Line Integrals

Mongi BLEL

King Saud University

March 25, 2024

Consider a plane curve given by the parametric equations

$$\gamma(t) = (x(t), y(t)), \quad t \in [a, b].$$

Definition

Let f be a continuous function on $\mathbb{R}^2$. If γ is continuously differentiable, the line integral of f on γ with respect to the arc length is defined by:

$$\int_a^b f \circ \gamma(t) \sqrt{(x'(t))^2 + (y'(t))^2} dt = \int_a^b f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

Remarks

1. If $f = 1$, $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt$ is the length of γ.
   Note that $\sqrt{(x'(t))^2 + (y'(t))^2} = \|\gamma'(t)\|$. We denote $ds = \sqrt{(x'(t))^2 + (y'(t))^2}dt$.
2. The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

Example

(Integrating along an arc of circle)
Consider the arc of circle C parametrized by $(\cos t, \sin t)$, with $t \in [0, \frac{\pi}{2}]$. In this case $ds = \sqrt{\cos^2 t + \sin^2 t}\, dt = dt$

$$
\begin{aligned}
\int_C (x + 4xy^2) ds &= \int_0^{\frac{\pi}{2}} (\cos t + 4 \cos t \sin^2 t) dt \\
&= \int_0^{\frac{\pi}{2}} \cos t (1 + 4 \sin^2 t) dt \\
&\overset{u=\cos t}{=} \int_0^1 (1 + 4u^2) du = \frac{7}{3}.
\end{aligned}
$$

Definition

Let f be a continuous function on $\mathbb{R}^2$ and let γ be piecewise-smooth curve, that is, γ is a union of a finite number of smooth curves $\gamma_1, \ldots, \gamma_k$, such that the initial point of γ_{j+1} is the terminal point of γ_j. Then we define the integral of a continuous function f along γ with respect to the arc length by:

$$\int_\gamma f(x,y)ds = \sum_{j=1}^{k} \int_{\gamma_j} f(x,y)ds.$$

Definition

[Center of mass of a wire]
If $\rho(x, y)$ is the linear density at a point (x, y) of a thin wire shaped like a curve $\gamma\colon [a, b] \longrightarrow \mathbb{R}^2$. The mass of the thin is

$$m = \int_a^b \rho(\gamma(t))\|\gamma'(t)\| dt$$

and the center of mass of the thin

$$(x_0, y_0) = \left(\int_a^b x(t)\rho(\gamma(t))\|\gamma'(t)\| dt, \int_a^b y(t)\rho(\gamma(t))\|\gamma'(t)\| dt\right).$$

Example

A wire takes the shape of an arc of circle $(\cos t, \sin t)$, with $t \in [0, \pi]$. If the density of the thin is $\rho(x, y) = x^2 + y^2$. Then the mass of the thin is

$$m = \int_0^\pi dt = \pi$$

and the center of mass of the this

$$\left(\int_0^\pi \cos t dt, \int_0^\pi \sin t dt\right) = (0, 2).$$

Consider a space curve given by the parametric equations

$$\gamma(t) = (x(t), y(t), z(t)), \quad t \in [a, b].$$

Definition

Let f be a continuous function on $\mathbb{R}^3$. If γ is continuously differentiable, the line integral of f on γ with respect to the arc length is defined by:

$$\int_a^b f \circ \gamma(t) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt.$$

Remarks

1. If $f = 1$, $\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt$ is the length of γ.
   Note that $\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \|\gamma'(t)\|$ and we denote $ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$.
2. The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

Example

Consider the curve γ parametrized by $\gamma(t) = (\cos t, \sin t, 1)$, with $t \in [0, \frac{\pi}{2}]$. In this case $ds = \sqrt{\cos^2 t + \sin^2 t}dt = dt$

$$\begin{aligned}
\int_C (2xz + 5xy^2 + z)ds &= \int_0^{\frac{\pi}{2}} (2\cos t + 5\cos t \sin^2 t + 1)dt \\
&= \frac{\pi}{2} + \int_0^{\frac{\pi}{2}} \cos t(2 + 5\sin^2 t)dt \\
\frac{\pi}{2} + &\overset{u=\cos t}{=} \int_0^1 (2 + 5u^2)du = \frac{\pi}{2} + \frac{11}{3}.
\end{aligned}$$

Definition

Let f be a continuous function on $\mathbb{R}^3$ and let γ be piecewise-smooth curve, that is, γ is a union of a finite number of smooth curves $\gamma_1, \dots, \gamma_k$, such that the initial point of γ_{j+1} is the terminal point of γ_j. Then we define the integral of a continuous function f along γ with respect to the arc length as

$$\int_{\gamma} f(x, y, z)ds = \sum_{j=1}^{k} \int_{\gamma_j} f(x, y, z)ds.$$

Definition

Let f be a continuous function on $D \subset \mathbb{R}^3$ and let C be piecewise-smooth curve on D parametrized by $(x(t), y(t), z(t)),\ t \in [a, b]$:

1. The line integral of $f(x, y, z)$ with respect to x along the oriented curve C is written $\displaystyle\int_C f(x, y, z)dx$ and defined by:

$$\int_C f(x, y, z)dx = \int_a^b f(x(t), y(t), z(t))x'(t)dt$$

2. The line integral of $f(x, y, z)$ with respect to y along the oriented curve C is written $\displaystyle\int_C f(x, y, z)dy$ and defined by:

$$\int_C f(x, y, z)dy = \int_a^b f(x(t), y(t), z(t))y'(t)dt$$

3. The line integral of $f(x, y, z)$ with respect to z along the oriented curve C is written $\int_C f(x, y, z)dz$ and defined by:

$$\int_C f(x, y, z)dz = \int_a^b f(x(t), y(t), z(t))z'(t)dt$$

Work of a Force Field

If $F = (f, g, h)$ is a force field defined on a domain $D \subset \mathbb{R}^3$ and let C be piecewise-smooth curve on D parametrized by $(x(t), y(t), z(t)),\ t \in [a, b]$: The work of F along the curve C is defined by:

$$\begin{aligned} W &= \int_a^b f(x(t), y(t), z(t))x'(t)dt + \int_a^b g(x(t), y(t), z(t))y'(t)dt \\ &\quad + \int_a^b h(x(t), y(t), z(t))z'(t)dt \\ &= \int_a^b \langle F \circ C(t), C'(t)\rangle dt. \end{aligned}$$

$\int_a^b \langle F \circ C(t), C'(t)\rangle dt$ is denoted also $\int_C F(x, y, z).dr$