Local Asymptotic Normality complexity arising in a parametric statistical Lévy Model

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Abstract We consider statistical experiments associated with a Lévy process X observed along a deterministic scheme $(i\,u_n,\,1\le i\le n)$. We assume that under a probability $\mathbb{P}_\theta$, at each $t>0$, X_t has a density g_t^θ regular enough relative to a parameter $\theta\in(0,+\infty)$. We prove that the sequence of the associated statistical models has the LAN property at each θ, and we investigate the case when X is the product of an unknown parameter θ by an another Lévy process Y with known characteristics, by giving examples with Y attracted by a stable process.

Key words: Lévy processes, stable processes, local asymptotic normality, Fisher information quantity, discrete observations

1 Introduction

Motivated by mathematical finance problems (see [4]), this work is a part of an ambitious program consisting in the estimation of the parameter θ intervening in the stochastic differential equation driven by a known Lévy process Y:

$$\mathrm{d}X_t = b(\theta,X)\,\mathrm{d}t + a(\theta,X)\,\mathrm{d}Y_t, \tag{1}$$

In these kind of models, the property of local asymptotic normality property (LAN) has become an important issue, cf. Lecam [15]. This property is described as follows: a sequence of families of probabilities $(\mathbb{P}^n_\theta)_{\theta\in\Theta}$ indexed by an open set $\Theta\subset\mathbb{R}$ is said to have the LAN property at each point $\theta_0\in\Theta$ with speed $\sqrt{n}$, if the sequence of probabilities localized around θ_0, $\left(\mathbb{P}^n_{\theta_0+n^{-1/2}\,\theta}\right)_{\theta\in\{\xi\,/\,\theta_0+n^{-1/2}\,\xi\in\Theta\}}$, converges, in the sense of weak convergence of the associated likelihood processes, to a Gaussian shift $(\mathbb{P}'_\theta)_{\theta\in\mathbb{R}}$, see Subsection 2.2 for a precise definition. The LAN property allows to recover the so-called asymptotic Fisher information quantity $I(\theta_0)$, which is crucial in any estimation procedure since $1/I(\theta_0)$ provides the lower bound of the variance of any estimator of θ_0.

The LAN property was investigated by Akritas [3] in models associated with a Lévy processes X observed continuously in time over the interval $[0,n]$, $n\to\infty$. He obtains the LAN property under differentiability assumptions on the characteristics $(b_\theta,c_\theta,\nu_\theta)$ of X. With the same asymptotic, and under some conditions, Luschgy [16] obtained the Local Asymptotic Mixed Normality (LAMN) property on models associated with semimartingales. Notice that LAMN property is a more general notion than the LAN one, since it allows the Fisher information quantity to be random. Study of Lévy models are motivated amongst others by mathematical finance (see [4] for instance). With the asymptotic $[0,n]$, $n\to+\infty$, the estimation methods do not seem to be feasible in practice. Recent tendency focuses on discretized schemes, i.e. observations of the process X along the discrete scheme

$$X_{i u_n},\quad 1\le i\le n,\quad n\to+\infty, \tag{2}$$

In practice, the most interesting case of the discretization path $u_n=1/n$ turns out to be relatively difficult. Our work is a part of an ambitious program consisting in the study of the LAN or LAMN properties and the estimation of the parameter θ

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intervening in discrete observations of the process X solution of the stochastic differential equation

$$\mathrm{d}X_t = b(\theta, X)\,\mathrm{d}t + a(\theta, X)\,\mathrm{d}Y_t, \tag{3}$$

driven by a known Lévy process Y. The classical case of a Brownian motion Y in (3) has been widely treated, see Genon-Catalot and Jacod [8] for instance. When Y in (3) is a Lévy process attracted by a symmetric stable process with index $\alpha \in (1,2]$, we refer to Clément and Gloter [6]. In case of constant coefficients, i.e. models of the form

$$\mathrm{d}X_t = \theta_1\,\mathrm{d}t + \theta_2\,\mathrm{d}Y_t, \tag{4}$$

we refer to the works of Aït-Sahalia and Jacod [1], Masuda [17], Kawai and Masuda [13, 14]. Our investigation goes to same direction of Aït-Sahalia and Jacod [2], who studied the LAN property and the problem of estimation of the parameter (θ_1, θ_2) involved in the model of a log-asset price X, solution of (4), in the case when Y is a standard symmetric stable process with index $\alpha \in (0,2]$. Section 4 below completes their situation in case where Y is a general stable process, eventually mixed. The last direction was initiated Rammeh [20] with observations according to random schemes $(T(i,n),\, 1 \le i \le n)$ for the scale model

$$X = \theta Y \tag{5}$$

where θ is a real unknown real parameter and Y is a symmetrical standard α-stable process. He showed that the LAN property always occurs and his main arguments strongly rely to the linearity in θ, to the fact that stable processes have the temporal scaling property and to the asymptotic behavior of the stable densities. Theorem 4.1 below, generalizes Rammeh's results in the context of deterministic discrete scheme $T(i,n) = i\,u_n$.

Because of the intricacy of the case 3, we first focus on the following model which contains (5) and intercepts (3): we assume that for all $\theta \in \Theta$, under $\mathbb{P}_\theta$, X is a Lévy process, null at $t = 0$, having the Lévy exponent

$$\varphi_\theta(u) = \log \mathbb{E}_{\mathbb{P}_\theta}\left[\exp i u X_1\right] = i u b_\theta - \frac{{c_\theta}^2 u^2}{2} + \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\,1_{|y|\le 1}\right)\mu_\theta(\mathrm{d}y), \tag{6}$$

where $b_\theta \in \mathbb{R}$, $c_\theta \in \mathbb{R}_+$ and μ_θ is a positive measure on $\mathbb{R}$ which integrates $\min(y^2,1)$. For sake of clarity, we take Θ is open interval $\mathbb{R}$. We always assume, as done in the pre-cited literature,

- the existence of densities g_t^θ such that $\theta \mapsto g_t^\theta$ is regular enough;
- the convergence, as $n \to \infty$, of some integrals depending on $g_{u_n}^\theta$,

Theorem 3.1 and Corollary 3.3 below provide conditions ensuring the LAN property for the model (6), when the process X is observed along the discrete scheme (2). Denoting $\bar{g}_{u_n}^\theta$ the logarithmic derivative of $g_{u_n}^\theta$ relative to θ, the asymptotic Fisher information quantity at each θ, when it is finite non-null, should be equal to

$$I(\theta) = \lim_{n\to\infty} \int (\bar{g}_{u_n}^\theta)^2(x)\, g_{u_n}^\theta(x)\,\mathrm{d}x. \tag{7}$$

It is still be difficult to find Lévy processes fulfilling (7), because even though the densities g_t^θ exists, they are not explicit in general, and moreover degenerate when $t \to 0$. For this reason, Corollary 3.3 focuses on the linear dependance (4) of the characteristics relative to θ, and the previous conditions are slightly simplified. For this case, without loss of generality, we consider that Θ contains a reference value, 1 for example, and we exclude the value 0 in order to avoid trivialities. We only need to assume, in this case, some regularity on $x \mapsto g_{u_n}^1(x)$ and conditions of the kind (7) for $\theta = 1$. Denoting $h_n = g_{u_n}^1$ and $\bar{h}_n$ the logarithmic derivative of $h_n(x)$, the asymptotic Fisher information quantity becomes

$$I(\theta) = \theta^{-2} \lim_{n\to\infty} \int (1 + x\bar{h}_n(x))^2\, h_n(x)\,\mathrm{d}x \in (0,+\infty). \tag{8}$$

The case of the asymptotic ($u_n = u \in (0,+\infty)$, $\forall n \in \mathbb{N}$), which corresponds to n equally spaced observations, is quite obvious. Indeed, in this case, we treat Lévy processes with the appropriate conditions on the densities $g_u^1(x)$ for a fixed u, and then, the scale model (5) becomes a regular i.i.d. one, that is to say $I(\theta)$ is finite and non-null. When $u_n \to 0$, the situation is more intricate, always because h_n degenerates when $n \to \infty$. It turns out that even the linear model (5) is falsely simple to handle. Intuitively, one looks at special Lévy processes Y attracted by stable processes on the sense of (41) below. The price to pay is to exhibit refined controls on the density of Y_t, $t > 0$. This the object of Section 4. Theorems 4.1 and 4.3 below provide non-trivial examples of LAN models.

Far [7] focused on the LAMN property for the model (5) discretized along the scheme $i\,u_n = i/n$, $1 \le i \le n$, when the process Y of the form $Y = W + N$, the sum of a standard Brownian motion and an independent compound Poisson process. She obtained LAMN property under the condition that the Lévy measure ν of N has no diffuse singular part and that if ν is absolutely continuous, then the model has the LAN property. Our development in Section 5 constitutes a complement to Corollary 3.3 for the scale model (5) and also to Far's work [7] and illustrates how to build a LAN scale model from an another LAN scale model.

2 The model and Definition of LAN property

2.1 The model

The sample space is $\Omega = \mathbb{D}(\mathbb{R}+, \mathbb{R})$, the Skorokhod space endowed with its Borel σ-field $\mathcal{D}$ and the canonical process $X = (X_t)_{t\ge 0}$.

As a first step, we assume the model (6). Notice that if in (6), $c_\theta = 0$, μ_θ integrates $|y| \wedge 1$ and $\mu_\theta(0, +\infty) = 0$ (respectively $\mu_\theta(-\infty, 0) = 0$) then the support of the law of X_t is $[d_\theta\, t, +\infty)$ (respectively $(-\infty, d_\theta\, t]$), with $d_\theta = b_\theta - \int_{\mathbb{R}} y\, 1_{|y|\le 1}\, \mu_\theta(\mathrm{d}y)$. Otherwise the law of X_t has a support equal to $\mathbb{R}$. There are many situations in which for all $t > 0$, X_t has a probability density $g_t^\theta(x)$ which is infinitely differentiable in x. It is true if for example $c_\theta > 0$ or $\int_{|y|\le\varepsilon} \min(y^2, 1)\, \mu_\theta(dy) \ge K_\theta\, \varepsilon^\alpha$, for any $\varepsilon \in [0, 1]$ and for some $K_\theta > 0$ and some $\alpha \in (0, 2)$. See [18], and for a general account concerning Lévy processes, the reader is referred to [10] or [19]. As a second step, we restrict our attention to the scale model (5). For simplicity's sake, it is easier in this case, to express the probabilities $(\mathbb{P}_\theta)_{\theta\in\Theta}$ in the form (5) rather than considering them as solutions of martingale problems associated with the family of characteristics $(b_\theta, c_\theta, \mu_\theta)_{\theta\in\Theta}$ because of the intricacy inherent in the truncation functions, (see [10]). Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers. We aim to provide in Section 3 some theoretical results on models associated with observations, at times $i\,u_n$, of the process X, and to illustrate by some examples. For that, we need to consider the sequence of i.i.d. random variables and the family of σ-fields:

$$X_j^n = X_{(j+1)\,u_n} - X_{j\,u_n}, \qquad \mathcal{G}_i^n = \sigma(X_j^n,\, 0 \le j \le i-1). \tag{9}$$

Denoting $\mathcal{H}^n = \mathcal{G}_n^n$ and $\mathcal{H}_t^n = \mathcal{G}_{[nt]}^n$, $t \in [0,1]$, we introduce the sequence of filtered statistical models:

$$E^n = (\Omega, \mathcal{H}^n, (\mathcal{H}_t^n)_{t\in[0,1]}, (\mathbb{P}_\theta)_{\theta\in\Theta}). \tag{10}$$

