Course Notes: Math 481

Contents

Chapter 1

Real Numbers

Supremum and Infimum

Bounded Sets

A subset $A \subset \mathbb{R}$ is:

- *bounded above* if there exists $K \in \mathbb{R}$ such that $x \leq K$ for all $x \in A$;
- *bounded below* if there exists $K \in \mathbb{R}$ such that $x \geq K$ for all $x \in A$;
- *bounded* if it is both bounded above and bounded below.

Remarks.

1. A is bounded if and only if there exists $M \geq 0$ such that $|x| \leq M$ for all $x \in A$.
2. A sequence is bounded above (resp. below) if and only if the set of its values is bounded above (resp. below).

Supremum and Infimum

Let $A \subset \mathbb{R}$.

- The *supremum* $\sup(A)$ is the *least upper bound* of A:
  1. $\sup(A)$ is an upper bound of A;
  2. if K' is any other upper bound of A, then $\sup(A) \leq K'$.
- The *infimum* $\inf(A)$ is the *greatest lower bound*, defined analogously.

If they exist, $\sup(A)$ and $\inf(A)$ are unique.

Theorem 1 *Every nonempty subset $A \subset \mathbb{R}$ that is bounded above (resp. below) has a supremum (resp. infimum).*

Examples

1. Closed interval: $\sup([a,b]) = b, \quad \inf([a,b]) = a$.
2. Open interval: $\sup((a,b)) = b, \quad \inf((a,b)) = a$. Proof: b is an upper bound. If K is any upper bound, take $x_n = b - 2^{-n}(b-a) \in (a,b)$. Then $x_n \leq K$ and $x_n \to b$, so $b \leq K$.
3. $A = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$: $\sup(A) = 1$.
4. **Example 9.10:** $A = \left\{ \frac{n^2}{2^n} : n \in \mathbb{N} \right\}$, $\sup(A) = \frac{9}{8}$ because $\frac{n^2}{2^n} \leq 1 < \frac{9}{8}$ for $n \neq 3$, and $\frac{3^2}{2^3} = \frac{9}{8} \in A$.

Maximum and Minimum

If $\sup(A) \in A$, it is called the *maximum* of A; if $\inf(A) \in A$, it is called the *minimum* of A.

In any case, there exists a sequence $(x_n) \subset A$ with $x_n \to \sup(A)$, and a sequence $(y_n) \subset A$ with $y_n \to \inf(A)$. If A is unbounded above (resp. below), we write $\sup(A) = +\infty$ (resp. $\inf(A) = -\infty$).

Limit Superior and Limit Inferior

Definition 2 *For a sequence $(a_n)_{n\in\mathbb{N}}$:*

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \left(\sup_{k\geq n} a_k \right), \quad \liminf_{n\to\infty} a_n := \lim_{n\to\infty} \left(\inf_{k\geq n} a_k \right).$$

Remark. $(\sup_{k\geq n} a_k)$ is decreasing (or $+\infty$), $(\inf_{k\geq n} a_k)$ is increasing (or $-\infty$). Thus, $\limsup a_n$ and $\liminf a_n$ always exist in $\mathbb{R} \cup \{\pm\infty\}$.

Examples

1. $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$:
$$\limsup_{n\to\infty} a_n = 1, \quad \liminf_{n\to\infty} a_n = -1.$$
2. $a_n = n$: $\limsup a_n = +\infty, \quad \liminf a_n = +\infty$.

Theorem 3 (Characterization of $\limsup$) *Let $a \in \mathbb{R}$. Then:*

$$\limsup_{n\to\infty} a_n = a$$

if and only if, for every $\varepsilon > 0$:

(i) $a_n < a + \varepsilon$ for all but finitely many n,

(ii) $a_n > a - \varepsilon$ for infinitely many n.

Analogously: $\liminf_{n\to\infty} a_n = a$ iff, for every $\varepsilon > 0$:

(i) $a_n > a - \varepsilon$ for all but finitely many n,

(ii) $a_n < a + \varepsilon$ for infinitely many n.

Chapter 2

The Riemann Integral

1. Partition and Refinement of an Interval

Let $[a, b]$ be a closed and bounded interval with $a < b$. A **partition** P of $[a, b]$ is a finite ordered set of points

$$P = \{x_0, x_1, \ldots, x_n\}, \qquad a = x_0 < x_1 < \cdots < x_n = b,$$

which subdivides $[a, b]$ into the n subintervals

$$[x_{k-1}, x_k], \qquad k = 1, 2, \ldots, n.$$

These subintervals are pairwise disjoint in their interiors and their union is $[a, b]$.

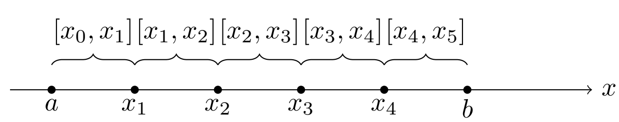


Figure 2.1: Partition P of $[a, b]$ into subintervals.

Let

$$P = \{x_0, \ldots, x_n\} \quad \text{with} \quad a = x_0 < \cdots < x_n = b.$$

A partition Q of $[a, b]$ is called a **refinement** of P if $P \subseteq Q$; that is, every point of P also appears in Q, and Q may contain additional points inside the subintervals determined by P.

Example

Suppose

$$P = \{a, x_1, x_2, b\}, \qquad a < x_1 < x_2 < b,$$

and we insert three additional points

$$q_1 \in (a, x_1), \qquad q_2 \in (x_1, x_2), \qquad q_3 \in (x_2, b).$$

Then the refinement Q is

$$Q = P \cup \{q_1, q_2, q_3\} = \{a, q_1, x_1, q_2, x_2, q_3, b\},$$

listed in strictly increasing order.

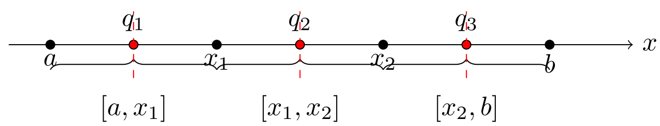


Figure 2.2: Refinement Q of P by inserting q_1, q_2, and q_3.

2. Lower and Upper Sums

Definition 4 (Lower and Upper Sums) *Let $f : [a,b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, \ldots, x_n\}$ a partition of $[a,b]$. For each subinterval $[x_{k-1}, x_k]$, define:*

$$m_k := \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}, \quad M_k := \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}.$$

*Then the **lower sum** of f with respect to P is:*

$$L(f,P) = \sum_{k=1}^{n} m_k \cdot (x_k - x_{k-1}),$$

*and the **upper sum** is:*

$$U(f,P) = \sum_{k=1}^{n} M_k \cdot (x_k - x_{k-1}).$$

3. Properties of Riemann Sums

Lemma 5 (Properties of Lower and Upper Sums) *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then:*

1. *For every partition P,*
$$L(f,P) \le U(f,P).$$
2. *If Q is a refinement of P, then*
$$L(f,P) \le L(f,Q) \quad \text{and} \quad U(f,P) \ge U(f,Q).$$
3. *For any two partitions P_1, P_2,*
$$L(f,P_1) \le U(f,P_2).$$

Proof.

1. **Lower sum is always less than or equal to upper sum.**
For each subinterval $[x_{k-1}, x_k]$, we define:
$$m_k := \inf\{f(x) : x \in [x_{k-1}, x_k]\}, \quad M_k := \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$
Since $m_k \le M_k$ for all k, it follows that:
$$L(f,P) = \sum_{k=1}^{n} m_k \Delta x_k \le \sum_{k=1}^{n} M_k \Delta x_k = U(f,P).$$
Example: Let $f(x) = x^2$ on $[0,1]$, and let $P = \{0, 0.5, 1\}$. Then:
$L(f,P) = 0^2 \cdot 0.5 + (0.5)^2 \cdot 0.5 = 0 + 0.125 = 0.125, U(f,P) = (0.5)^2 \cdot 0.5 + (1)^2 \cdot 0.5 = 0.125 + 0.5 = 0.625.$

So $L(f, P) < U(f, P)$.

2. **Refining increases lower sum and decreases upper sum.**
A refinement Q of P adds points to subdivide the interval more finely. The infimum over a smaller subinterval is at least as large as over the larger one (because we're minimizing over fewer values), and similarly, the supremum over a smaller subinterval is at most as large.

Hence:
$$L(f, Q) \geq L(f, P), \qquad U(f, Q) \leq U(f, P).$$

Example: Use the same $f(x) = x^2$ on $[0, 1]$, but refine $P = \{0, 0.5, 1\}$ to $Q = \{0, 0.25, 0.5, 0.75, 1\}$. You will find:
$$L(f, Q) > L(f, P), \quad U(f, Q) < U(f, P).$$

3. **Lower sum of one partition is less than upper sum of another.**
Let $R = P_1 \cup P_2$, which is a common refinement of both P_1 and P_2. Then, by part (2):

$$L(f, P_1) \leq L(f, R) \leq U(f, R) \leq U(f, P_2),$$

so:
$$L(f, P_1) \leq U(f, P_2).$$

Example: Let $P_1 = \{0, 0.5, 1\}$, $P_2 = \{0, 0.25, 1\}$. Their union is $R = \{0, 0.25, 0.5, 1\}$. Again using $f(x) = x^2$, you can compute and verify the inequality numerically:

$$L(f, P_1) \leq L(f, R) \leq U(f, R) \leq U(f, P_2).$$

■

Definition 6 *A bounded function $f : [a, b] \to \mathbb{R}$ is said to be* ***Riemann integrable*** *(or simply* ***integrable****) if its lower integral*

$$L(f) = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}$$

coincides with its upper integral

$$U(f) = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}.$$

The common value of $L(f)$ and $U(f)$ is called the ***Riemann integral*** *of f over the interval $[a, b]$, and is denoted by*
$$\int_a^b f \quad \text{or more explicitly} \quad \int_a^b f(x)\, dx.$$

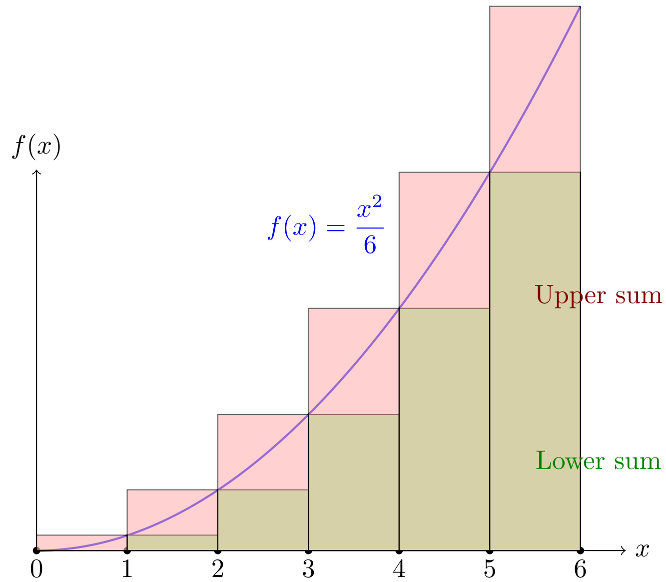


Figure 2.3: Lower and upper sums for the function $f(x) = \frac{x^2}{6}$ on $[0, 6]$.

Intuitively, a bounded function f is Riemann integrable if we can approximate the area under its graph from below (using lower sums) and from above (using upper sums) in such a way that both approximations can be made arbitrarily close to each other by refining the partition.

In the figure above:

- The **green rectangles** represent the *lower sum* $L(f, P)$, constructed using the minimum value of f on each subinterval.
- The **red translucent rectangles** represent the *upper sum* $U(f, P)$, constructed using the maximum value of f on each subinterval.
- The **blue curve** shows the graph of the function $f(x) = \frac{x^2}{6}$.

As the partition becomes finer (i.e., we divide $[a, b]$ into smaller subintervals), the lower and upper rectangles better approximate the area under the curve. The difference between the total areas of the upper and lower sums decreases.

This leads to the following fundamental characterization of Riemann integrability:

> A bounded function f is Riemann integrable on $[a, b]$ if and only if for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that:
> $$U(f, P) - L(f, P) < \varepsilon.$$

This ensures that all upper and lower sums are squeezed around a single unique value — the Riemann integral of the function.

Lemma 7 *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that*

$$U(f, P) - L(f, P) < \varepsilon.$$

Example 8 (Direct applications of the lemma) *We illustrate the lemma with two explicit examples.*

(a) ***Constant function:*** *Let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = c$, where $c \in \mathbb{R}$ is constant.*

Since f is constant, on every subinterval $[x_{k-1}, x_k]$ of any partition P, the infimum and supremum satisfy:

$$m_k = M_k = c.$$

Therefore, both the lower sum and the upper sum are equal:

$$L(f, P) = \sum_{k=1}^{n} m_k \Delta x_k = c(b-a), \quad U(f, P) = \sum_{k=1}^{n} M_k \Delta x_k = c(b-a).$$

It follows that

$$U(f, P) - L(f, P) = 0 < \varepsilon \quad \text{for all } \varepsilon > 0,$$

so the lemma is satisfied trivially. Thus, f is Riemann integrable and its integral is:

$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b-a).$$

(b) ***Quadratic function:*** *Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = x^2$.*

We construct a sequence of uniform partitions:

$$P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\right\}, \quad n \in \mathbb{N}.$$

Each subinterval has width $\Delta x = \frac{1}{n}$. On the interval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$, the function $f(x) = x^2$ is increasing, so:

$$m_k = \left(\frac{k-1}{n}\right)^2, \quad M_k = \left(\frac{k}{n}\right)^2.$$

The lower and upper sums are:

$$L(f, P_n) = \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} (k-1)^2 = \frac{1}{n^3} \cdot \frac{(n-1)n(2n-1)}{6},$$

$$U(f, P_n) = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}.$$

Therefore, the difference between the upper and lower sums is:

$$U(f, P_n) - L(f, P_n) = \frac{1}{n^3} \left(\frac{n(n+1)(2n+1) - (n-1)n(2n-1)}{6}\right).$$

This expression tends to 0 as $n \to \infty$, hence for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$U(f, P_n) - L(f, P_n) < \varepsilon.$$

By the lemma, $f(x) = x^2$ is Riemann integrable on $[0, 1]$, and we have:

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = \frac{1}{3}.$$

Theorem 9 *Every monotone function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.*

Proof. Suppose f is monotone increasing on $[a, b]$. Then f is bounded, since

$$f(a) \le f(x) \le f(b) \quad \text{for all } x \in [a, b].$$

Let $\varepsilon > 0$ be given. We want to find a partition P such that $U(f, P) - L(f, P) < \varepsilon$. Choose $\delta > 0$ such that

$$\delta(f(b) - f(a)) < \varepsilon.$$

Now select a partition $P = \{x_0, x_1, \ldots, x_n\}$ such that the width of every subinterval satisfies:

$$x_k - x_{k-1} < \delta \quad \text{for all } k = 1, \ldots, n.$$

Since f is increasing, on each subinterval $[x_{k-1}, x_k]$ we have:

$$m_k = f(x_{k-1}), \quad M_k = f(x_k),$$

so the difference between the upper and lower sums becomes:

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} (f(x_k) - f(x_{k-1}))(x_k - x_{k-1}).$$

Using the fact that $x_k - x_{k-1} < \delta$, we estimate:

$$U(f, P) - L(f, P) \le \delta \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) = \delta(f(b) - f(a)) < \varepsilon.$$

Hence, by the integrability criterion (Lemma), f is Riemann integrable. ■

Theorem 10 *Every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.*

Proof. Since f is continuous on the closed interval $[a, b]$, which is compact, the **Extreme Value Theorem** guarantees that f is bounded and attains its maximum and minimum on each subinterval of any partition. Furthermore, by the **Uniform Continuity Theorem**, f is uniformly continuous on $[a, b]$. Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ such that:

$$x_k - x_{k-1} < \delta \quad \text{for all } k = 1, \ldots, n.$$

On each subinterval $[x_{k-1}, x_k]$, the function f attains both its maximum M_k and minimum m_k (by continuity), and we have:

$$M_k - m_k < \frac{\varepsilon}{b - a}.$$

Thus,

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) < \frac{\varepsilon}{b - a} \sum_{k=1}^{n} (x_k - x_{k-1}) = \frac{\varepsilon}{b - a}(b - a) = \varepsilon.$$

Hence, by the integrability criterion (Lemma 7.4), f is Riemann integrable. ■

Generalization: Even though continuity guarantees integrability, the converse is not true. A function can be Riemann integrable without being continuous everywhere.

Theorem 11 (Generalization) *Let $f : [a, b] \to \mathbb{R}$ be bounded and have only* ***finitely many points of discontinuity****. Then f is Riemann integrable.*

Sketch of proof. Let $D = \{c_1, c_2, \ldots, c_m\} \subset [a, b]$ be the (finite) set of discontinuities of f. Around each c_i, construct an interval of length less than δ/m such that the total contribution to the upper-lower sum difference over these intervals is less than $\varepsilon/2$. On the complement of these intervals, f is continuous, so we apply the previous theorem to choose a partition on that region giving error less than $\varepsilon/2$. Combining both partitions yields a global partition P such that $U(f, P) - L(f, P) < \varepsilon$. ■

Example 12 (Discontinuous but integrable vs non-integrable) *This example illustrates how the nature and number of discontinuities affect integrability.*

(a) ***Integrable with one discontinuity:*** *Define $f : [-1, 1] \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

This function is discontinuous only at a single point $x = 0$, and is zero elsewhere. For any partition that isolates a small interval around 0, say $P_n = \{-1, -\frac{1}{2n}, \frac{1}{2n}, 1\}$, we have:

$$L(f, P_n) = 0, \quad U(f, P_n) = \frac{1}{n} \to 0.$$

Hence,

$$\int_{-1}^{1} f(x)\, dx = 0,$$

and f is integrable even though discontinuous at one point.

