24.3 Applications of Gauss' Law

One of the approaches used to determine the electric field is *Gauss's law.*

What conditions should the Gauss's surface satisfy when calculating the electric field due to charge distributions?

Each portion of the surface satisfies one or more of the following conditions:

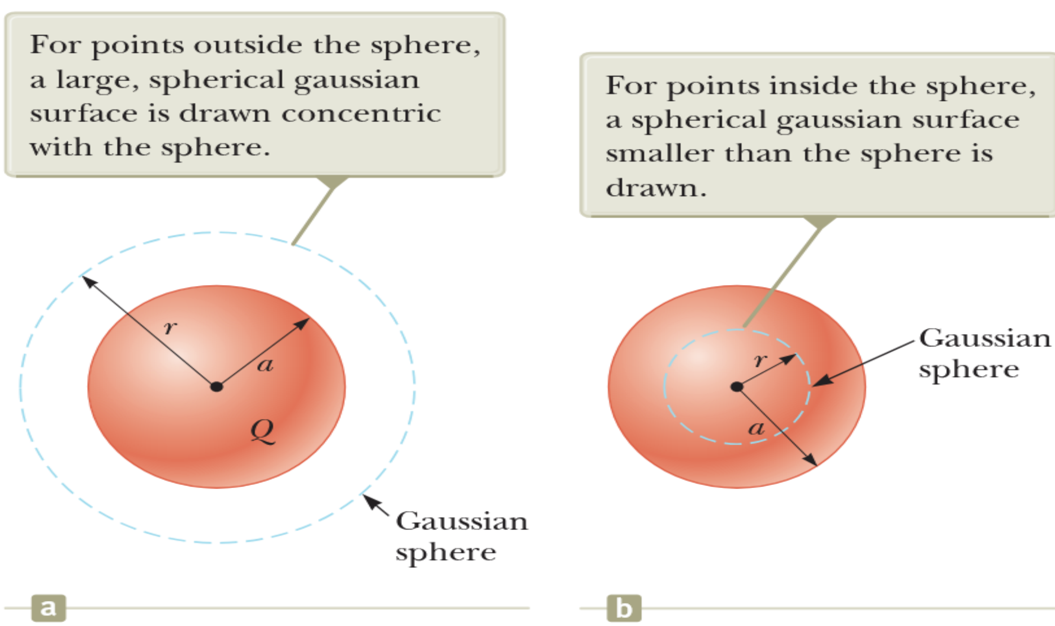
- The value of the electric field can be argued from symmetry to be constant over the surface.
- The dot product of $\vec{\boldsymbol{E}}.\overrightarrow{\Delta \boldsymbol{A}}$ can be expressed as a simple algebraic product EdA because $\vec{E}$ and $\overrightarrow{\Delta A}$ are parallel.
- The dot product is 0 because $\vec{E}$ and $\overrightarrow{\Delta A}$ are perpendicular.
- The electric field is constant over the portion of the surface.

I- Sphere of Uniform Charge

An insulating solid sphere of radius a has a uniform volume charge density **r** and carries a total positive charge **Q**.

(a) Calculate the magnitude of the electric field at a point outside the sphere.

(b) Find the magnitude of the electric field at a point inside the sphere.



(a) To determine the electric field outside a charged sphere, we use a spherical Gaussian surface with radius **r** centered on the sphere. According to Gauss's law, the total electric flux through this surface is proportional to the total charge **Q** enclosed by the surface:

$$\boldsymbol{\phi} = \oint \vec{\boldsymbol{E}}.\overrightarrow{\boldsymbol{dA}}$$

Since the electric field **E** is uniform in magnitude at all points on the spherical Gaussian surface and directed radially outward (or inward, depending on the charge), and the area vector **dA** is also radial, the dot product simplifies to:

$$\boldsymbol{\phi} = \mathbf{E}\oint \boldsymbol{dA}$$

The total surface area of a sphere is $A=4\pi r^2$. Substituting this into the equation gives:

$$\boldsymbol{\phi} = \mathbf{E}\oint \boldsymbol{dA} = \boldsymbol{EA} = \boldsymbol{E}(\mathbf{4}\boldsymbol{\pi r}^{\mathbf{2}})$$

According to Gauss's law, the total flux is also equal to the charge enclosed Q divided by the permittivity of free space ε_0 :

$$\boldsymbol{\phi} = \frac{\boldsymbol{Q}}{\boldsymbol{\varepsilon}_{\mathbf{0}}}$$

Equating the two expressions for Φ:

$$\boldsymbol{\phi} = \mathbf{E}\oint \boldsymbol{dA} = \boldsymbol{EA} = \boldsymbol{E}(\mathbf{4}\boldsymbol{\pi r}^{\mathbf{2}}) = \frac{\boldsymbol{Q}}{\boldsymbol{\varepsilon}_{\mathbf{0}}}$$