For any fixed $\theta_0 \in \Theta$, we denote

$$\Theta_n = \{\theta \in \mathbb{R} : \theta_0 + \theta/\sqrt{n} \in \Theta\}, \quad [\theta]_n = \theta_0 + \theta/\sqrt{n} \quad \text{and} \quad \mathbb{P}_\theta^n = \mathbb{P}_{[\theta]_n} | \mathcal{H}^n, \tag{11}$$

and we introduce the statistical experiments localized around θ_0:

$$\mathcal{E}^n(\theta_0) = (\Omega, \mathcal{H}^n, (\mathcal{H}_t^n)_{t\in[0,1]}, (\mathbb{P}_\theta^n)_{\theta\in\Theta_n}), \qquad \mathcal{E}'(\theta_0) = (\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\in[0,1]}, (\mathbb{P}'_\theta)_{\theta\in\mathbb{R}}), \tag{12}$$

where the last statistical experiment is a Gaussian Shift. That means, that for all $\theta \in \mathbb{R}$, $\mathbb{P}'_\theta$ is the unique probability on $(\Omega', \mathcal{F}')$ equivalent to
pr'_0 on each $\mathcal{F}'_t$ and that its associated likelihood process is the geometric Brownian motion defined by:

$$Z_t'^{\theta} = \frac{\mathrm{d}\mathbb{P}'_\theta \mid \mathcal{F}'_t}{\mathrm{d}\mathbb{P}'_{\theta_0} \mid \mathcal{F}'_t} = \exp\{\theta\sqrt{I(\theta_0)}\,X'_t - \frac{\theta^2}{2} I(\theta_0)\,t\}, \quad t \in [0,1],$$

where $(X'_t)_{t\in[0,1]}$ is a Wiener process and then, under $\mathbb{P}'_\theta$, $X'_t - t\,\theta\sqrt{I(\theta_0)}$ is again a Wiener process. $I(\theta_0)$ is the asymptotic Fisher information quantity, i.e. a positive constant relative to the sequence of statistical experiments $\mathcal{E}^n(\theta_0)$ given by (12), that has to determined. As announced in the beginning of this work asymptotic Fisher information quantity is crucial in estimation procedure since it inverse gives, under the LAN property, the lower bound of the variance of any estimator ϑ_n of θ_0. More precisely, HAJEK' convolution theorem [21], says that if ϑ_n is such that the convergence in law

$$\mathcal{L}aw\Big((\sqrt{n}\,(\vartheta_n - (\theta_0 + n^{-1/2}\,\theta)) \,|\, \mathbb{P}^n_\theta) \Big) \to \mathcal{L}_{\theta_0}, \quad \text{as} n \to \infty,$$

holds, then necessarily $\mathcal{L}_{\theta_0}$ is the convolution

$$\mathcal{L}_{\theta_0} = \mathcal{L}^1_{\theta_0} \star \mathcal{L}^2_{\theta_0}$$

and $\mathcal{L}^1_{\theta_0} = \mathcal{N}ormal(0, I(\theta_0)^{-1})$ et $\mathcal{L}^2_{\theta_0}$ is a probability measure on $\mathbb{R}$.

2.2 LAN property and weak functional convergence of the likelihood processes

Local asymptotic normality of the sequence of models E^n in (12) in a value $\theta_0 \in \Theta$ is actually equivalent to the weak functional convergence in time of the sequence of statistical experiments $\mathcal{E}^n(\theta_0)$ to the gaussian shift $\mathcal{E}(\theta_0)$ (12). This fact is explained as follows: let $Z^{'\eta\xi}$ et $Z^{n,\eta\xi}$ be the likelihood processes defined, for all $\eta, \xi \in \Theta_n$ and at each time $t \in [0,1]$, by

$$Z_t^{'\eta\xi} = \frac{\mathrm{d}\mathbb{P}'_\eta|\mathcal{F}'_t}{\mathrm{d}\mathbb{P}'_\xi\Big|\mathcal{F}'_t} = \mathbb{E}_{\mathbb{P}'_\xi}\left[\frac{\mathrm{d}\mathbb{P}'_\eta}{\mathrm{d}\mathbb{P}'_\xi}\Big|\mathcal{F}'_t\right] = \frac{Z_t^{'\eta}}{Z_t^{'\xi}} \quad \text{and} \quad Z_t^{n,\eta\xi} = \frac{\mathrm{d}\mathbb{P}^n_\eta|\mathcal{H}^n_t}{\mathrm{d}\mathbb{P}^n_\xi\Big|\mathcal{H}^n_t} = \mathbb{E}_{\mathbb{P}^n_\xi}\left[\frac{\mathrm{d}\mathbb{P}^n_\eta}{\mathrm{d}\mathbb{P}^n_\xi}\,\Big|\,\mathcal{H}^n_t\right], \tag{13}$$

with the convention $a/0 = 0$, $\forall a \in [0,+\infty)$. According to [8], the likelihood processes $Z^{n,\eta\xi}$ of the statistical experiment $\mathcal{E}^n(\theta_0)$ is represented by

$$Z_t^{n,\eta\xi} = \prod_{j=1}^{[nt]} \frac{g^\eta_{u_n}}{g^\xi_{u_n}}(X^n_j). \tag{14}$$

The notion of weak functional convergence in time was introduced by Lecam [15] and developed by Strasser [21] and Jacod [9]. It is expressed as follows: for every finite subset J of $\mathbb{R} = \cup_{n\geqslant 1}\Theta_n$, and every $\xi \in \Theta$, we have

$$\mathcal{L}aw\big(\, (Z^{n,\eta\xi})_{\eta\in J} \,|\, \mathbb{P}^n_{[\xi]_n} \,\big) \longrightarrow \mathcal{L}aw\big(\, (Z^{'\eta\xi})_{\eta\in J} \,|\, \mathbb{P}'_\xi \,\big), \quad \text{as } n \to +\infty. \tag{15}$$

in the sense of the weak convergence for the Skorohod topology.

3 When does LAN property hold for Lévy models?

Our aim is to give sufficient conditions on g^θ_t, the density of X_t under $\mathbb{P}_\theta$, in order to obtain the LAN property for the sequence of filtered statistical models E^n.

3.1 LAN property for the model (6)

Later on, we may assume the following:

(H0): *For all $\theta \in \Theta$ and $t > 0$, under $\mathbb{P}_\theta$, the support of the law of X_t is an interval K_t, independent from θ, of the form $K_t = \mathbb{R}$ or $(-\infty, dt]$ or $[dt, +\infty)$, for some $d \in \mathbb{R}$, and X_t has a probability density $x \mapsto g^\theta_t(x)$ which is of class C^2, relative to θ.*

We denote $\mathcal{K}_n = K_{u_n}$ and define on the interior of $\mathcal{K}_n$ the following functions:

$$h^\theta_n = g^\theta_{u_n}, \quad \bar{h}^\theta_n = \frac{\partial}{\partial_\theta}\log h^\theta_n, \quad \ddot{h}^\theta_n = \frac{\partial^2}{\partial^2_\theta}h^\theta_n, \quad i^\theta_n = h^\theta_n\,|\bar{h}^\theta_n|^2 \quad \text{and} \quad j^\theta_n = i^\theta_n + |\ddot{h}^\theta_n|. \tag{16}$$

When the number $\chi > 0$ appears, it is always understood that n is big enough so that χ and $-\chi$ are in Θ_n. For all $\theta \in \Theta$, $\rho \in (0,1)$ and $\rho' = 1 - \rho$, denote

$$I_n^\theta := \int_{\mathcal{K}_n} i_n^\theta(x)\,\mathrm{d}x, \qquad \widetilde{I}_n(\chi) = \sup_{|\varepsilon|\le\chi} I_n^{[\varepsilon]_n}, \tag{17}$$

$$\widetilde{J}_n^\rho(\chi) := \sup_{|\zeta|,|\varepsilon|\le\chi} \int_{\mathcal{K}_n} \frac{j_n^{[\zeta]_n}(x)\, j_n^{[\varepsilon]_n}(x)}{\left(h_n^{[\zeta]_n}(x)\right)^\rho \left(h_n^{[\varepsilon]_n}(x)\right)^{\rho'}}\,\mathrm{d}x = \widetilde{J}_n^{\rho'}(\chi). \tag{18}$$

For statisticians, $I_n^{\theta_0}$ is a familiar quantity and corresponds to a Fisher Information quantity at stage n. The quantity $\widetilde{J}_n^\rho(\chi)$ is less intuitive. It is a localized quantity around the true value θ_0 and corresponds to the rest of Taylor approximations at the order 1 of Hellinger integrals of the model.

We are now able to state our first result, that is, the LAN property for the model (6).

Theorem 3.1 *Assume* **(H0)** *and*

$$\textbf{(H1)}: \lim_{n\to+\infty} I_n^{\theta_0} = I(\theta_0) \in (0,+\infty) \text{ and for all } \chi>0,\ \limsup_{n\to+\infty} \widetilde{I}_n(\chi) < +\infty,$$

$$\textbf{(H2)}: \text{There exists } a\in(0,1/2) \text{ such that for all } \chi>0, \text{ one has}$$

$$\lim_{n\to+\infty} \frac{\widetilde{J}_n^\rho(\chi)}{n} = 0 \quad \text{for} \quad \rho\in\{1/2,a,1-a\}.$$

Then, the sequence of sequence of filtered statistical models E^n (10) corresponding to (6) has the LAN property at θ_0 with speed $\sqrt{n}$ and the asymptotic Fisher information quantity $I(\theta_0)$.

Remark 3.2 *Cauchy-Schwarz inequality, gives $I_n^\theta \le [\widetilde{J}_n^{1/2}(|\theta|)]^{1/2}$, and both* **(H1)** *and* **(H2)** *are implied by:*

$$\textbf{(H3)}: \lim_{n\to+\infty} I_n^{\theta_0} = I(\theta_0) \text{ and there exists } a\in(0,1/2) \text{ such that for all } \chi>0, \text{ one has}$$

$$\limsup_{n\to+\infty} \widetilde{J}_n^\rho(\chi) < +\infty \quad \text{for} \quad \rho\in\{1/2,a,1-a\}.$$

Genon-Catalot and Jacod [8] exhibited discretized models according to random sampling schemes $(T(i,n), 1\le i\le n)$ associated with a diffusion processes X driven by Brownian motions (with coefficients dependent on θ and by an homogeneous way on X) and proved the LAMN property under conditions similar to **(H0)**, that is differentiability to the third order relative to θ, and integrability of the densities of the processes. Their proofs have a general vocation in the sense that they only use the Markovian property of the processes and are based on a method of approximation of the log-likelihood. Because of the intricate form 14 of the likelihood processes, we show the weak functional convergence of $\mathcal{E}^n(\theta_0)$ to $\mathcal{E}'(\theta_0)$ via the convergence of the Hellinger processes, and according to a tool one can find in [9].

Proof (Proof of Theorem 3.1). Fix θ_0. The Hellinger process of order $\rho\in(0,1)$ between pr'_η and $\mathbb{P}'_\xi$, relative to $(\mathcal{F}'_t)_{t\in[0,1]}$, is deterministic and has the form: $\mathcal{H}'^{\eta\xi}(\rho)_t = \rho(1-\rho)(\eta-\xi)^2 I(\theta_0)t/2$. According to Theorem 5.3 [9], it is enough to show that the Hellinger processes $\mathcal{H}^{n,\eta\xi}(\rho)$ between $\mathbb{P}_\eta^n$ and $\mathbb{P}_\xi^n$, relative to $(\mathcal{G}_t^n)_{t\in[0,1]}$, satisfy the following: there exists $a\in(0,1/2)$ such that for every $\forall\eta,\xi\in\mathbb{R}$, $\rho\in\{1/2,a,1-a\}$ and $t\in[0,1]$, the convergence in law

$$\mathcal{H}^{n,\eta\xi}(\rho)_t \xrightarrow{\mathbb{P}_\xi^n} \mathcal{H}'^{\eta\xi}(\rho)_t, \quad \text{as } n\to\infty. \tag{19}$$

holds. We will use this method, because in our framework the processes $\mathcal{H}^{n,\eta\xi}$ are also deterministic and have the following quite simple form one can find in [8]: $[nt]$ being the integer part of nt, we have

$$\mathcal{H}^{n,\eta\xi}(\rho)_t = [nt]\left(1 - \int_{\mathcal{K}_n} \left(h_n^{[\eta]_n}\right)^\rho \left(h_n^{[\xi]_n}\right)^{1-\rho}(y)\,\mathrm{d}y\right). \tag{20}$$

1) For $\rho\in(0,1)$, $\rho' = 1-\rho$ take $\Phi_\rho(u,v) = \rho u + \rho' v - u^\rho v^{\rho'}$, $u,v\ge 0$ and observe that

$$\mathcal{H}^{n,\eta\xi}(\rho)_1 = \int_{\mathcal{K}_n} \Phi_\rho\left(h_n^{[\eta]_n}, h_n^{[\xi]_n}\right)(y)\mathrm{d}y.$$

