Introduction to Real Analysis

Limit of Functions

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Limit of a fucntion

Definition

If $f : D \longrightarrow \mathbb{R}$ and $c \in \widehat{D}$. We say the limit of f at c is L If

$$\begin{aligned} &\forall \varepsilon > 0 \;\; \exists\, \delta > 0 : \\ &x \in D,\; 0 < |x - c| < \delta \qquad \Rightarrow |f(x) - L| < \varepsilon \end{aligned}$$

We write

$$\lim_{x \to c} f(x) = L$$

or $f(x) \longrightarrow L$ as $x \longrightarrow c$.

Limit of a fucntion

Examples

1. $$\lim_{x\to 3} 4x+1=13$$

2. If
$$f(x)=\begin{cases} x^2 & x\neq 3 \\ 0 & x=3 \end{cases}$$
then
$$\lim_{x\to 3} f(x)=9$$

3. $$\lim_{x\to 3} \frac{1}{x}=\frac{1}{3}$$

Remarks

1. If we can find $\delta > 0$ such that $|f(x) - L| < \varepsilon$ then any $\delta' \in (0, \delta)$ will work.
2. To prove that $\lim_{x \to c} f(x) = L$ It suffices to show

$$\begin{aligned} \forall \varepsilon > 0 \;\; \exists\, \delta > 0 : \quad & x \in D, \\ & 0 < |x - c| < \delta \qquad \Rightarrow |f(x) - L| < a\varepsilon \end{aligned}$$

where $a > 0$ does not depend on x nor ε.

Limits and Sequences

Theorem

If $f : D \longrightarrow \mathbb{R}$ and $c \in \widehat{D}$. Then the following statements are equivalent.

1. $\lim_{x \to c} f(x) = L$
2. For every sequence (x_n) in D sucht that $x_n \neq c$ for all $n \in \mathbb{N}$ and $x_n \to c$ then the sequence $(f(x_n))$ converges to L.

Limits and Sequences

Corollary

If $f : D \longrightarrow \mathbb{R}$ and $c \in \widehat{D}$, and let we define $S := \{(x_n) \subset D : x_n \neq c, x_n \to c\}$ then

1. If there is a sequence (x_n) in S such that $(f(x_n))$ diverges, then $\lim\limits_{x \to c} f(x)$ does not exist.
2. If there are two sequences (x_n) and (y_n) in S such that $\lim\limits_{x \to c} f(x_n) \neq \lim\limits_{x \to c} f(y_n)$ then $\lim\limits_{x \to c} f(x)$ does not exist.

Limits and Sequences

Examples

1. $$\lim_{x\to 0} \frac{1}{x^2}$$

2. $$\lim_{x\to 0} \mathsf{sgn}(x)$$

$$\mathsf{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

Limits and Sequences

Examples

1.
$$\lim_{x\to 0} \sin \frac{1}{x}$$

2.
$$\lim_{x\to c} f(x)$$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{Q}^c \end{cases}$$

Basic Theorems

Theorem

Let $f : D \longrightarrow \mathbb{R}$ and $c \in \widehat{D}$. If $\lim_{x \to c} f(x)$ exists then it is unique.

Basic Theorems

Theorem

Let $f : D \longrightarrow \mathbb{R} \quad c \in \widehat{D}$. If f has a limit at c then f is bounded in a neighborhood U of c,
i.e., there is an $M > 0$ such that

$$|f(x)| \leq M \qquad \forall x \in U \cap D \backslash \{c\}$$

Basic Theorems

Theorem

Let $f : D \longrightarrow \mathbb{R} \quad c \in \widehat{D}$. If $\lim_{x \to c} f(x) = L$ and $L \neq 0$ then there is a neighborhood U of c, and $M > 0$ such that

$$|f(x)| > M \qquad \forall x \in U \cap D \backslash \{c\}$$

Basic Theorems

Theorem

Let $f, g : D \longrightarrow \mathbb{R}$, $c \in \widehat{D}$. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = K$ then

1. $\lim_{x\to c}[f(x) + g(x)] = L + K$
2. $\lim_{x\to c}[f(x) \cdot g(x)] = LK$
3. If $K \neq 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}$

Basic Theorems

Theorem

Let $f, g : D \longrightarrow \mathbb{R}$ and $c \in \widehat{D}$. If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$ and there is a neighborhood U of c, such that

$$f(x) \leq g(x) \qquad \forall x \in U \cap D \backslash \{c\}$$

then

$$L \leq K$$

Sequeeze Theorems

Theorem

Let $f, g, h : D \longrightarrow \mathbb{R}$ and $c \in \widehat{D}$. If there is a neighborhood U of c, such that

$$f(x) \leq g(x) \leq h(x) \qquad \forall x \in U \cap D \backslash \{c\}$$

and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$ then

$$\lim_{x \to c} g(x) = L$$

Squeeze Theorem

Examples

1. If p is a polynomial then $\lim_{x\to c} f(x) = f(c)$ for all $c \in \mathbb{R}$
2. $\lim_{x\to 1} \frac{x^2+2}{x+1}$
3. $\lim_{x\to 3} \frac{x^2-5x+6}{x^2-9}$
4. $\lim_{x\to 0} x \sin \frac{1}{x}$