Finally, solving for E:

===➔ $$\boldsymbol{E} = \frac{\boldsymbol{Q}}{\mathbf{4}\boldsymbol{\pi\varepsilon}_{\mathbf{0}}\boldsymbol{r}^{\mathbf{2}}} = \boldsymbol{K}\frac{\boldsymbol{Q}}{\boldsymbol{r}^{\mathbf{2}}} \propto \frac{\mathbf{1}}{\boldsymbol{r}^{\mathbf{2}}}$$

This equation shows that the electric field outside the charged sphere behaves like that of a point charge, with the field decreasing as the square of the distance from the center of the sphere.

(b) **Calculation of the Electric Field inside a Uniformly Charged Sphere**

To find the electric field at a point inside the sphere at a distance **r** from the center (**r<a**), we use a spherical Gaussian surface of radius **r**.

Step 1: Total charge enclosed by the Gaussian surface

The charge enclosed by the Gaussian surface depends only on the portion of the sphere within the radius r. The total charge **Q** is uniformly distributed, so the **charge density** ρ is given by:

$$\rho = \frac{Q}{4/3\,\pi a^3}$$

The charge enclosed by the Gaussian surface of radius **r** is then:

$$q_{inside} = \rho V^{`}$$

The volume of the smaller sphere (with radius r) is:

$$V^{`} = \frac{4}{3}\pi r^3$$

Thus:

$$q_{inside} = \frac{Q}{\frac{4}{3}\pi a^3} . \frac{4}{3}\pi r^3 = Q\frac{r^3}{a^3}$$

Step 2: Electric flux through the Gaussian surface

By Gauss's law, the total electric flux through the Gaussian surface is:

$$\boldsymbol{\Phi} = \oint \vec{\boldsymbol{E}}.\overrightarrow{\boldsymbol{dA}}$$

Since the electric field E is radially symmetric and constant over the Gaussian surface, and the area of the Gaussian surface is $A=4\pi r^2$, the flux becomes:

$$\boldsymbol{\Phi} = \mathbf{E}\oint \boldsymbol{dA} = \boldsymbol{EA} = \boldsymbol{E}(\mathbf{4}\boldsymbol{\pi r^2})$$

Step 3: Relating flux to charge enclosed

By Gauss's law: $\phi = \frac{q_{inside}}{\varepsilon_0}$

Substitute : $q_{inside} = Q\frac{r^3}{a^3}$

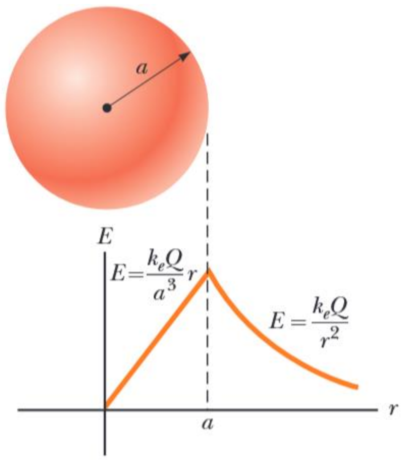
$$\Rightarrow \boldsymbol{\phi = E(4\pi r^2)} = \frac{q_{inside}}{\varepsilon_0} = Q\frac{r^3}{\varepsilon_0\, a^3}$$

Step 4: Solve for E

Simplify to find the electric field inside the sphere:

$$\boldsymbol{E = \frac{Q}{4\pi\varepsilon_0 a^3} r = K\frac{Q}{a^3} r \;\; \propto \; r}$$

The electric field inside a uniformly charged sphere increases linearly with the distance r from the center.



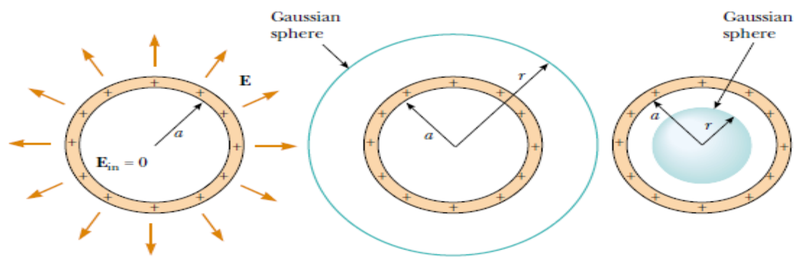
Comments:

1. The electric field inside the sphere is **proportional to the distance** from the center (**r**). At the very center (**r=0**), the field is zero.
2. The maximum electric field occurs at the surface of the sphere (**r=a**), and it matches the formula for the electric field outside the sphere.