(b) ***Not integrable:*** *Define $f : [0, 1] \to \mathbb{R}$ by:*

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

This function is known as the Dirichlet function and is discontinuous at ***every point*** *in $[0, 1]$. On every subinterval of any partition:*

$$\inf f = 0, \quad \sup f = 1,$$

so:

$$L(f, P) = 0, \quad U(f, P) = 1 \quad \text{for all } P.$$

Therefore,

$$U(f, P) - L(f, P) = 1 \not\to 0,$$

and f is not Riemann integrable.

Theorem 13 *Let $f : [a, b] \to \mathbb{R}$ be bounded and let $c \in (a, b)$. Then f is integrable on $[a, b]$ if and only if f is integrable on both $[a, c]$ and $[c, b]$. In that case:*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Remark 14 *If f is integrable on $[a, b]$, we define:*

$$\int_a^b f = -\int_b^a f.$$

Also, for any $c \in [a, b]$, we define:

$$\int_c^c f = 0.$$

Then, for any three points $a, b, c \in I$, where $I \subseteq \mathbb{R}$ is a compact interval and $f : I \to \mathbb{R}$ is integrable, we have:

$$\int_a^b f + \int_b^c f = \int_a^c f.$$

We leave the verification as an exercise.

Theorem 15 (Linearity, Order, and Absolute Value Properties of the Riemann Integral) *Suppose f and g are Riemann integrable on $[a, b]$, and let $k \in \mathbb{R}$. Then:*

1. *The function $f + g$ is integrable, and*

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

2. *The function kf is integrable, and*

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

3. *If $f(x) \leq g(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

4. *The function $|f|$ is integrable, and*

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Proof. We prove parts (1) and (4). Parts (2) and (3) follow from similar arguments and are left as exercises.

(1) Linearity of the integral. Let f and g be integrable on $[a, b]$, and let P be any partition of $[a, b]$ into subintervals $[x_{k-1}, x_k]$, $k = 1, \ldots, n$.

Define:

$$m_k^f = \inf_{x \in [x_{k-1}, x_k]} f(x), \quad M_k^f = \sup_{x \in [x_{k-1}, x_k]} f(x),$$

and similarly for g, and for $f + g$:

$$m_k^{f+g} = \inf_{x \in [x_{k-1}, x_k]} (f(x) + g(x)), \quad M_k^{f+g} = \sup_{x \in [x_{k-1}, x_k]} (f(x) + g(x)).$$

From basic properties of infima and suprema over sets:

$$m_k^f + m_k^g \leq m_k^{f+g}, \quad M_k^{f+g} \leq M_k^f + M_k^g.$$

Multiplying by the subinterval length $\Delta x_k = x_k - x_{k-1}$, and summing over all k, we obtain:

$$L(f,P) + L(g,P) \leq L(f+g,P), \quad U(f+g,P) \leq U(f,P) + U(g,P).$$

Let $\varepsilon > 0$. Since f and g are integrable, there exist partitions P_1 and P_2 such that:

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}, \quad U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$, a common refinement. Then using monotonicity of upper and lower sums under refinement:

$$U(f,P) \leq U(f,P_1), \quad L(f,P) \geq L(f,P_1), \quad \text{and similarly for } g.$$

Then:

$$U(f+g,P) \leq U(f,P) + U(g,P) \leq U(f,P_1) + U(g,P_2) < U(f) + U(g) + \varepsilon,$$
$$L(f+g,P) \geq L(f,P) + L(g,P) \geq L(f,P_1) + L(g,P_2) > L(f) + L(g) - \varepsilon.$$

Thus:

$$U(f+g) \leq U(f) + U(g), \quad L(f+g) \geq L(f) + L(g),$$

and since:

$$L(f+g) \leq U(f+g),$$

we conclude that:

$$L(f+g) = U(f+g) = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

So $f+g$ is integrable and its integral is the sum of the integrals.

(4) Integrability of $|f|$ and inequality.

First, note that since f is integrable, it is bounded, say $|f(x)| \leq M$ for all $x \in [a,b]$. Let P be a partition of $[a,b]$. Define:

$$m_k^{|f|} = \inf_{x \in [x_{k-1},x_k]} |f(x)|, \quad M_k^{|f|} = \sup_{x \in [x_{k-1},x_k]} |f(x)|.$$

Since $|f(x)|$ is Lipschitz continuous with respect to $f(x)$ (triangle inequality), we have:

$$M_k^{|f|} - m_k^{|f|} \leq M_k^f - m_k^f.$$

Summing over all subintervals gives:

$$U(|f|,P) - L(|f|,P) \leq U(f,P) - L(f,P).$$

Now, since f is integrable, for any $\varepsilon > 0$, there exists a partition P such that:

$$U(f,P) - L(f,P) < \varepsilon \quad \Rightarrow \quad U(|f|,P) - L(|f|,P) < \varepsilon.$$

So $|f|$ is also integrable.

To prove the inequality:

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx,$$

observe that for all $x \in [a,b]$:

$$-|f(x)| \leq f(x) \leq |f(x)|.$$

Integrating all parts and using the order property (proved in part 3), we get:

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx,$$

which implies:

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

■

Riemann Sums

Let $f : [a, b] \to \mathbb{R}$ be a function, and let

$$a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of the interval $[a, b]$. For each $k \in \{1, \ldots, n\}$, choose a point $\xi_k \in [x_{k-1}, x_k]$, called a *sample point* (or *tag*). We denote the collection of partition points and sample points by:

$$Z := \big((x_k)_{0 \le k \le n},\, (\xi_k)_{1 \le k \le n}\big).$$

Definition 16 (Riemann Sum) *The* Riemann sum *of f with respect to Z is defined as:*

$$S(Z, f) := \sum_{k=1}^{n} f(\xi_k)\,(x_k - x_{k-1}).$$

Geometrically, this is the integral of a step function that interpolates f at the sample points ξ_k.

Definition 17 (Mesh of a Partition) *The* mesh size *(or* fineness*) of Z is defined by:*

$$\|Z\| := \max_{1 \le k \le n} (x_k - x_{k-1}).$$

Theorem 18 *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every choice of partition and sample points Z with $\|Z\| \le \delta$, we have:*

$$\left| \int_a^b f(x)\,dx - S(Z, f) \right| \le \varepsilon.$$

Equivalently,

$$\lim_{\|Z\| \to 0} S(Z, f) = \int_a^b f(x)\,dx.$$

Proof. Let φ, ψ be step functions such that $\varphi \le f \le \psi$. Then for all partitions Z:

$$S(Z, \varphi) \le S(Z, f) \le S(Z, \psi).$$

Thus, it suffices to prove the theorem for step functions.

Suppose f is a step function with respect to the partition:

$$a = t_0 < t_1 < \cdots < t_m = b.$$

Since f is bounded, let:

$$M := \sup\{|f(x)| : x \in [a, b]\} < \infty.$$

Let Z be any partition with sample points, and define a step function $F \in \mathcal{T}[a, b]$ by:

$$F(a) = f(a), \quad F(x) = f(\xi_k) \quad \text{for } x_{k-1} < x \le x_k.$$

Then:

$$S(Z, f) = \int_a^b F(x)\,dx.$$

Hence:

$$\left| \int_a^b f(x)\,dx - S(Z, f) \right| \le \int_a^b |f(x) - F(x)|\,dx.$$

The functions f and F agree on all subintervals (x_{k-1}, x_k) that do not contain any partition point t_j. At most $2m$ subintervals differ, and their total length is at most $2m\|Z\|$. Since $|f(x) - F(x)| \le 2M$, we have:

$$\int_a^b |f(x) - F(x)|\,dx \le 4mM\|Z\|.$$

As $\|Z\| \to 0$, this expression tends to 0, proving the claim. ■

Example 18.4

We compute:

$$\int_0^a x\,dx, \quad (a > 0)$$

using Riemann sums.

For $n \ge 1$, choose the equally spaced partition:

$$x_k := \frac{ka}{n}, \quad k = 0, 1, \ldots, n,$$

with mesh size $\frac{a}{n}$. Take $\xi_k = x_k$ as sample points. The Riemann sum is:

$$S_n = \sum_{k=1}^{n} \frac{ka}{n} \cdot \frac{a}{n} = \frac{a^2}{n^2} \sum_{k=1}^{n} k = \frac{a^2}{n^2} \cdot \frac{n(n+1)}{2} = \frac{a^2}{2}\left(1 + \frac{1}{n}\right).$$

Taking the limit as $n \to \infty$:

$$\int_0^a x\,dx = \lim_{n\to\infty} S_n = \frac{a^2}{2}.$$

This corresponds to the area of a right triangle with base a and height a.

4.The Fundamental Theorem of Calculus

This central theorem states that the operations of differentiation and integration are, in a sense, inverses of each other.

Theorem 19 (Fundamental Theorem of Calculus)

1. *Let $f : [a,b] \to \mathbb{R}$ be integrable and let $F : [a,b] \to \mathbb{R}$ be differentiable with $F'(x) = f(x)$ for all $x \in [a,b]$. Then*
$$\int_a^b f = F(b) - F(a).$$
2. *Let $g : [a,b] \to \mathbb{R}$ be integrable and define*
$$G(x) := \int_a^x g(t)\,dt, \quad x \in [a,b].$$
Then G is continuous on $[a,b]$. Moreover, if g is continuous at $c \in [a,b]$, then G is differentiable at c, and
$$G'(c) = g(c).$$

In part (1), the function F is called an **antiderivative** of f. In part (2), the function G is called the **indefinite integral** of g.

Remark 20 *Not every derivative is continuous . However, Theorem 19 guarantees that every continuous function is the derivative of some function.*

Proof of Theorem 19. **(1)** Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. By the Mean Value Theorem, for each interval $[x_{k-1}, x_k]$, there exists $t_k \in (x_{k-1}, x_k)$ such that

$$F(x_k) - F(x_{k-1}) = F'(t_k)(x_k - x_{k-1}) = f(t_k)(x_k - x_{k-1}).$$

Since $m_k \le f(t_k) \le M_k$, we get

$$L(f, P) \le \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \le U(f, P).$$

The sum $\sum_{k=1}^{n} f(t_k)(x_k - x_{k-1})$ is a telescoping sum equal to $F(b) - F(a)$, hence

$$\int_a^b f = F(b) - F(a).$$

(2) Suppose $|g(x)| \le M$ on $[a, b]$. For any $x, y \in [a, b]$,

$$|G(x) - G(y)| = \left| \int_a^x g - \int_a^y g \right| = \left| \int_y^x g \right| \le \left| \int_y^x |g| \right| \le M|x - y|.$$

This shows that G is uniformly continuous.

Now suppose g is continuous at $c \in [a, b]$. Then for $x \ne c$:

$$\frac{G(x) - G(c)}{x - c} = \frac{1}{x - c} \int_c^x g(t)\, dt.$$

Given $\varepsilon > 0$, by continuity of g at c, there exists $\delta > 0$ such that $|g(t) - g(c)| < \varepsilon$ whenever $|t - c| < \delta$. Then for $0 < |x - c| < \delta$:

$$\left| \frac{G(x) - G(c)}{x - c} - g(c) \right| = \left| \frac{1}{x - c} \int_c^x (g(t) - g(c))\, dt \right| \le \varepsilon.$$

Hence $G'(c) = g(c)$. ■

Remark 21 *Computing integrals directly from the definition is usually not feasible in practice. The power of the Fundamental Theorem lies in allowing us to compute definite integrals using antiderivatives.*

Theorem 22 (Mean Value Theorem for Integrals) *If $g : [a, b] \to \mathbb{R}$ is continuous, then there exists $c \in (a, b)$ such that*

$$\int_a^b g = (b - a)g(c).$$

Proof. Apply the Mean Value Theorem to the function $x \mapsto \int_a^x g$, which by the Fundamental Theorem of Calculus is an antiderivative of g. ■

Improper Integrals

In this section, we study improper integrals, which arise in two main situations:

- One of the integration limits is infinite,
- The function becomes unbounded (e.g., has a vertical asymptote) at a boundary point.

We will consider these two cases in detail.

Case 1: Integration over an Infinite Interval

Definition 23 *Let $f : [a, \infty) \to \mathbb{R}$ be a function that is Riemann integrable over every finite interval $[a, R]$, for $a < R < \infty$. If the limit*

$$\lim_{R\to\infty} \int_a^R f(x)\, dx$$

exists and is finite, then the improper integral is said to converge, and we define

$$\int_a^\infty f(x)\, dx := \lim_{R\to\infty} \int_a^R f(x)\, dx.$$

Similarly, for a function $f : (-\infty, a] \to \mathbb{R}$, we define

$$\int_{-\infty}^a f(x)\, dx := \lim_{R\to-\infty} \int_R^a f(x)\, dx,$$

provided the limit exists.

Example

Consider the integral

$$\int_1^\infty \frac{1}{x^s}\, dx.$$

We compute:

$$\int_1^R \frac{1}{x^s}\, dx = \begin{cases} \frac{1}{s-1}\left(1 - \frac{1}{R^{s-1}}\right), & s \neq 1, \\ \log R, & s = 1. \end{cases}$$

Taking the limit as $R \to \infty$, we get:

$$\int_1^\infty \frac{1}{x^s}\, dx = \begin{cases} \frac{1}{s-1}, & \text{if } s > 1, \\ \text{diverges}, & \text{if } s \leq 1. \end{cases}$$

Case 2: The Function is Unbounded at an Endpoint

Definition 24 *Let $f : (a, b] \to \mathbb{R}$ be a function that is Riemann integrable over every interval $[a+\varepsilon, b]$, for $0 < \varepsilon < b - a$. If the limit*

$$\lim_{\varepsilon\searrow 0} \int_{a+\varepsilon}^b f(x)\, dx$$

exists and is finite, then the improper integral is said to converge, and we define

$$\int_a^b f(x)\, dx := \lim_{\varepsilon\searrow 0} \int_{a+\varepsilon}^b f(x)\, dx.$$

Example

Let us evaluate

$$\int_0^1 \frac{1}{x^s}\, dx.$$

For $s \neq 1$, we compute:

$$\int_\varepsilon^1 \frac{1}{x^s}\, dx = \frac{1}{1-s}\left(1 - \varepsilon^{1-s}\right).$$

Now take the limit as $\varepsilon \to 0^+$:

$$\lim_{\varepsilon \to 0^+} \varepsilon^{1-s} = \begin{cases} 0, & s < 1, \\ \infty, & s > 1. \end{cases}$$

Hence,

$$\int_0^1 \frac{1}{x^s}\, dx = \begin{cases} \dfrac{1}{1-s}, & \text{if } s < 1, \\ \text{diverges}, & \text{if } s \geq 1. \end{cases}$$

We now consider the general case of improper integrals over open intervals.

Definition 25 *Let $f : (a,b) \to \mathbb{R}$, where $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$, be a function that is Riemann integrable over every compact subinterval $[\alpha, \beta] \subset (a,b)$. Let $c \in (a,b)$ be arbitrary. If both of the improper integrals*

$$\int_a^c f(x)\, dx := \lim_{\alpha \searrow a} \int_\alpha^c f(x)\, dx, \quad \int_c^b f(x)\, dx := \lim_{\beta \nearrow b} \int_c^\beta f(x)\, dx$$

converge, then the integral over the full interval is called convergent, and we define:

$$\int_a^b f(x)\, dx := \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Note that this definition is independent of the choice of the intermediate point $c \in (a,b)$.

Examples

Example 1

According to previous examples, the integral

$$\int_0^\infty \frac{1}{x^s}\, dx$$

diverges for all $s \in \mathbb{R}$.

Example 2

The integral

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\, dx$$

converges. We compute:

$$\begin{aligned} \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\, dx &= \lim_{\varepsilon \searrow 0} \int_{-1+\varepsilon}^0 \frac{1}{\sqrt{1-x^2}}\, dx + \lim_{\varepsilon \searrow 0} \int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\, dx \\ &= -\lim_{\varepsilon \searrow 0} \sin^{-1}(-1+\varepsilon) + \lim_{\varepsilon \searrow 0} \sin^{-1}(1-\varepsilon) \\ &= -(-\tfrac{\pi}{2}) + \tfrac{\pi}{2} = \pi. \end{aligned}$$

Example 3

The integral

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\, dx$$

also converges:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{R\to\infty}\int_{-R}^{0} \frac{1}{1+x^2}\,dx + \lim_{R\to\infty}\int_{0}^{R} \frac{1}{1+x^2}\,dx \\ &= -\lim_{R\to\infty} \tan^{-1}(-R) + \lim_{R\to\infty} \tan^{-1}(R) \\ &= -(-\tfrac{\pi}{2}) + \tfrac{\pi}{2} = \pi.\end{aligned}$$

Example 4: Evaluation of the Dirichlet Integral

We evaluate the improper integral:

$$\int_0^\infty \frac{\sin x}{x}\,dx.$$

Although the integrand is undefined at $x = 0$, we extend it continuously by defining:

$$\frac{\sin x}{x}\Big|_{x=0} := \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

This makes the function continuous on $[0, \infty)$. We define the sine integral function:

$$\mathrm{Si}(x) := \int_0^x \frac{\sin t}{t}\,dt.$$

The function $\mathrm{Si}(x)$ is continuous for all $x \geq 0$, although it cannot be written using elementary functions.