According to (19) and (20), it is enough to show that $\forall\eta,\xi\in\mathbb{R}$ and $\rho\in\{1/2,a,1-a\}$, we have

$$\lim_{n\to+\infty} n \int_{\mathcal{K}_n} \Phi_\rho\left(h_n^{[\eta]_n}, h_n^{[\xi]_n}\right)(y)\,\mathrm{d}y = \frac{\rho\rho'}{2}(\eta-\xi)^2 I(\theta_0)\,. \tag{21}$$

2) Assume **(H0)**, **(H1)**, and **(H2)** for a fixed $a \in (0,1/2)$. Applying Taylor expansion at the first order of $\theta \mapsto (h_n^\theta)^\rho$ for $\eta \in \mathbb{R}$ and n big enough, we get for $\eta \in \Theta_n$, the representation of $\left(h_n^{[\eta]_n}\right)^\rho$ on $\mathcal{K}_n$:

$$\left(h_n^{[\eta]_n}\right)^\rho = \left(h_n^{\theta_0}\right)^\rho + \frac{\rho\,\eta}{\sqrt{n}}\,k_n^{\theta_0,\rho} + \frac{\rho\,\eta}{\sqrt{n}}\,V_n^{\eta,\rho}, \tag{22}$$

where for all $\theta \in \Theta$, the functions:

$$k_n^{\theta,\rho} = \bar{h}_n^\theta\,\left(h_n^\theta\right)^\rho, \qquad \dot{k}_n^{\theta,\rho} = \left(h_n^\theta\right)^\rho\left[\frac{\partial}{\partial\theta}\bar{h}_n^\theta + \rho\left(\bar{h}_n^\theta\right)^2\right] = \left(h_n^\theta\right)^\rho\left[\frac{\ddot{h}_n^\theta}{h_n^\theta} - \rho'\left(\bar{h}_n^\theta\right)^2\right], \tag{23}$$

$$V_n^{\eta,\rho} = \int_0^1\left[k_n^{[\eta r]_n,\rho} - k_n^{\theta_0,\rho}\right]\mathrm{d}r = \frac{\eta}{\sqrt{n}}\int_0^1 (1-r)\,\dot{k}_n^{[\eta r]_n,\rho}\,\mathrm{d}r\,. \tag{24}$$

are defined on the interior of $\mathcal{K}_n$. Also, observe that:

$$k_n^{\theta,\rho}\left(h_n^\theta\right)^{\rho'} = k_n^{\theta,\rho'}\left(h_n^\theta\right)^\rho = k_n^{\theta,1}, \quad k_n^{\theta,\rho}\,k_n^{\theta,\rho'} = i_n^\theta \quad \text{and} \quad \dot{k}_n^{\theta,\rho}\left(h_n^\theta\right)^{\rho'} = \dot{k}_n^{\theta,1} - \rho'\,i_n^\theta\,. \tag{25}$$

Because of (24) and (25), one has

$$\begin{aligned} V_n^{\eta,1} - \left(h_n^{\theta_0}\right)^{\rho'} V_n^{\eta,\rho} &= \frac{\eta}{\sqrt{n}}\int_0^1 (1-r)\left[\dot{k}_n^{[\eta r]_n,1} - \left(h_n^{\theta_0}\right)^{\rho'}\dot{k}_n^{[\eta r]_n,\rho}\right]\mathrm{d}r \\ &= \frac{\eta}{\sqrt{n}}\int_0^1 (1-r)\,\dot{k}_n^{[\eta r]_n,\rho}\left[\left(h_n^{[\eta r]_n}\right)^{\rho'} - \left(h_n^{\theta_0}\right)^{\rho'}\right]\mathrm{d}r + \frac{\eta\rho'}{\sqrt{n}}\int_0^1 (1-r)\,i_n^{[\eta r]_n}\mathrm{d}r. \end{aligned}$$

Using (22), finally write

$$V_n^{\eta,1} - \left(h_n^{\theta_0}\right)^{\rho'} V_n^{\eta,\rho} = \frac{\eta^2\rho'}{n}\int_0^1 (1-r)\,\dot{k}_n^{[\eta r]_n,\rho}\left[k_n^{\theta_0,\rho'} + V_n^{\eta r,\rho'}\right]\mathrm{d}r + \frac{\eta\rho'}{\sqrt{n}}\int_0^1 (1-r)\,i_n^{[\eta r]_n}\mathrm{d}r. \tag{26}$$

3) Write for all $\eta,\xi \in \mathbb{R}$ and n big enough so that $\eta,\xi \in \Theta_n$:

$$\begin{aligned} \Phi_\rho\left(h_n^{[\eta]_n}, h_n^{[\xi]_n}\right) = \rho&\left[h_n^{\theta_0} + \frac{\eta}{\sqrt{n}}\,k_n^{\theta_0,1} + \frac{\eta}{\sqrt{n}}\,V_n^{\eta,1}\right] + \rho'\left[h_n^{\theta_0} + \frac{\xi}{\sqrt{n}}\,k_n^{\theta_0,1} + \frac{\xi}{\sqrt{n}}\,V_n^{\xi,1}\right] \\ &-\left[\left(h_n^{\theta_0}\right)^\rho + \frac{\rho\,\eta}{\sqrt{n}}\,k_n^{\theta_0,\rho} + \frac{\rho\,\eta}{\sqrt{n}}\,V_n^{\eta,\rho}\right]\left[\left(h_n^{\theta_0}\right)^{\rho'} + \frac{\rho'\,\xi}{\sqrt{n}}\,k_n^{\theta_0,\rho'} + \frac{\rho'\,\xi}{\sqrt{n}}\,V_n^{\xi,\rho'}\right]. \end{aligned}$$

and (25) yields

$$\begin{aligned} n\,\Phi_\rho\left(h_n^{[\eta]_n}, h_n^{[\xi]_n}\right) = \sqrt{n}\,\rho\,\eta&\left[V_n^{\eta,1} - \left(h_n^{\theta_0}\right)^{\rho'} V_n^{\eta,\rho}\right] + \sqrt{n}\,\rho'\,\xi\left[V_n^{\xi,1} - \left(h_n^{\theta_0}\right)^\rho V_n^{\eta,\rho'}\right] \\ &-\rho\,\rho'\,\eta\,\xi\left[i_n^{\theta_0} + V_n^{\eta,\rho}V_n^{\xi,\rho'} + k_n^{\theta_0,\rho}V_n^{\xi,\rho'} + k_n^{\theta_0,\rho'}V_n^{\eta,\rho}\right]. \end{aligned}$$

According to (26)we have $n\,\Phi_\rho\left(h_n^{[\eta]_n}, h_n^{[\xi]_n}\right) = \rho\,\rho'\left[A_n^{\eta,\xi} + B_n^{\eta,\xi}\right]$, where

$$\begin{aligned} A_n^{\eta,\xi} &= \eta^2\int_0^1 (1-r)\,i_n^{[\eta r]_n}\mathrm{d}r + \xi^2\int_0^1 (1-r)\,i_n^{[\xi r]_n}\mathrm{d}r - \eta\,\xi\,i_n^{\theta_0}, \\ B_n^{\eta,\xi} &= \frac{\eta^3}{\sqrt{n}}\int_0^1 (1-r)\,\dot{k}_n^{[\eta r]_n,\rho}\left[k_n^{\theta_0,\rho'} + V_n^{\eta r,\rho'}\right]\mathrm{d}r + \frac{\xi^3}{\sqrt{n}}\int_0^1 (1-r)\,\dot{k}_n^{[\xi r]_n,\rho'}\left[k_n^{\theta_0,\rho} + V_n^{\xi r,\rho}\right]\mathrm{d}r \\ &\quad -\eta\,\xi\left[k_n^{\theta_0,\rho}V_n^{\xi,\rho'} + k_n^{\theta_0,\rho'}V_n^{\eta,\rho} + V_n^{\eta,\rho}V_n^{\xi,\rho'}\right]. \end{aligned} \tag{27}$$

4) a) We are now able to prove that for all $\eta,\xi \in \mathbb{R}$,

$$\lim_{n\to+\infty}\int_{\mathcal{K}_n} A_n^{\eta,\xi}(x)\,\mathrm{d}x = \frac{1}{2}(\eta-\xi)^2 I(\theta_0)\,, \tag{28}$$

$$\lim_{n\to+\infty}\int_{\mathcal{K}_n} B_n^{\eta,\xi}(x)\,\mathrm{d}x = 0\,, \tag{29}$$

which gives (21).

4) b) Fix $\eta, \xi \in \mathbb{R}$ and take $\chi = |\eta| \vee |\xi|$.To prove (28), we use both (16)), (23) and for all $\rho \in (0,1)$ and $\theta \in \Theta$, we have the representation and the control

$$|k_n^{\theta,\rho}| = \sqrt{i_n^{\theta}}\,(h_n^{\theta})^{\rho-1/2} \qquad \text{and} \qquad |\dot{k}_n^{\theta,\rho}| \le \frac{j_n^{\theta}}{(h_n^{\theta})^{\rho'}}\,. \tag{30}$$

Since $r \in [0,1]$, then (16), (23) and Taylor expansion at the first order of $\theta \mapsto \left(k_n^{\theta,1/2}\right)^2$, yield

$$I_n^{[\eta r]_n} - I_n^{\theta_0} = \int_{\mathcal{K}_n}\left[k_n^{[\eta r]_n,1/2}(x)^2 - k_n^{\theta_0,1/2}(x)^2\right]\mathrm{d}x = \frac{2\eta r}{\sqrt{n}}\int\int_0^1 k_n^{[\eta rs]_n,1/2}(x)\dot{k}_n^{[\eta rs]_n,1/2}(x)\mathrm{d}s\,\mathrm{d}x.$$

If we express (30) with $\rho = 1/2$, then Cauchy-Schwarz inequality, **(H1)** and then **(H2)** give:

$$\sup_{r\in[0,1]} |I_n^{[\eta r]_n} - I_n^{\theta_0}| \le 2|\eta|\widetilde{I}_n(\chi)\left(\frac{\widetilde{J}_n^{1/2}(\chi)}{n}\right)^{1/2} \longrightarrow 0 \quad \text{as } n\to+\infty\,. \tag{31}$$

Now, we can write

$$\begin{aligned}\int_{\mathcal{K}_n} A_n^{\eta,\xi}(x)\,\mathrm{d}x &= \eta^2\int_0^1(1-r)\,I_n^{[\eta r]_n}\,\mathrm{d}r + \xi^2\int_0^1(1-r)\,I_n^{[\xi r]_n}\,\mathrm{d}r - \eta\,\xi\,I_n^{\theta_0}\\ &= \eta^2\int_0^1(1-r)\,(I_n^{[\eta r]_n} - I_n^{\theta_0})\,\mathrm{d}r + \xi^2\int_0^1(1-r)\,(I_n^{[\xi r]_n} - I_n^{\theta_0})\,\mathrm{d}r + \frac{1}{2}\,(\eta^2+\xi^2-2\eta\,\xi)\,I_n^{\theta_0},\end{aligned}$$

and it is clear that (31) yields (28).

