111 ريض - حساب التكامل
الفصل الدراسي الأول 1444 هـ
حل الاختبار الفصلي
د. طارق عبدالرحمن مُحَمَّد الفاضل

السؤال الأول (12 درجة) :

(1) استخدم مجموع ريمان لحساب التكامل المحدد $\int_{-1}^{4}(2x+1)\,dx$.

الحل : $[a,b]=[-1,4]$ و $f(x)=2x+1$.

$$\Delta_x=\frac{b-a}{n}=\frac{4-(-1)}{n}=\frac{4+1}{n}=\frac{5}{n}$$

$$x_k=a+k\,\Delta_x=-1+k\left(\frac{5}{n}\right)=-1+\frac{5k}{n}$$

$$f\left(x_k\right)=2\left(-1+\frac{5k}{n}\right)+1=-2+\frac{10k}{n}+1=\frac{10k}{n}-1$$

$$R_n=\sum_{k=1}^{n}f\left(x_k\right)\,\Delta_x=\sum_{k=1}^{n}\left(\frac{10k}{n}-1\right)\,\left(\frac{5}{n}\right)=\sum_{k=1}^{n}\left(\frac{50k}{n^2}-\frac{5}{n}\right)=\sum_{k=1}^{n}\frac{50k}{n^2}-\sum_{k=1}^{n}\frac{5}{n}$$

$$=\frac{50}{n^2}\sum_{k=1}^{n}k-\frac{5}{n}\sum_{k=1}^{n}1=\frac{50}{n^2}\,\frac{n(n+1)}{2}-\frac{5}{n}\,(n)=25\left(\frac{n+1}{n}\right)-5$$

$$\int_{-1}^{4}(2x+1)\,dx=\lim_{n\to\infty}R_n=\lim_{n\to\infty}\left[25\left(\frac{n+1}{n}\right)-5\right]=25(1)-5=20$$

(2) جد $F'(x)$ إذا كانت $F(x)=\int_{\sqrt{3x}}^{\ln 2x}\frac{1}{\sqrt{t^2-2t}}\,dt$

الحل :

$$F'(x)=\frac{d}{dx}\int_{\sqrt{3x}}^{\ln 2x}\frac{1}{\sqrt{t^2-2t}}\,dt$$

$$=\frac{1}{\sqrt{(\ln 2x)^2-2\ln 2x}}\,\left(\frac{2}{2x}\right)-\frac{1}{\sqrt{\left(\sqrt{3x}\right)^2-2\sqrt{3x}}}\,\left(\frac{1}{2\sqrt{3x}}\,(3)\right)$$

$$=\frac{1}{x\,\sqrt{(\ln 2x)^2-2\ln 2x}}-\frac{3}{2\sqrt{3x}\,\sqrt{3x-2\sqrt{3x}}}$$

احسب $\frac{dy}{dx}$ فيمايلي :

$$y = 7^{x^2}\ \cosh^{-1}\left(x^2\right) \quad (3)$$

الحل :

$$\frac{dy}{dx} = \left(7^{x^2}\ (2x)\ln 7\right)\ \cosh^{-1}\left(x^2\right) + 7^{x^2}\left[\frac{1}{\sqrt{\left(x^2\right)^2 - 1}}\ (2x)\right]$$

$$= 2x\ 7^{x^2}\ \ln 7\ \cosh^{-1}\left(x^2\right) + \frac{2x\ 7^{x^2}}{\sqrt{x^4 - 1}}$$

$$y = \sin^{-1}(2x)\ \log_5|3 - \tan 6x| \quad (4)$$

الحل :

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1-(2x)^2}}\ (2)\right)\log_5|3-\tan 6x| + \sin^{-1}(2x)\left(\frac{-\sec^2(6x)\ (6)}{3-\tan 6x}\ \frac{1}{\ln 5}\right)$$

$$= \frac{2\log_5|3-\tan 6x|}{\sqrt{1-4x^2}} - \frac{6\sec^2(6x)\ \sin^{-1}(2x)}{(3-\tan 6x)\ \ln 5}$$

$$y = (\tan x)^{\sin x} \quad (5)$$

الحل :

$$y = (\tan x)^{\sin x} \implies \ln|y| = \ln\left|(\tan x)^{\sin x}\right| = \sin x\ \ln|\tan x|$$

باشتقاق الطرفين

$$\frac{y'}{y} = (\cos x)\ \ln|\tan x| + \sin x\left(\frac{\sec^2 x}{\tan x}\right)$$

$$y' = y\left[\cos x\ \ln|\tan x| + \frac{\sin x\ \sec^2 x}{\tan x}\right]$$

$$y' = (\tan x)^{\sin x}\ \ [\cos x\ \ln|\tan x| + \sec x]$$

$$y = \coth^{-1}\left(\frac{1}{x}\right) \quad (6)$$

الحل :

$$\frac{dy}{dx} = \frac{-1}{1-\left(\frac{1}{x}\right)^2}\ \left(\frac{-1}{x^2}\right) = \frac{1}{x^2\left[1-\frac{1}{x^2}\right]} = \frac{1}{x^2-1}$$

السؤال الثاني (18 درجة) : أحسب التكاملات التالية

$$\int \frac{\csc^2\left(e^{2x}\right)}{e^{-2x}}\,dx \quad (1)$$

الحل :

$$\int \frac{\csc^2\left(e^{2x}\right)}{e^{-2x}}\,dx = \int \csc^2\left(e^{2x}\right)\ e^{2x}\,dx = \frac{1}{2}\int \csc^2\left(e^{2x}\right)\ e^{2x}\,(2)\,dx$$

$$= \frac{1}{2}\left(-\cot\left(e^{2x}\right)\right) + c = -\frac{1}{2}\cot\left(e^{2x}\right) + c$$

باستخدام القانون

$$\int \csc^2\left(f(x)\right)\ f'(x)\,dx = -\cot\left(f(x)\right) + c$$

$$\int \left(x^3+4\right)^3\ x^2\,dx \quad (2)$$

الحل :

$$\int \left(x^3+4\right)^3\ x^2\,dx = \frac{1}{3}\int \left(x^3+4\right)^3\ (3x^2)\,dx = \frac{1}{3}\ \frac{(x^3+4)^4}{4} + c$$

باستخدام القانون $\int [f(x)]^n\ f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$ حيث $n \neq -1$.

$$\int_0^1 x\ 10^{2-x^2}\,dx \quad (3)$$

الحل :

$$\int_0^1 x\ 10^{2-x^2}\,dx = \frac{1}{-2}\int_0^1 10^{2-x^2}(-2x)\,dx = \frac{1}{-2}\left[\frac{10^{2-x^2}}{\ln 10}\right]_0^1$$

$$= \frac{1}{-2}\left[\frac{10^{2-(1)^2}}{\ln 10} - \frac{10^{2-(0)^2}}{\ln 10}\right] = \frac{1}{-2}\left[\frac{10^1}{\ln 10} - \frac{10^2}{\ln 10}\right] = \frac{1}{-2}\left(\frac{-90}{\ln 10}\right) = \frac{45}{\ln 10}$$

باستخدام القانون $\int a^{f(x)}\ f'(x)\,dx = \frac{a^{f(x)}}{\ln a} + c$.

$$\int \frac{x+2}{x+1}\,dx \quad (4)$$

الحل :

$$\int \frac{x+2}{x+1}\,dx = \int \frac{(x+1)+1}{x+1}\,dx = \int \left(\frac{x+1}{x+1} + \frac{1}{x+1}\right)\,dx$$

$$= \int \left(1 + \frac{1}{x+1}\right)\,dx = \int 1\,dx + \int \frac{1}{x+1}\,dx = x + \ln|x+1| + c$$

باستخدام القانون $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$.

(5) $\int \frac{e^{2x}}{\sqrt{4+e^{4x}}}\,dx$

الحل :

$$\int \frac{e^{2x}}{\sqrt{4+e^{4x}}}\,dx = \int \frac{e^{2x}}{\sqrt{(2)^2+(e^{2x})^2}}\,dx = \frac{1}{2}\int \frac{2\,e^{2x}}{\sqrt{(2)^2+(e^{2x})^2}}\,dx$$

$$= \frac{1}{2}\sinh^{-1}\left(\frac{e^{2x}}{2}\right) + c$$

باستخدام القانون

$$\int \frac{f'(x)}{\sqrt{a^2+[f(x)]^2}}\,dx = \sinh^{-1}\left(\frac{f(x)}{a}\right) + c$$

(6) $\int \frac{\tanh(\sqrt{x})}{\sqrt{x}}\,dx$

الحل :

$$\int \frac{\tanh(\sqrt{x})}{\sqrt{x}}\,dx = \int \tanh\left(\sqrt{x}\right)\ \left(\frac{1}{\sqrt{x}}\right)\,dx = 2\int \tanh\left(\sqrt{x}\right)\ \left(\frac{1}{2\sqrt{x}}\right)\,dx$$

$$= 2\ln\left|\cosh\left(\sqrt{x}\right)\right| + c$$

باستخدام القانون

$$\int \tanh(f(x))\ f'(x)\,dx = \ln|\cosh(f(x))| + c$$

(7) $\int \frac{3}{9x^2+4}\,dx$

الحل :

$$\int \frac{3}{9x^2+4}\,dx = \int \frac{3}{(3x)^2+(2)^2}\,dx = \frac{1}{2}\ \tan^{-1}\left(\frac{3x}{2}\right) + c$$

باستخدام القانون

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\ \tan^{-1}\left(\frac{f(x)}{a}\right)+c\ .$$