Exercise: (Thin Spherical Shell)

II- The Electric Field Due to a Thin Spherical Shell

A thin spherical shell of radius a has a total charge **Q** distributed uniformly over its surface. Find the electric field at points (A) outside and (B) inside the shell.

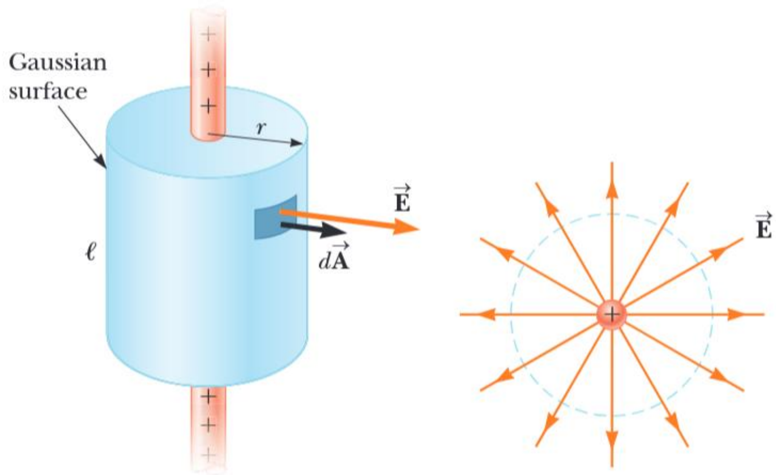


(a) $\boldsymbol{E = K\frac{Q}{r^2} \propto \frac{1}{r^2}}$ **r > a**

(b) **E= zero.** The Gaussian surface encloses no charge: $Q_{enc}=0$.

III- Electric Field of Line Charge

Find the electric field a distance **r** from a line of positive charge of infinite length and constant charge per unit length **λ**.



- Consider a long, straight line charge with **linear charge density λ** (charge per unit length, in C/m).
- We are interested in finding the electric field E at a perpendicular distance **r** from the line charge.
- The system has cylindrical symmetry, so the electric field is radially outward from the line charge and depends only on r, the distance from the line charge.

Step-by-Step Calculation Using Gauss's Law

Step 1: Gaussian Surface

To apply Gauss's law, we choose a cylindrical Gaussian surface:

- The axis of the cylinder aligns with the line charge.
- The cylinder has radius **r** and length ***l***.

The total electric flux Φ through the Gaussian surface comes from the **curved surface** of the cylinder, as the electric field is perpendicular to it and parallel to the area vectors. The ends of the cylinder (flat surfaces) contribute zero flux because the electric field is parallel to those surfaces.

The area of the curved surface is:

$A=2\pi r l$

Step 2: Flux through the Gaussian Surface

The electric flux through the curved surface is:

$$\emptyset = \oint \vec{E}.\overrightarrow{dA} = EA = E(2\pi rl)$$

Step 3: Charge Enclosed

The charge enclosed by the Gaussian surface is:

$$q_{inside} = \lambda l$$

Step 4: Gauss's Law

According to Gauss's law:

$$\emptyset = \oint \vec{E}.\overrightarrow{dA} = EA = E(2\pi rl) = \frac{q_{inside}}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0}$$

Step 5: Solve for EE

Cancel *l* (since the electric field is independent of the cylinder's length) and solve for E:

$$\boldsymbol{E = \frac{\lambda}{2\pi r\varepsilon_0} = 2K\frac{\lambda}{r} \propto \frac{1}{r}}$$

Comments:

1. **Radial Dependence**: The electric field decreases inversely with distance (**E∝1/r**).
2. **Cylindrical Symmetry**: The electric field is radial and uniform along any circle of radius r around the line charge.
3. **Dependence on λ**: The field is directly proportional to the linear charge density λ.

IV- A plan of charge

The electric field due to a non-conducting, infinite plane of positive charge with uniform surface charge density **σ**.

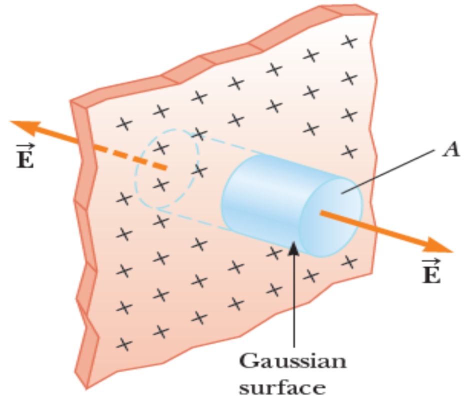


Figure 23.17 (Example 23.8) A cylindrical gaussian surface penetrating an infinite plane of charge. The flux is *EA* through each end of the gaussian surface and zero through its curved surface.

