \int_{-2}^{0} (x+1)^{\frac{1}{3}}\ dx$$

   $$2\sqrt[3]{c+1} = \left[\frac{3}{4}\ (x+1)^{\frac{4}{3}}\right]_{-2}^{0} = \frac{3}{4}\ (0+1)^{\frac{4}{3}} - \frac{3}{4}\ (-2+1)^{\frac{4}{3}}$$

   $$2\sqrt[3]{c+1} = \frac{3}{4} - \frac{3}{4} = 0 \implies 2\sqrt[3]{c+1} = 0$$

   $$\sqrt[3]{c+1} = 0 \implies c+1 = 0 \implies c = -1\ .$$

   Note that $c = -1 \in (-2, 0)$ .

   The desired value of $c$ is $-1$ .

2. Find $F'(x)$ , if $F(x) = \int_{\sqrt{3x}}^{e^{x^2}} \tan(t^2+1)\ dt$ . [2]

   **Solution :** $F'(x) = \frac{d}{dx} \int_{\sqrt{3x}}^{e^{x^2}} \tan(t^2+1)\ dt$

   $$F'(x) = \tan\left(\left(e^{x^2}\right)^2 + 1\right)\ e^{x^2}\ (2x) - \tan\left(\left(\sqrt{3x}\right)^2 + 1\right)\ \frac{1}{2\sqrt{3x}}\ (3)$$

   $$= 2x\ e^{x^2}\ \tan\left(e^{2x^2} + 1\right) - \frac{3}{2\sqrt{3x}}\ \tan(3x+1)\ .$$

3. Find $f'(x)$ , if $f(x) = \cosh^{-1}\left(3^{2x-1}\right) + \log\left|csch\left(\frac{x}{2}\right)\right|$ . [2]

   **Solution :**

   $$f'(x) = \frac{1}{\sqrt{\left(3^{2x-1}\right)^2 - 1}}\ \left(3^{2x-1}\ (2)\ \ln 3\right) + \frac{-csch\left(\frac{x}{2}\right)\ \coth\left(\frac{x}{2}\right)\ \left(\frac{1}{2}\right)}{csch\left(\frac{x}{2}\right)}\ \frac{1}{\ln 10}$$

   $$= \frac{2\ 3^{2x-1} \ln 3}{\sqrt{3^{4x-2} - 1}} - \frac{\coth\left(\frac{x}{2}\right)}{2 \ln 10}\ .$$

**Second Part** [16 marks] .
Evaluate the following integrals :

1. $\displaystyle\int \frac{2}{\sqrt{4x^2+8x+20}}\, dx$ . [3]

   **Solution :** Using completing the square .

   $4x^2+8x+20 = 4(x^2+2x+5) = 4\left[(x^2+2x+1)+4\right] = 4\left[(x+1)^2+(2)^2\right]$

   $$\int \frac{2}{\sqrt{4x^2+8x+20}}\, dx = \int \frac{2}{\sqrt{4\left[(x+1)^2+(2)^2\right]}}\, dx = \int \frac{2}{2\sqrt{(x+1)^2+(2)^2}}\, dx$$

   $$= \int \frac{1}{\sqrt{(x+1)^2+(2)^2}}\, dx = \sinh^{-1}\left(\frac{x+1}{2}\right) + c \ .$$

   using the formula

   $$\int \frac{f'(x)}{\sqrt{[f(x)]^2+a^2}}\, dx = \sinh^{-1}\left(\frac{f(x)}{a}\right) + c \ , \text{ where } a > 0 \ .$$

2. $\displaystyle\int \tan^{-1} x \ dx$ . [3]

   **Solution :** Using integration by parts.

   $u = \tan^{-1} x \qquad dv = dx$

   $du = \dfrac{1}{1+x^2}\, dx \qquad v = x$

   $$\int \tan^{-1} x \ dx = x \ \tan^{-1} x - \int x \ \frac{1}{1+x^2}\, dx$$

   $$= x \ \tan^{-1} x - \frac{1}{2}\int \frac{2x}{x^2+1}\, dx = x \ \tan^{-1} x - \frac{1}{2} \ \ln(x^2+1) + c \ .$$

3. $\displaystyle\int \frac{4}{x^2\sqrt{x^2-4}}\, dx$ . [3]

   **Solution :** Using Trigonometric substitutions.

   Put $x = 2\sec\theta \implies \sec\theta = \dfrac{x}{2} \implies \cos\theta = \dfrac{2}{x}$ .

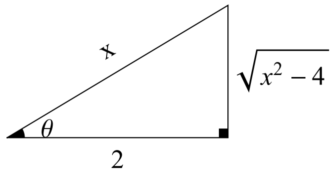
   $dx = 2\sec\theta \ \tan\theta \ d\theta$ .