(8) $\int (x+3)\,\sqrt{x+2}\,dx$

الحل : بوضع $u=x+2 \implies u+1=x+3$

$$dx=du$$

$$\int (x+3)\,\sqrt{x+2}\,dx=\int (u+1)\sqrt{u}\,du=\int (u+1)u^{\frac{1}{2}}\ du$$

$$=\int \left(u^{\frac{3}{2}}+u^{\frac{1}{2}}\right)\ du=\frac{u^{\frac{5}{2}}}{\frac{5}{2}}+\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+c=\frac{2}{5}\ (x+2)^{\frac{5}{2}}+\frac{2}{3}\ (x+2)^{\frac{3}{2}}+c$$

(9) $\int e^{3x}\cosh x\,dx$

الحل :

$$\int e^{3x}\cosh x\,dx=\int e^{3x}\left(\frac{e^{x}+e^{-x}}{2}\right)\ dx=\int \left(\frac{e^{4x}+e^{2x}}{2}\right)\ dx$$

$$=\int \left(\frac{e^{4x}}{2}+\frac{e^{2x}}{2}\right)\ dx=\frac{1}{2}\,\frac{1}{4}\int e^{4x}\,(4)\,dx+\frac{1}{2}\,\frac{1}{2}\int e^{2x}(2)\,dx$$

$$=\frac{1}{8}\,e^{4x}+\frac{1}{4}\,e^{2x}+c=\frac{e^{4x}}{8}+\frac{e^{2x}}{4}+c$$

باستخدام القانون $\int e^{f(x)}\ f'(x)\,dx=e^{f(x)}+c$

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الجزء الأول (7 درجات) :

(1) جد قيمة c التي تحقق نظرية القيمة المتوسطة للدالة $f(x) = \sqrt{x-1}$ على الفترة $[1,5]$. [3]

الحل : باستخدام العلاقة $(b-a)f(c) = \int_a^b f(x)\,dx$

حيث $f(x) = \sqrt{x-1}$ و $[a,b] = [1,5]$

$$(5-1)\,\sqrt{c-1} = \int_1^5 \sqrt{x-1}\,dx = \int_1^5 (x-1)^{\frac{1}{2}}dx$$

$$4\sqrt{c-1} = \left[\frac{(x-1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^5 = \left[\frac{2}{3}\,(x-1)^{\frac{3}{2}}\right]_1^5 = \frac{2}{3}\,(5-1)^{\frac{3}{2}} - \frac{2}{3}\,(1-1)^{\frac{3}{2}}$$

$$4\sqrt{c-1} = \frac{2}{3}(4)^{\frac{3}{2}} - 0 = \frac{2}{3}(8) = \frac{16}{3} \implies \sqrt{c-1} = \frac{1}{4}\,\frac{16}{3} = \frac{4}{3}$$

$$c-1 = \frac{16}{9} \implies c = \frac{16}{9} + 1 = \frac{25}{9} \in (1,5)$$

قيمة c المطلوبة هي $\frac{25}{9}$.

(2) جد $F'(x)$ إذا كانت $F(x) = \int_{\sqrt{2x}}^{e^{3x}} \cos(t^2+1)\,dt$. [2]

الحل : $F'(x) = \frac{d}{dx}\int_{\sqrt{2x}}^{e^{3x}} \cos(t^2+1)\,dt$

$$F'(x) = \cos\left(\left(e^{3x}\right)^2+1\right)\,\left(e^{3x}\,(3)\right) - \cos\left(\left(\sqrt{2x}\right)^2+1\right)\,\left(\frac{1}{2\sqrt{2x}}\,(2)\right)$$

$$= 3\,e^{3x}\cos\left(e^{6x}+1\right) - \frac{\cos\left(2x+1\right)}{\sqrt{2x}}$$

(3) جد $f'(x)$ إذا كانت $f(x) = \sinh^{-1}\left(5^{x^2-1}\right) + \log_6|sech(4x)|$. [2]

الحل :

$$f'(x) = \frac{1}{\sqrt{\left(5^{x^2-1}\right)^2+1}}\left(5^{x^2-1}\,(2x)\,\ln 5\right) + \frac{-sech(4x)\tanh(4x)\,(4)}{sech(4x)}\,\frac{1}{\ln 6}$$

$$= \frac{2x\, 5^{x^2-1}\ \ln 5}{\sqrt{5^{2x^2-2}+1}} - \frac{4\tanh(4x)}{\ln 6}$$

الجزء الثاني (14 درجة) : أحسب التكاملات التالية :

$$(1)\quad \int \frac{2}{\sqrt{x^2-6x+8}}\, dx \ . \ [3]$$

الحل : باستخدام الإكمال إلى مربع كامل

$$\int \frac{2}{\sqrt{x^2-6x+8}}\, dx = \int \frac{2}{\sqrt{(x^2-6x+9)-1}}\, dx = 2\int \frac{1}{\sqrt{(x-3)^2-(1)^2}}\, dx$$

$$= 2\ \cosh^{-1}\left(\frac{x-3}{1}\right) + c = 2\ \cosh^{-1}(x-3) + c$$

باستخدام القانون $\int \frac{f'(x)}{\sqrt{[f(x)]^2-a^2}}\, dx = \cosh^{-1}\left(\frac{f(x)}{a}\right) + c$ حيث $a>0$

$$(2)\quad \int x^2\ \ln|x|\, dx \ . \ [3]$$

الحل : باستخدام طريقة التكامل بالتجزئ

$$u = \ln|x| \qquad dv = x^2\, dx$$
$$du = \frac{1}{x}\, dx \qquad v = \frac{x^3}{3}$$

$$\int x^2\ \ln|x|\, dx = \frac{x^3}{3}\ \ln|x| - \int \frac{x^3}{3}\,\frac{1}{x}\, dx$$

$$= \frac{x^3}{3}\ \ln|x| - \frac{1}{3}\int x^2\, dx = \frac{x^3}{3}\ \ln|x| - \frac{1}{3}\,\frac{x^3}{3} + c$$

$$(3)\quad \int \frac{1}{x^2\sqrt{9-x^2}}\, dx \ . \ [3]$$

الحل : باستخدام التعويضات المثلثية

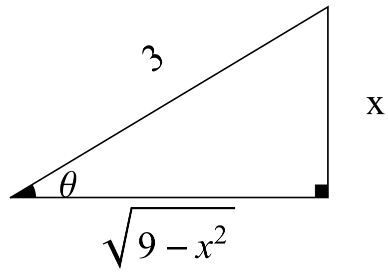
ضع $x = 3\sin\theta \implies \sin\theta = \frac{x}{3}$

$$dx = 3\cos\theta\, d\theta$$

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\left(1-\sin^2\theta\right)} = \sqrt{9\cos^2\theta} = 3\cos\theta$$

$$\int \frac{1}{x^2\sqrt{9-x^2}}\, dx = \int \frac{3\cos\theta}{(3\sin\theta)^2\, 3\cos\theta}\, d\theta = \int \frac{1}{9\sin^2\theta}\, d\theta$$

$$= \frac{1}{9}\int \frac{1}{\sin^2\theta}\, d\theta = \frac{1}{9}\int \csc^2\theta\, d\theta = \frac{1}{9}\left(-\cot\theta\right) + c = -\frac{1}{9}\, \cot\theta + c$$



من المثلث نجد أن : $\cot\theta = \dfrac{\sqrt{9-x^2}}{x}$

$$\int \frac{1}{x^2\sqrt{9-x^2}}\, dx = -\frac{1}{9}\, \frac{\sqrt{9-x^2}}{x} + c$$

(4) $\displaystyle\int \frac{2x^2+9x-9}{x^3-9x}\, dx$. [3]

الحل : باستخدام طريقة الكسور الجزئية

$$\frac{2x^2+9x-9}{x^3-9x} = \frac{2x^2+9x-9}{x(x^2-9)} = \frac{2x^2+9x-9}{x(x-3)(x+3)} = \frac{A_1}{x} + \frac{A_2}{x-3} + \frac{A_3}{x+3}$$

$$\frac{2x^2+9x-9}{x^3-9x} = \frac{A_1(x-3)(x+3)}{x(x-3)(x+3)} + \frac{A_2 x(x+3)}{(x-3)x(x+3)} + \frac{A_3 x(x-3)}{(x+3)x(x-3)}$$

$$2x^2+9x-9 = A_1(x-3)(x+3) + A_2x(x+3) + A_3x(x-3)$$

$$= A_1\left(x^2-9\right) + A_2x^2 + 3A_2x + A_3x^2 - 3A_3x$$

$$= A_1x^2 - 9A_1 + A_2x^2 + 3A_2x + A_3x^2 - 3A_3x$$

$$= \left(A_1+A_2+A_3\right)x^2 + \left(3A_2-3A_3\right)x - 9A_1$$

بمساواة معاملات كثيرتي الحدود

$$\begin{array}{lcl} A_1+A_2+A_3 = 2 & \longrightarrow & (1) \\ 3A_2-3A_3 = 9 & \longrightarrow & (2) \\ -9A_1 = -9 & \longrightarrow & (3) \end{array}$$