The integrand $\frac{\sin x}{x}$ changes sign on each interval $[n\pi, (n+1)\pi]$, and we define:

$$a_n := \left| \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x}\,dx \right|.$$

Then (a_n) is a decreasing sequence with $a_n \to 0$, and:

$$\mathrm{Si}(n\pi) = \sum_{k=0}^{n-1} (-1)^k a_k.$$

By the Leibniz criterion (alternating series test), this sum converges, so:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{n\to\infty} \mathrm{Si}(n\pi)$$

exists.

To evaluate the limit, we consider:

$$\mathrm{Si}\left(\frac{\lambda\pi}{2}\right) = \int_0^{\pi/2} \frac{\sin(\lambda x)}{x}\,dx,$$

by the substitution $t = \lambda x$.

Define the auxiliary function:

$$g(x) := \begin{cases} \frac{1}{x} - \frac{1}{\sin x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then g is continuous on $[0, \pi/2]$, and we decompose the integrand:

$$\frac{\sin(\lambda x)}{x} = \frac{\sin(\lambda x)}{\sin x} + \sin(\lambda x) \cdot g(x).$$

We now use the following key result:

Theorem 26 (Riemann's Lemma) *Let $f \in C^1([a,b])$. Then:*

$$\lim_{|k|\to\infty} \int_a^b f(x)\sin(kx)\,dx = 0.$$

Proof. Let $F(k) := \int_a^b f(x)\sin(kx)\,dx$. For $k \neq 0$, we integrate by parts:

$$F(k) = -\frac{f(x)\cos(kx)}{k}\Big|_a^b + \frac{1}{k}\int_a^b f'(x)\cos(kx)\,dx.$$

If $|f(x)| \leq M$ and $|f'(x)| \leq M$, then:

$$|F(k)| \leq \frac{2M}{|k|} + \frac{M(b-a)}{|k|} = \frac{2M + M(b-a)}{|k|} \to 0 \quad \text{as } |k| \to \infty.$$

■

We apply this lemma with $f(x) = g(x) \in C^1([0,\pi/2])$, which gives:

$$\lim_{\lambda\to\infty} \int_0^{\pi/2} \sin(\lambda x)\cdot g(x)\,dx = 0.$$

Hence,

$$\lim_{\lambda\to\infty} \int_0^{\pi/2} \frac{\sin(\lambda x)}{x}\,dx = \lim_{\lambda\to\infty} \int_0^{\pi/2} \frac{\sin(\lambda x)}{\sin x}\,dx.$$

We now evaluate the remaining limit. For every integer $n \geq 1$, the following identity holds:

$$\frac{\sin((2n+1)x)}{\sin x} = 1 + 2\sum_{k=1}^{n} \cos(2kx).$$

Integrating term-by-term over $[0,\pi/2]$, and noting that each $\cos(2kx)$ integrates to zero, we get:

$$\int_0^{\pi/2} \frac{\sin((2n+1)x)}{\sin x}\,dx = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}.$$

Taking the limit $n \to \infty$, we conclude:

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}. \quad ■$$

When all functions in a sequence share the same domain $D \subset \mathbb{R}$, convergence of a function sequence can be studied using the concept of pointwise convergence. That is, we say $f_n(x) \to f(x)$ as $n \to \infty$ for each $x \in D$.

However, pointwise convergence alone is often insufficient when we want to deduce properties of the limit function f from the approximating functions f_n. In many cases, we require the stronger notion of *uniform convergence*, which roughly means that the convergence occurs at the same rate for all $x \in D$.

For example, uniform convergence ensures that if all functions f_n are continuous, then the limit function f is also continuous. It also plays a crucial role in determining when we can interchange limits with differentiation or integration. Power series provide many important examples of uniformly convergent function sequences.

Chapter 3

Sequences of functions

Definition

Let $D \subset \mathbb{R}$, and let $f_n : D \to \mathbb{R}$ be a sequence of functions.

(a) The sequence (f_n) **converges pointwise** to a function $f : D \to \mathbb{C}$ if, for every $x \in D$, the sequence $f_n(x) \to f(x)$. That is:

$$\forall x \in K,\ \forall \varepsilon > 0\ \exists N = N(x, \varepsilon) \text{ such that } |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N.$$

(b) The sequence (f_n) **converges uniformly** to a function $f : D \to \mathbb{C}$ if:

$$\forall \varepsilon > 0\ \exists N = N(\varepsilon) \text{ such that } |f_n(x) - f(x)| < \varepsilon \quad \forall x \in D,\ \forall n \geq N.$$

Remark:

Uniform convergence implies pointwise convergence, but not vice versa. The key difference is that for uniform convergence, N depends only on ε, not on the specific point x.

Example: A sequence of functions that converges pointwise but not uniformly

Let (f_n) be a sequence of functions defined on $[0, 1]$ by

$$f_n(x) = \begin{cases} 1 - nx, & 0 \leq x \leq \frac{1}{n}, \\ 0, & \frac{1}{n} \leq x \leq 1\ . \end{cases}$$

We will prove that:

1. $f_n \to f$ pointwise, where
$$f(x) = \begin{cases} 1, & x = 0, \\ 0, & 0 < x \leq 1, \end{cases}$$
2. The convergence is not uniform on $[0, 1]$.

Pointwise convergence.
Fix $x \in [0, 1]$. We distinguish two cases:

- If $x = 0$: For all $n \in \mathbb{N}$, $f_n(0) = 1$. Hence
$$\lim_{n\to\infty} f_n(0) = 1 = f(0).$$

- If $x > 0$: By the Archimedean property, there exists $N \in \mathbb{N}$ such that

$$N \geq \frac{1}{x} \quad \Rightarrow \quad \frac{1}{N} \leq x.$$

For every $n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} \leq x$, and by definition of f_n,

$$f_n(x) = 0.$$

Therefore, the sequence $(f_n(x))$ is eventually zero and

$$\lim_{n\to\infty} f_n(x) = 0 = f(x).$$

We have shown that $f_n \to f$ pointwise on $[0, 1]$.

The convergence is not uniform.

Recall the definition: $f_n \to f$ uniformly on $[0, 1]$ if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ :\ \forall n \geq N,\ \forall x \in [0, 1],\ |f_n(x) - f(x)| < \varepsilon.$$

We will show that this statement is false by proving its negation:

$$\exists \varepsilon > 0\ :\ \forall N \in \mathbb{N},\ \exists n \geq N,\ \exists x \in [0, 1],\ |f_n(x) - f(x)| \geq \varepsilon.$$

Take $\varepsilon = \frac{1}{3}$. Let $N \in \mathbb{N}$ be arbitrary. Choose $n \geq N$ and consider the point

$$x = \frac{2}{3n} > 0.$$

Since $x \leq \frac{1}{n}$, we are in the first branch of f_n:

$$f_n(x) = 1 - nx = 1 - \frac{2}{3} = \frac{1}{3}.$$

Moreover, $f(x) = 0$ for every $x > 0$. Thus

$$|f_n(x) - f(x)| = |f_n(x)| = \frac{1}{3} = \varepsilon.$$

We have found, for every N, some $n \geq N$ and some $x \in [0, 1]$ for which the error does not go below ε.
Conclusion. The sequence (f_n) converges pointwise to f on $[0, 1]$, but the convergence is not uniform.

Uniform Convergence and Continuity

Theorem 27 *Let $D \subset \mathbb{R}$ and let $f_n : D \to \mathbb{R}$ be a sequence of continuous functions that converges uniformly to a function $f : D \to \mathbb{C}$. Then f is also continuous.*
In other words: the limit of a uniformly convergent sequence of continuous functions is itself continuous.

Proof. Let $x \in D$. We aim to show that f is continuous at x, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(x')| < \varepsilon \quad \text{for all } x' \in D \text{ with } |x - x'| < \delta.$$

Since (f_n) converges uniformly to f, there exists $N \in \mathbb{N}$ such that

$$|f_N(\xi) - f(\xi)| < \frac{\varepsilon}{3} \quad \text{for all } \xi \in D.$$

Because f_N is continuous at x, there exists $\delta > 0$ such that

$$|f_N(x) - f_N(x')| < \frac{\varepsilon}{3} \quad \text{whenever } |x - x'| < \delta.$$

Then for such $x' \in K$, we estimate:

$$\begin{aligned} |f(x) - f(x')| &\le |f(x) - f_N(x)| + |f_N(x) - f_N(x')| + |f_N(x') - f(x')| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This proves that f is continuous at x. Since x was arbitrary, f is continuous on D. ■

Remark. If a sequence of continuous functions converges only *pointwise*, then the limit function need not be continuous.

Example

(a) Consider the sequence of functions $f_n : [0, 1] \to \mathbb{R}$ defined by

$$f_n(x) = x^n.$$

We determine its pointwise limit f:

$$f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1\ . \end{cases}$$

Indeed, if $0 \le x < 1$, then $x^n \to 0$ as $n \to \infty$, while $f_n(1) = 1^n = 1$ for all n.

The convergence $f_n \to f$ is *not uniform* on $[0, 1]$. Recall that uniform convergence would require:

$$\forall \varepsilon > 0,\ \exists N :\ n \ge N \implies |f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in [0, 1].$$

Take $\varepsilon = \frac{1}{2}$. For every N, choose

$$x = \left(\frac{1}{2}\right)^{1/N} \in (0, 1),$$

so that for $n = N$,

$$f_N(x) = x^N = \frac{1}{2}.$$

Hence,

$$|f_N(x) - f(x)| = \frac{1}{2} = \varepsilon,$$

showing that the condition for uniform convergence fails.

Furthermore, even though each f_n is continuous on $[0, 1]$, the pointwise limit function f is **not continuous** at $x = 1$. This illustrates that pointwise convergence of continuous functions does not, in general, preserve continuity.

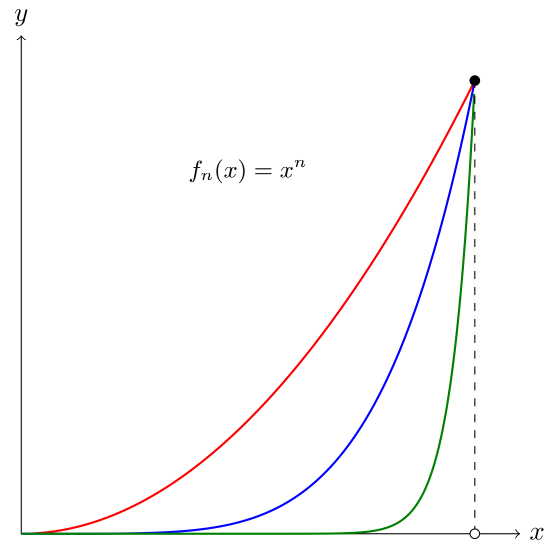


Figure 3.1: Pointwise convergence of $f_n(x) = x^n$ to a discontinuous function f.

Theorem 28 *Suppose (f_n) is a sequence of differentiable functions on $[a, b]$ which converges at some point $x_0 \in [a, b]$. If the sequence (f'_n) is uniformly convergent on $[a, b]$, then (f_n) is also uniformly convergent on $[a, b]$ to a function f, which is differentiable on $[a, b]$, and:*

$$f'_n \to f'.$$

Example

Consider the sequence of functions $g_n : \mathbb{R} \to \mathbb{R}$ defined by

$$g_n(x) = x^{1+\frac{1}{2n-1}} = x \cdot \sqrt[2n-1]{x}.$$

We study its pointwise limit and differentiability.

Pointwise limit

For each $x \in \mathbb{R}$:

$$g_n(x) = \begin{cases} x^{1+\frac{1}{2n-1}}, & x > 0, \\ 0, & x = 0, \\ x \cdot (-x)^{\frac{1}{2n-1}}, & x < 0 \ . \end{cases}$$

We consider three cases:

- **Case 1: $x > 0$** $x^{\frac{1}{2n-1}} \to 1$ as $n \to \infty$. Hence:

$$\lim_{n\to\infty} g_n(x) = x.$$

- **Case 2: $x = 0$** $g_n(0) = 0$ for all n, so:

$$\lim_{n\to\infty} g_n(0) = 0.$$

- **Case 3: $x < 0$** We have:

$$g_n(x) = x(-x)^{\frac{1}{2n-1}} = -|x|(-x)^{\frac{1}{2n-1}-1}.$$

Since $(-x)^{\frac{1}{2n-1}} \to 1$, it follows that:

$$\lim_{n\to\infty} g_n(x) = -x = |x|.$$

Combining all cases:

$$g(x) = \begin{cases} x, & x > 0, \\ 0, & x = 0, \\ -x, & x < 0, \end{cases} \qquad \text{i.e.,} \qquad g(x) = |x|.$$

Differentiability of the pointwise limit
Each g_n is differentiable on $\mathbb{R}$. The limit function $g(x) = |x|$ is not differentiable at $x = 0$, because:

$$g'_{-}(0) = \lim_{h \to 0^-} \frac{|h| - 0}{h} = -1, \qquad g'_{+}(0) = \lim_{h \to 0^+} \frac{|h| - 0}{h} = 1.$$

Since $g'_{-}(0) \neq g'_{+}(0)$, the derivative does not exist at $x = 0$.

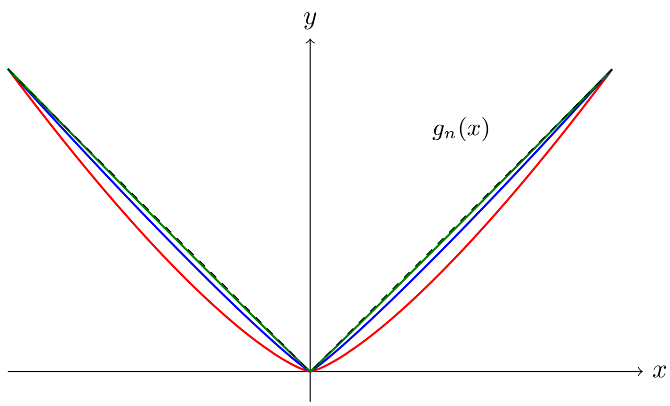


Figure 3.2: Pointwise convergence of $g_n(x) = x \cdot \sqrt[2n-1]{x}$ to $g(x) = |x|$.

Exercise 29 *Find the limit of the sequence* $f_n(x) = \frac{x^n}{1+x^n}$ *on the interval* $[0, 2]$ *and determine whether the convergence is uniform.*

Proof.
We analyze the pointwise limit of $f_n(x)$ for different values of $x \in [0, 2]$.

- **For** $x = 0$**:**

$$f_n(0) = \frac{0^n}{1 + 0^n} = 0 \quad \text{for all } n.$$

  Hence, $\lim_{n \to \infty} f_n(0) = 0$.

- **For** $x \in (0, 1)$**:** As $n \to \infty$, $x^n \to 0$ for $x \in (0, 1)$, so:

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{x^n}{1 + x^n} = 0.$$

- **For** $x = 1$**:**

$$f_n(1) = \frac{1^n}{1 + 1^n} = \frac{1}{2} \quad \text{for all } n,$$

  so $\lim_{n \to \infty} f_n(1) = \frac{1}{2}$.

- **For** $x \in (1, 2]$**:** As $n \to \infty$, $x^n \to \infty$ for $x > 1$, so:

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{x^n}{1 + x^n} = 1.$$

Thus, the pointwise limit function $f(x)$ of the sequence $f_n(x)$ on $[0, 2]$ is:

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ \frac{1}{2} & \text{if } x = 1, \\ 1 & \text{if } x \in (1,2]. \end{cases}$$

The convergence is **not uniform** on $[0, 2]$. ■

Proposition 30 *(Cauchy Criterion for Uniform Convergence) Let $D \subseteq \mathbb{R}$ and (f_n) be a sequence of functions $f_n : D \to \mathbb{R}$. The sequence (f_n) converges uniformly over D to a function $f : D \to \mathbb{R}$ if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have:*

$$\sup_{x \in D} |f_n(x) - f_m(x)| < \varepsilon.$$

Proof. We prove the implications separately.

(⇒):

Fix $\varepsilon > 0$. Since $f_n \xrightarrow{u} f$ on D, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in D$, we have:

$$|f_n(x) - f(x)| < \frac{\varepsilon}{4}.$$

Thus, if $m, n \geq N$, for all $x \in D$, by the triangle inequality, we have:

$$|f_m(x) - f_n(x)| = |f_m(x) - f(x) + f(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f_n(x) - f(x)| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},$$

and taking the supremum over all $x \in D$, we get:

$$\sup_{x \in X} |f_m(x) - f_n(x)| \leq \frac{\varepsilon}{2} < \varepsilon,$$

for any $m, n \geq N$, which is what we wanted.