   $\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = \sqrt{4\,(\sec^2\theta - 1)} = \sqrt{4\tan^2\theta} = 2\tan\theta$ .

   $$\int \frac{4}{x^2\sqrt{x^2-4}}\, dx = \int \frac{4 \ (2\sec\theta \ \tan\theta)}{(2\sec\theta)^2 \ 2\tan\theta}\, d\theta = \int \frac{8\sec\theta \ \tan\theta}{8\sec^2\theta \ \tan\theta}\, d\theta$$

   $$= \int \frac{1}{\sec\theta}\, d\theta = \int \cos\theta \ d\theta = \sin\theta + c$$

$\cos\theta = \dfrac{2}{x}$ .




From the triangle :

$\sin\theta = \dfrac{\sqrt{x^2-4}}{x}$ .

$$\int \frac{1}{x^2\sqrt{x^2-4}}\,dx = \frac{\sqrt{x^2-4}}{x} + c \ .$$

4. $\displaystyle\int \frac{3x+1}{x^2+2x-8}\,dx$ . [3]

**Solution :** Using the method of partial fractions.

$$\frac{3x+1}{x^2+2x-8} = \frac{3x+1}{(x-2)(x+4)} = \frac{A_1}{x-2} + \frac{A_2}{x+4}$$

$$\frac{3x+1}{(x-2)(x+4)} = \frac{A_1\ (x+4)}{(x-2)(x-4)} + \frac{A_2\ (x-2)}{(x+4)(x-2)} \ .$$

$3x+1 = A_1\ (x+4) + A_2\ (x-2)$ .

Put $x=2$ , then : $3(2)+1 = A_1\ (2+4) \implies 7 = 6A_1 \implies A_1 = \dfrac{7}{6}$ .

Put $x=-4$ , then : $3(-4)+1 = A_2\ (-4-2) \implies -11 = -6A_2$

$\implies A_2 = \dfrac{11}{6}$ .

$$\int \frac{3x+1}{x^2+2x-8}\,dx = \int \left(\frac{\frac{7}{6}}{x-2} + \frac{\frac{11}{6}}{x+4}\right)\,dx$$

$$= \frac{7}{6}\int \frac{1}{x-2}\,dx + \frac{11}{6}\int \frac{1}{x+4}\,dx$$

$$= \frac{7}{6}\ln|x-2| + \frac{11}{6}\ln|x+4| + c \ .$$

5. $\displaystyle\int \frac{1}{x+\sqrt[3]{x}}\,dx$ . [2]

**Solution :** $\displaystyle\int \frac{dx}{x+\sqrt[3]{x}} = \int \frac{1}{x+x^{\frac{1}{3}}}\,dx$

using the substitution $x = u^3$ , hence $u = x^{\frac{1}{3}}$ .

$dx = 3u^2\ du$ .

$$\int \frac{dx}{x+x^{\frac{1}{3}}} = \int \frac{3u^2}{u^3+(u^3)^{\frac{1}{3}}}\,du = \int \frac{3u^2}{u^3+u}\,du = \int \frac{3u^2}{u\ (u^2+1)}\,du$$

$$= 3\int \frac{u}{u^2+1}\,du = \frac{3}{2}\int \frac{2u}{u^2+1}\,du = \frac{3}{2}\ \ln(u^2+1)+c = \frac{3}{2}\ \ln\left(x^{\frac{2}{3}}+1\right)+c$$

6. $\int \sin^5 x \ \cos^6 x \ dx$ . [2]

**Solution :**

$$\int \sin^5 x \ \cos^6 x \ dx = \int \sin^4 x \ \cos^6 x \ \sin x \ dx$$

$$= \int \left(\sin^2 x\right)^2 \ \cos^6 x \ \sin x \ dx = \int \left(1 - \cos^2 x\right)^2 \ \cos^6 x \ \sin x \ dx$$

Using the substitution $u = \cos x$ .

$du = -\sin x \implies (-1) \ du = \sin x \ dx$ .

$$\int \left(1 - \cos^2 x\right)^2 \ \cos^6 x \ \sin x \ dx = \int \left(1 - u^2\right)^2 \ u^6 \ (-1) \ du$$

$$= - \int \left(1 - 2u^2 + u^4\right) \ u^6 \ du = - \int \left(u^6 - 2u^8 + u^{10}\right) \ du$$

$$= - \left[\frac{u^7}{7} - 2 \ \frac{u^9}{9} + \frac{u^{11}}{11}\right] + c = -\frac{\cos^7 x}{7} + 2 \ \frac{\cos^9 x}{9} - \frac{\cos^{11} x}{11} + c \ .$$