من المعادلة (3) نجد أن : $A_1 = 1$

المعادلة (1) تصبح : (4) $A_2 + A_3 = 1 \quad \longrightarrow$

المعادلة (2) تصبح : (5) $A_2 - A_2 = 3 \quad \longrightarrow$

بجمع المعادلتين (5) , (4) نجد أن : $A_2 = 2 \impliedby 2A_2 = 4$

من المعادلة (4) نجد أن : $2 + A_3 = 1 \implies A_3 = -1$

$$\int \frac{2x^2+9x-9}{x^3-9x}\,dx = \int \left(\frac{1}{x} + \frac{2}{x-3} + \frac{-1}{x+3}\right)\,dx$$
$$= \int \frac{1}{x}\,dx + 2\int \frac{1}{x-3}\,dx - \int \frac{1}{x+3}\,dx$$
$$= \ln|x| + 2\ln|x-3| - \ln|x+3| + c$$

(5) $\int \frac{1}{x+\sqrt{x}}\,dx$. [2]

الحل :

ضع $u = \sqrt{x} \implies x = u^2$

$dx = 2u\,du$

$$\int \frac{1}{x+\sqrt{x}}\,dx = \int \frac{2u}{u^2+u}\,du = 2\int \frac{u}{u(u+1)}\,du = 2\int \frac{1}{u+1}\,du$$
$$= 2\ln|u+1| + c = 2\ln\left|\sqrt{x}+1\right| + c$$

الجزء الثالث (19 درجة) :

(1) احسب $\lim\limits_{x\to 0} \frac{x+1-e^x}{3x^2}$. [2]

الحل : $\left(\frac{0}{0}\right) \quad \lim\limits_{x\to 0} \frac{x+1-e^x}{3x^2}$

باستخدام قاعدة لوبيتال

$$\lim_{x\to 0} \frac{1-e^x}{6x} \quad \left(\frac{0}{0}\right)$$

باستخدام قاعدة لوبيتال

$$\lim_{x\to 0} \frac{-e^x}{6} = \frac{-e^0}{6} = -\frac{1}{6}$$

إذاً $\lim\limits_{x\to 0} \frac{x+1-e^x}{3x^2} = -\frac{1}{6}$

(2) بين فيما إذا كان التكامل المعتل $\int_{-\infty}^{0} x\, e^{x^2+1}\,dx$ متقارباً أم متباعداً . [3]

الحل :

$$\int_{-\infty}^{0} x\, e^{x^2+1}\, dx = \lim_{t\to-\infty} \int_{t}^{0} x\, e^{x^2+1}\, dx = \lim_{t\to-\infty} \left(\frac{1}{2} \int_{t}^{0} e^{x^2+1}(2x)\, dx \right)$$

$$= \lim_{t\to-\infty} \left(\frac{1}{2} \left[e^{x^2+1} \right]_{t}^{0} \right) = \lim_{t\to-\infty} \left(\frac{1}{2} \left[e^{1} - e^{t^2+1} \right] \right) = \frac{1}{2} [e - \infty] = -\infty$$

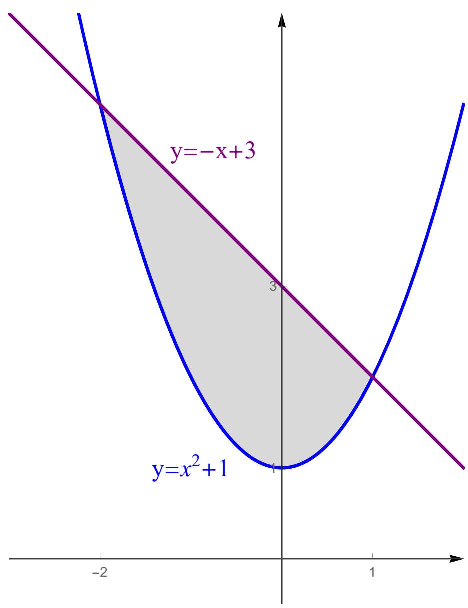
التكامل المعتل متباعد .

(3) أرسم المنطقة المحصورة بين المنحنيين $y = 3 - x$ و $y = x^2 + 1$
وجد مساحتها . [3]

الحل :

المنحنى $y = 3 - x$ يمثل خط مستقيم ميله -1 ويمر بالنقطة $(0,3)$.

المنحنى $y = x^2 + 1$ يمثل قطع مكافئ رأسه $(0,1)$ وفتحته للأعلى .



إيجاد نقاط تقاطع المنحنيين $y = 3 - x$ و $y = x^2 + 1$:

$$x^2 + 1 = 3 - x \implies x^2 + x - 2 = 0 \implies (x-1)(x+2) = 0 \implies x = 1\,,\ x = -2$$

$$\text{A} = \int_{-2}^{1} \left[(3-x) - (x^2+1) \right]\, dx = \int_{-2}^{1} \left(-x^2 - x + 2 \right)\, dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^{1}$$

$$= \left(-\frac{(1)^3}{3} - \frac{(1)^2}{2} + 2(1) \right) - \left(-\frac{(-2)^3}{3} - \frac{(-2)^2}{2} + 2(-2) \right)$$

$$= -\frac{1}{3} - \frac{1}{2} + 2 - \frac{8}{3} + 2 + 4 = 8 - \frac{9}{3} - \frac{1}{2} = \frac{9}{2}$$

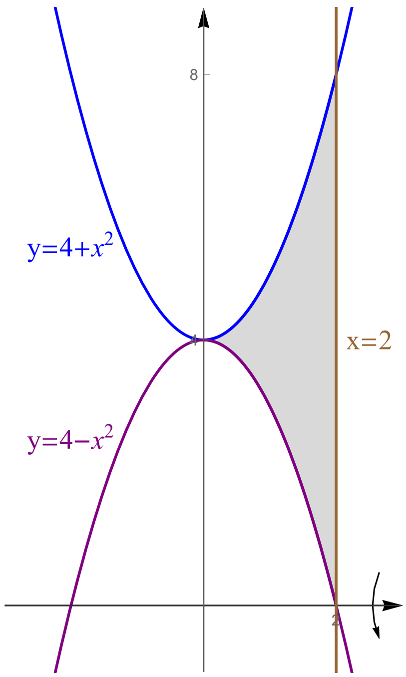
(4) ارسم المنطقة المحصورة بين المنحنيات $y = 4 - x^2$ و $y = 4 + x^2$ و $x = 2$ ، ثم جد حجم الجسم الناتج من دوران هذه المنطقة حول محور x . [3]

الحل :

$y = 4 - x^2$ يمثل قطع مكافئ رأسه $(0, 4)$ وفتحته للأسفل .

$y = 4 + x^2$ يمثل قطع مكافئ رأسه $(0, 4)$ وفتحته للأعلى .

$x = 2$ يمثل خط مستقيم يوازي محور y ويمر بالنقطة $(2, 0)$.



باستخدام طريقة الوردات :

$$\text{V} = \pi \int_0^2 \left[\left(4 + x^2\right)^2 - (4 - x^2)^2 \right] \, dx = \pi \int_0^2 \left[\left(16 + 8x^2 + x^4\right) - (16 - 8x^2 + x^4) \right] \, dx$$

$$= \pi \int_0^2 \left(16 + 8x^2 + x^4 - 16 + 8x^2 - x^4\right) \, dx = \pi \int_0^2 \left(16x^2\right) \, dx$$

$$= \pi \left[16 \, \frac{x^3}{3} \right]_0^2 = \pi \left[16 \, \frac{2^3}{3} - 16 \, \frac{0^3}{3} \right] = \frac{128\pi}{3}$$

(5) جد طول المنحنى $y = 1 + \cosh x$ من $x = 0$ إلى $x = \ln 4$. [3]

الحل :

$$y' = 0 + \sinh x = \sinh x$$

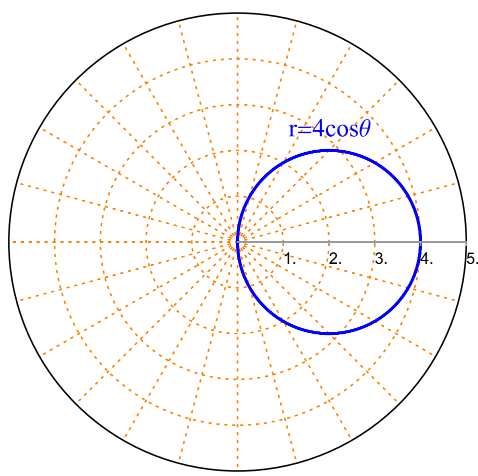
$$\begin{aligned} \mathrm{L} &= \int_0^{\ln 4} \sqrt{1 + (\sinh x)^2}\, dx = \int_0^{\ln 4} \sqrt{1 + \sinh^2 x}\, dx = \int_0^{\ln 4} \sqrt{\cosh^2 x}\, dx \\ &= \int_0^{\ln 4} |\cosh x|\ dx = \int_0^{\ln 4} \cosh x\, dx = [\sinh x]_0^{\ln 4} = \sinh(\ln 4) - \sinh(0) \\ &= \frac{e^{\ln 4} - e^{-\ln 4}}{2} - 0 = \frac{4 - \frac{1}{4}}{2} = \frac{15}{8} \end{aligned}$$

(6) حول المعادلة القطبية $r = 4\cos\theta$ إلى معادلة كارتيزية وارسمها . [2]

الحل :

$$\begin{aligned} r = 4\cos\theta &\implies r^2 = 4(r\cos\theta) \implies x^2 + y^2 = 4x \implies x^2 - 4x + y^2 = 0 \\ &\implies \left(x^2 - 4x + 4\right) + y^2 = 4 \implies (x-2)^2 + y^2 = 2^2 \end{aligned}$$

تمثل دائرة مركزها النقطة $(2, 0)$ ونصف قطرها 2 .