(⇐):

Assume that for every $\tilde{\varepsilon} > 0$, there exists $\tilde{N} \in \mathbb{N}$ such that:

$$\sup_{x \in X} |f_n(x) - f_m(x)| < \tilde{\varepsilon} \quad \text{for all } m, n \geq \tilde{N}.$$

This means that for every $x \in D$, we have:

$$|f_n(x) - f_m(x)| \leq \sup_{x \in D} |f_n(x) - f_m(x)| < \tilde{\varepsilon}.$$

Thus, for each $x \in D$, the real sequence $(f_n(x))$ is Cauchy, and hence convergent. The sequence of functions (f_n) converges pointwise to some function $f : D \to \mathbb{R}$.

We now show that the pointwise convergence $f_n \xrightarrow{\text{pw}} f$ is actually uniform. Fix $\varepsilon > 0$. By our assumption, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ and $x \in D$, we have:

$$|f_n(x) - f_m(x)| \leq \sup_{x \in D} |f_n(x) - f_m(x)| < \frac{\varepsilon}{2}.$$

By fixing n, we take the limit as $m \to \infty$. Since (f_m) converges pointwise to f, for all $x \in X$, we have $f_m(x) \to f(x)$ as $m \to \infty$. Using limits and preserving inequalities, we obtain:

$$\lim_{m \to \infty} |f_n(x) - f_m(x)| = |f_n(x) - \lim_{m \to \infty} f_m(x)| = |f_n(x) - f(x)|,$$

for all $x \in D$. Taking the supremum over $x \in D$, we get:

$$\sup_{x \in X} |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon,$$

for all $n \geq N$. Therefore, we conclude that $f_n \xrightarrow{u} f$ on D.

■

Theorem 31 *Let $X \subseteq \mathbb{R}$ and (f_n) be a sequence of functions $f_n : X \to \mathbb{R}$ that converges uniformly over X to a function $f : X \to \mathbb{R}$. Assume that for $x_0 \in X$, both $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} f_n(x)$ for all $n \in \mathbb{N}$ exist. Then:*

$$\lim_{x\to x_0}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right).$$

Proof. First, note that since the sequence (f_n) converges uniformly to f, this convergence is also pointwise, meaning $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in X$. Therefore, we want to prove the equality:

$$\lim_{x\to x_0} f(x) = \lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right).$$

Let $p_n = \lim_{x\to x_0} f_n(x)$ for each $n \in \mathbb{N}$, and let $p = \lim_{x\to x_0} f(x)$. Proving this equation is equivalent to showing the convergence of the real sequence $p_n \to p$.

Fix $\varepsilon > 0$. Since (f_n) converges uniformly to f, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have:

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \text{for any } x \in X.$$

Now, take the limit as $x \to x_0$ on both sides. Since limits preserve weak inequalities (as seen in Exercise 9.10), we get:

$$\lim_{x\to x_0} |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon.$$

By applying the algebra of limits, we then have:

$$|p_n - p| = \left|\lim_{x\to x_0} f_n(x) - \lim_{x\to x_0} f(x)\right| = \left|\lim_{x\to x_0} (f_n(x) - f(x))\right| = \lim_{x\to x_0} |f_n(x) - f(x)| < \varepsilon.$$

Thus, for all $n \geq N$, we have $|p_n - p| < \varepsilon$, which is what we wanted to prove. Therefore

$$\lim_{x\to x_0}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right).$$

■ As a consequence, knowing that the functions in (f_n) are continuous everywhere guarantees that their uniform limit is also continuous everywhere.

Theorem 32 *Suppose $f_n \in \mathcal{R}(a,b)$ for each $n \in \mathbb{N}$. If $f_n \xrightarrow{u} f$ on $[a,b]$, then $f \in \mathcal{R}(a,b)$ and:*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

Proof. Let $\varepsilon > 0$. To prove that $f \in \mathcal{R}(a,b)$, we need to show there exists a partition P such that:

$$U(f,P) - L(f,P) < C\varepsilon,$$

where C is independent of ε.

Since f_n converges uniformly to f, we can find a positive integer N such that:

$$n \geq N \implies f_n(x) - \varepsilon < f(x) < f_n(x) + \varepsilon \quad \text{for all } x \in [a,b].$$

Since $f_N \in \mathcal{R}(a,b)$, there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that:

$$U(f_N, P) - L(f_N, P) < \varepsilon.$$

But, since $f_N(x) - \varepsilon < f(x) < f_N(x) + \varepsilon$ for all $x \in P$, we have:

$$L(f_N, P) - \varepsilon(b-a) \leq L(f,P),$$

$$U(f,P) \leq U(f_N,P) + \varepsilon(b-a).$$

Hence,

$$U(f,P) - L(f,P) \leq U(f_N,P) - L(f_N,P) + 2\varepsilon(b-a) < \varepsilon + 2\varepsilon(b-a) = C\varepsilon,$$

where $C = 1 + 2(b-a)$, so $f \in \mathcal{R}(a,b)$.

Furthermore, we have:

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \leq \int_a^b |f_n(x) - f(x)|\,dx,$$

$$\leq \int_a^b \sup_{x\in[a,b]} |f_n(x) - f(x)|\,dx \leq (b-a) \sup_{x\in[a,b]} |f_n(x) - f(x)|.$$

The uniform convergence of (f_n) now ensures that:

$$\sup_{x\in[a,b]} |f_n(x) - f(x)| \to 0 \quad \text{as} \quad n \to \infty.$$

■

Chapter 4

Real Series

Convergent Series

Definition 33 (Convergent Series) *A real series $\sum_{j=1}^{\infty} a_j$ is called a convergent series if the sequence of its partial sums (s_n) converges. We define the value of the series as the limit of its partial sums. In other words, if*

$$\lim_{n\to\infty} s_n = s,$$

we assign the value s to the series:

$$\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} \sum_{j=1}^{n} a_j = \lim_{n\to\infty} s_n = s.$$

Otherwise, if the sequence of the partial sums (s_n) diverges, the series $\sum_{n=1}^{\infty} a_n$ is called a divergent series.

Consider the series $\sum_{j=1}^{\infty} \frac{1}{j(j+1)}$. The terms of this series can be rewritten as:

$$\frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1}, \quad \text{for every } j \in \mathbb{N}.$$

Thus, the series can be expressed as:

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)} = \sum_{j=1}^{\infty} \left(\frac{1}{j} - \frac{1}{j+1} \right).$$

Now, considering the sequence of partial sums (s_n) for this series, many terms cancel out due to the nature of the expression. Specifically, we have:

$$s_n = \sum_{j=1}^{n} \left(\frac{1}{j} - \frac{1}{j+1} \right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

Applying the limit to the sequence of partial sums, we obtain:

$$\lim_{n\to\infty} s_n = 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1.$$

Thus, the sequence of partial sums converges, and we conclude:

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)} = \lim_{n\to\infty} s_n = 1.$$

In general, a real series that can be written in the form $\sum_{j=1}^{\infty}(f(j)-f(j+1))$ for some function $f:\mathbb{N}\to\mathbb{R}$ is known as a *telescoping series*. The partial sums of such a series simplify to:

$$s_n = f(1) - f(n+1), \quad \text{for all } n \in \mathbb{N},$$

which makes the series easier to analyze due to the cancellation of terms.

Proposition 34 *If the real series $\sum_{j=1}^{\infty} a_j$ converges, then $\lim_{j\to\infty} a_j = 0$.*

Proof. Since the series converges, by definition, the sequence of partial sums $s_n = \sum_{j=1}^{n} a_j$ also converges, say $s_n \to s \in \mathbb{R}$. Note that $a_n = s_n - s_{n-1}$. Taking the limit as $n\to\infty$ on both sides and applying the algebra of limits, we get:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

Thus, we have shown that $\lim_{n\to\infty} a_n = 0$, completing the proof. ■

Proposition 35 *Let $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ be convergent real series. Then:*

1. *For any $\lambda \in \mathbb{R}$, the series $\sum_{j=1}^{\infty} \lambda a_j$ converges, and its sum is equal to $\lambda \sum_{j=1}^{\infty} a_j$.*
2. *The series $\sum_{j=1}^{\infty}(a_j + b_j)$ converges, and its sum is equal to $\sum_{j=1}^{\infty} a_j + \sum_{j=1}^{\infty} b_j$.*

Since the convergence of a series is determined by its limiting behavior, we can safely ignore or add any finite number of terms at the beginning of the series without affecting its convergence. This leads to the following proposition:

Proposition 36 *Let $\sum_{j=1}^{\infty} a_j$ be a real series.*

1. *If there exists $N \in \mathbb{N}$ such that the series $\sum_{j=N}^{\infty} a_j$ converges, then the series $\sum_{j=1}^{\infty} a_j$ also converges, and its sum is given by*

$$\sum_{j=1}^{\infty} a_j = \sum_{j=1}^{N-1} a_j + \sum_{j=N}^{\infty} a_j.$$

2. *If the series $\sum_{j=1}^{\infty} a_j$ converges, then for any $N \in \mathbb{N}$, the series $\sum_{j=N}^{\infty} a_j$ also converges.*

Proof. We prove each assertion separately.

1. For $n \geq N$, consider the sequence of partial sums (t_n) where $t_n = \sum_{j=N}^{n} a_j$. Let (s_n) be the sequence of partial sums where $s_n = \sum_{j=1}^{n} a_j$. For $n \geq N$, we have

$$s_n = \sum_{j=1}^{N-1} a_j + t_n = K + t_n,$$

where $K = \sum_{j=1}^{N-1} a_j \in \mathbb{R}$ is a real constant. Since (t_n) converges as $n \to \infty$, by the algebra of limits, we conclude that (s_n) also converges. Moreover, we have

$$\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(K + t_n) = K + \lim_{n\to\infty} t_n = \sum_{j=1}^{N-1} a_j + \sum_{j=N}^{\infty} a_j.$$

2. Fix $N \in \mathbb{N}$ and for $n \geq N$, consider the sequence of partial sums (t_n) where $t_n = \sum_{j=N}^{n} a_j$. Let (s_n) be the sequence of partial sums for the series $\sum_{j=1}^{\infty} a_j$ where $s_n = \sum_{j=1}^{n} a_j$. Then, for any $n \geq N$, we have
$$t_n = s_n - \sum_{j=1}^{N-1} a_j = s_n - K,$$
where $K = \sum_{j=1}^{N-1} a_j \in \mathbb{R}$ is a real constant. Since (s_n) converges, by the algebra of limits, we conclude that the sequence (t_n) also converges.

■

Absolute and Conditional Convergence

Proposition 37 (Cauchy Criterion for Convergence of a Series) *The real series $\sum_{j=1}^{\infty} a_j$ converges if and only if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n > m \geq N$, we have:*
$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Definition 38 (Absolute Convergence) *A real series $\sum_{j=1}^{\infty} a_j$ is said to be absolutely convergent if the corresponding series of absolute values, $\sum_{j=1}^{\infty} |a_j|$, converges.*

Definition 39 (Conditional Convergence) *A real series $\sum_{j=1}^{\infty} a_j$ is called conditionally convergent if $\sum_{j=1}^{\infty} a_j$ converges but $\sum_{j=1}^{\infty} |a_j|$ diverges to infinity.*

An important distinction between absolutely convergent and conditionally convergent series is that the terms of an absolutely convergent series can be rearranged without changing the value of the series. However, in the case of a conditionally convergent series, the terms can be rearranged in such a way that the rearranged series converges to any real number in $\mathbb{R}$ or even diverges to $\pm\infty$. This result is known as the *Riemann rearrangement theorem.*

Alternating Series

To illustrate an example of a conditionally convergent series, we define *alternating series.* As the name suggests, an alternating series is a real series where the terms alternate in sign.

Definition 40 (Alternating Series) *A real series is called alternating if it takes one of the following forms:*
$$\sum_{j=1}^{\infty} (-1)^j b_j \quad \text{or} \quad \sum_{j=1}^{\infty} (-1)^{j-1} b_j,$$
where $b_j > 0$ for all $j \in \mathbb{N}$.

Theorem 41 (Alternating Series Test) *An alternating series of the form $\sum_{j=1}^{\infty} (-1)^j b_j$ or $\sum_{j=1}^{\infty} (-1)^{j-1} b_j$, with $b_j > 0$, converges if the terms (b_j) are decreasing and $b_j \to 0$.*

Proof. Without loss of generality (WLOG), consider the alternating series of the form $\sum_{j=1}^{\infty} (-1)^{j-1} b_j$, where the first term in the series is positive. Let (s_n) denote the sequence of partial sums. We analyze the subsequences of even-indexed and odd-indexed partial sums, namely (s_{2n}) and (s_{2n-1}).

For the subsequence of even-indexed partial sums, by grouping some consecutive terms together, we have:

$$s_{2n} = b_1 - b_2 + b_3 - b_4 + \cdots + b_{2n-1} - b_{2n} = b_1 - (b_2 - b_3) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n} \leq b_1,$$

since $b_j \geq b_{j+1}$ for all $j \in \mathbb{N}$. Additionally, we observe that:

$$s_{2(n+1)} - s_{2n} = -b_{2n+2} + b_{2n+1} \geq 0,$$

which implies that $s_{2(n+1)} \geq s_{2n}$ for all $n \in \mathbb{N}$. Thus, the subsequence of even-indexed partial sums (s_{2n}) is increasing and bounded above. By the monotone convergence theorem, the subsequence (s_{2n}) converges.

Using similar arguments, we can show that the subsequence of odd-indexed partial sums (s_{2n-1}) is bounded below and decreasing. Therefore, by the monotone convergence theorem, the subsequence (s_{2n-1}) also converges.

Furthermore, since $-b_{2n} = s_{2n} - s_{2n-1}$, taking the limit as $n \to \infty$ and applying the algebra of limits, we obtain:

$$0 = -\lim_{n\to\infty} b_{2n} = \lim_{n\to\infty} (s_{2n} - s_{2n-1}) = \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} s_{2n-1}.$$

Thus, $\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n-1}$, say s. Finally, by Exercise 5.7(a), the entire sequence of partial sums (s_n) converges to the same limit s. Hence, the series converges. ■

Comparison Tests

For real sequences, we have seen that limits preserve weak inequalities, as demonstrated by the sandwich lemma. These results can help us compare or bound the limits of a sequence with those of commonly known sequences. We now extend this idea to series. By comparing series that converge or diverge, we can apply these standard examples to test the behavior of other series.

Direct Comparison Test

The first convergence test is the *direct comparison test* for series. The idea is simple and intuitive: suppose we have two series with non-negative terms such that one series is term-wise larger than the other. If the series with the larger terms converges, then the series with the smaller terms must also converge. Similarly, if the series with the smaller terms diverges, the series with the larger terms must diverge as well. We state the following proposition:

Proposition 42 (Direct Comparison Test) *Let $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ be two real series such that $0 \leq a_j \leq b_j$ for all $j \in \mathbb{N}$.*

1. *If the series $\sum_{j=1}^{\infty} b_j$ converges, then the series $\sum_{j=1}^{\infty} a_j$ also converges.*
2. *If the series $\sum_{j=1}^{\infty} a_j$ diverges to ∞, then the series $\sum_{j=1}^{\infty} b_j$ diverges to ∞ as well.*

Proof. Let $s_n = \sum_{j=1}^{n} a_j$ and $t_n = \sum_{j=1}^{n} b_j$ be the sequences of partial sums for the series $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$, respectively. Since $a_j, b_j \geq 0$, both sequences (s_n) and (t_n) are increasing. Moreover, the condition $0 \leq a_j \leq b_j$ implies that $0 \leq s_n \leq t_n$ for all $n \in \mathbb{N}$.

We now prove the two assertions separately.

1. Since the series $\sum_{j=1}^{\infty} b_j$ converges, the sequence (t_n) is bounded, say $t_n \leq M$ for all $n \in \mathbb{N}$ and some $M > 0$. Thus, $s_n \leq t_n \leq M$ for every $n \in \mathbb{N}$. By the boundedness of (s_n), the sequence (s_n) must also converge, and therefore the series $\sum_{j=1}^{\infty} a_j$ converges.
2. Since $\sum_{j=1}^{\infty} a_j$ diverges to ∞, the sequence (s_n) diverges to ∞. As $s_n \leq t_n$ for all $n \in \mathbb{N}$, the sequence (t_n) must also diverge to ∞. Therefore, the series $\sum_{j=1}^{\infty} b_j$ diverges to ∞.