**Third Part** [17 marks] .

1. Determine whether the improper integral $\int_0^\infty x \ e^{1-x^2} \ dx$ converges or diverges. [3]

   **Solution :**

   $$\int_0^\infty x \ e^{1-x^2} \ dx = \lim_{t \to \infty} \int_0^t x \ e^{1-x^2} \ dx = \lim_{t \to \infty} \left( \frac{-1}{2} \ \int_0^t e^{1-x^2} (-2x) \ dx \right)$$

   $$= \lim_{t \to \infty} \left( \frac{-1}{2} \ \left[ e^{1-x^2} \right]_0^t \right) = \lim_{t \to \infty} \left( \frac{-1}{2} \ \left[ e^{1-t^2} - e^{1-0^2} \right] \right)$$

   $$= \frac{-1}{2} \ [0 - e] = \frac{e}{2} \ .$$

   The improper integral converges.

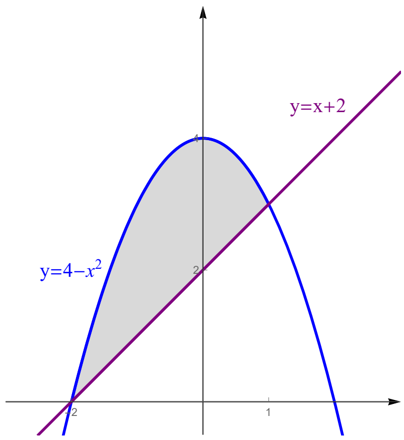
   Note that $\lim_{t \to \infty} e^{1-t^2} = 0$ .

2. Sketch the region bounded by the graphs of the curves $y = 4 - x^2$ and $y = x + 2$ , then find its area . [3]

**Solution :**

$y = 4 - x^2$ represents a parabola opens downwards, with vertex $(0,4)$
$y = x + 2$ represents a straight line passes through $(0,2)$ , with slope equals 1 .

Points of intersection between $y = 4 - x^2$ and $y = x + 2$ :
$x + 2 = 4 - x^2 \implies x^2 + x - 2 = 0$
$\implies (x+2)(x-1) = 0 \implies x = -2 \ , \ x = 1$




$$\mathbf{A} = \int_{-2}^{1} \left[(4 - x^2) - (x + 2)\right] \ dx = \int_{-2}^{1} (4 - x^2 - x - 2) \ dx$$

$$= \int_{-2}^{1} (-x^2 - x + 2) \ dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1}$$

$$= \left(-\frac{1^3}{3} - \frac{1^2}{2} + 2(1)\right) - \left(-\frac{(-2)^3}{3} - \frac{(-2)^2}{2} + 2(-2)\right)$$

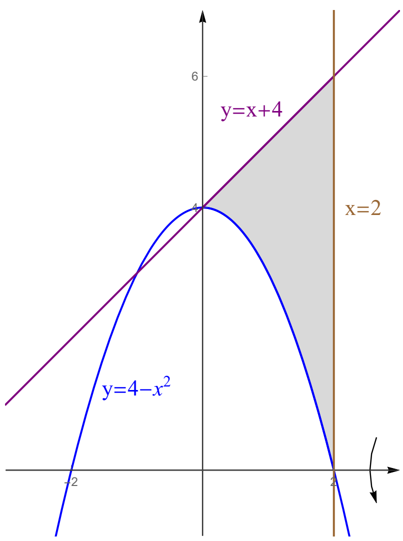
$$= -\frac{1}{3} - \frac{1}{2} + 2 - \left(\frac{8}{3} - 2 - 4\right) = -\frac{1}{3} - \frac{1}{2} + 2 - \frac{8}{3} + 6 = -3 + 8 - \frac{1}{2} = \frac{9}{2} \ .$$

3. Sketch the region bounded by the graphs of the curves $y = 4 - x^2$ , $y = x + 4$ and $x = 2$ , then find the volume of the solid generated by revolving this region about the $x$-axis. [3]

**Solution :**

$y = 4 - x^2$ represents a parabola opens downwards, with vertex $(0,4)$.
$y = x + 4$ represents a straight line passes through $(0,4)$ , with slope equals 1 .
$x = 2$ represents a straight line parallel to the $y$-axis and passes through $(2,0)$ .

Note that $x = 2$ intersects $y = 4 - x^2$ at the point $(2,0)$ .