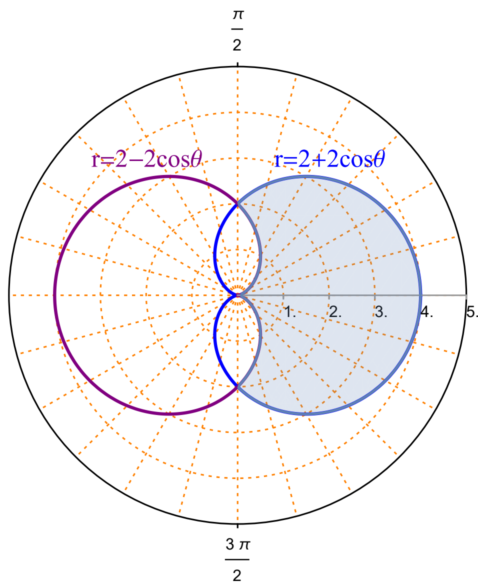


(7) ارسم المنطقة الواقعة داخل المنحنى $r = 2 + 2\cos\theta$ وخارج المنحنى $r = 2 - 2\cos\theta$ وجد مساحتها. [3]

الحل :

المنحنى $r = 2 + 2\cos\theta$ يمثل منحنى قلبي رأسه إلى اليمين ومتناظر حول المحور القطبي .

المنحنى $r = 2 - 2\cos\theta$ يمثل منحنى قلبي رأسه إلى اليسار ومتناظر حول المحور القطبي .



نقاط تقاطع المنحنى $r = 2 + 2\cos\theta$ مع المنحنى $r = 2 - 2\cos\theta$:

$$2 + 2\cos\theta = 2 - 2\cos\theta \implies 4\cos\theta = 0 \implies \cos\theta = 0 \implies \theta = \frac{\pi}{2} \ , \ \theta = \frac{3\pi}{2}$$

لاحظ أن المنطقة المطلوبة متناظرة حول المحور القطبي

$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \left[(2 + 2\cos\theta)^2 - (2 - 2\cos\theta)^2\right] \, d\theta\right)$$

$$= \int_0^{\frac{\pi}{2}} \left[\left(4 + 8\cos\theta + 4\cos^2\theta\right) - \left(4 - 8\cos\theta + 4\cos^2\theta\right)\right] \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(4 + 8\cos\theta + 4\cos^2\theta - 4 + 8\cos\theta - 4\cos^2\theta\right) \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 16\cos\theta \, d\theta = 16\left[\sin\theta\right]_0^{\frac{\pi}{2}} = 16\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right] = 16(1 - 0) = 16$$

111 ريض - حساب التكامل
الفصل الدراسي الثاني 1444 هـ
حل الاختبار الفصلي
د. طارق عبدالرحمن مُحَمَّد الفاضل

السؤال الأول (12 درجة) :

(1) استخدم مجموع ريمان لحساب التكامل المحدد $\int_1^3 (5x-6)\,dx$.

الحل : $[a,b]=[1,3]$ و $f(x)=5x-6$.

$$\Delta_x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

$$x_k = a + k\,\Delta_x = 1 + k\left(\frac{2}{n}\right) = 1 + \frac{2k}{n}$$

$$f\left(x_k\right) = 5\left(1+\frac{2k}{n}\right) + 6 = 5 + \frac{10k}{n} - 6 = \frac{10k}{n} - 1$$

$$R_n = \sum_{k=1}^{n} f\left(x_k\right)\ \Delta_x = \sum_{k=1}^{n}\left(\frac{10k}{n}-1\right)\ \left(\frac{2}{n}\right) = \sum_{k=1}^{n}\left(\frac{20k}{n^2}-\frac{2}{n}\right) = \sum_{k=1}^{n}\frac{20k}{n^2} - \sum_{k=1}^{n}\frac{2}{n}$$

$$= \frac{20}{n^2}\sum_{k=1}^{n} k - \frac{2}{n}\sum_{k=1}^{n} 1 = \frac{20}{n^2}\ \frac{n(n+1)}{2} - \frac{2}{n}\ (n) = 10\left(\frac{n+1}{n}\right) - 2$$

$$\int_1^3 (5x-6)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[10\left(\frac{n+1}{n}\right) - 2\right] = 10(1) - 2 = 8$$

(2) جد $F'(x)$ إذا كانت $F(x) = \int_{\sqrt{\sin x}}^{4^{2x}} \left(\sqrt{2t^3-2t^2}\right)\,dt$

الحل :

$$F'(x) = \frac{d}{dx}\int_{\sqrt{\sin x}}^{4^{2x}} \left(\sqrt{2t^3-2t^2}\right)\,dt$$

$$= \sqrt{2\left(4^{2x}\right)^3 - 2\left(4^{2x}\right)^2}\ \left(4^{2x}(2)\ln 4\right) - \sqrt{2\left(\sqrt{\sin x}\right)^3 - 2\left(\sqrt{\sin x}\right)^2}\ \left(\frac{1}{2\sqrt{\sin x}}\ \cos x\right)$$

$$= 2\ln 4\left(4^{2x}\right)\sqrt{2\left(4^{6x}\right) - 2\left(4^{4x}\right)} - \frac{\cos x\sqrt{2\left(\sin x\right)^{\frac{3}{2}} - 2\sin x}}{2\sqrt{\sin x}}$$

احسب $\frac{dy}{dx}$ فيمايلي :

$$y = 3^{\sqrt{x}}\ \sinh^{-1}(x) \quad (3)$$

الحل :

$$\frac{dy}{dx} = \left(3^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)\ln 3\right)\sinh^{-1}(x) + 3^{\sqrt{x}}\ \frac{1}{\sqrt{1+x^2}}$$
$$= \frac{3^{\sqrt{x}}\sinh^{-1}(x)\ln 3}{2\sqrt{x}} + \frac{3^{\sqrt{x}}}{\sqrt{1+x^2}}$$

$$y = \tan^{-1}(x)\ \log_4|1-\cot(3x)| \quad (4)$$

الحل :

$$\frac{dy}{dx} = \left(\frac{1}{1+x^2}\right)\log_4|1-\cot(3x)| + \tan^{-1}(x)\left(\frac{0-\left(-\csc^2(3x)\ (3)\right)}{1-\cot(3x)}\ \frac{1}{\ln 4}\right)$$
$$= \frac{\log_4|1-\cot(3x)|}{1+x^2} + \frac{3\csc^2(3x)\ \tan^{-1}(x)}{(1-\cot(3x))\ln 4}$$

$$y = x^{\sin x} \quad (5)$$

الحل :

$$y = x^{\sin x} \implies \ln|y| = \ln\left|x^{\sin x}\right| = \sin x\ \ln|x|$$

باشتقاق الطرفين

$$\frac{y'}{y} = \cos x\ \ln|x| + \sin x\ \left(\frac{1}{x}\right)$$
$$y' = y\ \left[\cos x\ \ln|x| + \frac{\sin x}{x}\right]$$
$$y' = x^{\sin x}\ \left[\cos x\ \ln|x| + \frac{\sin x}{x}\right]$$

$$y = sech^{-1}\left(\sqrt{x}\right) \quad (6)$$

الحل :

$$\frac{dy}{dx} = \frac{-1}{\sqrt{x}\ \sqrt{1-\left(\sqrt{x}\right)^2}}\ \left(\frac{1}{2\sqrt{x}}\right) = \frac{-1}{2x\ \sqrt{1-x}}$$

السؤال الثاني (18 درجة) : أحسب التكاملات التالية

$$\int \frac{\cos(\ln|x|)}{x}\, dx \quad (1)$$

الحل :

$$\int \frac{\cos(\ln|x|)}{x}\, dx = \int \cos(\ln|x|)\ \frac{1}{x}\, dx = \sin(\ln|x|) + c$$

باستخدام القانون

$$\int \cos(f(x))\ f'(x)\, dx = \sin(f(x)) + c$$

$$\int \sqrt{(4-2x^2)^3}\, x\, dx \quad (2)$$

الحل :

$$\int \sqrt{(4-2x^2)^3}\, x\, dx = \int \left[(4-2x^2)^3\right]^{\frac{1}{2}}\ x\, dx = \int (4-2x^2)^{\frac{3}{2}}\ x\, dx$$

$$= \frac{1}{-4}\int (4-2x^2)^{\frac{3}{2}}\ (-4x)\, dx = -\frac{1}{4}\ \frac{(4-2x^2)^{\frac{5}{2}}}{\frac{5}{2}} + c = -\frac{(4-2x^2)^{\frac{5}{2}}}{10} + c$$

باستخدام القانون $\int [f(x)]^n\, f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c$ حيث $n \neq -1$.

$$\int_0^1 3x\, 5^{2-x^2}\, dx \quad (3)$$

الحل :

$$\int_0^1 3x\, 5^{2-x^2}\, dx = \frac{3}{-2}\int_0^1 5^{2-x^2}(-2x)\, dx = \frac{3}{-2}\left[\frac{5^{2-x^2}}{\ln 5}\right]_0^1$$

$$= \frac{3}{-2}\left[\frac{5^{2-(1)^2}}{\ln 5} - \frac{5^{2-(0)^2}}{\ln 5}\right] = \frac{3}{-2}\left[\frac{5^1}{\ln 5} - \frac{5^2}{\ln 5}\right] = \frac{3}{-2}\left(\frac{-20}{\ln 5}\right) = \frac{30}{\ln 5}$$

باستخدام القانون $\int a^{f(x)}\, f'(x)\, dx = \frac{a^{f(x)}}{\ln a} + c$.