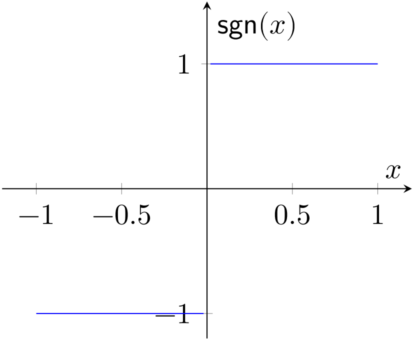
Squeeze Theorem

Examples

1. $\lim_{\theta \to 0} \sin \theta = 0$
2. $\lim_{\theta \to 0} \cos \theta = 1$
3. $\lim_{\theta \to \theta_0} \sin \theta = \sin \theta_0, \quad \lim_{\theta \to \theta_0} \cos \theta = \cos \theta_0$
4. $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$

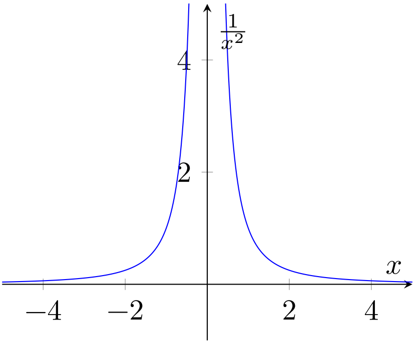
Extensions of Limits

$$\lim_{x \to 0} \mathsf{sgn}(x)$$



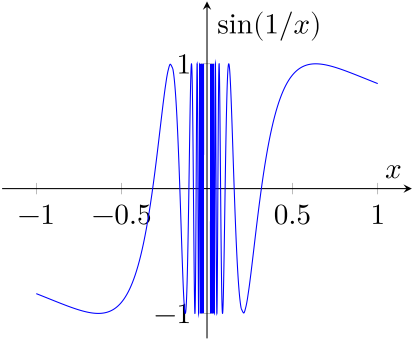
Extensions of Limits

$$\lim_{x\to 0} \frac{1}{x^2}$$



Extensions of Limits

$$\lim_{x\to 0} \sin \frac{1}{x}$$



Right Limit

Definition

If $f : D \longrightarrow \mathbb{R}$ and c is a cluster point of $D \cap (c, \infty)$. Then the right hand limit of f at c equals L if

$$\forall \varepsilon > 0 \;\; \exists \, \delta > 0 : \quad x \in D, \\ 0 < x - c < \delta \qquad \Rightarrow |f(x) - L| < \varepsilon$$

and we Write

$$\lim_{x \to c^+} f(x) = L$$

Left Limit

Definition

If $f : D \longrightarrow \mathbb{R}$ and c is a cluster point of $D \cap (-\infty, c)$. Then the left hand limit of f at c equals K if

$$\forall \varepsilon > 0 \ \ \exists\, \delta > 0 : \quad x \in D, \quad 0 < c - x < \delta \quad \Rightarrow |f(x) - K| < \varepsilon$$

and we write

$$\lim_{x \to c^-} f(x) = K$$

The Limit

Theorem

If $f : D \longrightarrow \mathbb{R}$ and $c \in \widehat{D}$ then

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$$

Right and Left Limits

Examples

1. $$f(x) = \begin{cases} \dfrac{x-4}{\sqrt{x}-2} & x > 4 \\ 0 & x = 4 \\ 2x-4 & x < 4 \end{cases}$$

   what is $\lim_{x\to 4} f(x)$?

2. $$f(x) = \begin{cases} 1-x & x < 0 \\ x & x > 0 \end{cases}$$

   does $\lim_{x\to 0} f(x)$ exist ?

Infinite limits

Definition

Let $f : D \longrightarrow \mathbb{R}$ and $c \in \widehat{D}$ then

1. $\lim_{x \to c} f(x) = \infty$ if

$$\forall M \in \mathbb{R} \;\; \exists\, \delta > 0 : \quad x \in D, \quad 0 < |x - c| < \delta \quad \Rightarrow f(x) > M$$

2. $\lim_{x \to c} f(x) = -\infty$ if

$$\forall M \in \mathbb{R} \;\; \exists\, \delta > 0 : \quad x \in D, \quad 0 < |x - c| < \delta \quad \Rightarrow f(x) < M$$

Limits at infinity

Definition

Let $f : D \longrightarrow \mathbb{R}$

1. If $(a, \infty) \subset D$ then $\lim_{x \to \infty} f(x) = L$ if

$$\forall \varepsilon > 0 \ \ \exists \ M \in \mathbb{R} : \quad \begin{array}{l} x \in D, \\ x > M \end{array} \quad \Rightarrow |f(x) - L| < \varepsilon$$

2. If $(-\infty, b) \subset D$ then $\lim_{x \to -\infty} f(x) = L$ if

$$\forall \varepsilon > 0 \ \ \exists \ M \in \mathbb{R} : \quad \begin{array}{l} x \in D, \\ x < M \end{array} \quad \Rightarrow |f(x) - L| < \varepsilon$$

Infinite Limits

Examples

1. $\lim_{x \to 0} \frac{1}{x^2}$
2. $\lim_{x \to \infty} \frac{1}{x^2}$