Using Washer method :

$$\mathbf{V} = \pi \int_0^2 \left[(x+4)^2 - (4-x^2)^2\right] \; dx = \pi \int_0^2 \left[(x^2+8x+16) - (16-8x^2+x^4)\right] \; dx$$

$$= \pi \int_0^2 \left(x^2+8x+16-16+8x^2-x^4\right) \; dx = \pi \int_0^2 \left(-x^4+9x^2+8x\right) \; dx$$

$$= \pi \left[-\frac{x^5}{5} + 3x^3 + 4x^2\right]_0^2 = \pi \left[\left(-\frac{2^5}{5} + 3(2)^3 + 4(2)^2\right) - \left(-\frac{0^5}{5} + 3(0)^3 + 4(0)^2\right)\right]$$

$$= \pi \left(-\frac{32}{5} + 24 + 16\right) = \pi \left(-\frac{32}{5} + 40\right) = \pi \left(\frac{-32+200}{5}\right) = \frac{168\pi}{5} \; .$$

4. Find the arc length of $y = \sqrt{1-x^2}$, from $x = -1$ to $x = 1$ . [3]

**Solution :**

$$y' = \frac{1}{2\sqrt{1-x^2}} \; (-2x) = \frac{-x}{\sqrt{1-x^2}} \; .$$

$$\mathbf{L} = \int_{-1}^1 \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2} \; dx = \int_{-1}^1 \sqrt{1 + \frac{x^2}{1-x^2}} \; dx = \int_{-1}^1 \sqrt{\frac{(1-x^2)+x^2}{1-x^2}} \; dx$$

$$= \int_{-1}^1 \sqrt{\frac{1}{1-x^2}} \; dx = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \; dx = \left[\sin^{-1} x\right]_{-1}^1$$

$$= \sin^{-1}(1) - \sin^{-1}(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi \; .$$

5. Convert the polar equation $r = 8\sin\theta + 6\cos\theta$ into a Cartesian equation. [2]

**Solution :**

$$r = 8\sin\theta + 6\cos\theta \implies r^2 = 8(r\sin\theta) + 6(r\cos\theta) \implies x^2 + y^2 = 8y + 6x$$

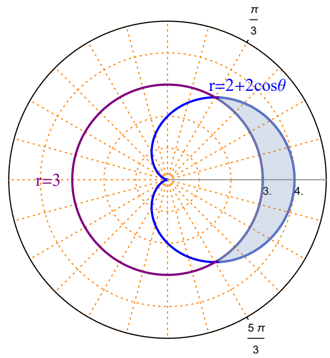
$$\implies x^2 - 6x + y^2 - 8y = 0 \implies (x^2 - 6x + 9) + (y^2 - 8y + 16) = 9 + 16$$

$$\implies (x-3)^2 + (y-4)^2 = 5^2 \; .$$

It represents a circle centered at $(3,4)$ , with radius equals 5 .

6. Sketch the region inside the curve $r = 2 + 2\cos\theta$ and outside the curve $r = 3$ , then find its area. [3]

**Solution :**




$r = 2+2\cos\theta$ represents a cardioid symmetric with respect to the polar axis.
$r = 3$ represents a circle centered at the pole, with radius equals 3 .

Points of intersection between $r = 2+2\cos\theta$ and $r = 3$ :
$2+2\cos\theta = 3 \implies 2\cos\theta = 1$
$\implies \cos\theta = \frac{1}{2} \implies \theta = \frac{\pi}{3}$ , $\theta = \frac{5\pi}{3}$ .

Note that the shaded region is symmetric with respect to the polar axis.

$$\mathbf{A} = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{3}}\left[(2+2\cos\theta)^2-(3)^2\right]\,d\theta\right) = \int_0^{\frac{\pi}{3}}\left[4+8\cos\theta+4\cos^2\theta-9\right]\,d\theta$$

$$= \int_0^{\frac{\pi}{3}}\left[-5+8\cos\theta+4\cos^2\theta\right]\,d\theta = \int_0^{\frac{\pi}{3}}\left[-5+8\cos\theta+4\left(\frac{1+\cos 2\theta}{2}\right)\right]\,d\theta$$

$$= \int_0^{\frac{\pi}{3}}\left[-5+8\cos\theta+2+2\cos 2\theta\right]\,d\theta = \int_0^{\frac{\pi}{3}}\left[-3+8\cos\theta+2\cos 2\theta\right]\,d\theta$$

$$= \left[-3\theta+8\sin\theta+\sin 2\theta\right]_0^{\frac{\pi}{3}} = \left[-3\theta+8\sin\theta+2\sin\theta\,\cos\theta\right]_0^{\frac{\pi}{3}}$$

$$= \left[-3\left(\frac{\pi}{3}\right)+8\sin\left(\frac{\pi}{3}\right)+2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)\right]-\left[-3\,(0)+8\sin\,(0)+2\sin\,(0)\cos\,(0))\right]$$

$$= \left[-\pi+8\left(\frac{\sqrt{3}}{2}\right)+2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)\right]-[0+0+0]$$

$$= -\pi+\frac{8\sqrt{3}}{2}+\frac{\sqrt{3}}{2} = -\pi+\frac{9\sqrt{3}}{2}\ .$$