$$\int \frac{1-2x}{\sqrt{1-x^2}}\, dx \quad (4)$$

الحل :

$$\int \frac{1-2x}{\sqrt{1-x^2}}\,dx = \int \left(\frac{1}{\sqrt{1-x^2}} + \frac{-2x}{\sqrt{1-x^2}}\right)\,dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{-2x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx + \int \left(1-x^2\right)^{-\frac{1}{2}}\,(-2x)\,dx$$

$$= \sin^{-1}(x) + \frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = \sin^{-1}(x) + 2\sqrt{1-x^2} + c$$

$$\int \frac{\cos 2x}{4+\sin^2 2x}\,dx \quad (5)$$

الحل :

$$\int \frac{\cos 2x}{4+\sin^2 2x}\,dx = \int \frac{\cos 2x}{(2)^2+(\sin 2x)^2}\,dx = \frac{1}{2}\int \frac{\cos 2x\,(2)}{(2)^2+(\sin 2x)^2}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\tan^{-1}\left(\frac{\sin 2x}{2}\right)\right] + c = \frac{1}{4}\,\tan^{-1}\left(\frac{\sin 2x}{2}\right) + c$$

باستخدام القانون

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\,\tan^{-1}\left(\frac{f(x)}{a}\right) + c$$

$$\int \frac{sech\left(\sqrt{x}\right)\tanh\left(\sqrt{x}\right)}{\sqrt{x}}\,dx \quad (6)$$

الحل :

$$\int \frac{sech\left(\sqrt{x}\right)\tanh\left(\sqrt{x}\right)}{\sqrt{x}}\,dx = \int sech\left(\sqrt{x}\right)\tanh\left(\sqrt{x}\right)\,\left(\frac{1}{\sqrt{x}}\right)\,dx$$

$$= 2\int sech\left(\sqrt{x}\right)\tanh\left(\sqrt{x}\right)\,\left(\frac{1}{2\sqrt{x}}\right)\,dx = 2\left[-sech\left(\sqrt{x}\right)\right] + c = -2sech\left(\sqrt{x}\right) + c$$

باستخدام القانون

$$\int sech\left(f(x)\right)\tanh\left(f(x)\right)\,f'(x)\,dx = -sech\left(f(x)\right) + c$$

$$\int \frac{2}{\sqrt{9-e^{6x}}}\,dx \quad (7)$$

الحل :

$$\int \frac{2}{\sqrt{9-e^{6x}}}\,dx = 2\int \frac{1}{\sqrt{(3)^2-\left(e^{3x}\right)^2}}\,dx = \frac{2}{3}\int \frac{e^{3x}\,(3)}{e^{3x}\,\sqrt{(3)^2-\left(e^{3x}\right)^2}}\,dx$$

$$= \frac{2}{3}\,\left[-\frac{1}{3}sech^{-1}\left(\frac{e^{3x}}{3}\right)\right] + c = -\frac{2}{9}\,sech^{-1}\left(\frac{e^{3x}}{3}\right) + c$$

باستخدام القانون

$$\int \frac{f'(x)}{f(x)\,\sqrt{a^2-[f(x)]^2}}\,dx = -\frac{1}{a}\,sech^{-1}\left(\frac{f(x)}{a}\right) + c \;.$$

(8) $\int (x+1)\,\sqrt{x-2}\,dx$

الحل : بوضع $u+3 = x+1 \implies u = x-2$

$dx = du$

$$\int (x+1)\,\sqrt{x-2}\,dx = \int (u+3)\sqrt{u}\,du = \int (u+3)u^{\frac{1}{2}}\,du$$

$$= \int \left(u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\right)\,du = \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + 3\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{5}\,\left(x-2\right)^{\frac{5}{2}} + 2\,\left(x-2\right)^{\frac{3}{2}} + c$$

(9) $\int e^{3x}\sinh 2x\,dx$

الحل :

$$\int e^{3x}\sinh 2x\,dx = \int e^{3x}\left(\frac{e^{2x}-e^{-2x}}{2}\right)\,dx = \int \left(\frac{e^{5x}-e^{x}}{2}\right)\,dx$$

$$= \int \left(\frac{e^{5x}}{2} - \frac{e^{x}}{2}\right)\,dx = \frac{1}{2}\,\frac{1}{5}\int e^{5x}\,(5)\,dx - \frac{1}{2}\int e^{x}\,dx$$

$$= \frac{1}{10}\,e^{5x} - \frac{1}{2}\,e^{x} + c = \frac{e^{5x}}{10} - \frac{e^{x}}{2} + c$$

باستخدام القانون $\int e^{f(x)}\,f'(x)\,dx = e^{f(x)} + c$

111 ريض - حساب التكامل
الفصل الدراسي الثاني 1444 هـ
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الجزء الأول (7 درجات):

(1) جد قيمة c التي تحقق نظرية القيمة المتوسطة للدالة $f(x)=\sqrt{x+1}$ على الفترة $[-1,8]$. [3]

الحل : باستخدام العلاقة $(b-a)f(c)=\int_a^b f(x)\,dx$

حيث $f(x)=\sqrt{x+1}$ و $[a,b]=[-1,8]$

$$(8-(-1))\sqrt{c+1}=\int_{-1}^{8}\sqrt{x+1}\,dx=\int_{-1}^{8}(x+1)^{\frac{1}{2}}dx$$

$$9\sqrt{c+1}=\left[\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-1}^{8}=\left[\frac{2}{3}(x+1)^{\frac{3}{2}}\right]_{-1}^{8}=\frac{2}{3}(8+1)^{\frac{3}{2}}-\frac{2}{3}(-1+1)^{\frac{3}{2}}$$

$$9\sqrt{c+1}=\frac{2}{3}(9)^{\frac{3}{2}}-0=\frac{2}{3}(27)=2(9)=18\implies\sqrt{c+1}=\frac{18}{9}=2$$

$$\implies c+1=4\implies c=4-1=3\in(-1,8)$$

قيمة c المطلوبة هي 3 .

(2) جد $F'(x)$ إذا كانت $F(x)=\int_{5^{2x}}^{\ln 3x}\frac{t^2}{2t^2+1}\,dt$. [2]

الحل : $F'(x)=\frac{d}{dx}\int_{5^{2x}}^{\ln 3x}\frac{t^2}{2t^2+1}\,dt$

$$F'(x)=\frac{(\ln 3x)^2}{2(\ln 3x)^2+1}\left(\frac{3}{3x}\right)-\frac{\left(5^{2x}\right)^2}{2\left(5^{2x}\right)^2+1}\left(5^{2x}\,(2)\,\ln 5\right)$$

$$=\frac{(\ln 3x)^2}{x\left[2(\ln 3x)^2+1\right]}-\frac{2\,5^{6x}\,\ln 5}{2\,5^{4x}+1}$$

(3) جد $f'(x)$ إذا كانت $f(x)=\tanh^{-1}\left(e^{3x}\right)+\log_6\left|\cosh 2x+2^{x+1}\right|$. [2]

الحل :

$$f'(x)=\frac{e^{3x}\,(3)}{1-\left(e^{3x}\right)^2}+\frac{\sinh 2x\,(2)+2^{x+1}\,(1)\ln 2}{\cosh 2x+2^{x+1}}\,\frac{1}{\ln 6}$$

$$f'(x) = \frac{3\,e^{3x}}{1-e^{6x}} + \frac{2\sinh 2x + 2^{x+1}\ln 2}{\ln 6\left[\cosh 2x + 2^{x+1}\right]}$$

الجزء الثاني (13 درجة) : أحسب التكاملات التالية :

(1) $\int \frac{2}{\sqrt{x^2-4x+13}}\,dx$. [3]

الحل : باستخدام الإكمال إلى مربع كامل

$$\int \frac{2}{\sqrt{x^2-4x+13}}\,dx = \int \frac{2}{\sqrt{(x^2-4x+4)+9}}\,dx$$

$$= 2\int \frac{1}{\sqrt{(x-2)^2+(3)^2}}\,dx = 2\,\sinh^{-1}\left(\frac{x-2}{3}\right) + c$$

باستخدام القانون $\int \frac{f'(x)}{\sqrt{\left[f(x)\right]^2+a^2}}\,dx = \sinh^{-1}\left(\frac{f(x)}{a}\right) + c$ حيث $a > 0$

(2) $\int \sec^{-1} x\,dx$. [2]

الحل : باستخدام طريقة التكامل بالتجزئ

$$u = \sec^{-1} x \qquad dv = 1\,dx$$
$$du = \frac{1}{x\sqrt{x^2-1}}\,dx \qquad v = x$$

$$\int \sec^{-1} x\,dx = x\sec^{-1} x - \int x\,\frac{1}{x\sqrt{x^2-1}}\,dx$$

$$= x\sec^{-1} x - \int \frac{1}{\sqrt{x^2-1}}\,dx = x\sec^{-1} x - \cosh^{-1} x + c$$

(3) $\int \frac{1}{(9-x^2)^{\frac{3}{2}}}\,dx$. [3]