■

Proposition 43 (Limit Comparison Test) *Let $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ be two real series such that $a_j \geq 0$ and $b_j > 0$ for all $j \in \mathbb{N}$. Suppose that*

$$\lim_{j\to\infty} \frac{a_j}{b_j} = L$$

for some $0 < L < \infty$. Then, either both series converge or both series diverge. In other words:

$$\sum_{j=1}^{\infty} a_j \text{ converges} \quad \Leftrightarrow \quad \sum_{j=1}^{\infty} b_j \text{ converges.}$$

Proof. Since $\lim_{j\to\infty} \frac{a_j}{b_j} = L$, for $\epsilon = \frac{L}{2} > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| \frac{a_j}{b_j} - L \right| < \frac{L}{2} \quad \text{for all } j \geq N.$$

Equivalently, for any $j \geq N$, we have:

$$\frac{L}{2} b_j < a_j < \frac{3L}{2} b_j.$$

We now prove the two implications separately:

- ($\Rightarrow$):
  Assume that the series $\sum_{j=1}^{\infty} a_j$ converges. Therefore, the series $\frac{2}{L}\sum_{j=1}^{\infty} a_j = \sum_{j=1}^{\infty} \frac{2}{L} a_j$ also converges by Proposition 7.2.8. Since $0 < b_j < \frac{2}{L} a_j$ for all $j \geq N$, by the direct comparison test, the series $\sum_{j=N}^{\infty} b_j$ converges. Finally, by Proposition 7.2.9, the full series $\sum_{j=1}^{\infty} b_j$ converges.
- ($\Leftarrow$):
  Similarly, suppose that the series $\sum_{j=1}^{\infty} b_j$ converges. Since $0 \leq a_j < \frac{3L}{2} b_j$ for all $j \geq N$, and the series $\sum_{j=N}^{\infty} \frac{3L}{2} b_j$ converges, the series $\sum_{j=N}^{\infty} a_j$ also converges by the direct comparison test. Therefore, the full series $\sum_{j=1}^{\infty} a_j$ converges as well.

■

Theorem 44 (Ratio Test) *Let $\sum_{j=1}^{\infty} a_j$ be a real series such that $a_j \neq 0$ for all $j \in \mathbb{N}$. Let $L = \lim_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| \geq 0$.*

1. *If $L < 1$, then the series converges absolutely.*
2. *If $L > 1$, then the series diverges.*

Proof. We prove the assertions separately.

1. Suppose that $\lim_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| = L < 1$. Then, for $\epsilon = \frac{1-L}{2} > 0$, there exists an $N \in \mathbb{N}$ such that

$$\left| \frac{a_{n+1}}{a_n} - L \right| < \frac{1-L}{2} \quad \text{for all } n \geq N.$$

   This implies that

$$\left| \frac{a_{n+1}}{a_n} \right| < \frac{1+L}{2} \quad \text{for all } n \geq N.$$

   Denote $r = \frac{1+L}{2} < 1$, so that $|a_{n+1}| < r\,|a_n|$ for all $n \geq N$. By induction, we can show that $|a_{k+N}| < r^k\,|a_N|$ for all $k \in \mathbb{N}$.

Let us compare the tail of the series $\sum_{j=N+1}^{\infty} |a_j| = \sum_{k=1}^{\infty} |a_{k+N}|$ with the geometric series $\sum_{k=1}^{\infty} r^k |a_N|$. Clearly, the geometric series converges since $r < 1$. By the direct comparison test, since $|a_{k+N}| < r^k |a_N|$ for all $k \in \mathbb{N}$, the tail of the series $\sum_{j=N+1}^{\infty} |a_j|$ also converges. Proposition 7.2.9 then implies that the entire series $\sum_{j=1}^{\infty} |a_j|$ converges, meaning the series converges absolutely.

2. Suppose that $\lim_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| = L > 1$. By a similar argument as in the previous case, if we choose $\epsilon = \frac{L-1}{2} > 0$, we can show that there exists $N \in \mathbb{N}$ such that

$$\frac{1+L}{2} < \left|\frac{a_{n+1}}{a_n}\right| \quad \text{for all } n \geq N.$$

Denote $r = \frac{1+L}{2} > 1$, so that $0 < |a_N| < r^k |a_N| < |a_{k+N}|$ for all $k \in \mathbb{N}$. Since $r^k |a_N| \to \infty$, we have $|a_{k+N}| \to \infty$ as well. This implies that $\lim_{j\to\infty} |a_j| \neq 0$ and, by Lemma 5.9.3, $\lim_{j\to\infty} a_j \neq 0$. Thus, the series $\sum_{j=1}^{\infty} a_j$ cannot converge by Proposition 7.2.5.

■

Chapter 5

Series of Functions

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions, where each $f_n : D \to \mathbb{R}$. A **series of functions** is the formal infinite sum:

$$\sum_{n=1}^{\infty} f_n(x).$$

The n-th **partial sum** of this series is the function:

$$s_n(x) = \sum_{k=1}^{n} f_k(x), \qquad x \in D.$$

We define the sum of the series at a point $x \in D$ as the pointwise limit:

$$\sum_{n=1}^{\infty} f_n(x) = \lim_{n\to\infty} s_n(x).$$

Definition 45 (Domain of Convergence) *The **domain of convergence** of the series $\sum_{n=1}^{\infty} f_n(x)$ is the set:*

$$D_{conv} = \left\{ x \in D \;\middle|\; \sum_{n=1}^{\infty} f_n(x) \text{ converges} \right\}.$$

Examples

1. **Geometric Series:**

$$\sum_{n=1}^{\infty} x^n$$

   Converges for $|x| < 1$. So:

$$D_{\text{conv}} = (-1, 1).$$

2. **Factorial Series:**

$$\sum_{n=0}^{\infty} n!\, x^n$$

   Using the ratio test:

$$\left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right| = (n+1)|x| \to \infty \quad \text{as } n \to \infty.$$

   So the series diverges for all $x \neq 0$, and:

$$D_{\text{conv}} = \{0\}.$$

3. **Power of Index Series:**

$$\sum_{n=1}^{\infty} n^x$$

Set $x = -p$, then the series becomes:

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

which converges if and only if $p > 1$, i.e., $x < -1$. Hence:

$$D_{\text{conv}} = (-\infty, -1).$$

Uniform Convergence of Function Series

Definition 46 (Uniform Convergence) *The series $\sum_{n=1}^{\infty} f_n$ with partial sums s_n converges **uniformly** to a function $s : D \to \mathbb{R}$ if:*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow \sup_{x \in D} |s_n(x) - s(x)| < \varepsilon.$$

We denote this as $s_n \xrightarrow{u} s$.

Proposition 47 (Characterization of Uniform Convergence) *The series $\sum_{n=1}^{\infty} f_n$ converges uniformly to s on D if and only if:*

$$\sup_{x \in D} |s_n(x) - s(x)| \to 0 \quad \text{as } n \to \infty.$$

Theorem 48 (Weierstrass M-Test) *Let $\{f_n\}$ be a sequence of functions $f_n : D \to \mathbb{R}$. Suppose there exist constants $M_n \geq 0$ such that:*

1. *$\forall n,\ \sup_{x \in D} |f_n(x)| \leq M_n$, and*
2. *The series $\sum_{n=1}^{\infty} M_n$ converges.*

Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on D, and:

$$\sup_{x \in D} \left| \sum_{n=1}^{\infty} f_n(x) \right| \leq \sum_{n=1}^{\infty} M_n.$$

Example 49 *Consider the series:*

$$\sum_{n=1}^{\infty} \frac{n \sin(nx)}{e^n}, \qquad x \in \mathbb{R}.$$

We note:

$$\sup_{x \in \mathbb{R}} \left| \frac{n \sin(nx)}{e^n} \right| \leq \frac{n}{e^n}.$$

Since $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converges, the Weierstrass M-test implies that the function series converges uniformly on $\mathbb{R}$.

Dirichlet's Test for Uniform Convergence

Theorem 50 (Dirichlet's Test) *Let $\{f_n\}$ and $\{g_n\}$ be sequences of real-valued functions defined on a common domain $D \subseteq \mathbb{R}$, with:*

$$f_n, g_n : D \to \mathbb{R}.$$

Suppose:

1. *The partial sums $S_n(x) = \sum_{k=1}^{n} f_k(x)$ are uniformly bounded on D, i.e., there exists $M > 0$ such that:*
$$|S_n(x)| \leq M \quad \text{for all } x \in D \text{ and all } n \in \mathbb{N}.$$
2. *The sequence $\{g_n(x)\}$ is monotonic in n for every $x \in D$.*
3. *$\{g_n(x)\}$ converges pointwise to 0 on D.*

Then the series $\sum_{n=1}^{\infty} f_n(x)g_n(x)$ converges uniformly on D.

Example 51 *Consider the function series:*

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{3n+1}, \qquad x \in [0,1].$$

We define:

$$f_n(x) = (-1)^n, \qquad g_n(x) = \frac{x^{3n+1}}{3n+1}.$$

Then:

- *The partial sums $\sum_{k=0}^{n} (-1)^k$ are bounded by 1.*
- *$\{g_n(x)\}$ is decreasing for each fixed $x \in [0,1]$.*
- *$g_n(x) \to 0$ uniformly on $[0,1]$ since:*
$$\sup_{x\in[0,1]} |g_n(x)| = \frac{1}{3n+1} \to 0.$$

Therefore, by Dirichlet's Test, the series converges uniformly on $[0,1]$.

Abel's Test for Uniform Convergence

Theorem 52 (Abel's Test) *Let $\{f_n\}$ and $\{g_n\}$ be sequences of real-valued functions defined on D, with:*

$$f_n, g_n : D \to \mathbb{R}.$$

Suppose:

1. *The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on D.*
2. *The sequence $\{g_n(x)\}$ is uniformly bounded and monotonic in n for each $x \in D$, i.e., there exists $M > 0$ such that:*
$$|g_n(x)| \leq M \quad \text{for all } x \in D \text{ and all } n \in \mathbb{N}.$$

Then the series $\sum_{n=1}^{\infty} f_n(x)g_n(x)$ converges uniformly on D.

Example 53 *Let:*

$$f_n(x) = \frac{x^n}{n}, \qquad g_n(x) = (-1)^n, \quad \textit{for } x \in [0,1].$$

- $\sum_{n=1}^{\infty} \frac{x^n}{n}$ *converges uniformly on* $[0,1]$ *(this is the Taylor series of* $-\log(1-x)$*, and it converges uniformly on* $[0,a]$ *for any* $a<1$*).*
- *The sequence* $g_n(x) = (-1)^n$ *is bounded and monotonic (since it oscillates and is fixed in absolute value).*

Hence, by Abel's Test, the series $\sum_{n=1}^{\infty} \frac{x^n}{n}(-1)^n$ *converges uniformly on* $[0,a]$ *for any* $a<1$*.*

Chapter 6

Lebesuge Integral

Lecture 1: Outer measure

The length $\ell(I)$ of an open interval $I \subset \mathbb{R}$ is defined as:

$$\ell(I) = \begin{cases} b-a & \text{if } I=(a,b) \text{ for some } a<b\in\mathbb{R}, \\ 0 & \text{if } I=\emptyset, \\ \infty & \text{if } I=(-\infty,a) \text{ or } I=(a,\infty), \\ \infty & \text{if } I=(-\infty,\infty). \end{cases}$$

This notion of length can be extended to a finite or infinite disjoint union of open intervals. Suppose

$$A = \bigcup_n I_n, \quad \text{with } I_n \cap I_m = \emptyset \text{ for } n \neq m,$$

then the total length of A is defined as:

$$\ell(A) = \sum_n \ell(I_n),$$

where $\ell(A) = \infty$ if the series diverges—this includes the case where at least one I_n is unbounded.

Definition 54 *The **outer measure** of a set $A \subset \mathbb{R}$, denoted $m^*(A)$, is defined by:*

$$m^*(A) = \inf\left\{\sum_{k=1}^{\infty}\ell(I_k) \;:\; A \subset \bigcup_{k=1}^{\infty} I_k,\ I_k \text{ are open intervals in } \mathbb{R}\right\}.$$

This means:

- We look at **all possible countable collections** of open intervals $I_1, I_2, I_3, \ldots$ that cover the set A.
- For each such collection, we calculate the total length:

$$\sum_{k=1}^{\infty}\ell(I_k).$$

- The outer measure $m^*(A)$ is the **smallest possible total length** (i.e., the infimum over all such sums).

Let's calculate the outer measure of the closed interval $A = [0,1]$.

- To do this, we cover $[0,1]$ using open intervals. One simple choice is to take a slightly larger open interval that contains all of $[0,1]$. For any small $\varepsilon > 0$, let:

$$I_1 = (-\varepsilon, 1+\varepsilon), \quad \text{and set } I_2 = I_3 = \cdots = \emptyset.$$

- The total length of this cover is:

$$\sum_{k=1}^{\infty}\ell(I_k) = \ell(I_1) = (1+\varepsilon) - (-\varepsilon) = 1 + 2\varepsilon.$$

- Since ε can be made arbitrarily small, we take the infimum over all such covers:

$$m^*([0,1]) \leq \inf\{1 + 2\varepsilon : \varepsilon > 0\} = 1.$$

To prove the opposite inequality, let $I_1, I_2, \dots$ be a countable collection of open intervals such that:

$$[0,1] \subset \bigcup_{k=1}^{\infty} I_k.$$

By the Heine–Borel Theorem, there exists a finite subcover; that is, there exists $n \in \mathbb{N}$ such that:

$$[0,1] \subset I_1 \cup \cdots \cup I_n.$$

We will now show by induction on n that this implies:

$$\sum_{k=1}^{n} \ell(I_k) \geq 1.$$

Since this finite sum is a lower bound for the total infinite sum, it follows that:

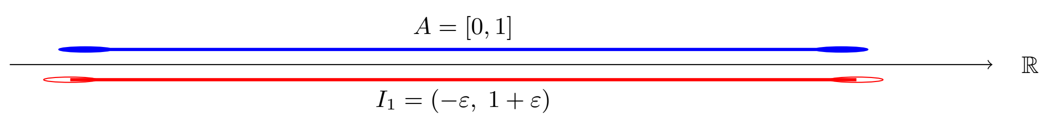
$$\sum_{k=1}^{\infty} \ell(I_k) \geq \sum_{k=1}^{n} \ell(I_k) \geq 1.$$

Thus, for every such cover:

$$m^*([0,1]) \geq 1.$$

Combining both inequalities, we conclude:

$$m^*([0,1]) = 1.$$



This example shows how outer measure works: we cover the set with open intervals and try to minimize the total length. Since every subset $A \subset \mathbb{R}$ can be covered by a countable union of bounded open intervals, and since all interval lengths are nonnegative (or infinite), the outer measure $m^*(A)$ is always well-defined. If every covering gives an infinite total length, then $m^*(A) = \infty$.

Properties of Outer Measure

- **Countable Sets Have Zero Measure:**
  If $A \subset \mathbb{R}$ is countable (finite or infinite), then:

$$m^*(A) = 0.$$

Why? Let $A = \{a_1, a_2, a_3, \dots\}$. For any $\varepsilon > 0$, surround each point a_n with an open interval:

$$I_n = \left(a_n - \frac{\varepsilon}{2^{n+1}},\, a_n + \frac{\varepsilon}{2^{n+1}}\right), \quad \text{so} \quad \ell(I_n) = \frac{\varepsilon}{2^n}.$$

These intervals cover A, and the total length is:

$$\sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

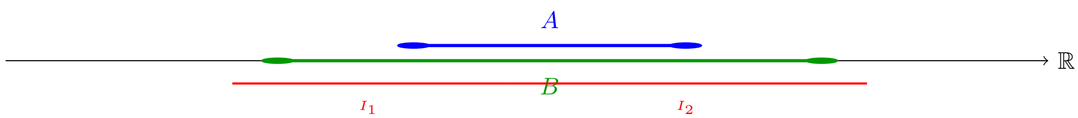
Since ε can be made arbitrarily small, the outer measure must be zero:

$$m^*(A) = 0.$$

Examples: Finite sets and $\mathbb{Q} \cap [0,1]$ are countable, so they have outer measure zero.

- **Monotonicity:** If $A \subset B$, then:
$$m^*(A) \leq m^*(B).$$

  Why? Any collection of open intervals that covers B also covers A. Since outer measure is defined as the smallest such total length, the measure of A can't exceed that of B.



  Interpretation:

  - The red intervals cover B (green), so they also cover A (blue).
  - The total length needed to cover A is at most the length needed to cover B.

- **Countable Subadditivity:**
  For any sequence of sets $E_1, E_2, E_3, \dots \subset \mathbb{R}$:
$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

  Example 55 *Let $\mathbb{Q} \cap [0,1] = \{q_1, q_2, q_3, \dots\}$, and set $E_k = \{q_k\}$. Then:*

  - $m^*(E_k) = 0$ *for all* k,
  - $\sum m^*(E_k) = 0$,
  - $\bigcup E_k = \mathbb{Q} \cap [0,1]$, *so:*
$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) = 0.$$

  Here, we get equality.

  Now consider $A = \mathbb{Q} \cap [0,1]$ and $B = [0,1] \setminus \mathbb{Q}$. Then:

  - $A \cup B = [0,1]$, *and* $A \cap B = \emptyset$,
  - $m^*(A) = 0$, $m^*(B) = 1$,
  - *So:*
$$m^*(A \cup B) = 1 = m^*(A) + m^*(B).$$

  Again, we have equality, but this is not always true.