4) c) To prove (29), we use arguments similar to 4) b), that is Taylor expansion at the first order of $\theta \mapsto k_n^{\theta,1/2}$, and we use the representation

$$\dot{k}_n^{[\eta r]_n,\rho}\,k_n^{\theta_0,\rho'} = \dot{k}_n^{[\eta r]_n,\rho}\,k_n^{[\eta r]_n,\rho'} - \frac{\eta r}{\sqrt{n}}\int_0^1 \dot{k}_n^{[\eta r]_n,\rho}\,\dot{k}_n^{[\eta rs]_n,\rho'}\,\mathrm{d}s, \qquad \rho\in(0,1),\ r\in[0,1]\,.$$

By (30) and Cauchy-Schwarz inequality, we have for all $r \in [0,1]$,

$$\int_{\mathcal{K}_n} |\dot{k}_n^{[\eta r]_n,\rho}(x)\,k_n^{\theta_0,\rho'}(x)|\,\mathrm{d}x \le \int_{\mathcal{K}_n}\sqrt{i_n^{[\eta r]_n}(x)}\,\frac{j_n^{[\eta r]_n}(x)}{\sqrt{h_n^{[\eta r]_n}(x)}}\,\mathrm{d}x + \frac{|\eta|r}{\sqrt{n}}\widetilde{J}_n^{\rho}(\chi)\,.$$

This implies

$$\sup_{r\in[0,1]}\int_{\mathcal{K}_n}|\dot{k}_n^{[\eta r]_n,\rho}(x)k_n^{\theta_0,\rho'}(x)|\mathrm{d}x \le (\widetilde{I}_n(\chi)\widetilde{J}_n^{1/2}(\chi))^{1/2} + \frac{\chi}{\sqrt{n}}\widetilde{J}_n^{\rho}(\chi) := \delta_n^{\rho}(\chi) = \delta_n^{\rho'}(\chi). \tag{32}$$

Because of (30), we also have

$$\int_{\mathcal{K}_n}|\dot{k}_n^{[\eta r]_n,\rho}(x)\,V_n^{\eta r,\rho'}(x)|\,\mathrm{d}x \le \frac{|\eta|r}{\sqrt{n}}\int_0^1(1-s)\int_{\mathcal{K}_n}|\dot{k}_n^{[\eta r]_n,\rho}(x)\,\dot{k}_n^{[\eta rs]_n,\rho'}(x)|\,\mathrm{d}x\mathrm{d}s.$$

Then, reproducing the method we used for (32), we get

$$\sup_{r\in[0,1]}\int_{\mathcal{K}_n}|\dot{k}_n^{[\eta r]_n,\rho}(x)\,V_n^{\eta r,\rho'}(x)|\,\mathrm{d}x \le \delta_n^{\rho}(\chi). \tag{33}$$

According to (24) and (32), we also have

$$\int_{\mathcal{K}_n}|k_n^{\theta_0,\rho}(x)\,V_n^{\eta,\rho'}(x)|\,\mathrm{d}x \le \frac{|\eta|}{\sqrt{n}}\int_0^1(1-r)\int_{\mathcal{K}_n}|\dot{k}_n^{[\eta r]_n,\rho'}(x)\,k_n^{\theta_0,\rho}(x)|\,\mathrm{d}x\,\mathrm{d}r \le \frac{\chi}{\sqrt{n}}\,\delta_n^{\rho'}(\chi). \tag{34}$$

Further, (24), (30) and Cauchy-Schwarz inequality give

$$\int_{\mathcal{K}_n} |V_n^{\eta,\rho}(x)\, V_n^{\xi,\rho'}(x)|\, \mathrm{d}x \le \frac{|\eta\,\xi|}{n} \int_0^1\!\!\int_0^1 (1-r)(1-s) \int \frac{j_n^{[\eta r]_n}(x)\, j_n^{[\xi s]_n}(x)}{h_n^{[\eta r]_n}(x)^{\rho'}\, h_n^{[\xi s]_n}(x)^{\rho}}\, \mathrm{d}x\,\mathrm{d}s\,\mathrm{d}r \le \frac{|\chi|}{\sqrt{n}}\, \delta_n^{\rho'}(\chi). \tag{35}$$

Finally, according to (27), (32), (33), (34) and (35), we obtain that

$$\int_{\mathcal{K}_n} B_n^{\eta,\xi}(x)\, \mathrm{d}x \le 7\chi^3\, \frac{\delta_n^{\rho}(\chi)}{\sqrt{n}} = 7\chi^3 \left[\left(\widetilde{I}_n(\chi)\, \frac{\widetilde{J}_n^{1/2}(\chi)}{n} \right)^{1/2} + \chi\, \frac{\widetilde{J}_n^{\rho}(\chi)}{n} \right],$$

and we conclude with the fact that assumptions **(H1)** and **(H2)** imply (29).

3.2 LAN property for the scale model (5)

As a consequence of Theorem 3.1, we obtain a result for the scale model (5) based on the equivalence between the following assertions:

- X_t has a density $g_t^1(.)$ under $\mathbb{P}_1$, $\forall\, t > 0$;
- X_t has the density $g_t^\theta(.) = \dfrac{1}{\theta}\, g_t^1\left(\dfrac{\cdot}{\theta}\right)$ under $\mathbb{P}_\theta$, $\forall\,\theta \in \Theta$, $t > 0$.

The functions h_n, h_n', and h_n'' denote respectively h_n^1 the probability density of X_{u_n} under $\mathbb{P}_1$, the first and the second derivatives of $x \mapsto h_n(x)$. Notice that for all $\theta \in \Theta$ and $n \in \mathbb{N}^*$, if $x \in \mathcal{K}_n = Support(h_n^\theta)$, then $h_n^\theta(x) = h_n(x/\theta)/\theta$ and $\mathcal{K}_n = \theta\,\mathcal{K}_n = \{\,\theta x,\ x \in \mathcal{K}_n\,\}$. Therefore, if we want **(H0)** to be satisfied, we need $K = \mathcal{K}_n = \mathbb{R}$ or $\mathbb{R}_+$ or $\mathbb{R}_-$, and then, for all $x \in K$:

$$i_n^\theta(x) = \frac{1}{\theta^3}\, i_n(x/\theta) \quad \text{and} \quad j_n^\theta(x) = \frac{1}{\theta^3}\, j_n(x/\theta),$$

where the functions $i_n = i_n^1$ and $j_n = j_n^1$ are defined by

$$i_n(x) = |1 + x\,\frac{h_n'}{h_n}(x)|^2\, h_n(x) \quad \text{and} \quad j_n(x) = i_n(x) + |2 + 4x\,\frac{h_n'}{h_n}(x) + x^2\,\frac{h_n''}{h_n}(x)|\, h_n(x). \tag{36}$$

After a change of variables, the quantities I_n^θ and $\widetilde{I}_n(\chi)$ defined in (17) satisfy

$$I_n^\theta = \frac{1}{\theta^2} \int_K i_n(x)\, \mathrm{d}x = \frac{I_n}{\theta^2} \quad \text{and} \quad \widetilde{I}_n(\chi) = \sup_{|\zeta| \le \chi} \frac{I_n}{([\zeta]_n)^2} \le \frac{I_n}{(|\theta_0| - \chi/\sqrt{n})^2}\,. \tag{37}$$

For $\rho \in (0,1)$, denote

$$J_n(\rho) = \int_K \frac{j_n(x)^2}{h_n(x)^{2\rho}}\, \mathrm{d}x.$$

Using Cauchy-Schwarz inequality and again a change of variables, on has: for all $\chi > 0$, the quantity $\widetilde{J}_n^{\rho}(\chi)$ defined in (18), satisfies

$$\begin{aligned}
\widetilde{J}_n^{\rho}(\chi) &\le \sup_{|\zeta|,|\xi| \le \chi} \left[\int_K \frac{j_n^{[\zeta]_n}(x)^2}{h_n^{[\zeta]_n}(x)^{2\rho}}\, \mathrm{d}x \right]^{1/2} \left[\int_K \frac{j_n^{[\varepsilon]_n}(x)^2}{h_n^{[\varepsilon]_n}(x)^{2\rho'}}\, \mathrm{d}x \right]^{1/2} \\
&= \sup_{|\zeta|,|\varepsilon| \le \chi} \left[\frac{1}{|[\zeta]_n|^{5-2\rho}\, |[\varepsilon]_n|^{5-2\rho}} \int_K \frac{j_n(x)^2}{h_n(x)^{2\rho}}\, \mathrm{d}x \int_K \frac{j_n(x)^2}{h_n(x)^{2\rho'}}\, \mathrm{d}x \right]^{1/2} \\
&\le \left| |\theta_0| - \frac{\chi}{\sqrt{n}} \right|^{4\rho - 10} [\, J_n(\rho)\, J_n(\rho')\,]^{1/2}.
\end{aligned} \tag{38}$$

Corollary 3.3 *For the scale model (5), the sequence of statistical models E^n has the LAN property with speed $\sqrt{n}$ at each $\theta \in \Theta$ if the following conditions are satisfied:*

(C0) : *Under $\mathbb{P}_1$, the support K of the law of X_1 is either $\mathbb{R}$ or $\mathbb{R}_+$ or $\mathbb{R}_-$ and for all $t > 0$, X_t has a probability density $x \mapsto g_t^1(x)$ of class C^2 on the interior of K*

(C1) : *The sequence $I_n = \int_K i_n(x)\,\mathrm{d}x$, satisfies $\lim_{n\to+\infty} I_n = I \in (0,+\infty)$*

(C2) : *There exists $a \in (0,1/2)$ such that $\lim_{n\to+\infty} \frac{1}{n} J_n(\rho) = 0$ for $\rho \in \{1/2, a, 1-a\}$.*

In this case, the asymptotic Fisher information quantity is $I(\theta) = I/\theta^2$.

Proof. It is immediate that **(C0)** implies **(H0)**. Representation (37) shows that **(C1)** implies **(H1)** and (38) shows that **(C2)** implies **(H2)**.

Remark 3.4 *The effect of the discretization path u_n is hidden in the assumptions* **(C1)** *and* **(C2)**. *Recall that the density h_n is the density of X_{u_n}. We anticipate a little (on Theorem 4.1 below) by saying that in some favorable cases (scaling property of stable processes), the path u_n has a quite negligible effect. In general, the LAN property depends strongly on the limit of u_n when n goes to infinity.*

4 Example: LAN property for scale models associated to Lévy processes attracted by stable processes.

In this Section, we provide examples of Lévy processes X satisfying Corollary 3.3 which treats the scale model (5), i.e. $X = \theta Y$. and where the LAN property was obtained under regularity and integrability conditions on the probability densities of $Y_t, t > 0$. Usually, these densities, if they exist, are not explicit. For this reason, we focus on processes Y which belong to the domain of attraction of stable processes. We recall that a stable process $S^{\alpha,\beta,\gamma,\delta}$ is a Lévy process, characterized in [22] by its Lévy exponent given, for all $t > 0$, $u \in \mathbb{R}$, by

$$\varphi(u) = \frac{1}{t}\log \mathbb{E}[e^{\mathrm{i}uS_t^{\alpha,\beta,\gamma,\delta}}] = \begin{cases} \mathrm{i}\,\delta u - \gamma|u|^\alpha \exp(-\mathrm{i}\,\frac{\pi}{2}\,\beta K(\alpha)\,\mathrm{sgn}(u)), & \text{if } \alpha \neq 1, \\ \mathrm{i}\,\delta u - \gamma|u|\left(1 + \mathrm{i}\,\frac{2\beta}{\pi}\,\mathrm{sgn}(u)\log|u|\right), & \text{if } \alpha = 1. \end{cases} \tag{39}$$

where $\alpha \in (0,2]$ is the *stability* coefficient, $\beta \in [-1,1]$ is the *skewness* coefficient, $\gamma > 0$ is the *scale* coefficient, the real number δ is the *drift* and $K(\alpha) = \alpha - 1 + \mathrm{sgn}(1-\alpha)$. When $\alpha \in (0,1)$ and $\delta \geq 0$, the process $S^{\alpha,1,\gamma,\delta}$ is a subordinator, i.e. a positive increasing Lévy process and for all $t > 0$, the law of $S_t^{\alpha,1,\gamma,\delta}$ has a support equal to $[\delta t, \infty)$. In all the cases, for all $t > 0$, the r.v. $S_t^{\alpha,\beta,\gamma,\delta}$ has an infinitely differentiable probability density $G_t^{\alpha,\beta,\gamma,\delta}$ which is explicit only for particular values of the coefficients $(\alpha,\beta) = (1/2,1), (1,0), (2,0)$, corresponding respectively to the processes: First passage times of the Brownian Motion, Cauchy Process and Brownian Motion. Otherwise, it is expressed only via the inverse Fourier transform of $\exp\varphi(u)$. The *scaling property* for stable processes reads as follows:

$$S_t^{\alpha,\beta,\gamma,\delta} \stackrel{d}{=} \delta_{t,\alpha} + (\gamma t)^{-1/\alpha} S_1^{\alpha,\beta,1,0}, \quad \text{where } \delta_{t,\alpha} = t\,[\delta + \frac{2\beta}{\pi}\gamma\log(\gamma t)\,1_{\alpha=1}]. \tag{40}$$

As announced if the introduction, we focus here on the case where the Lévy processes Y satisfies the following: there exist measurable functions $b(t) \in \mathbb{R}$, $a(t) > 0$ and a non-degenerate law ν such that the following convergence in distribution holds:

$$\widehat{Y}_t = \frac{Y_t - b(t)}{a(t)} \xrightarrow{d} \nu, \quad \text{as } t \to 0 \text{ or } +\infty. \tag{41}$$

A known result, see Bertoin and Doney [5] for $t \to 0$, says that the process Y is attracted by a stable law. More precisely, if (41) holds, then necessarily

- $b(t) = bt$, $a(t) = t^{1/\alpha} l(t)$, where $b \in \mathbb{R}$, $\alpha \in (0,2]$ and $l(t)$ is a slowly varying function, i.e. $l(\lambda t)/l(t) \to 1$, $\forall \lambda > 0$;
- ν is the law of $S_1^{\alpha,\beta,\gamma,0}$ for some $\beta \in [-1,1]$, $\gamma > 0$

- $\widehat{Y}_t, t>0$, admits an infinitely differentiable probability density G_t such that: for all $k\in\mathbb{N}$, and $t\to 0$ or $+\infty$:

$$(G_t)^{(k)}(x) \to G^{(k)}(x), \quad \text{uniformly in } x\in Support(G), \tag{42}$$

  where G is the density of $S_1^{\alpha,\beta,1,0}$.