الحل : باستخدام التعويضات المثلثية

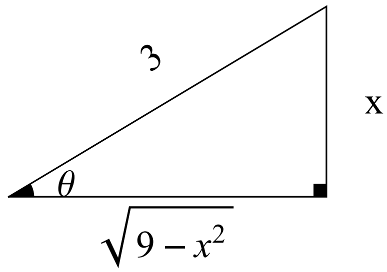
ضع $\sin\theta = \frac{x}{3} \implies x = 3\sin\theta$

$$dx = 3\cos\theta\,d\theta$$

$$\left(9-x^2\right)^{\frac{3}{2}} = \left(9-9\sin^2\theta\right)^{\frac{3}{2}} = \left[9\left(1-\sin^2\theta\right)\right]^{\frac{3}{2}} = \left[9\,\cos^2\theta\right]^{\frac{3}{2}}$$

$$= (9)^{\frac{3}{2}}\,\left(\cos^2\theta\right)\frac{3}{2} = 27\cos^3\theta$$

$$\int \frac{1}{(9-x^2)^{\frac{3}{2}}}\,dx = \int \frac{3\cos\theta}{27\cos^3\theta}\,d\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\,d\theta$$
$$= \frac{1}{9}\int \sec^2\theta\,d\theta = \frac{1}{9}\ \tan\theta + c$$



من المثلث نجد أن : $\tan\theta = \dfrac{x}{\sqrt{9-x^2}}$

$$\int \frac{1}{(9-x^2)^{\frac{3}{2}}}\,dx = \frac{1}{9}\,\frac{x}{\sqrt{9-x^2}} + c$$

(4) $\int \dfrac{5x^2+x-1}{x^3+x^2}\,dx$. [3]

الحل : باستخدام طريقة الكسور الجزئية

$$\frac{5x^2+x-1}{x^3+x^2} = \frac{5x^2+x+1}{x^2(x+1)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x+1}$$
$$\frac{5x^2+x-1}{x^2+x^3} = \frac{A_1x(x+1)}{x^2(x+1)} + \frac{A_2(x+1)}{x^2(x+1)} + \frac{A_3x^2}{(x+1)x^2}$$
$$5x^2+x-1 = A_1x(x+1) + A_2(x+1) + A_3x^2$$
$$= A_1\left(x^2+x\right) + A_2x + A_2 + A_3x^2$$
$$= A_1x^2 + A_1x + A_2x + A_2 + A_3x^2$$
$$= \left(A_1+A_3\right)x^2 + \left(A_1+A_2\right)x + A_2$$

بمساواة معاملات كثيرتي الحدود

$$\begin{array}{lcl} A_1 + A_3 = 5 & \longrightarrow & (1) \\ A_1 + A_2 = 1 & \longrightarrow & (2) \\ A_2 = -1 & \longrightarrow & (3) \end{array}$$

من المعادلة (3) نجد أن : $A_2 = -1$

من المعادلة (2) نجد أن : $A_1 - 1 = 1 \implies A_1 = 1 + 1 = 2$

من المعادلة (1) نجد أن : $2 + A_3 = 5 \implies A_3 = 5 - 2 = 3$

$$\int \frac{5x^2 + x - 1}{x^3 + x^2}\,dx = \int \left(\frac{2}{x} + \frac{-1}{x^2} + \frac{3}{x+1}\right) dx$$

$$= 2\int \frac{1}{x}\,dx - \int x^{-2}\,dx + 3\int \frac{1}{x+1}\,dx$$

$$= 2\ln|x| - \frac{x^{-1}}{-1} + 3\ln|x+1| + c = 2\ln|x| + \frac{1}{x} + 3\ln|x+1| + c$$

(5) $\int \frac{1}{\sqrt{3+\sqrt{x}}}\,dx$. [2]

الحل :

ضع $u = \sqrt{3+\sqrt{x}} \implies u^2 = 3 + \sqrt{x} \implies u^2 - 3 = \sqrt{x} \implies \left(u^2 - 3\right)^2 = x$

$$2\left(u^2-3\right) 2u\,du = dx \implies 4u\left(u^2-3\right)\,du = dx$$

$$\int \frac{1}{\sqrt{3+\sqrt{x}}}\,dx = \int \frac{4u\left(u^2-3\right)}{u}\,du = \int 4\left(u^2-3\right)\,du = \int \left(4u^2 - 12\right)\,du$$

$$= 4\,\frac{u^3}{3} - 12u + c = 4\,\frac{\left(\sqrt{3+\sqrt{x}}\right)^3}{3} - 12\left(\sqrt{3+\sqrt{x}}\right) + c$$

الجزء الثالث (20 درجة) :

(1) احسب $\lim_{x\to 0} \frac{e^x + e^{-x} - 2\cos x}{x^2 + \sin x}$. [2]

الحل : $\left(\frac{0}{0}\right)$ $\quad \lim_{x\to 0} \frac{e^x + e^{-x} - 2\cos x}{x^2 + \sin x}$

باستخدام قاعدة لوبيتال

$$\lim_{x\to 0} \frac{e^x + e^{-x}(-1) - 2(-\sin x)}{2x + \cos x} = \lim_{x\to 0} \frac{e^x - e^{-x} + 2\sin x}{2x + \cos x}$$

$$= \frac{e^0 - e^0 + 2\sin(0)}{2(0) + \cos(0)} = \frac{1 - 1 + 2(0)}{0 + 1} = \frac{0}{1} = 0$$

إذاً $\lim_{x\to 0} \frac{e^x + e^{-x} - 2\cos x}{x^2 + \sin x} = 0$

(2) بين فيما إذا كان التكامل المعتل $\int_5^\infty \frac{1}{x-4}\,dx$ متقارباً أم متباعداً . [3]

الحل :

$$\int_5^\infty \frac{1}{x-4}\,dx = \lim_{t\to\infty}\int_5^t \frac{1}{x-4}\,dx = \lim_{t\to\infty}\left[\ln|x-4|\right]_5^t$$
$$= \lim_{t\to\infty}\left[\ln|t-4| - \ln|5-4|\right] = \lim_{t\to\infty}\left[\ln|t-4| - \ln|1|\right]$$
$$= \lim_{t\to\infty}\ln|t-4| = \infty$$

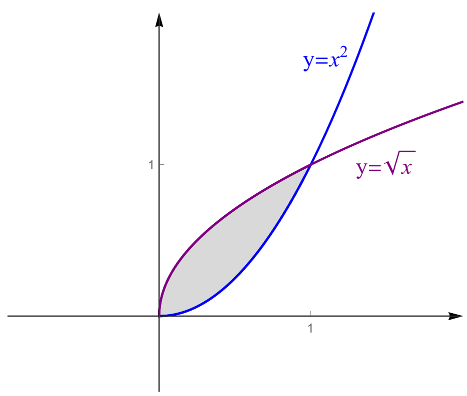
التكامل المعتل متباعد .

(3) أرسم المنطقة المحصورة بين المنحنيين $y = x^2$ و $y = \sqrt{x}$
وجد مساحتها . [3]

الحل :

المنحنى $y = x^2$ يمثل قطع مكافئ رأسه $(0,0)$ وفتحته للأعلى .

المنحنى $y = \sqrt{x}$ يمثل النصف العلوي من القطع المكافئ الذي رأسه $(0,0)$ وفتحته إلى اليمين .



إيجاد نقاط تقاطع المنحنيين $y = x^2$ و $y = \sqrt{x}$:

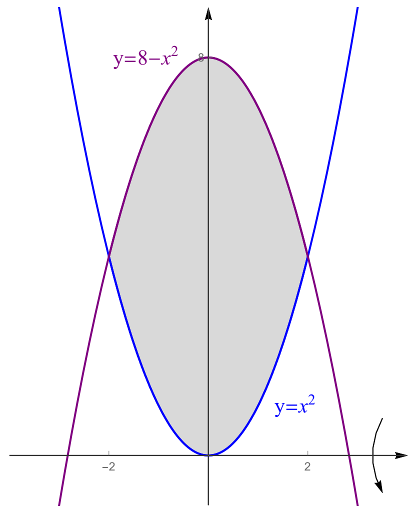
$$x^2 = \sqrt{x} \implies x^4 = x \implies x^4 - x = 0 \implies x(x^3-1) = 0$$
$$\implies x = 0\,,\ x^3 - 1 = 0 \implies x = 0\,,\ x^3 = 1 \implies x = 0\,,\ x = 1$$
$$\mathrm{A} = \int_0^1 \left[\sqrt{x} - x^2\right]\,dx = \int_0^1 \left[x^{\frac{1}{2}} - x^2\right]\,dx = \left[\frac{2}{3}\,x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$
$$= \left(\frac{2}{3}\,(1)^{\frac{3}{2}} - \frac{1^3}{3}\right) - \left(\frac{2}{3}\,(0)^{\frac{3}{2}} - \frac{0^3}{3}\right) = \left(\frac{2}{3} - \frac{1}{3}\right) - (0-0) = \frac{1}{3}$$

(4) ارسم المنطقة المحصورة بين المنحنيات $y = x^2$ و $y = 8 - x^2$ ، ثم جد حجم الجسم الناتج من دوران هذه المنطقة حول محور x . [3]

الحل :

$y = x^2$ يمثل قطع مكافئ رأسه $(0,0)$ وفتحته للأعلى .

$y = 8 - x^2$ يمثل قطع مكافئ رأسه $(0,8)$ وفتحته للأسفل .