  Important: *Outer measure is* not *always additive! Even for disjoint sets A and B, it can happen that:*
$$m^*(A \cup B) \neq m^*(A) + m^*(B).$$

  So while outer measure is always ***countably subadditive****, it is not generally* ***countably additive****.*

Lecture 2: σ-algebra

Definition 56 (Sigma-Algebra and Measurable Space) *Let X be a set, and let $\mathcal{S}$ be a collection of subsets of X.*
*We say that $\mathcal{S}$ is a σ-**algebra** on X if it satisfies the following:*

- $\emptyset \in \mathcal{S}$ *(the empty set is included),*
- *If $E \in \mathcal{S}$, then the complement $X \setminus E \in \mathcal{S}$,*
- *If $E_1, E_2, E_3, \dots \in \mathcal{S}$, then the union*
$$\bigcup_{k=1}^{\infty} E_k \in \mathcal{S}$$
(closed under countable unions).

*If $\mathcal{S}$ is a σ-algebra on X, then the pair $(X, \mathcal{S})$ is called a **measurable space**.*

Example 57

- $\{\emptyset, X\}$*: the smallest possible σ-algebra on X,*
- $\mathcal{P}(X)$*: the power set of X, containing all subsets — the largest possible σ-algebra,*
- *The collection of all subsets $E \subseteq X$ such that either E is countable or $X \setminus E$ is countable.*

Proposition 58 *Let $\mathcal{S}$ be a σ-algebra on a set X. Then:*

(a) $X \in \mathcal{S}$

(b) If $D, E \in \mathcal{S}$, then:
$$D \cup E \in \mathcal{S}, \quad D \cap E \in \mathcal{S}, \quad D \setminus E \in \mathcal{S}$$

(c) If $E_1, E_2, E_3, \dots \in \mathcal{S}$, then:
$$\bigcap_{k=1}^{\infty} E_k \in \mathcal{S}$$

Proof. **(a)** Since $\emptyset \in \mathcal{S}$ (by definition), and $X = X \setminus \emptyset$, closure under complements gives $X \in \mathcal{S}$.

(b) Suppose $D, E \in \mathcal{S}$. Then:

- $D \cup E \in \mathcal{S}$ because $\mathcal{S}$ is closed under countable unions.
- For $D \cap E$, use De Morgan's law:
$$X \setminus (D \cap E) = (X \setminus D) \cup (X \setminus E)$$
The right-hand side is in $\mathcal{S}$, so the left-hand side is too. Taking its complement shows $D \cap E \in \mathcal{S}$.
- For $D \setminus E$, note that:
$$D \setminus E = D \cap (X \setminus E)$$
Both sets on the right are in $\mathcal{S}$, so their intersection is too.

(c) Let $E_1, E_2, \dots \in \mathcal{S}$. Then by De Morgan's law:
$$X \setminus \left(\bigcap_{k=1}^{\infty} E_k \right) = \bigcup_{k=1}^{\infty} (X \setminus E_k)$$

Since each $X \setminus E_k \in \mathcal{S}$ and $\mathcal{S}$ is closed under countable unions, the right-hand side is in $\mathcal{S}$. Taking the complement, we conclude:

$$\bigcap_{k=1}^{\infty} E_k \in \mathcal{S}$$

■

Borel σ-Algebra on $\mathbb{R}$

Definition 59 *The* ***Borel*** σ***-algebra*** *on* $\mathbb{R}$*, denoted by* $\mathcal{B}(\mathbb{R})$*, is the smallest* σ*-algebra that contains all open intervals* (a, b)*, where* $a, b \in \mathbb{R}$*.*

- It includes many familiar sets in real analysis: open, closed, half-open intervals, countable sets, and more.
- It is the foundation for defining measures (like Lebesgue measure) on subsets of $\mathbb{R}$.
- Any set in $\mathcal{B}(\mathbb{R})$ is called a **Borel set**.

Examples of Borel Sets:

- **Open intervals:** $(a, b) \in \mathcal{B}(\mathbb{R})$ by definition.
- **Half-open intervals:**
$$[a, b) = \bigcap_{k=1}^{\infty} \left(a - \frac{1}{k}, b\right).$$
Since each interval on the right is open, and Borel sets are closed under countable intersections, $[a, b) \in \mathcal{B}(\mathbb{R})$.
- **Unbounded intervals:**
$$(a, \infty) = \bigcup_{k=1}^{\infty} (a + k, a + k + 1).$$
- **Closed intervals:**
$$[a, b] = \mathbb{R} \setminus ((-\infty, a) \cup (b, \infty)).$$
Since open sets are Borel, so are their complements.
- **Countable sets:** Any countable set, like the rationals in $[0, 1]$, is Borel. For example:
$$B = \{x_1, x_2, x_3, \dots\}, \quad B = \bigcup_{k=1}^{\infty} \{x_k\},$$
where each $\{x_k\}$ is a closed set.
- **Continuity sets of functions:** If $f : \mathbb{R} \to \mathbb{R}$, then the set where f is continuous is a Borel set, because it can be written as a countable intersection of open sets.

How is the Borel σ-algebra built?

- Start with all open intervals (a, b),
- Add all their complements to get closed sets,
- Then include all countable unions and intersections of those sets.

The Borel σ-algebra is large enough to cover most useful sets in analysis, but not all subsets of $\mathbb{R}$. Some sets are too “wild” to be Borel and require Lebesgue theory to handle.

Measure

Definition 60 *Let $\mathcal{S}$ be a σ-algebra on a set X. A function*

$$\mu : \mathcal{S} \to [0, \infty]$$

*is called a **measure** if it satisfies the following properties:*

1. ***Empty Set Has Zero Measure:***
$$\mu(\emptyset) = 0.$$

2. ***Countable Additivity (or σ-Additivity):***
If $\{E_n\}_{n=1}^{\infty}$ is a countable collection of pairwise disjoint sets in $\mathcal{S}$ (i.e., $E_i \cap E_j = \emptyset$ for $i \neq j$), then:
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

*A triple $(X, \mathcal{S}, \mu)$ is called a **measure space**.*

Let's explore several types of measures to better understand what a measure is and what properties it must satisfy.

(i) Counting Measure (Finite Case):

Define a function μ on all subsets of $\mathbb{R}$ by:

$$\mu(E) = \begin{cases} \text{Number of elements in } E, & \text{if } E \text{ is finite,} \\ \infty, & \text{if } E \text{ is infinite.} \end{cases}$$

- This measure simply counts how many elements are in a set.
- If the set is infinite (for example, the set of all natural numbers), we define its measure to be ∞.
- For instance:
$$\mu(\{1, 2, 4\}) = 3, \qquad \mu(\mathbb{N}) = \infty.$$

Why this is a measure:

- $\mu(\emptyset) = 0$, which satisfies the null empty set property.
- For any countable collection of disjoint finite sets $E_1, E_2, \ldots$, we have:
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n),$$
because union just adds up all the elements with no overlap.

(ii) Dirac Measure at a Point $c \in \mathbb{R}$:

Define:

$$\mu_c(E) = \begin{cases} 1, & \text{if } c \in E, \\ 0, & \text{if } c \notin E. \end{cases}$$

Explanation:

- This measure concentrates all the "mass" at a single point c.
- Think of placing a unit of "weight" only at point c. Any set containing c will have measure 1; otherwise, 0.

- For example:

$$\mu_5([4,6]) = 1, \quad \mu_5((0,4)) = 0.$$

Why this is a measure:

- $\mu_c(\emptyset) = 0$ since $c \notin \emptyset$.
- For disjoint sets $E_1, E_2, \ldots$, only one of them (at most) can contain c, so:

$$\mu_c\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu_c(E_n),$$

which is either 1 or 0 depending on whether $c \in \bigcup E_n$.

(iii) Weighted Dirac Measures (Discrete Probability Model):

Let $c_1, c_2, \dots \in \mathbb{R}$ be points, and $p_1, p_2, \dots \geq 0$ be corresponding weights (think of probabilities or masses). Define:

$$\mu(E) = \sum_{\{i: c_i \in E\}} p_i.$$

Explanation:

- Each point c_i has a fixed weight $p_i \geq 0$.
- To measure a set E, we sum up all the weights of those c_i that lie in E.
- Example:

$$\text{If } c_1 = 1,\ p_1 = 0.3;\ c_2 = 2,\ p_2 = 0.7;\ \text{then } \mu(\{1,2\}) = 1.$$

Why this is a measure:

- $\mu(\emptyset) = 0$, because none of the c_i are in $\emptyset$.
- Countable additivity holds: if $E_1, E_2, \ldots$ are disjoint, the weights of points in each are disjoint too, so:

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Note: This kind of measure is used in probability theory to model discrete random variables with weighted outcomes.

(iv) Define a set function μ by:

$$\mu(E) = \begin{cases} 0, & \text{if } E \text{ is finite,} \\ \infty, & \text{if } E \text{ is infinite.} \end{cases}$$

Why this fails to be a measure:

- It satisfies $\mu(\emptyset) = 0$, and is **finitely additive**, meaning:

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2),$$

when E_1, E_2 are disjoint and finite.
- However, it is **not countably additive**. For example, take the disjoint sets:

$$E_n = \{n\}, \quad n = 1, 2, 3, \ldots$$

Then each $\mu(E_n) = 0$, so:

$$\sum_{n=1}^{\infty} \mu(E_n) = 0.$$

But their union is the infinite set $\mathbb{N}$, so:

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu(\mathbb{N}) = \infty.$$

This contradicts countable additivity.

Lecture 3: Lebesgue measure

Lebesgue Measurable Sets

Definition 61 *A set $E \subset \mathbb{R}$ is called* ***Lebesgue measurable*** *if, for every subset $A \subset \mathbb{R}$, the following equality holds:*

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c),$$

where:

- *$m^*(\cdot)$ denotes the* ***outer measure****, and*
- *$E^c = \mathbb{R} \setminus E$ is the* ***complement*** *of E.*

This condition is known as the ***Carathéodory criterion****.*

The intuition behind this definition is that a Lebesgue measurable set E splits any other set $A \subset \mathbb{R}$ into two disjoint parts—$A \cap E$ and $A \cap E^c$—in a way that preserves the total outer measure. That is, measuring the parts separately and adding the results gives exactly the same outer measure as measuring the whole set A directly.

From the properties of outer measure, we always have the inequality:

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c),$$

since $A \subset (A \cap E) \cup (A \cap E^c)$ and outer measure is **countably subadditive**.

Therefore, to verify that E is measurable, we only need to check the **reverse inequality**:

$$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A) \quad \text{for all } A \subset \mathbb{R}.$$

If this inequality holds, then equality follows automatically from the previous inequality, and E is Lebesgue measurable.

Summary: A set $E \subset \mathbb{R}$ is Lebesgue measurable if splitting any set A using E and its complement does not increase the outer measure. This ensures that E behaves well with respect to measure and integration.

Properties of Measurable Sets

- **The empty set $\emptyset$ and the real line $\mathbb{R}$ are measurable.**

  Why? For any set $A \subset \mathbb{R}$:

  $$A \cap \emptyset = \emptyset, \quad A \cap \emptyset^c = A \quad \Rightarrow \quad m^*(A) = 0 + m^*(A).$$

  Similarly, for $E = \mathbb{R}$:

  $$A \cap \mathbb{R} = A, \quad A \cap \mathbb{R}^c = \emptyset \quad \Rightarrow \quad m^*(A) = m^*(A) + 0.$$

- **A set is measurable if and only if its complement is measurable.**

  Why? If E is measurable, then for all $A \subset \mathbb{R}$:

  $$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

  This expression is symmetric in E and E^c, so E^c is also measurable.

- **Every set of outer measure zero is measurable.**

  Why? If $m^*(E) = 0$, then for any $A \subset \mathbb{R}$:

  $$m^*(A \cap E) \leq m^*(E) = 0 \quad \Rightarrow \quad m^*(A \cap E) = 0.$$

  Hence,

  $$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

  So E satisfies the measurability condition.

- **The union of two measurable sets is measurable.**

  Why? Let $E, F \in \mathcal{M}$, and let $A \subset \mathbb{R}$. Define:

  $$A_1 = A \cap E, \quad A_2 = A \cap E^c \cap F, \quad A_3 = A \cap E^c \cap F^c.$$

  These three parts are disjoint and cover A, and since E and F are measurable:

  $$m^*(A) = m^*(A_1) + m^*(A_2) + m^*(A_3).$$

  Notice:

  $$A \cap (E \cup F) = A_1 \cup A_2, \quad A \cap (E \cup F)^c = A_3,$$

  so:

  $$m^*(A) = m^*(A \cap (E \cup F)) + m^*(A \cap (E \cup F)^c),$$

  and thus $E \cup F$ is measurable.

- **The interval (a, ∞) is measurable for any $a \in \mathbb{R}$.**

  Why? Let $A \subset \mathbb{R}$. Define:

  $$A_1 = A \cap (a, \infty), \quad A_2 = A \cap (-\infty, a].$$

  These cover A, and are disjoint:

  $$A = A_1 \cup A_2, \quad A_1 \cap A_2 = \emptyset.$$

  If $m^*(A) = \infty$, then the inequality

  $$m^*(A_1) + m^*(A_2) \leq m^*(A)$$

  holds trivially. Otherwise, for any $\varepsilon > 0$, choose an open cover $\{I_n\}$ of A such that:

  $$\sum \ell(I_n) \leq m^*(A) + \varepsilon.$$

  Define:

  $$J_n = I_n \cap (a, \infty), \quad K_n = I_n \cap (-\infty, a].$$

  Then $A_1 \subset \bigcup J_n$, $A_2 \subset \bigcup K_n$, and:

  $$\ell(I_n) = \ell(J_n) + \ell(K_n) \Rightarrow \sum \ell(J_n) + \sum \ell(K_n) = \sum \ell(I_n).$$

  Therefore:

  $$m^*(A_1) + m^*(A_2) \leq m^*(A) + \varepsilon.$$

  Letting $\varepsilon \to 0$, we conclude:

  $$m^*(A) = m^*(A_1) + m^*(A_2),$$

  proving that (a, ∞) is measurable.

Theorem 62 *The collection of Lebesgue measurable sets $\mathcal{M}$ is a* $\boldsymbol{\sigma}$***-algebra***.

Proof. We have already established that:

- $\mathcal{M}$ contains $\emptyset$ and $\mathbb{R}$,
- $\mathcal{M}$ is closed under complements,
- $\mathcal{M}$ is closed under finite unions.

Now let $E_1, E_2, \dots \in \mathcal{M}$ be a countable collection of pairwise disjoint measurable sets, and define:

$$F_n = \bigcup_{i=1}^{n} E_i, \quad F = \bigcup_{i=1}^{\infty} E_i.$$

Since each F_n is a finite union of measurable sets, $F_n \in \mathcal{M}$ for all n. For any $A \subset \mathbb{R}$, we have:

$$m^*(A) = \lim_{n\to\infty} \left[m^*(A \cap F_n) + m^*(A \cap F_n^c) \right].$$

As $F_n \uparrow F$, we get:

$$\lim_{n\to\infty} m^*(A \cap F_n) = m^*(A \cap F),$$

and since $A \cap F_n^c \downarrow A \cap F^c$, we also have:

$$\lim_{n\to\infty} m^*(A \cap F_n^c) = m^*(A \cap F^c).$$

Thus:

$$m^*(A) = m^*(A \cap F) + m^*(A \cap F^c),$$

which shows $F \in \mathcal{M}$. Therefore, $\mathcal{M}$ is closed under countable unions, and hence is a σ-algebra. ■

Theorem 63 *The Borel σ-algebra $\mathcal{B}$ is contained in the collection of Lebesgue measurable sets $\mathcal{M}$, i.e., $\mathcal{B} \subset \mathcal{M}$.*

Proof. We previously showed that every interval of the form (a, ∞) is measurable.

Now consider an open interval of the form $(-\infty, b)$. Observe that:

$$(-\infty, b) = \bigcup_{n=1}^{\infty} \left(-\infty, b - \frac{1}{n} \right),$$

and since each $(b - \frac{1}{n}, \infty)$ is measurable, their complements $(-\infty, b - \frac{1}{n})$ are also measurable. Therefore, $(-\infty, b)$ is measurable as a countable union of measurable sets.

Consequently, any open interval (a, b) can be written as:

$$(a, b) = (a, \infty) \cap (-\infty, b),$$

which is an intersection of two measurable sets, and hence also measurable.

Since any open set in $\mathbb{R}$ can be written as a countable union of open intervals, and $\mathcal{M}$ is a σ-algebra, it follows that all open sets are measurable.

Therefore, the Borel σ-algebra $\mathcal{B}$, which is generated by open intervals, is a subset of $\mathcal{M}$. ■

Theorem 64 *The restriction of the outer measure m^* to the collection $\mathcal{M}$ of Lebesgue measurable sets defines a measure. That is,*

$$m := m^*|_{\mathcal{M}}$$

is a measure on the measurable space $(\mathbb{R}, \mathcal{M})$.

$$(\mathbb{R}, \mathcal{M}, m)$$

is called the ***Lebesgue measure space****.*

Idea of the Proof. To prove that m is a measure, we must verify the two key properties of a measure:

1. $m(\emptyset) = 0$,
2. Countable additivity: For any disjoint collection $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$,

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

Since $m = m^*$ on $\mathcal{M}$, and $m^*(\emptyset) = 0$, the first property is automatically satisfied.
Now we focus on countable additivity.
Let $\{E_i\}_{i=1}^{\infty}$ be a disjoint family of measurable sets. We want to show:

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i).$$

Step 1: Lower bound (using monotonicity and finite additivity).
For any $n \in \mathbb{N}$, we have

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \geq m^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(E_i),$$

because the sets E_i are disjoint and measurable.