The convergence (41) can be entirely expressed with the behavior of the tail of the Lévy measure of Y (case $\alpha<2$) or by existence or a Brownian component in Y (case $\alpha=2$, $l(t)$ constant). Observe that h_n, the probability density function of Y_{u_n} is expressed by $h(x)=G_{u_n}(a(u_n)^(1)(x-b(u_n)))$. As for $\overline{h}_n$ in (8), if $\bar{G}_{u_n}$ (respectively $\bar{G}$) denotes the derivative of $x\mapsto \log G_{u_n}$ (respectively $\log G$), then the corresponding asymptotic Fisher information quantity which is the limit in (37), becomes after the change of variable $x\mapsto a(u_n)x+b(u_n)$,

$$I(\theta)=\theta^{-2}\lim_{n\to\infty}\int\left[1+\left(x+\frac{b(u_n)}{a(u_n)}\right)\bar{G}_{u_n}(x)\right]^2 G_{u_n}(x)\,\mathrm{d}x.$$

Of course, the convergence (42) is not sufficient to ensure that $I(\theta)$ is in $(0,+\infty)$, but at least it ensures that if it is true, then necessarily $l=\lim_{n\to\infty} b(u_n)/a(u_n)\in\mathbb{R}$ and

$$I(\theta)=\frac{I}{\theta^2} \quad \text{with} \quad I=\int\left[1+(x+l)\,\bar{G}(x)\right]^2 G(x)\,\mathrm{d}x\in(0,+\infty) \tag{43}$$

Actually, what one expects is to have more than (42). An additional control of the type $\limsup_{|x|\to+\infty,n\to+\infty}|x\bar{G}_{u_n}(x)|<+\infty$ would be sufficient to prove that $I(\theta)=I/\theta^2$. Unfortunately, here also, there is a lack in the literature concerning such controls. We will see that to all stable processes, with eventually conditions on the drifts, and we illustrate by the non-trivial case of the sum of independent stable processes or time changed stable processes. The proofs of these examples are mainly based on tools developed in [11] and [12] giving examples of controls, of Lévy densities $G_t(x)$, in the space variable x, and uniformly in small or big time t. Our aim is, according to the asymptotic $(u_n)_{n\in\mathbb{N}^*}$, to study the LAN property with different paths u_n for processes of type

$$Y=\sum_{i=k}^{N} S^{a_k,b_k,c_k,d_k} \quad \text{or} \quad Y=S^{\alpha,\beta,1,0}\circ Z, \tag{44}$$

where the coefficients (a_k,b_k,c_k,d_k) vary in a set $\mathcal{D}_N$, the processes S^{a_k,b_k,c_k,d_k} are independent and $S^{\alpha,\beta,1}$ is independent from Z which is contained in a special class of subordinators attracted by stable subordinators. For example Z could be the sum of a stable subordinator and a Poisson process. Notice that in the situation (44), we loose the scaling property (40). Nevertheless, the processes Y belongs to the class (41). In the first situation, in case of pairwise distinct stability coefficients, we have the following property, which we call *asymptotic scaling*: if $i_\wedge=$ Argmin$\{\alpha_i,\ 1\le i\le N\}$ and $i_\vee=$ Argmax$\{\alpha_i,\ 1\le i\le N\}$, then the following convergence in distribution hold:

$$(\gamma_{i_0}t)^{-1/a_{i_0}}\left(Y_t-\sum_{k=1}^{N}\delta_{t,a_k}\right)\longrightarrow S_1^{a_{i_0},b_{i_0},1,0}, \tag{45}$$

$i_0=i_\vee$ (respectively $i_\wedge$), when $t\to 0$ (respectively when $t\to+\infty$). We will see in Subsection 4.2 how much is this property useful for the path $u_n\to 0$ (respectively $u_n\to+\infty$). For the second situation we treat subordinators such that for some $0<\varepsilon<1$ and some speed r_t (deterministic), this convergence in distribution holds: $Z_t/r_t\to S_1^{\varepsilon,1,1,0}$ when $t\to 0$ (respectively ∞). In fact r_t is necessarily regularly varying of order $1/\varepsilon$ at 0 (respectively ∞), recall it means $r_t=t^{1/\varepsilon}\,l(t)$ and l is slowly varying, i.e. $l(\lambda t)/l(t)\to 1$, $\forall\lambda>0$. In this situation, there exist $\beta\in(-1,1)$, $\gamma'>0$ such that

$$\frac{Y_t}{r_t^{1/\alpha}}\longrightarrow S_1^{\varepsilon\alpha,\beta',\gamma',0}, \tag{46}$$

4.1 Scale models associated with stable processes have the LAN property

The first application of Corollary 3.3:

Theorem 4.1 *For the scale model (5), assume $Y = S^{\alpha,\beta,\gamma,\delta}$ is a stable process such that $\delta = 0$ is null if $\alpha < 1$ and $|\beta| = 1$ and G the density of $S_1^{\alpha,\beta,1,0}$. Let $(u_n)_{n\in\mathbb{N}^*}$ be any sequence such that $l_n = (\gamma u_n)^{-1/\alpha}\,\delta_{u_n,\alpha} \to l \in \mathbb{R}$, where $\delta_{t,\alpha}$ is defined in (40), and such that one of the following holds:*

(i) $u_n \to L \in (0,+\infty)$;
(ii) $u_n \to L = 0, \quad \exists R > 0$ s.t. $n^R u_n \to +\infty$;
(iii) $u_n \to L = +\infty, \quad \exists S > 0$ s.t. $n^{-S} u_n \to 0$.

Then, the sequence of sequence of filtered statistical scale models E^n (10) have the LAN property with speed $\sqrt{n}$ at each $\theta \in \Theta$ and the asymptotic Fisher information quantity is given by

$$I(\theta) = \theta^{-2}\left[\int (y+l)^2\,\frac{G'(y)^2}{G(y)}\,\mathrm{d}y - 1\right].$$

Remark 4.2 *If $\alpha \in (0,1)$, then support of the law of $S_1^{\alpha,\beta,\gamma,\delta}$ is $[\delta,+\infty)$ (respectively $(-\infty,\delta]$) if $\beta = 1$ (respectively $\beta = -1$), otherwise the support is whole $\mathbb{R}$. The assumption $\delta = 0$ ensures that the support of the law of $S_1^{\alpha,\beta,\gamma,\delta}$ satisfies the condition* **(C0)** *of Corollary 3.3. In [12], we described in depth the behavior of the density $G_t^{\alpha,\beta,\gamma,\delta}$ of $S_t^{\alpha,\beta,\gamma,\delta}$ and considerations like support, asymptotic behavior in x uniformly in t,β,γ,δ are given. There,all the tools needed to treat any asymptotic with discretization path u_n are available. The cases $n u_n \to 0$ and $u_n \to +\infty$ are statistically not very realistic, nevertheless, they are treated since proofs cost more than the case $u_n \to 0$. The most interesting cases are:*

1. *$u_n = u \in (0,+\infty)$. Necessarily, $L = u$ and the assumptions of Theorem 4.1 are then immediately satisfied. This is an essentially trivial result because we treat then a regular i.i.d. model.*
2. *$u_n = u/n$, $u \in (0,+\infty)$. Necessarily, $L = 0$ and the assumptions of Theorem 4.1 are satisfied if and only if one of the following holds:*
   (i) $\alpha < 1, \delta = 0$ and then $l = 0$;
   (ii) $\alpha = 1, \beta = 0$ and then $ll = \delta/\gamma$;
   (iii) $\alpha > 1$ and then $ll = 0$.

Proof (Proof of Theorem 4.1). 1) a) Let h_n be the C^∞ density of $S_{u_n}^{\alpha,\beta,\gamma,\delta}$. The scaling property (40) reads on h_n and its derivatives as follows: for all $k \in \mathbb{N}$ and $x \in Support(h_n)$,

$$(h_n)^{(k)}(x) = (\gamma u_n)^{-(k+1)/\alpha}\left(G_1^{\alpha,\beta,1,0}\right)^{(k)}\left((\gamma u_n)^{-1/\alpha}(x-\delta_{u_n,\alpha})\right), \tag{47}$$

where

$$Support(h_n) = \begin{cases} \mathbb{R}_+ & \text{if } 0<\alpha<1,\ \beta = 1,\\ \mathbb{R}_- & \text{if } 0<\alpha<1,\ \beta=-1,\\ \mathbb{R} & \text{otherwise.}\end{cases} \tag{48}$$

1) b) In [11] and [12], we provided several properties of the density $G = G_1^{\alpha,\beta,1,0}$. For instance, there exist $A,B,C,D>0$, constants depending explicitly on α,β, such that with functions

$$\chi(x) := \frac{D}{|x|^{\alpha+1}}, \quad \xi(x) := B\,|x|^{(2-\alpha)/2(\alpha-1)}\,e^{-A|x|^{\alpha/(\alpha-1)}}, \quad \eta(x) = C\exp(-e^{\pi|x|/2} + \pi|x|/4),$$

we have

$$\begin{aligned} G(x) &\overset{0+}{\sim} (\textit{respectively} \overset{0-}{\sim})\ \xi(x), \text{ if } \beta = 1\ (\textit{respectively} -1) \text{ and } 0<\alpha<1\\ &\overset{+\infty}{\sim} (\textit{respectively} \overset{-\infty}{\sim})\ \chi(x), \text{ if } \beta \neq -1\,(\textit{respectively } 1) \text{ and } 0<\alpha<2\\ &\overset{+\infty}{\sim} (\textit{respectively} \overset{-\infty}{\sim})\ \eta(x), \text{ if } \beta = -1\,(\textit{respectively } 1) \text{ and } \alpha = 1\\ &\overset{+\infty}{\sim} (\textit{respectively} \overset{-\infty}{\sim})\ \xi(x), \text{ if } \beta = -1\,(\textit{respectively } 1) \text{ and } 1<\alpha\le 2.\end{aligned}$$

where $G(x)\overset{l}{\sim}H(x)$ means $\lim_{x\to l} G(x)/H(x) = 1$. Further, with the convention $0/0 = 0$, the functions

$$F_k(x) := |G^{(k)}(x)/G(x)|, k \in \mathbb{N}$$

are continuous on the support defined in (48) and there exist $a, b, c > 0$, constants depending explicitly on α and β and k, such that

$$\begin{array}{llll} F_k(x) & \overset{0+}{\sim} \ (respectively \overset{0-}{\sim}) & a\,|x|^{k/(\alpha-1)}, & \text{if } \beta = 1 \ (respectively -1) \text{ and } 0<\alpha<1 \\ & \overset{+\infty}{\sim} \ (respectively \overset{-\infty}{\sim}) & b|x|^{-k}, & \text{if } \beta \neq -1 \ (respectively \ 1) \text{ and } 0<\alpha<2 \\ & \overset{+\infty}{\sim} \ (respectively \overset{-\infty}{\sim}) & c\ \exp(k\pi|x|/2), & \text{if } \beta = -1 \ (respectively \ 1) \text{ and } \alpha = 1 \\ & \overset{+\infty}{\sim} \ (respectively \overset{-\infty}{\sim}) & a\,|x|^{k/(\alpha-1)}, & \text{if } \beta = -1 \ (respectively \ 1) \text{ and } 1<\alpha\le 2, \end{array}$$