إيجاد نقاط تقاطع المنحنيين $y = x^2$ و $y = 8 - x^2$:

$$x^2 = 8 - x^2 \implies 2x^2 - 8 = 0 \implies x^2 - 4 = 0 \implies (x-2)(x+2) = 0 \implies x = -2\,,\ x = 2$$

باستخدام طريقة الوردات :

$$\mathrm{V} = \pi \int_{-2}^{2} \left[\left(8 - x^2\right)^2 - (x^2)^2 \right] \, dx = \pi \int_{-2}^{2} \left[\left(64 - 16x^2 + x^4\right) - x^4 \right] \, dx$$

$$= \pi \int_{-2}^{2} \left(64 - 16x^2\right) \, dx = \pi \left[64x - \frac{16}{3}\, x^3 \right]_{-2}^{2}$$

$$= \pi \left[\left(64(2) - \frac{16}{3}\,(2)^3 \right) - \left(64(-2) - \frac{16}{3}\,(-2)^3 \right) \right] = \pi \left[\left(128 - \frac{128}{3} \right) - \left(-128 + \frac{128}{3} \right) \right]$$

$$\pi \left(128 - \frac{128}{3} + 128 - \frac{128}{3} \right) = \pi \left(256 - \frac{256}{3} \right) = \pi \left[256 \left(1 - \frac{1}{3} \right) \right] = \frac{512\pi}{3}$$

(5) جد طول المنحنى $y = \frac{1}{3}x^{\frac{3}{2}} - \sqrt{x}$ من $x = 1$ إلى $x = 4$. [3]

الحل :

$$y' = \frac{3}{2}\,\frac{1}{3}\,x^{\frac{1}{2}} - \frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2} - \frac{1}{2\sqrt{x}}$$

$$\mathrm{L} = \int_1^4 \sqrt{1 + \left(\frac{\sqrt{x}}{2} - \frac{1}{2\sqrt{x}}\right)^2}\, dx = \int_1^4 \sqrt{1 + \left(\frac{x}{4} - \frac{1}{2} + \frac{1}{4x}\right)}\, dx$$

$$= \int_1^4 \sqrt{\frac{x}{4} + \frac{1}{2} + \frac{1}{4x}}\, dx = \int_1^4 \sqrt{\left(\frac{\sqrt{x}}{2} - \frac{1}{2\sqrt{x}}\right)^2}\, dx = \int_1^4 \left|\frac{\sqrt{x}}{2} - \frac{1}{2\sqrt{x}}\right| dx$$

$$= \int_1^4 \left(\frac{\sqrt{x}}{2} - \frac{1}{2\sqrt{x}}\right) dx = \frac{1}{2}\int_1^4 \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx = \frac{1}{2}\int_1^4 \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x^{\frac{3}{2}} + 2\sqrt{x}\right]_1^4 = \frac{1}{2}\left[\left(\frac{2}{3}(4)^{\frac{3}{2}} + 2\sqrt{4}\right) - \left(\frac{2}{3}(1)^{\frac{3}{2}} + 2\sqrt{1}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{2}{3}(8) + 2(2)\right) - \left(\frac{2}{3}(1) + 2(1)\right)\right] = \frac{1}{2}\left[\frac{16}{3} + 4 - \frac{2}{3} - 2\right]$$

$$= \frac{1}{2}\left[\frac{14}{3} + 2\right] = \frac{1}{2}\left(\frac{20}{3}\right) = \frac{10}{3}$$

(6) حول المعادلة الكارتيزية (الديكارتية) $\frac{x}{\sqrt{x^2+y^2}} = 5y$ إلى معادلة قطبية . [2]

الحل :

$$\frac{x}{\sqrt{x^2+y^2}} = 5y \implies \frac{r\cos\theta}{r} = 5(r\sin\theta) \implies 5r\sin\theta = \cos\theta$$

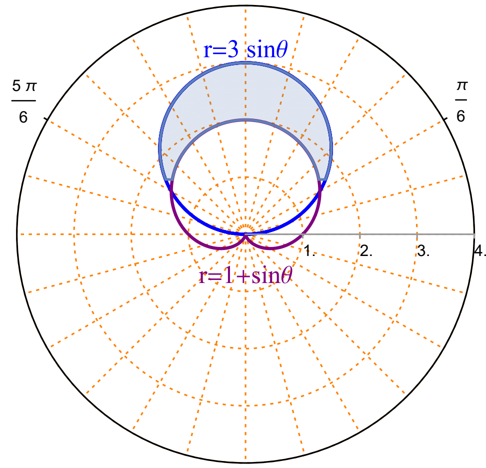
$$\implies r = \frac{1}{5}\left(\frac{\cos\theta}{\sin\theta}\right) \implies r = \frac{1}{5}\cot\theta = \frac{\cot\theta}{5}$$

(7) ارسم المنطقة الواقعة داخل المنحنى $r = 3\sin\theta$ وخارج المنحنى $r = 1 + \sin\theta$ وجد مساحتها. [4]

الحل :

المنحنى $r = 3\sin\theta$ يمثل دائرة تمر بالقطب ومركزها $(r,\theta) = \left(\frac{3}{2}, \frac{\pi}{2}\right)$ ونصف قطرها $\frac{3}{2}$.

المنحنى $r = 1 + \sin\theta$ يمثل منحنى قلبي رأسه إلى الأعلى ومتناظر حول الخط $\theta = \frac{\pi}{2}$.



نقاط تقاطع المنحنى $r = 3\sin\theta$ مع المنحنى $r = 1 + \sin\theta$:

$$3\sin\theta = 1 + \sin\theta \implies 2\sin\theta = 1 \implies \sin\theta = \frac{1}{2} \implies \theta = \frac{\pi}{6} \ , \ \theta = \frac{5\pi}{6}$$

لاحظ أن المنطقة المطلوبة متناظرة حول المحور القطبي

$$A = 2\left(\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[(3\sin\theta)^2 - (1+\sin\theta)^2\right] \, d\theta\right) = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[(3\sin\theta)^2 - (1+\sin\theta)^2\right] \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[9\sin^2\theta - \left(1 + 2\sin\theta + \sin^2\theta\right)\right] \, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[8\sin^2\theta - 2\sin\theta - 1\right] \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[8\left(\frac{1-\cos 2\theta}{2}\right) - 2\sin\theta - 1\right] \, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[4 - 4\cos 2\theta - 2\sin\theta - 1\right] \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[3 - 4\cos 2\theta - 2\sin\theta\right] \, d\theta = \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(3\left(\frac{\pi}{2}\right) - 2\sin\left(2\,\frac{\pi}{2}\right) + 2\cos\left(\frac{\pi}{2}\right)\right) - \left(3\left(\frac{\pi}{6}\right) - 2\sin\left(2\,\frac{\pi}{6}\right) + 2\cos\left(\frac{\pi}{6}\right)\right)$$

$$= \left(\frac{3\pi}{2} - 2\sin(\pi) + 2\cos\left(\frac{\pi}{2}\right)\right) - \left(\frac{\pi}{2} - 2\sin\left(\frac{\pi}{3}\right) + 2\cos\left(\frac{\pi}{6}\right)\right)$$

$$= \left(\frac{3\pi}{2} - 2\sin(0) + 2(0)\right) - \left(\frac{\pi}{2} - 2\left(\frac{\sqrt{3}}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= \frac{3\pi}{2} - 0 + 0 - \frac{\pi}{2} + \sqrt{3} - \sqrt{3} = \pi$$

111 ريض - حساب التكامل
الفصل الدراسي الثالث 1444 هـ
حل الاختبار الفصلي
د. طارق عبدالرحمن مُحَمَّد الفاضل

السؤال الأول (12 درجة) :

(1) استخدم مجموع ريمان لحساب التكامل المحدد $\int_{-1}^{4} (2x+3)\, dx$.

الحل : $[a,b]=[-1,4]$ و $f(x)=2x+3$.

$$\Delta_x = \frac{b-a}{n} = \frac{4-(-1)}{n} = \frac{5}{n}$$

$$x_k = a + k\,\Delta_x = -1 + k\left(\frac{5}{n}\right) = -1 + \frac{5k}{n}$$

$$f(x_k) = 2\left(-1+\frac{5k}{n}\right) + 3 = -2 + \frac{10k}{n} + 3 = \frac{10k}{n} + 1$$

$$R_n = \sum_{k=1}^{n} f(x_k)\ \Delta_x = \sum_{k=1}^{n}\left(\frac{10k}{n}+1\right)\ \left(\frac{5}{n}\right) = \sum_{k=1}^{n}\left(\frac{50k}{n^2}+\frac{5}{n}\right) = \sum_{k=1}^{n}\frac{50k}{n^2} + \sum_{k=1}^{n}\frac{5}{n}$$

$$= \frac{50}{n^2}\sum_{k=1}^{n} k + \frac{5}{n}\sum_{k=1}^{n} 1 = \frac{50}{n^2}\ \frac{n(n+1)}{2} + \frac{5}{n}\,(n) = 25\left(\frac{n+1}{n}\right) + 5$$

$$\int_{-1}^{4}(2x+3)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[25\left(\frac{n+1}{n}\right)+5\right] = 25(1)+5 = 30$$

(2) جد $F'(x)$ إذا كانت $F(x) = \int_{\frac{1}{x}}^{5^{-2x}} \frac{t+1}{\sqrt{t^2+2}}\, dt$

الحل :

$$F'(x) = \frac{d}{dx}\int_{\frac{1}{x}}^{5^{-2x}} \frac{t+1}{\sqrt{t^2+2}}\, dt$$

$$= \frac{5^{-2x}+1}{\sqrt{\left(5^{-2x}\right)^2+2}}\ \left[5^{-2x}\,(-2)\ \ln 5\right] - \frac{\frac{1}{x}+1}{\sqrt{\left(\frac{1}{x}\right)^2+2}}\ \left(\frac{-1}{x^2}\right)$$

$$= \frac{-2\ 5^{-2x}\ \ln 5\left(5^{-2x}+1\right)}{\sqrt{5^{-4x}+2}} + \frac{\frac{1}{x}+1}{x^2\sqrt{\frac{1}{x^2}+2}}$$