Taking the limit as $n \to \infty$ gives:

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m^*(E_i).$$

Step 2: Upper bound (using subadditivity of m^*).
By the definition of outer measure and countable subadditivity:

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i).$$

Conclusion: Since we have both inequalities ($\leq$ and $\geq$, equality holds:

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i).$$

Therefore, m is countably additive on $\mathcal{M}$, so it is indeed a measure. ■

Key Remark: Although the outer measure m^* is defined on all subsets of $\mathbb{R}$, it is not countably additive in general. However, when restricted to the collection $\mathcal{M}$ of Lebesgue measurable sets, it becomes a proper measure. Importantly, $\mathcal{M}$ contains all Borel sets, which are sufficient for most practical applications in analysis .

Theorem 65 *Then the following properties hold:*

(a) *Monotonicity:* *If $E, F \in \mathcal{M}$ with $E \subset F$, then*

$$m(E) \le m(F).$$

(b) *Countable Sub-additivity:* *For any countable collection $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$,*

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} m(E_n).$$

(c) *Continuity from Below:* *If $E_1 \subset E_2 \subset \cdots$ (increasing sequence), then*

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n).$$

(d) *Continuity from Above:* *If $E_1 \supset E_2 \supset \cdots$ (decreasing sequence) and $m(E_k) < \infty$ for some k, then*

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n).$$

We prove each property individually.

(a) Monotonicity: Suppose $E \subset F$. Then the set difference $F \setminus E \in \mathcal{M}$, and the sets E and $F \setminus E$ are disjoint. Since $E \cup (F \setminus E) = F$, we get:

$$m(F) = m(E) + m(F \setminus E) \ge m(E).$$

(b) Countable Sub-additivity: Let $\{E_n\}$ be any sequence of measurable sets. Define:

$$F_1 = E_1, \quad F_n = E_n \setminus \bigcup_{k=1}^{n-1} E_k \quad (n \ge 2).$$

Then the F_n are disjoint, and:

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n.$$

Thus:

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(F_n) \le \sum_{n=1}^{\infty} m(E_n).$$

(c) Continuity from Below: Let $E_1 \subset E_2 \subset \cdots$, and set:

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Define $A_1 = E_1$, and $A_n = E_n \setminus E_{n-1}$ for $n \ge 2$. Then $E = \bigsqcup_{n=1}^{\infty} A_n$, so:

$$m(E) = \sum_{n=1}^{\infty} m(A_n), \quad \text{and} \quad m(E_k) = \sum_{n=1}^{k} m(A_n).$$

Hence,

$$\lim_{k\to\infty} m(E_k) = m(E).$$

(d) Continuity from Above: Let $E_1 \supset E_2 \supset \cdots$, and assume $m(E_k) < \infty$ for some k. Let:

$$E = \bigcap_{n=1}^{\infty} E_n = \bigcap_{n=k}^{\infty} E_n,$$

and set $A_n = E_k \setminus E_n$. Then $A_n \subset A_{n+1}$ and:

$$E_k \setminus E = \bigcup_{n=k}^{\infty} A_n.$$

By continuity from below:

$$m(E_k \setminus E) = \lim_{n\to\infty} m(E_k \setminus E_n).$$

Therefore:

$$\lim_{n\to\infty} m(E_n) = m(E_k) - \lim_{n\to\infty} m(E_k \setminus E_n) = m(E).$$

Lecture 4: Lebesgue Measurable Function

Definition 66 *Let $f : E \to \mathbb{R}$ be a function, where $E \subseteq \mathbb{R}$ is a measurable set.*
We say that f is ***Lebesgue measurable*** *(or simply* ***measurable****) if for every real number $\alpha \in \mathbb{R}$, the set*

$$\{x \in E : f(x) > \alpha\}$$

belongs to $\mathcal{M}$; that is, it is a measurable set.

We now present several examples to illustrate the concept of measurable functions. In each case, we examine whether the set $\{x \in \mathbb{R} : f(x) > \alpha\}$ belongs to $\mathcal{M}$ (e.g., Lebesgue measurable set). If this condition holds for every $\alpha \in \mathbb{R}$, then f is measurable.

1. **Constant function:** Let $f(x) \equiv c$, a constant function for some $c \in \mathbb{R}$. Consider the set

$$\{x \in \mathbb{R} : f(x) > \alpha\}.$$

   - If $\alpha \geq c$, then $f(x) > \alpha$ is never true, so the set is empty: $\emptyset$. - If $\alpha < c$, then $f(x) > \alpha$ for all $x \in \mathbb{R}$, so the set is $\mathbb{R}$.

   Since both $\emptyset$ and $\mathbb{R}$ are elements of $\mathcal{M}$, this shows that constant functions are always measurable .

2. **Continuous functions:** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. For any $\alpha \in \mathbb{R}$, the set

$$\{x \in \mathbb{R} : f(x) > \alpha\}$$

   is an open set, because the preimage of an open interval (α, ∞) under a continuous function is open. Since every open set is a Borel set, it follows that every continuous function is Borel measurable.

3. **Characteristic function of a measurable set:** If $E, F \subset \mathbb{R}$ are two measurable sets, then the indicator function $\chi_F : E \to \mathbb{R}$, defined by

$$\chi_F(x) = \begin{cases} 1, & x \in F, \\ 0, & x \notin F, \end{cases}$$

   is measurable.

   This can be verified by direct computation. For any $\alpha \in \mathbb{R}$, the preimage $\chi_F^{-1}((\alpha, \infty])$ is given by

$$\{x \in \mathbb{R} : \chi_F(x) > \alpha\} = \begin{cases} \emptyset, & \alpha > 1, \\ E \cap F, & 0 \leq \alpha < 1, \\ E, & \alpha < 0. \end{cases}$$

   Since E and F are measurable, each of these preimages is measurable, thus making χ_F measurable.

4. **Monotone functions:** Let $f : \mathbb{R} \to \mathbb{R}$ be any monotone increasing function, and let $\alpha \in \mathbb{R}$.
   Then the set

$$\{x \in \mathbb{R} : f(x) > \alpha\}$$

   is one of the following:

   - a right-open half-line of the form $\{x \in \mathbb{R} : x > \gamma\}$,
   - a right-closed half-line $\{x \in \mathbb{R} : x \geq \gamma\}$,
   - the entire real line $\mathbb{R}$, or
   - the empty set $\emptyset$,

   Conclusion. These examples illustrate that the class of measurable functions includes:

- all continuous functions,
- all characteristic functions of measurable sets,
- and all monotone functions.

Remark. The collection of measurable functions is closed under arithmetic operations (addition, subtraction, scalar multiplication, etc.), pointwise limits, and taking absolute values. This makes them very useful in integration theory and probability.

Theorem 67 *Let $E \subset \mathbb{R}$ be measurable, and suppose $f, g : E \to \mathbb{R}$ are two measurable functions, and let $c \in \mathbb{R}$ be a constant. Then the following functions are also measurable:*

$$cf, \quad f^2, \quad f+g, \quad f \cdot g, \quad |f|.$$

Proof. We verify measurability for each case:

1. **Scalar multiplication:** Assume $c > 0$ (the case $c < 0$ is similar and $c = 0$ is trivial). For any $\alpha \in \mathbb{R}$, we have:
$$\{x \in \mathbb{R} : cf(x) > \alpha\} = \{x \in \mathbb{R} : f(x) > \alpha/c\}.$$
Since f is measurable, the right-hand side is in $\mathcal{M}$, hence cf is measurable.

2. **Square function:** Assume $\alpha > 0$ (for $\alpha \leq 0$, the set $\{f^2 > \alpha\}$ is either $\mathbb{R}$ or empty, and thus measurable). Then:
$$\{x \in \mathbb{R} : f^2(x) > \alpha\} = \{x \in \mathbb{R} : f(x) > \sqrt{\alpha}\} \cup \{x \in \mathbb{R} : f(x) < -\sqrt{\alpha}\}.$$
Both sets on the right are measurable since f is measurable. Therefore, f^2 is measurable.

3. **Sum $f + g$:** Fix $\alpha \in \mathbb{R}$. For each rational number $r \in \mathbb{Q}$, define:
$$S_r = \{x \in \mathbb{R} : f(x) > r\} \cap \{x \in \mathbb{R} : g(x) > \alpha - r\}.$$
Each set $S_r \in \mathcal{M}$, since f and g are measurable. Moreover,
$$\{x \in \mathbb{R} : f(x) + g(x) > \alpha\} = \bigcup_{r \in \mathbb{Q}} S_r,$$
which is a countable union of measurable sets, hence measurable. Thus $f + g$ is measurable.

4. **Product $f \cdot g$:** Using the identity:
$$f \cdot g = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right],$$
and since sums, differences, and squares of measurable functions are measurable (as shown above), it follows that $f \cdot g$ is measurable.

5. **Absolute value:** For $\alpha > 0$, we write:
$$\{x \in \mathbb{R} : |f(x)| > \alpha\} = \{x \in \mathbb{R} : f(x) > \alpha\} \cup \{x \in \mathbb{R} : f(x) < -\alpha\}.$$
Each set on the right is measurable, hence $|f|$ is measurable.

■

Suppose f is a function. We define the positive part f^+ and the negative part f^- of f as functions from Ω to $[0, \infty]$ as follows:

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{if } f(x) < 0, \end{cases}$$

and

$$f^-(x) = \begin{cases} 0 & \text{if } f(x) \geq 0, \\ -f(x) & \text{if } f(x) < 0. \end{cases}$$

Note that

$$f = f^+ - f^-, \quad |f| = f^+ + f^-.$$

Theorem 68 *The function f is measurable if and only if f^+ and f^- are both measurable.*

In dealing with sequences of measurable functions, it is often convenient to consider operations such as suprema, infima, lim sup, lim inf, and pointwise limits. These operations naturally lead us to consider functions that may take infinite values. Therefore, it is useful—and often necessary—to allow functions to take values in the *extended real line*, that is, to take the values $+\infty$ and $-\infty$ in addition to the usual real values.

We denote the set of **extended real numbers** by:

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}.$$

Definition 69 (Measurable Extended Real-Valued Function) *Let $f : E \to \overline{\mathbb{R}}$, where $E \subseteq \mathbb{R}$ is a measurable set and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ denotes the extended real line.*
*We say that f is **measurable** (with respect to a σ-algebra $\mathcal{M}$) if the following conditions are satisfied:*

- *For every $\alpha \in \mathbb{R}$, the set*
$$\{x \in E : f(x) > \alpha\} \in \mathcal{M}.$$
- *The sets*
$$\{x \in E : f(x) = +\infty\} \quad \text{and} \quad \{x \in E : f(x) = -\infty\}$$
also belong to $\mathcal{M}$.

Definition 70 *Let $E \subset \mathbb{R}$ be a measurable set. A statement $P(x)$ is said to hold **almost everywhere (a.e.)** on E if*

$$m(\{x \in E : P(x) \text{ does not hold}\}) = 0.$$

In other words, the set where $P(x)$ does not hold has measure zero. Note that any set with outer measure zero also has measure zero, so using m^* instead of m in this definition would yield the same statement.

Theorem 71 *If two functions $f, g : E \to [-\infty, \infty]$ satisfy $f = g$ almost everywhere on E, and f is measurable, then g is also measurable.*

In other words, modifying a measurable function on a set of measure zero does not affect its measurability.
Proof. Let $N = \{x \in E : f(x) \neq g(x)\}$. By assumption, N has outer measure zero, so $m(N) = 0$. For any $\alpha \in \mathbb{R}$, define

$$N_\alpha = \{x \in N : g(x) > \alpha\} \subset N,$$

which also has measure zero since $m^*(N_\alpha) \leq m^*(N) = 0$.

Now, for each $\alpha \in \mathbb{R}$, we can express the preimage $g^{-1}((\alpha, \infty])$ as

$$g^{-1}((\alpha, \infty]) = \big(f^{-1}((\alpha, \infty]) \setminus N\big) \cup N_\alpha.$$

Since f is measurable, $f^{-1}((\alpha, \infty])$ is measurable, and both N and N_α have measure zero. Thus, $g^{-1}((\alpha, \infty])$ is a union of measurable sets, making it measurable as well. This proves that g is measurable. ■

Corollary 72 *If f and g are measurable, then the sets $\{x : f(x) < g(x)\}$, $\{x : f(x) \leq g(x)\}$, and $\{x : f(x) = g(x)\}$ are also measurable.*

Theorem 73 *Let $\{f_n(x)\}$ be a sequence of measurable functions. Then the functions*

$$\inf_n f_n(x), \quad \sup_n f_n(x), \quad \liminf_{n\to\infty} f_n(x), \quad \textit{and} \quad \limsup_{n\to\infty} f_n(x)$$

are all measurable.

Proof. Define $g(x) = \sup_n f_n(x)$ and let $a \in \mathbb{R}$. Then we can express the set $\{x : g(x) \leq a\}$ as

$$\{x : g(x) \leq a\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) \leq a\}.$$

This set is measurable, as it is the countable intersection of measurable sets, each $\{x : f_n(x) \leq a\}$ being measurable by the measurability of f_n.

Now, let $h(x) = \limsup_{n\to\infty} f_n(x)$. For $h(x) \leq a$ (where $a \in \mathbb{R}$), it is true if and only if for every $n \in \mathbb{N}$, there exists $m \geq n$ such that $f_m(x) \leq a$. This can be written as

$$\{x : h(x) \leq a\} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{x : f_m(x) \leq a\},$$

which is measurable as it is a countable intersection of countable unions of measurable sets.

The arguments for $\inf_n f_n$ and $\liminf_{n\to\infty} f_n$ follow similarly and are left as an exercise. ■

A function is called a **simple function** if it takes only a finite number of values and can be written as a finite linear combination of characteristic functions of measurable sets:

$$f(x) = \sum_{i=1}^{N} a_i \chi_{A_i}(x), \quad \text{where } A_i \in \mathcal{M}.$$

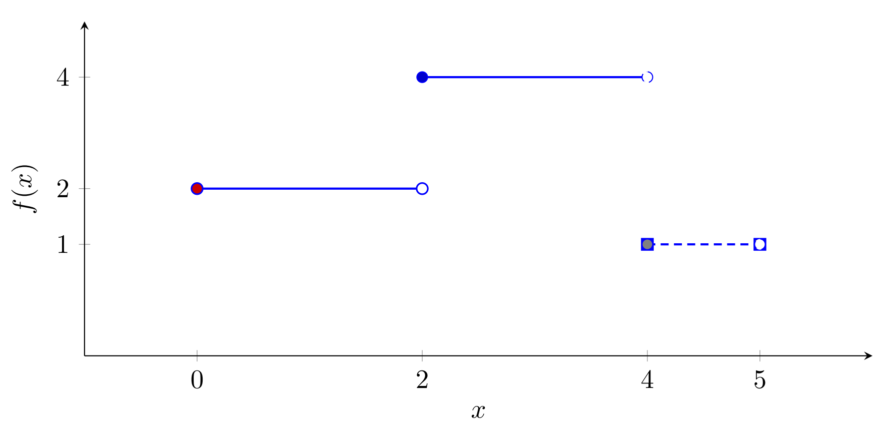
Here, $\chi_A(x)$ is the characteristic function of the set A, defined by:

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Example: Let

$$f(x) = 2\chi_{[0,2)}(x) + 4\chi_{[2,4)}(x) + 1\chi_{[4,5)}(x).$$

Then f is a simple function defined on $[0, 5]$, taking the values 2, 4, and 1 over disjoint intervals.



Theorem 74 *If $f : \Omega \to [0, \infty]$ is a Lebesgue measurable function, then there exists a sequence of non-negative simple functions (φ_n) such that:*

(i) $\varphi_{n+1}(x) \geq \varphi_n(x)$ *for all* $n \in \mathbb{N}$ *and* $x \in \Omega$,

(ii) $\lim_{n\to\infty} \varphi_n(x) = f(x)$ *for all* $x \in \Omega$.

We write $\varphi_n \uparrow f$ *to denote that* φ_n *increases to* f.

For each $n \in \mathbb{N}$, define the sets:

$$F_{n,i} = f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right), \quad i \in \{1, 2, \ldots, n2^n\},$$

$$F_{n,\infty} = f^{-1}([n, \infty]) \cup f^{-1}(\{\infty\}),$$

and the simple function

$$\varphi_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{F_{n,i}} + n\chi_{F_{n,\infty}}.$$

Each φ_n is measurable because each interval $\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)$ and $[n, \infty)$ is a Borel set.

(i) For any $x \in F_{n,i}$, we have $\frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}$. Then either: - $\frac{i-1}{2^n} \leq f(x) < \frac{2i-1}{2^{n+1}}$, so $x \in F_{n+1,2i-1}$ and $\varphi_{n+1}(x) = \frac{i-1}{2^n} = \varphi_n(x)$, - $\frac{2i-1}{2^{n+1}} \leq f(x) < \frac{i}{2^n}$, so $x \in F_{n+1,2i}$ and $\varphi_{n+1}(x) > \varphi_n(x)$.