Last equivalences, imply that for any nonnegative integer s, we have

$$0 \le r \le k \quad \Longrightarrow \quad \lim_{|x|\to+\infty} |x|^{2(1-\rho)(1+\alpha)}\,(|x|^r F_k)^s\,(G)^{2(1-\rho)} \in [0.+\infty) \tag{49}$$

and because $0 < \rho < 1 - 1/(2(1+\alpha)) \Leftrightarrow 2(1-\rho)(1+\alpha) > 1$, we have

$$0 \le r \le k,\ \ 0 < \rho < 1 - \frac{1}{2(\alpha+1)} \quad \Longrightarrow \quad x \mapsto (|x|^r F_k(x))^s\, G(x)^{2(1-\rho)} \in \mathrm{L}^1(dx)\,. \tag{50}$$

2) We need to verify the assumptions of Corollary 3.3, i.e. to check **(C1)**: as $n \to \infty$,

$$I_n = \int [1 + x\frac{h_n'}{h_n}(x)]^2\, h_n(x)\,dx \longrightarrow I = \int (y+l)^2\,\frac{(G')^2}{G}(y)\,dy \ \ -1 \ \ \in (0,+\infty) \tag{51}$$

and **(C2)**, that is to say: there exists $a \in (0,1/2)$, such that for $\rho \in \{1/2, a, 1-a\}$, one has

$$\frac{1}{n} J_n(\rho) = \frac{1}{n}\int \frac{(j_n)^2}{(h_n)^{2\rho}}(x)\,dx \longrightarrow 0, \tag{52}$$

$$j_n(x) = \left[\left|1 + x\frac{h_n'}{h_n}(x)\right|^2 + \left|2 + 4x\frac{h_n'}{h_n}(x) + x^2\frac{h_n''}{h_n}(x)\right|\right] h_n(x). \tag{53}$$

3) The scaling property (47) and the corresponding change of variables give the following representation of the quantity I_n in (51):

$$I_n = \int [1 + (y+l_n)\frac{G'(y)}{G(y)}]^2\, G(y)\,dy\,.$$

Because $l_n \to l \in \mathbb{R}$ and thanks to (49), we obtain

$$x \mapsto \sup_{n\in\mathbb{N}}\Big[1 + (y+l_n)\frac{G'}{G}(y)\Big]^2 G(y) \in \mathrm{L}^1(dy)\ \lim_{n\to\infty} I_n = \int [1 + (y+l)\frac{G'}{G}(y)]^2\, G(y)\,dy.$$

Developing last expression, integrating by parts and using the fact that $G(y)$ and $yG(y)$ both tend to 0 as y goes to each endpoint of the support (48), we recover (51).

4) Again, by the change of variables corresponding to (47), and by the representation (53), one has

$$J_n(\rho) = (\gamma u_n)^{(2\rho-1)/\alpha} \int \left[\left|1 + (y+l_n)\frac{G'}{G}(y)\right|^2 + \left|2 + 4(y+l_n)\frac{G'}{G}(y) + (y+l_n)^2\frac{G''}{G}(y)\right|\right]^2 G(y)^{2(1-\rho)}\,dy.$$

Thanks to (50) and that $l_n \to l \in \mathbb{R}$, it is clear that if $\rho \in (0, 1 - 1/2(\alpha+1))$, then

$$y \mapsto \sup_{n\in\mathbb{N}}\left[\left|1 + (y+l_n)\frac{G'}{G}(y)\right|^2 + \left|2 + 4(y+l_n)\frac{G'}{G}(y) + (y+l_n^2\frac{G''}{G}(y)\right|\right]^2 G(y)^{2(1-\rho)} \in \mathrm{L}^1(dy).$$

Denote $\varepsilon = 1/2 - 1/(2(\alpha+1))$. In order to prove the convergence (52), it is enough to have

$$\rho \in (0, \varepsilon + 1/2) \tag{54}$$

and

$$\lim_{n\to+\infty} \frac{{u_n}^{(2\rho-1)/\alpha}}{n} = 0\,. \tag{55}$$

5) Now, we distinguish between the values of $L = \lim_{n\to+\infty} u_n$.
5) a) $L = 0$: (55) is always true if $\rho \geq 1/2$. By assumptions $1/u_n \leq n^R$, if n is big enough. We deduce that if $\rho < 1/2$, we have (55) as soon as $R(1-2\rho)/\alpha < 1$ which is equivalent to $\rho > 1/2 - \alpha/(2R)$. We only have to choose $\varepsilon' = \varepsilon \wedge \alpha/(4R)$ and $a = 1/2 - \varepsilon'$, to get (54) and (55) for $\rho \in \{1/2, a, 1-a\}$.
5) b) $L \in (0, +\infty)$: (55) is always true.
5) c) $L = +\infty$: (55) is always true if $\rho \leq 1/2$. By the same way than 5) a), $u_n \leq n^S$ if n is big enough. We deduce that if $\rho > 1/2$, we have (55) as soon as $S(2\rho-1)/\alpha < 1$ which is equivalent to $\rho < 1/2 + \alpha/(2S)$. We only have to choose $\varepsilon'' = \varepsilon \wedge \alpha/(4S)$ and $a = 1/2 - \varepsilon''$, to get (54) and (55) for $\rho \in \{1/2, a, 1-a\}$.

4.2 Scale models associated with the sum of independent stables processes has the LAN property

This sub-section gives a second example which also generalizes the previous one and achieves the situation (41). Define $\bar{K}(\alpha) = 1$ if $\alpha \leq 1$ and $\bar{K}(\alpha) = (\alpha-2)/\alpha$ and assume the following restrictions on the skewness parameters:

$(\mathbf{S_{a,b}})$: *We have N independent processes $S^{a_k,b_k,c_k,0}$, such that*

(a) $a_1 < a_2 < \cdots < a_N < 2$ *and* $\mathcal{D} = \bigcap_{k=1}^{N} [\, a_k\,, 2/(1+|b_k \bar{K}(a_k)|)\,] \neq \varnothing$

(b) $b_k = 0$ *if* $a_k = 1$ *and* $B = \max\{\, |b_k \bar{K}(a_k)/\bar{K}(a_N)|,\ 1 \leq k \leq N\,\} < 1$.

Let $Y = \sum_{k=1}^{N} S^{a_k,b_k,c_k,0}$ and Y^i the processes defined by:

$$Y^i_t = \frac{Y_t}{t^{1/a_i}}, \quad i = 1 \text{ or } i = N \text{ and } t > 0. \tag{56}$$

The processes Y^i satisfy the *asymptotic scaling* property of the situation (45). Denote by H_t the density of Y_t

$$H_t = G_t^{a_1,b_1,c_1,0} \star \cdots \star G_t^{a_N,b_N,c_N,0}. \tag{57}$$

and by $H_{i,t}$ the one of Y^i_t. Then, $h_n = H_{u_n}$ satisfies

$$(h_n)^{(k)}(x) = (u_n c_i)^{-(k+1)/\alpha_i}\,(H_{i,u_n})^{(k)}\left((u_n c_i)^{-1/a_i} x\right). \tag{58}$$

In [11],[12], we showed that the following function H_0 and H_∞ have a meaning if defined by

$$H_L := \begin{cases} \lim_{t\to+\infty} H_{1,t}(x) = G_1^{a_1,b_1,1,0}(x) \text{ if } L = +\infty \\ \lim_{t\to 0+} H_{N,t}(x) = G_1^{a_N,b_N,1,0}(x) \text{ if } \ L = 0 \end{cases} \tag{59}$$

the convergence still hold for the successive derivatives, and what is more, uniformly in $x \in Support(G_1^{a_i,b_i,1,0})$. As one can guess, we are going to make the most of the identity (58) and have the theorem:

Theorem 4.3 *Let $(u_n)_{n\in\mathbb{N}^*}$ be a sequence satisfying one of the following:*

$$\begin{aligned} &\text{(i)}\ \ u_n \to L \in (0,+\infty) \\ &\text{(ii)}\ \ u_n \to L = 0, \quad \exists R > 0 \ s.t.\ n^R u_n \to +\infty \\ &\text{(iii)}\ \ u_n \to L = +\infty, \quad \exists S > 0 \ s.t.\ n^{-S} u_n \to 0\,. \end{aligned}$$

For the scale model (5) with $Y = \sum_{k=1}^{N} S^{a_k,b_k,c_k,0}$, assume $(\mathbf{S_{a,b}})$. Then, the sequence of filtered statistical scale models E^n (10) have the LAN property with speed $\sqrt{n}$ at each value $\theta \in \Theta$ and the asymptotic Fisher information quantity $I(\theta_0)$ is I_L/θ^2, and with the function H_L defined in (57) and (59), depending on the value of L, we have

$$I_L = \int y^2 \frac{(H_L')^2(y)}{H_L(y)}\,\mathrm{d}y - 1\,.$$

Remark 4.4 *Let us briefly explain the nature of the previous assumption. In [12], conditions of type* $(\mathbf{S_{a,b}})$ *allowed to show that* Y_t^i *is distributed as an* α*-stable variable mixed on the skewness and scale parameters by other processes. More precisely, for all* $t>0$*, we have these identities in distribution: for all* α *in the interior of* $\mathcal{D}$ *and* $t>0$*, there exist a r.v.* β_t *and* γ_t^i *such that*

$$Y_t^i \stackrel{d}{=} S_1^{\alpha,\beta_t,\gamma_t^i,0} \stackrel{d}{=} (\gamma_t^i)^{1/\alpha}\, S_1^{\alpha,\beta_t,1,0}. \tag{60}$$

The processes β *and* γ^i *are such that*

$$|\beta_t| \le B \quad \text{and} \quad C\,t^{-\alpha/a_i} Z_t \le \gamma_t^i \le D\,t^{-\alpha/a_i} Z_t\,, \tag{61}$$

where $Z_t = \sum_{k=1}^n S_t^{a_k/\alpha,1,1,0}$ *is a sum of independent standard stable subordinators and the non-negative numbers* $C \le D$ *depend only on* $(\alpha, a_1, b_1, c_1, \cdots, a_N, b_N, c_N)$*. In the case* $b_k \bar{K}(a_k)$ *constant for all* $k=1,\cdots N$*, then* $\beta_t = b_1 \bar{K}(a_1)$ *and* γ_t^i *is distributed as a normalized sum of independent stable subordinators. Moreover, they satisfy the following converges in distribution*

$$\left(\beta_t\,,\gamma_t^i\right) \to \left(b_i \bar{K}(a_i)/\bar{K}(\alpha)\,, S_1^{a_i/\alpha,1,1,0}\right), \quad \text{if } i=1 \text{ and } t\to+\infty, \text{ or if } i=N \text{ and } t\to 0.$$

Furthermore, notice that the assumption $(\mathbf{S_{a,b}})$ *is satisfied in the symmetrical cases*

$$b_1 = \cdots = b_N = 0 \quad \text{and then} \quad \beta_t = 0\,.$$

In general when β,γ,δ *are r.v.'s lying in the set of admissible parameters and* $\mathcal{F}$ *is the* σ*-field generated by them, we gave in [12] a structure of* $\mathcal{F}$*-conditional Lévy process to* $(S_r^{\alpha,\beta,\gamma,\delta})_{r\ge0}$*. See also [10] for the notion of* $\mathcal{F}$*-conditional Lévy process. If* β *is deterministic* $S_1^{\alpha,\beta,\gamma,0}$ *is simply distributed as* $\gamma^{1/\alpha} S_1^{\alpha,\beta,1,0}$ *and with our construction, we allow the same identity even if* β *is random and correlated with* γ*. We also considered in [12] the densities of some families of mixed stable variables* $(S_1^{\alpha,\beta_t,\gamma_t,0})_{t\in T}$ *and gave several examples when these densities and their derivatives behave like the proper stable densities and this uniformly in* $t\in T$*.*
In fact, it is also possible to state a version of the Theorem 4.3 with stable processes with drifts. It is enough to strengthen the conditions on the asymptotic as done in Theorem 4.1.We consider that Theorem 4.3 is far from being exhaustive. It is produced in the aim of illustrating the difficulty of this case. If one wants to reduce the assumption $(\mathbf{S_{a,b}})$ *on the coefficients, some more controls on the densities are needed.*