احسب $\frac{dy}{dx}$ فيمايلي :

$$y = 2^{\sqrt[3]{x}}\ \tanh^{-1}(x) \quad (3)$$

الحل :

$$\frac{dy}{dx} = \left(2^{\sqrt[3]{x}}\left(\frac{1}{3}\ x^{-\frac{2}{3}}\right)\ln 2\right)\tanh^{-1}(x) + 2^{\sqrt[3]{x}}\ \frac{1}{1-x^2}$$

$$= \frac{2^{\sqrt[3]{x}}\tanh^{-1}(x)\ln 2}{3\ x^{\frac{2}{3}}} + \frac{2^{\sqrt[3]{x}}}{1-x^2}$$

$$y = \sin^{-1}(2x)\ \log_7|1-\ln|3x|| \quad (4)$$

الحل :

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{1-(2x)^2}}\ (2)\right)\log_7|1-\ln|3x|| + \sin^{-1}(2x)\left(\frac{0-\frac{3}{3x}}{1-\ln|3x|}\ \frac{1}{\ln 7}\right)$$

$$= \frac{2\ \log_7|1-\ln|3x||}{\sqrt{1-4x^2}} - \frac{\sin^{-1}(2x)}{x\,(1-\ln|3x|)\ln 7}$$

$$y = (\tan x)^{\cos(e^x)} \quad (5)$$

الحل :

$$y = (\tan x)^{\cos(e^x)} \implies \ln|y| = \ln\left|(\tan x)^{\cos(e^x)}\right| = \cos(e^x)\ \ln|\tan x|$$

باشتقاق الطرفين

$$\frac{y'}{y} = -\sin(e^x)\ e^x\ \ln|\tan x| + \cos(e^x)\ \frac{\sec^2 x}{\tan x}$$

$$y' = y\ \left[-e^x\sin(e^x)\ln|\tan x| + \frac{\sec^2 x\ \cos(e^x)}{\tan x}\right]$$

$$y' = (\tan x)^{\cos(e^x)}\ \left[\frac{\sec^2 x\ \cos(e^x)}{\tan x} - e^x\sin(e^x)\ln|\tan x|\right]$$

$$y = \coth^{-1}\left(x^2\right) \quad (6)$$

الحل :

$$\frac{dy}{dx} = \frac{-1}{1-\left(x^2\right)^2}\ (2x) = \frac{-2x}{1-x^4}$$

السؤال الثاني (18 درجة) : أحسب التكاملات التالية

$$\int e^{3x}\ \tanh\left(e^{3x}\right)\ dx \quad (1)$$

الحل :

$$\int e^{3x}\ \tanh\left(e^{3x}\right)\ dx = \frac{1}{3}\int \tanh\left(e^{3x}\right)\ \left(3e^{3x}\right)\ dx = \frac{1}{3}\ln\left|\cosh\left(e^{3x}\right)\right| + c$$

باستخدام القانون

$$\int \tanh\left(f(x)\right)\ f'(x)\,dx = \ln\left|\cosh\left(f(x)\right)\right| + c$$

$$\int \sqrt{1-x^3}\,x^2\,dx \quad (2)$$

الحل :

$$\int \sqrt{1-x^3}\,x^2\,dx = -\frac{1}{3}\int \left(1-x^3\right)^{\frac{1}{2}}\ \left(-3x^2\right)dx$$

$$= -\frac{1}{3}\,\frac{\left(1-x^3\right)^{\frac{3}{2}}}{\frac{3}{2}} + c = -\frac{2}{9}\ \left(1-x^3\right)^{\frac{3}{2}} + c$$

باستخدام القانون $\int [f(x)]^n\ f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$ حيث $n \neq -1$.

$$\int_0^1 x\,4^{2-x^2}\ dx \quad (3)$$

الحل :

$$\int_0^1 x\,4^{2-x^2}\ dx = \frac{1}{-2}\int_0^1 4^{2-x^2}(-2x)\,dx = \frac{1}{-2}\left[\frac{4^{2-x^2}}{\ln 4}\right]_0^1$$

$$= \frac{1}{-2}\left[\frac{4^{2-(1)^2}}{\ln 4} - \frac{4^{2-(0)^2}}{\ln 4}\right] = \frac{1}{-2}\left[\frac{4^1}{\ln 4} - \frac{4^2}{\ln 4}\right] = \frac{1}{-2}\left(\frac{-12}{\ln 4}\right) = \frac{6}{\ln 4}$$

باستخدام القانون $\int a^{f(x)}\ f'(x)\,dx = \frac{a^{f(x)}}{\ln a} + c$.

$$\int \frac{x-1}{\sqrt{4-x^2}}\,dx \quad (4)$$

الحل :

$$\int \frac{x-1}{\sqrt{4-x^2}}\,dx = \int \left(\frac{x}{\sqrt{4-x^2}} - \frac{1}{\sqrt{4-x^2}}\right)\,dx$$

$$= \int \frac{x}{\sqrt{4-x^2}}\,dx - \int \frac{1}{\sqrt{4-x^2}}\,dx = -\frac{1}{2}\int \left(4-x^2\right)^{-\frac{1}{2}}\ (-2x)\,dx - \int \frac{1}{\sqrt{(2)^2-x^2}}\,dx$$

$$= -\frac{1}{2}\,\frac{\left(4-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} - \sin^{-1}\left(\frac{x}{2}\right) + c = -\left(4-x^2\right)^{\frac{1}{2}} - \sin^{-1}\left(\frac{x}{2}\right) + c$$

$$\int \frac{e^{2x}}{4+e^{4x}}\,dx \quad (5)$$

الحل :

$$\int \frac{e^{2x}}{4+e^{4x}}\,dx = \frac{1}{2}\int \frac{e^{2x}\ (2)}{(2)^2+\left(e^{2x}\right)^2}\,dx$$

$$= \frac{1}{2}\ \left(\frac{1}{2}\ \tan^{-1}\left(\frac{e^{2x}}{2}\right)\right) + c = \frac{1}{4}\ \tan^{-1}\left(\frac{e^{2x}}{2}\right) + c$$

باستخدام القانون

$$\int \frac{f'(x)}{a^2+\left[f(x)\right]^2}\,dx = \frac{1}{a}\ \tan^{-1}\left(\frac{f(x)}{a}\right) + c$$

$$\int \frac{3\,csch\left(\ln|x|\right)\coth\left(\ln|x|\right)}{x}\,dx \quad (6)$$

الحل :

$$\int \frac{3\,csch\left(\ln|x|\right)\coth\left(\ln|x|\right)}{x}\,dx = 3\int csch\left(\ln|x|\right)\coth\left(\ln|x|\right)\ \frac{1}{x}\,dx$$

$$= 3\left(-csch\left(\ln|x|\right)\right) + c = -3\,csch\left(\ln|x|\right) + c$$

باستخدام القانون

$$\int csch\left(f(x)\right)\coth\left(f(x)\right)\ f'(x)\,dx = -csch\left(f(x)\right) + c$$

$$\int \frac{2}{\sqrt{16-e^{2x}}}\,dx \quad (7)$$

الحل :

$$\int \frac{2}{\sqrt{16-e^{2x}}}\,dx = 2\int \frac{1}{\sqrt{(4)^2-\left(e^x\right)^2}}\,dx = 2\int \frac{e^x}{e^x\,\sqrt{(4)^2-\left(e^x\right)^2}}\,dx$$

$$= 2\ \left[-\frac{1}{4} sech^{-1}\left(\frac{e^x}{4}\right)\right] + c = -\frac{1}{2}\ sech^{-1}\left(\frac{e^x}{4}\right) + c$$

باستخدام القانون

$$\int \frac{f'(x)}{f(x)\sqrt{a^2 - [f(x)]^2}}\, dx = -\frac{1}{a}\ sech^{-1}\left(\frac{f(x)}{a}\right) + c\ .$$

$$\int (2x-1)\sqrt{x+2}\, dx \quad (8)$$

الحل : بوضع $u = x + 2 \implies x = u - 2$

$dx = du$ و $2x - 1 = 2(u-2) - 1 = 2u - 5$

$$\int (2x-1)\sqrt{x+2}\, dx = \int (2u-5)\sqrt{u}\, du = \int (2u-5)u^{\frac{1}{2}}\ du$$

$$= \int \left(2u^{\frac{3}{2}} - 5u^{\frac{1}{2}}\right)\ du = 2\ \frac{u^{\frac{5}{2}}}{\frac{5}{2}} - 5\ \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{4}{5}\ (x+2)^{\frac{5}{2}} - \frac{10}{3}\ (x+2)^{\frac{3}{2}} + c$$

$$\int e^{-2x}\cosh 2x\, dx \quad (9)$$

الحل :

$$\int e^{-2x}\cosh 2x\, dx = \int e^{-2x}\left(\frac{e^{2x} + e^{-2x}}{2}\right)\ dx = \int \left(\frac{e^0 + e^{-4x}}{2}\right)\ dx$$

$$= \int \left(\frac{1}{2} + \frac{e^{-4x}}{2}\right)\ dx = \frac{1}{2}\int 1\, dx + \frac{1}{2}\ \frac{1}{-4}\ \int e^{-4x}\ (-4)\, dx$$

$$= \frac{1}{2}\ x - \frac{1}{8}\ e^{-4x} + c = \frac{x}{2} - \frac{e^{-4x}}{8} + c$$

باستخدام القانون $\int e^{f(x)}\ f'(x)\, dx = e^{f(x)} + c$