Electric Field Due to an Infinite Plane Sheet of Charge

- Consider an **infinite plane sheet** with a uniform **surface charge density** σ (charge per unit area, in C/m^2).
- We are tasked with finding the electric field E at a point on either side of the sheet, at some distance from it.

Key Assumptions

1. **Symmetry**: An infinite sheet has translational and planar symmetry. The electric field will be:
   - Perpendicular to the plane at all points.
   - The same magnitude at any point equidistant from the sheet.
2. **Direction**:
   - For a positively charged sheet, the electric field points away from the plane.
   - For a negatively charged sheet, the field points toward the plane.

Step-by-Step Calculation Using Gauss's Law

Step 1: Gaussian Surface

To apply Gauss's law, we choose a **cylindrical Gaussian surface** (also called a **pillbox**):

- The **flat circular ends** of the cylinder are parallel to the plane sheet.
- The **curved side** of the cylinder contributes no flux because the electric field is perpendicular to the plane and parallel to the curved surface.

Step 2: Electric Flux through the Gaussian Surface

The electric field is uniform and perpendicular to the sheet on both flat ends of the cylinder. The total flux is the sum of the contributions from both ends:

$$\emptyset = EA + EA = 2EA$$

Here, **A** is the area of the flat circular ends of the Gaussian surface.

Step 3: Charge Enclosed by the Gaussian Surface

The total charge enclosed by the Gaussian surface is:

$$q_{inside} = \sigma A$$

Step 4: Gauss's Law

According to Gauss's law:

$$\emptyset = 2EA = \frac{q_{inside}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

Step 5: Solve for EE

Cancel A (since the field is uniform) and solve for E:

$$\boldsymbol{E = \frac{\sigma}{2\varepsilon_0}}$$

Comments

1. **Constant Field**:
   - The electric field due to an infinite plane sheet is **constant**, regardless of the distance from the sheet.
   - This is because the plane is infinite, so the contribution from all parts of the sheet remains the same at any distance.
2. **Direction**:
   - For a **positively charged sheet**, the field points **away** from the sheet.
   - For a **negatively charged sheet**, the field points **toward** the sheet.
3. **Surface Charge Density Dependence**:
   - The electric field is directly proportional to the surface charge density σ.

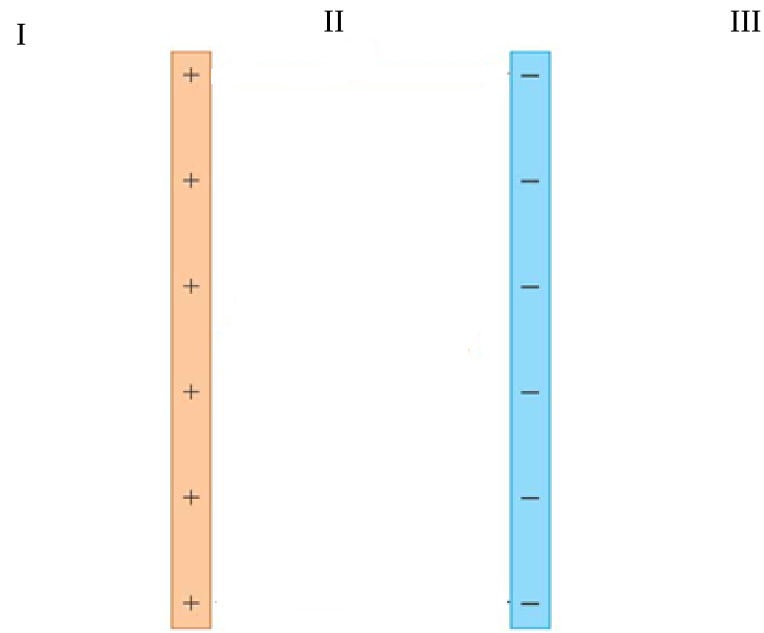
Applications

- **Parallel Plate Capacitor**: The result for an infinite plane sheet is used to derive the electric field between two oppositely charged parallel plates. The field between the plates is doubled (since both plates contribute to the field), giving:

  $$E_{\text{capacitor}} = \frac{\sigma}{\varepsilon_0}$$

Problem-1

In the figure below, can you calculate the electric field in the three regions?



Problem-2:

An infinite nonconducting wire carries a constant charge per unit length of +1.5 nC/m.

The electric field at a point 20 cm from the axis of the wire (in N/C) is:

A. 6.75

B. 13.5

C. 67.5

D. 135

Problem-3

For a conducting slab having surface charge density σ, the electric field just above the conductor is given by:

A. $\frac{\sigma}{2\varepsilon_0}$

B. $\frac{\sigma}{\varepsilon_0}$

C. $\frac{\varepsilon_0}{2\sigma}$

D. $\frac{\varepsilon_0}{\sigma}$