Before tackling the proof of the Theorem 4.3, we need tour following result borrowed from [12]:

Theorem 4.5 (Controls of the densities of some mixed stable variables $(S_1^{\alpha,\beta_r,\gamma_r,0})_{r\in\mathcal{R}}$**)**
Let $(\gamma_r)_{r\ge0}$ *a pure jump subordinator, i.e.*

$$\mathbb{E}[e^{-\lambda\,\gamma_1}] = \exp\int_{0,\infty}(e^{-\lambda x}-1)\,\nu(\mathrm{d}x),\ \lambda\ge0.$$

Assume $\nu(x)=\nu(x,\infty)=x^{-a}L(x)$*, with* $0<a<1$ *and* L *a slowly varying function. For all* $t,x>0$*, define*

$$v_r := \sup\{t>0:\ \nu(t)>1/r\},\quad \nu_r(x) := s\,\nu(x v_r,\infty) \quad \text{and} \quad \bar{\gamma}_r := \frac{\gamma_r}{v_r}\,. \tag{62}$$

a) When L *is slowly varying at infinity and* $r\to\infty$ *or* L *is slowly varying at zero then*

$$\nu_r(x) \longrightarrow \frac{1}{x^a},\ x>0, \qquad \text{and} \qquad \bar{\gamma}_r \stackrel{d}{\longrightarrow} S_1^{a,1,\Gamma(1-a)/a,0}, \qquad \text{as } r\to0+\,. \tag{63}$$

b) Moreover assume that there exist $0<c\le a\le d<1$ *and* $K\ge1$ *such that*

$$(y/x)^{a-c} \le L(y)/L(x) \le K(y/x)^{a-d}, \quad 0<y<x\,. \tag{64}$$

Let $R'>R>0$ *and* $(\gamma_r^p)_{t\in R_p}$ *denote one of these families:* $R_1=R_4=[0,R]$*,* $R_2=[R,R']$*,* $R_3=[R,\infty)$ *and* $\gamma_r^1=\gamma_r^2=\gamma_r$*,* $\gamma_r^3=\bar{\gamma}_r$ *with* L *slowly varying at* ∞*,* $\gamma_r^4=\bar{\gamma}_r$ *with* L *slowly varying at* 0*. Let* $\alpha\in(0,2)$ *and* $(\beta_r)_{r\ge0}$ *any family of r.v. such that* $\sup_{r\ge0}|\beta_r|\le B$*, for some* $B\in[0,1)$*, (*$B=0$*, if* $\alpha=1$*). Then the densities* G_r^p *of the mixed stable variables* $S_1^{\alpha,\beta_r,\gamma_r^p}$ *are infinitely differentiable and satisfy, for all* $k\in\mathbb{N}$ *and all* p*:*

$$0<\liminf_{|x|\to\infty}\inf_{r\in R_p}|x|^{1+\alpha d}G_r^p(x),\qquad \limsup_{|x|\to\infty}\sup_{r\in R_p}|x|^{1+\alpha c}G_r^p(x)<\infty\,, \tag{65}$$

$$\limsup_{|x|\to\infty}\sup_{r\in R_p,x\in\mathbb{R}}\frac{|x^k(G_r^p)^{(k)}(x)|}{G_r^p(x)}<\infty\,, \tag{66}$$

and for all $X>0$, $p\neq 1$,

$$\sup_{r\in R_p,x\in\mathbb{R}}|(G_r^p)^{(k)}(x)|<\infty \quad \text{and} \quad \inf_{r\in R_p}\inf_{|x|\le X}G_r^p(x)>0\,. \tag{67}$$

c) If $(\gamma'_r)_{r\ge 0}$ is a family of r.v.'s such that $\gamma_r/K\le\gamma'_r\le K\gamma_r$ for some $K>1$ and all $r\ge 0$, then the controls (65-66-67) still be true for the densities obtained by replacing γ_r, $\bar{\gamma}_r$ by γ'_r, γ'_r/v_r with v_r given by (62).

Remark 4.6 *If γ is a pure jump a-stable subordinator, then $\nu_r(x)=1/x^a$ and the scaling property gives $\bar{\gamma}_r\overset{d}{=}S_1^{a,1,\Gamma(1-a)/a}$. If furthermore for all $t>0$, $\beta_r=\beta$ deterministic, then $S_1^{\alpha,\beta_r,\bar{\gamma}_r}\overset{d}{=}S_1^{\alpha a,\beta',\gamma'}$, for some $\beta'\in(-1,1)$, $\gamma'>0$. The estimates (65-66-67) are an immediate consequence of the behavior of the stable densities given in the proof of Theorem 4.1. The following are examples of processes satisfying the conditions of Theorem 4.5. Let $0<b,b_1,\cdots,b_N<1$ and $c,c_1,\cdots c_N>0$. Let γ^1, γ^2 a pure jump subordinators having Lévy measures equal respectively to $\nu_1(\mathrm{d}x)=\sum_{k=1}^N c_i x^{-(b_k+1)}1_{x>0}\mathrm{d}x$ and $\nu_2(\mathrm{d}x)=cx^{-(b+1)}1_{x>0}\mathrm{d}x+\delta_1(\mathrm{d}x)$. γ^1 is the sum of independent stable subordinators, and γ^2 is the independent sum of a stable subordinator and a standard Poisson process. Let $b_\vee=\max b_i$, $b_\wedge=\min b_i$. Then $\nu_1(x)=x^{-b_\vee}L_1^\vee(x)=x^{-b_\wedge}L_1^\wedge(x)$, $\nu_2(x)=x^{-b}L_2(x)$ and it is easily seen that $L_1^\vee$, L_2 are slowly varying at 0, L_1 is slowly varying at ∞ and $0<y<x$ implies*

$$\left(\frac{y}{x}\right)^{b_\vee-b_\wedge}\le\frac{L_1^\vee(y)}{L_1^\vee(x)}\le 1,\qquad 1\le=\frac{L_1^\wedge(y)}{L_1^\wedge(x)}\le\left(\frac{y}{x}\right)^{b_\wedge-b_\vee}\quad\text{and}\quad\left(\frac{y}{x}\right)^{b/2}\le\frac{L_2(y)}{L_2(x)}\le 2\,.$$

As a consequence of Theorem (4.5), we can state:

Corollary 4.7 *Assume $(\mathbf{S_{a,b}})$. Denote for $i=1,N$, $t>0$, $k\in\mathbb{N}$ and $z\in\mathbb{R}$, $H_{i,t}(z)$ the density of Y_t^i, for $T>0$, $T_1=[T,\infty)$, $T_N=[0,T]$ and*

$$F_{i,t}^k(z)=\left|\frac{(H_{i,t})^{(k)}(z)}{H_{i,t}(z)}\right|.$$

Then, for every nonnegative integer s, we have

$$0\le r\le k,\ \ 0<\rho<1-\frac{1}{2(a_1+1)}\quad\Longrightarrow\quad z\mapsto\sup_{t\in T_i}\left(|z|^r F_{i,t}^k(z)\right)^s H_{i,t}(z)^{2(1-\rho)}\in\mathrm{L}^1(\mathrm{d}z)\,. \tag{68}$$

Proof. Using Remark 4.6, the subordinator Z in (61) has a Lévy measure satisfying the required conditions and they imply, according to Theorem 4.5 c), that the family $(H_{1,t})_{t\in T_1}$ behaves like $(G_r^3)_{r\in R_3}$ and $(H_{N,t})_{t\in T_N}$ behaves like $(G_r^4)_{r\in R_4}$ with $c=a_1/\alpha$ and $d=a_N/\alpha$.

Proof (Proof of Theorem 4.3). We take up a method analog to the one of the proof of Theorem 4.1, that is, we have to verify the assumptions of Corollary 3.3. Let

$$I_n=\int\left[1+z\frac{(H_{i,u_n})'}{H_{i,u_n}}(z)\right]^2 H_{i,u_n}(z)\,\mathrm{d}z,$$

$$J_n(\rho)=(\gamma u_n)^{(2\rho-1)/\alpha}\int\left[\left|1+z\frac{(H_{i,u_n})'}{H_{i,u_n}}(z)\right|^2+\left|2+4z\frac{(H_{i,u_n})'}{H_{i,u_n}}(z)+z^2\frac{(H_{i,u_n})''}{H_{i,u_n}}(z)\right|\right]^2 H_{i,u_n}(z)^{2(1-\rho)}\,\mathrm{d}z.$$

By Corollary 4.7, the functions $F_{i,t}^k$, satisfy

$$\sup_{n\in\mathbb{N}}\left(|z|^r F_{i,u_n}^k\right)^s(H_{i,u_n})^{2(1-\rho)}\in L^1(\mathrm{d}z),$$

for $s\in\{0,1,2,3,4\}$, $r\in\{0,1,\cdots,k\}$, $0\le k\le 2$, and $\rho\le 1-\frac{1}{2(a_1+1)}$. The rest is obtained by reproducing the proof of Theorem 4.1,

Remark 4.8 *Notice that the main argument for proving Theorem 4.3 is the behavior of the densities uniformly in time. Theorem 4.5 provides many other examples. For example, with the same proof as in Theorem 4.3, one could state a version with stable processes time changed by any independent nice subordinator. The time change process could be the sum of a stable subordinator and a Poisson process (see Remark 4.6). Finally, it appears that more investigation concerning the behavior in small time of densities of Lévy processes attracted by stable processes, would be extremely useful for the kind of statistical properties we are looking for.*

5 How to build a LAN model from another LAN model?

In this section we treat the following question: to which extent is crucial the choice of the asymptotic? More precisely, if we start from a LAN model associated to the observations along a discretization scheme $i u_n$, $1 \leq\leq n$, of a processus de Lévy X, how can we affirm that the model associated to the observations of $X + \widetilde{X}$, where $\widetilde{X}$ is another independent Lévy process also enjoys the LAN property with the same discretization scheme? For the answer, we need some preliminaries and two lemmas.

Consider two independent Lévy processes Y and N defined on some probability space $(\bar{\Omega}, \bar{F}, \bar{\mathbb{P}})$ with set values the Skorokhod space $\Omega = \mathbb{D}(\mathbb{R}+, \mathbb{R})$ (when the processes Y and N are seen as infinite-dimensional random variables). Assume that N is a non-drifted compound Poisson process, with Lévy measure ν and denote $\widetilde{Y} = Y + N$. Recall that the observed increment processes X^n along a scheme u_n of a process X is defined in (9) by

$$X^n_j = X_{(j+1)\,u_n} - X_{j\,u_n}, \qquad 0 \leq j \leq n-1.$$

For $\theta \in \Theta$, suppose we observe $X^n = \theta Y^n$, $\widetilde{X}^n = \theta \widetilde{Y}^n$ and let

$$\mathbb{P}_\theta = \mathcal{L}aw(\theta\, Y \,|\, \overline{\mathbb{P}}) \quad \text{and} \quad \widetilde{\mathbb{P}}_\theta = \mathcal{L}aw(\theta\, \widetilde{Y} \,|\, \overline{\mathbb{P}}).$$

The probability measure $\mathbb{P}^n_\theta$ (respectively $\widetilde{\mathbb{P}}^n_\theta$) and the scale models E^n (respectively $\widetilde{E}^n$) correspond to X (respectively $\widetilde{X}$) as in (10) and (11).

Recall that if Q, Q' are two probability measures on some sample space, then the total variation distance $\| Q - Q' \|$, is the quantity

$$\| Q - Q' \| = \sup_{\phi \in \Phi} | \mathbb{E}_Q(\phi) - \mathbb{E}_{Q'}(\phi) |, \qquad \Phi = \{ \phi : \Omega \to [-1, 1], \phi \text{ measurable} \}.$$

Lecam's Lemma [15] says:

Lemma 5.1 *For every probability measures Q, Q', R, R', we have the inequality*

$$\int 1 \wedge \left| \frac{\mathrm{d}R}{\mathrm{d}Q} - \frac{\mathrm{d}R'}{\mathrm{d}Q'} \right| \mathrm{d}(Q + Q') \leq \| Q - Q' \| + 2 \, \| R - R' \| + (2 \, \| Q - Q' \| \| R + R' \|)^{1/2}. \tag{69}$$

We have:

Lemma 5.2 *If $lim_{n \to +\infty} n\, u_n = 0$, then* $\lim_{n \to +\infty} \sup_{\theta \in \Theta} \| \mathbb{P}^n_\theta - \widetilde{\mathbb{P}}^n_\theta \| = 0.$

Proof. Since

$$\bar{\mathbb{P}}(N_{u_n} \in \mathrm{d}y) = e^{-\nu(\mathbb{R})\, u_n} \sum_{k=0}^{+\infty} \frac{{u_n}^k}{k!}\, \nu^{*k}(\mathrm{d}y) \quad \text{and} \quad \bar{\mathbb{P}}(N_{u_n} = 0) \geq e^{-\nu(\mathbb{R})\, u_n},$$

and since N has stationary and independent increments, we have

$$
\begin{aligned}
\left|\mathbb{P}^n_\theta - \widetilde{\mathbb{P}}^n_\theta\right| &= \sup_{\phi\in\Phi}\left|\mathbb{E}_{\mathbb{P}^n_\theta}[\phi(X)] - \mathbb{E}_{\widetilde{\mathbb{P}}^n_\theta}[\phi(X)]\right| = \sup_{\phi\in\Phi}\left|\mathbb{E}_{\mathbb{P}_\theta}[\phi(X^n)] - \mathbb{E}_{\widetilde{\mathbb{P}}_\theta}[\phi(X^n)]\right| \\
&= \sup_{\phi\in\Phi}\left|\mathbb{E}_{\bar{\mathbb{P}}}[\phi(\theta Y^n)] - \mathbb{E}_{\bar{\mathbb{P}}}[\phi(\theta \widetilde{Y}^n)]\right| = \sup_{\phi\in\Phi}\left|\mathbb{E}_{\bar{\mathbb{P}}}[\phi(\theta Y^n)] - \mathbb{E}_{\bar{\mathbb{P}}}\left[\phi\left(\theta(Y^n+N^n)\right)\right]\right| \\
&= \sup_{\phi\in\Phi}\left|\mathbb{E}_{\bar{\mathbb{P}}}[\phi(Y^n)] - \mathbb{E}_{\bar{\mathbb{P}}}[\phi(Y^n+N^n)]\right| = \sup_{\phi\in\Phi}\left|\mathbb{E}_{\bar{\mathbb{P}}}\left[\left(\phi(Y^n)-\phi(Y^n+N^n)\right)1_{N^n\not\equiv 0}\right]\right| \\
&\le 2\bar{\mathbb{P}}(N^n \not\equiv 0) = 2\left(1-\bar{\mathbb{P}}(N^n\equiv 0)\right) = 2\left(1-\bar{\mathbb{P}}(N_{(j+1)u_n} - N_{ju_n} = 0,\ \forall\, 0\le j\le n)\right) \\
&\le 2\left(1-\bar{\mathbb{P}}(N_{u_n}=0)^n\right) \le 2\left(1-e^{-\nu(\mathbb{R})\,n u_n}\right).
\end{aligned}
$$

It is now clear that $\| \mathbb{P}^n_\theta - \widetilde{\mathbb{P}}^n_\theta \|$ goes to 0, uniformly in θ as $nu_n \to 0$.

We can now able to complete the problem studied by Far [7] who treated the case where Y is a brownian motion and the discretization path is $u_n = 1/n$, i.e. $\lim_{n\to\infty} nu_n = 1$.

Theorem 5.3 *Assume* $\lim_{n\to\infty} nu_n = 0$. *If the scale model* (5) E^n *associated to the process X has the LAN property with speed* $\sqrt{n}$ *in a point* $\theta_0 \in \Theta$, *then so is the scale model* $\widetilde{E}^n$ *associated to the process* $\widetilde{X}$.

Proof (Proof of Theorem 5.3). 1) Fix $\theta_0 \in \Theta$, J, where J is a finite subset of $\mathbb{R}$ and $\xi \in \mathbb{R}$. We shall prove that the weak functional convergence (15) of the likelihood processes $(Z^{n,\eta\xi})_{\eta\in J}$ of E^n yields the one of the likelihood processes $(\widetilde{Z}^{n,\eta\xi})_{\eta\in J}$ of $\widetilde{E}_n$. The expression of the likelihood processes are given by given by (14) and for more convenience, we denote them from now on by

$$
\mathbf{Z}^n_t = (Z^{n,\eta\xi}_k)_{\eta\in J}, \quad \widetilde{\mathbf{Z}}^n_t = (\widetilde{Z}^{n,\eta\xi}_t)_{\eta\in J} \quad \text{and} \quad \mathbf{Z}'_t = (Z'^{,\eta\xi}_t)_{\eta\in J}, \quad t\in[0,1].
$$

We need to show the following convergence in laws: when $n\to\infty$,

$$
\mathcal{L}aw\left(\ \mathbf{Z}^n \,|\, \mathbb{P}^n_{[\xi]_n}\ \right) \longrightarrow \mathcal{L}aw\left(\ (Z'^{\eta\xi})_{\eta\in J} \,|\, \mathbb{P}'_\xi\ \right) \Longrightarrow \mathcal{L}aw\left(\ \widetilde{\mathbf{Z}}^n \,|\, \widetilde{\mathbb{P}}^n_{[\xi]_n}\ \right) \longrightarrow \mathcal{L}aw\left(\ (Z'^{\eta\xi})_{\eta\in J} \,|\, \mathbb{P}'_\xi\ \right),
$$

or, equivalently, for every K-Lipschitz function $f : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^J) \to \mathbb{R}$, bounded by a constant $C>0$, we need to show that

$$
\lim_{n\to+\infty} \mathbb{E}_{\mathbb{P}^n_{[\xi]_n}}[f(\mathbf{Z}^n)] = \mathbb{E}_{\mathbb{P}'_\xi}[f(\mathbf{Z}')] \Longrightarrow \lim_{n\to+\infty} \mathbb{E}_{\widetilde{\mathbb{P}}^n_{[\xi]_n}}[f(\widetilde{\mathbf{Z}}^n)] = \mathbb{E}_{\mathbb{P}'_\xi}[f(\mathbf{Z}')].
$$

2) For such functions f, we will control the difference

$$
\mathbb{E}_{\mathbb{P}^n_{[\xi]_n}}[f(\mathbf{Z}^n)] - \mathbb{E}_{\widetilde{\mathbb{P}}^n_{[\xi]_n}}[f(\widetilde{\mathbf{Z}}^n)] = \left(\mathbb{E}_{\mathbb{P}^n_{[\xi]_n}}[f(\mathbf{Z}^n)] - \mathbb{E}_{\widetilde{\mathbb{P}}^n_{[\xi]_n}}[f(\mathbf{Z}^n)]\right) + \left(\mathbb{E}_{\widetilde{\mathbb{P}}^n_{[\xi]_n}}[f(\mathbf{Z}^n)] - \mathbb{E}_{\widetilde{\mathbb{P}}^n_{[\xi]_n}}[f(\widetilde{\mathbf{Z}}^n)]\right).
$$

In virtue of Lemma (5.2), we have

$$
\left|\mathbb{E}_{\mathbb{P}^n_{[\xi]_n}}[f(\mathbf{Z}^n)] - \mathbb{E}_{\widetilde{\mathbb{P}}^n_{[\xi]_n}}[f(\widetilde{\mathbf{Z}}^n)]\right| \le C \sup_{\theta\in\Theta} \| \mathbb{P}^n_\theta - \widetilde{\mathbb{P}}^n_\theta \| \to 0, \quad \text{as} n\to\infty. \tag{70}
$$

For every $\varepsilon>0$, we have

$$
\begin{aligned}
\mathbb{E}_{\widetilde{\mathbb{P}}^n_{[\xi]_n}}\left[f(\mathbf{Z}^n) - f(\widetilde{\mathbf{Z}}^n)\right] &= \mathbb{E}_{\widetilde{\mathbb{P}}^n_{[\xi]_n}}\left[\left(f(\mathbf{Z}^n) - f(\widetilde{\mathbf{Z}}^n)\right) 1_{|\mathbf{Z}^n - \widetilde{\mathbf{Z}}^n|\le\varepsilon}\right] + \mathbb{E}_{\widetilde{\mathbb{P}}^n_{[\xi]_n}}\left[\left(f(\mathbf{Z}^n) - f(\widetilde{\mathbf{Z}}^n)\right) 1_{|\mathbf{Z}^n - \widetilde{\mathbf{Z}}^n|>\varepsilon}\right] \\
\left|\mathbb{E}_{\widetilde{\mathbb{P}}^n_{[\xi]_n}}[f(\mathbf{Z}^n)] - f(\widetilde{\mathbf{Z}}^n)]\right| &\le \varepsilon K + 2C\widetilde{\mathbb{P}}^n_{[\xi]_n}(|\,\mathbf{Z}^n - \widetilde{\mathbf{Z}}^n\,| > \varepsilon).
\end{aligned} \tag{71}
$$

With representation (14), observe that $\mathbf{Z}^n$ and $\widetilde{\mathbf{Z}}^n$ are step-processes and depend on time up to time $[nt], t\in[0,1]$. Then, denoting $\tau^n = \inf\{1\le j\le n\ s.t.\ |\,\mathbf{Z}^n_j - \widetilde{\mathbf{Z}}^n_j\,| > \varepsilon,\ \forall \eta\in J, \xi\in\mathbb{R}\}$, and using Markov in equality, we obtain

$$
\begin{aligned}
\widetilde{\mathbb{P}}^n_{[\xi]_n}(|\,\mathbf{Z}^n - \widetilde{\mathbf{Z}}^n\,| > \varepsilon) &= \widetilde{\mathbb{P}}^n_{[\xi]_n}(|\,\mathbf{Z}^n_{\tau^n} - \widetilde{\mathbf{Z}}^n_{\tau^n}\,| > \varepsilon) = \widetilde{\mathbb{P}}^n_{[\xi]_n}(|\,\mathbf{Z}^n_i - \widetilde{\mathbf{Z}}^n_j\,| > \varepsilon,\ \forall 1\le j\le n) \\
&\le \frac{1}{\varepsilon}\,\mathbb{E}_{\widetilde{\mathbb{P}}^n_{[\xi]_n}}\left[1\wedge |\,\mathbf{Z}^n_{\tau^n} - \widetilde{\mathbf{Z}}^n_{\tau^n}\,|\right].
\end{aligned}
$$

Using Lemma 5.1 with $R = \mathbb{P}^n_{[\eta]_n}$, $Q = \mathbb{P}^n_{[\xi]_n}$, $R' = \widetilde{\mathbb{P}}^n_{[\eta]_n}$, $Q' = \widetilde{\mathbb{P}}^n_{[\xi]_n}$, we get

$$\mathbb{E}_{\widetilde{\mathbb{P}}^n_{[\xi]_n}}\left[1\wedge|\mathbf{Z}^n_{\tau^n}-\widetilde{\mathbf{Z}}^n_{\tau^n}|\right]\leq 2\sup_{\theta\in\Theta}\left[3\parallel\mathbb{P}^n_\theta-\widetilde{\mathbb{P}}^n_\theta\parallel+(2\parallel\mathbb{P}^n_\theta-\widetilde{\mathbb{P}}^n_\theta\parallel)^{1/2}\right]$$

and Lemma 5.2 gives

$$\lim_{n\to+\infty}\sup_{\eta\in J,\,\xi\in\mathbb{R}}\widetilde{\mathbb{P}}^n_{[\xi]_n}(|\mathbf{Z}^n-\widetilde{\mathbf{Z}}^n|>\varepsilon)=0\,.$$

The latter, together with (71), allows to conclude that

$$\lim_{n\to+\infty}\mathbb{E}_{\mathbb{P}^n_{[\xi]_n}}[f(\mathbf{Z}^n)-f(\widetilde{\mathbf{Z}}^n)]=0.$$

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