(ii) If $f(x) < N$ for some $N \in \mathbb{N}$, then for all $n \geq N$ there is an integer i such that

$$\frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n},$$

which implies $0 \leq f(x) - \varphi_n(x) < \frac{1}{2^n}$. Hence $\varphi_n(x) \to f(x)$.

Lecture 5: Lebesgue Integral of Nonnegative Measurable Functions

We often work with functions that can take the value $+\infty$ and sets with infinite measure. For this reason, we adopt the following conventions:

$$a + \infty = \infty + a = \infty \quad \text{for } a \in [0, \infty],$$

$$a \cdot \infty = \infty \cdot a = \infty \quad \text{for } a \in (0, \infty],$$

$$0 \cdot \infty = \infty \cdot 0 = 0.$$

Integral of a Simple Function

Let $f : E \to [0, \infty]$ be a simple measurable function, which means it takes only finitely many values. Suppose these values are $\alpha_1, \ldots, \alpha_N$. For each $j = 1, \ldots, N$, define:

$$A_j = \{x \in E : f(x) = \alpha_j\}.$$

Then the Lebesgue integral of f over E is:

$$\int_E f \, dm = \sum_{j=1}^{N} \alpha_j \, m(A_j).$$

Alternatively, if f is written as a sum of characteristic functions:

$$f = \sum_{i=1}^{n} a_i \chi_{A_i},$$

then:

$$\int_E f \, dm = \sum_{i=1}^{n} a_i \, m(A_i).$$

Example: Define the function $f : \mathbb{R} \to \mathbb{R}$ by:

$$f(x) = \begin{cases} 2, & \text{if } -1 < x < 1, \\ 3, & \text{if } 3 < x < 7, \\ -1, & \text{if } -4 \le x < -3, \\ 0, & \text{otherwise.} \end{cases}$$

Then:

$$\int_{\mathbb{R}} f(x) \, dm = 2 \cdot 2 + 3 \cdot 4 + (-1) \cdot 1 = 4 + 12 - 1 = \boxed{15}.$$

Integral of a General Nonnegative Function

Let $f : E \to [0, \infty]$ be any nonnegative measurable function. We define:

$$\int_E f \, dm = \sup \left\{ \int_E \varphi \, dm \;\middle|\; \varphi \in S^+(E), \, 0 \le \varphi \le f \right\},$$

where $S^+(E)$ is the set of all nonnegative simple functions on E.

Basic Properties

If $f, g : E \to [0, \infty]$ are measurable, and $\lambda \geq 0$, then:

- If $f \leq g$, then $\int_E f\, dm \leq \int_E g\, dm$.
- $\int_E \lambda f\, dm = \lambda \int_E f\, dm$.
- If $F \subset E$, then $\int_F f\, dm = \int_E f \chi_F\, dm$.
- If $m(E) = 0$, then $\int_E f\, dm = 0$.

Monotone Convergence Theorem

Theorem 75 *Let $f_n : E \to [0, \infty]$ be an increasing sequence of measurable functions (i.e., $f_1 \leq f_2 \leq \cdots$), and let $f(x) = \lim_{n\to\infty} f_n(x)$. Then:*

$$\lim_{n\to\infty} \int_E f_n\, dm = \int_E f\, dm.$$

Idea of the proof: Since the sequence $\int_E f_n$ increases, its limit exists. Also, for any simple function $\phi \leq f$, eventually $f_n \geq \phi$, so:

$$\int_E \phi \leq \lim_{n\to\infty} \int_E f_n.$$

Taking the supremum over all such ϕ, we get the reverse inequality and conclude:

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

This theorem justifies interchanging limits and integrals for nonnegative functions that grow pointwise.

Note: Additivity of the integral over disjoint measurable subsets is not obvious and will be proved using the Monotone Convergence Theorem.

Theorem 76 (Monotone Convergence Theorem) *Let $\{f_n\}$ be a sequence of nonnegative measurable functions in E such that $f_1 \leq f_2 \leq \ldots$ pointwise on E, and suppose $f_n \to f$ pointwise on E for some f (which will also be a measurable functions is measurable). Then*

$$\lim_{n\to\infty} \int_E f_n\, dm = \int_E f\, dm.$$

Proof. Since $f_1 \leq f_2 \leq \cdots$, it follows that $\int_E f_1 \leq \int_E f_2 \leq \cdots$. Thus, $\int_E f_n$ forms a nonnegative, increasing sequence, which ensures that the limit $\lim_{n\to\infty} \int_E f_n$ exists within the interval $[0, \infty]$. Additionally, because $\lim_{n\to\infty} f_n(x) = f(x)$ for each x, we know $f_n \leq f$ for all n, implying that $\int_E f$ (a finite value in $[0, \infty]$) must satisfy

$$\int_E f_n \leq \int_E f \Rightarrow \lim_{n\to\infty} \int_E f_n \leq \int_E f.$$

To establish the reverse inequality (i.e., $\int_E f \leq \lim_{n\to\infty} \int_E f_n$), we will show that $\int_E \phi \leq \lim_{n\to\infty} \int_E f_n$ for every simple function $\phi \leq f$, noting that eventually, f_n will exceed ϕ.

Let $\epsilon \in (0, 1)$ be chosen as a "margin." For any simple function $\phi = \sum_{j=1}^m a_j \chi_{A_j}$ with $\phi \leq f$, we define the set

$$E_n = \{x \in E : f_n(x) \geq (1 - \epsilon)\phi(x)\}.$$

Since $(1 - \epsilon)\phi(x) < f(x)$ for all x (strict inequality holds as ϵ is positive) and $\lim_{n\to\infty} f_n(x) = f(x)$, each x must belong to some E_n. Thus, we have

$$\bigcup_{n=1}^{\infty} E_n = E.$$

Moreover, because $f_1 \leq f_2 \leq \cdots$, it follows that $E_1 \subset E_2 \subset \cdots$, so the sets E_n are nested by inclusion. Now, observe that

$$\int_E f_n \geq \int_{E_n} f_n \geq \int_{E_n} (1-\epsilon)\phi = (1-\epsilon)\int_{E_n} \phi = (1-\epsilon)\sum_{j=1}^{m} a_j m(A_j \cap E_n),$$

since the inequality holds on E_n, and the sets $A_j \cap E_n$ are measurable and disjoint. As E_n increases to E, the sets $E_1 \cap A_j \subset E_2 \cap A_j \subset \cdots$ expand to cover A_j. By the continuity of the Lebesgue measure, we conclude that as $n \to \infty$,

$$m(A_j \cap E_n) \to m(A_j).$$

Taking limits on both sides (noting that we have a finite sum on the right) gives, for all $\epsilon \in (0,1)$,

$$\lim_{n\to\infty} \int_E f_n \geq \lim_{n\to\infty} (1-\epsilon)\sum_{j=1}^{m} a_j m(A_j \cap E_n) = (1-\epsilon)\sum_{j=1}^{m} a_j m(A_j) = (1-\epsilon)\int_E \phi.$$

By letting $\epsilon \to 0$, we obtain the desired inequality $\int_E \phi \leq \lim_{n\to\infty} \int_E f_n$. Combining this with the initial inequality completes the proof. ■

Theorem 77 (Fatou's Lemma) *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative measurable functions on a measurable set E. Then:*

$$\int_E \liminf_{n\to\infty} f_n \, dm \leq \liminf_{n\to\infty} \int_E f_n \, dm.$$

Proof. We begin by expressing the pointwise lim inf using the identity:

$$\liminf_{n\to\infty} f_n(x) = \sup_{n\geq 1} \left(\inf_{k\geq n} f_k(x) \right).$$

Define:

$$g_n(x) := \inf_{k\geq n} f_k(x).$$

Then $g_n(x)$ is an increasing sequence of measurable functions (since $g_n(x) \leq g_{n+1}(x)$) and:

$$\lim_{n\to\infty} g_n(x) = \liminf_{n\to\infty} f_n(x).$$

Now apply the Monotone Convergence Theorem:

$$\int_E \liminf_{n\to\infty} f_n \, dm = \lim_{n\to\infty} \int_E g_n \, dm.$$

For each n, we know $g_n(x) \leq f_k(x)$ for all $k \geq n$, so:

$$\int_E g_n \, dm \leq \int_E f_k \, dm \quad \text{for all } k \geq n.$$

Hence,

$$\int_E g_n \, dm \leq \inf_{k\geq n} \int_E f_k \, dm.$$

Now take the limit as $n \to \infty$ on both sides:

$$\lim_{n\to\infty} \int_E g_n \, dm \leq \lim_{n\to\infty} \inf_{k\geq n} \int_E f_k \, dm = \liminf_{n\to\infty} \int_E f_n \, dm.$$

Putting it all together:

$$\int_E \liminf_{n\to\infty} f_n \, dm \leq \liminf_{n\to\infty} \int_E f_n \, dm.$$

This completes the proof. ■

Lebesgue Integrable Functions

Definition 78 (Lebesgue Integrable Function and Integral) *Let $E \subset \mathbb{R}$ be a measurable set, and let $f : E \to \mathbb{R}$ be a measurable function. Define the positive and negative parts of f as*

$$f^+(x) = \max(f(x), 0), \quad f^-(x) = \max(-f(x), 0).$$

These are nonnegative measurable functions, and satisfy

$$f = f^+ - f^-, \quad |f| = f^+ + f^-.$$

*We say that f is **Lebesgue integrable** over E if*

$$\int_E |f|\, dm = \int_E f^+\, dm + \int_E f^-\, dm < \infty.$$

*In this case, the **Lebesgue integral** of f over E is defined as*

$$\int_E f dm := \int_E f^+\, dm - \int_E f^-\, dm.$$

Proposition 79 *Let $f, g : E \to \mathbb{R}$ be Lebesgue integrable functions. Then:*

1. *For any scalar $c \in \mathbb{R}$, the function cf is integrable, and*

$$\int_E cf\, dm = c \int_E f\, dm.$$

2. *The sum $f + g$ is integrable, and*

$$\int_E (f + g)\, dm = \int_E f\, dm + \int_E g\, dm.$$

3. *If $A, B \subset E$ are disjoint measurable subsets, then*

$$\int_{A \cup B} f\, dm = \int_A f\, dm + \int_B f\, dm.$$

Proof. (1) Since $|cf| = |c| \cdot |f|$ and $f \in L^1(E)$, we know $|cf| \in L^1(E)$, so cf is integrable. Linearity of the integral gives:

$$\int_E cf\, dm = c \int_E f\, dm.$$

(2) By the triangle inequality:

$$|f + g| \le |f| + |g| \quad \Rightarrow \quad \int_E |f + g|\, dm \le \int_E |f|\, dm + \int_E |g|\, dm < \infty,$$

so $f + g \in L^1(E)$. Using the decomposition $f = f^+ - f^-$ and similarly for g, we get:

$$f + g = (f^+ + g^+) - (f^- + g^-),$$

and since all terms are nonnegative measurable functions, we apply linearity:

$$\int_E (f + g)\, dm = \int_E f^+\, dm + \int_E g^+\, dm - \int_E f^-\, dm - \int_E g^-\, dm = \int_E f\, dm + \int_E g\, dm.$$

(3) Since A and B are disjoint,

$$\chi_{A\cup B} = \chi_A + \chi_B.$$

Hence,

$$f\chi_{A\cup B} = f\chi_A + f\chi_B,$$

and since the product of a measurable function with an indicator function restricts the domain of integration:

$$\int_{A\cup B} f\,dm = \int_E f\chi_{A\cup B}\,dm = \int_E f\chi_A\,dm + \int_E f\chi_B\,dm = \int_A f\,dm + \int_B f\,dm.$$

■

Proposition 80 *Let $f, g : E \to \mathbb{R}$ be measurable functions. Then:*

1. *If f is Lebesgue integrable, then*

$$\left|\int_E f\,dm\right| \leq \int_E |f|\,dm.$$

2. *If $f = g$ almost everywhere and $g \in L^1(E)$, then $f \in L^1(E)$ and*

$$\int_E f\,dm = \int_E g\,dm.$$

3. *If $f, g \in L^1(E)$ and $f(x) \leq g(x)$ almost everywhere on E, then*

$$\int_E f\,dm \leq \int_E g\,dm.$$

Proof. **(1)** Since $f = f^+ - f^-$, we have

$$\left|\int_E f\,dm\right| = \left|\int_E f^+\,dm - \int_E f^-\,dm\right| \leq \int_E f^+\,dm + \int_E f^-\,dm.$$

Using the identity $|f| = f^+ + f^-$, it follows that

$$\int_E |f|\,dm = \int_E f^+\,dm + \int_E f^-\,dm.$$

(2) Since $f = g$ almost everywhere, we also have $|f| = |g|$ almost everywhere. Thus,

$$\int_E |f|\,dm = \int_E |g|\,dm < \infty,$$

so f is Lebesgue integralble. Also, $f - g = 0$ almost everywhere implies

$$\left|\int_E f\,dm - \int_E g\,dm\right| = \left|\int_E (f-g)\,dm\right| \leq \int_E |f-g|\,dm = 0,$$

which gives $\int_E f\,dm = \int_E g\,dm$.

(3) Define the function

$$h(x) = \begin{cases} g(x) - f(x), & \text{if } g(x) \geq f(x), \\ 0, & \text{otherwise.} \end{cases}$$

Then $h \geq 0$, measurable, and $h = g - f$ almost everywhere. Therefore,

$$\int_E h\,dm = \int_E (g-f)\,dm = \int_E g\,dm - \int_E f\,dm \geq 0,$$

which yields $\int_E f\,dm \leq \int_E g\,dm$. ■

Theorem 81 (Dominated Convergence Theorem) *Let $g : E \to [0, \infty)$ be a Lebesgue integrable function. Suppose $\{f_n\}$ is a sequence of measurable functions $f_n : E \to \mathbb{R}$ such that:*

1. *$|f_n(x)| \leq g(x)$ almost everywhere on E, for all $n \in \mathbb{N}$,*
2. *$f_n(x) \to f(x)$ pointwise almost everywhere on E, for some function $f : E \to \mathbb{R}$.*

Then f is Lebesgue integrable, and

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm.$$

Proof. Since $|f_n| \leq g$ and g is Lebesgue integrable, it follows that each f_n is also Lebesgue integrable. The pointwise limit f is measurable and satisfies $|f| \leq g$, so f is also Lebesgue integrable.

We aim to prove:

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm.$$

Apply Fatou's Lemma to the nonnegative functions $g - f_n$:

$$\int_E \liminf_{n\to\infty} (g - f_n) \, dm \leq \liminf_{n\to\infty} \int_E (g - f_n) \, dm.$$

Since $f_n \to f$ pointwise, the left-hand side becomes $\int_E (g - f) \, dm$, yielding:

$$\int_E (g - f) \, dm \leq \liminf_{n\to\infty} \left(\int_E g \, dm - \int_E f_n \, dm \right).$$

Rewriting this, we obtain:

$$\limsup_{n\to\infty} \int_E f_n \, dm \leq \int_E f \, dm.$$

Similarly, apply Fatou's Lemma to $g + f_n$:

$$\int_E \liminf_{n\to\infty} (g + f_n) \, dm \leq \liminf_{n\to\infty} \int_E (g + f_n) \, dm,$$

which gives:

$$\int_E (g + f) \, dm \leq \liminf_{n\to\infty} \left(\int_E g \, dm + \int_E f_n \, dm \right).$$

Rearranging:

$$\int_E f \, dm \leq \liminf_{n\to\infty} \int_E f_n \, dm.$$

Combining both inequalities:

$$\limsup_{n\to\infty} \int_E f_n \, dm \leq \int_E f \, dm \leq \liminf_{n\to\infty} \int_E f_n \, dm.$$

Since $\liminf \leq \limsup$ always holds, we conclude:

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm.$$

■

Proposition 82 *Let* (f_n) *be a bounded sequence of measurable functions on a set* E *with finite measure* $m(E) < \infty$*. If* $f_n \to f$ *almost everywhere on* E*, then the limit function* f *is Lebesgue integrable and*

$$\lim_{n\to\infty} \int_E f_n\, dm = \int_E f\, dm.$$

Proof. Assume there exists a constant $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in E$ and all n. Define $g(x) = M$, which is clearly Lebesgue integrable on E since $m(E) < \infty$. Then $|f_n(x)| \leq g(x)$ for all n, and $f_n \to f$ almost everywhere. The Dominated Convergence Theorem applies and yields the result. ■ ■

Theorem 83 (Term-by-Term Integration of a Series) *Let* $\{f_n\}$ *be a sequence of measurable functions on a measurable set* E*. Then:*

(i) The series of integrals of absolute values satisfies:

$$\int_E \left(\sum_{n=1}^{\infty} |f_n| \right) dm = \sum_{n=1}^{\infty} \int_E |f_n|\, dm.$$

Both sides may be infinite, or both are finite and equal.

(ii) If the right-hand side is finite, then each f_n *is Lebesgue integrable, the series*

$$\sum_{n=1}^{\infty} f_n(x)$$

converges almost everywhere on E*, and the sum defines a Lebesgue integrable function* F*. Moreover,*

$$\int_E \left(\sum_{n=1}^{\infty} f_n \right) dm = \sum_{n=1}^{\infty} \int_E f_n\, dm.$$

Proof. (i) Define the function

$$G(x) := \sum_{n=1}^{\infty} |f_n(x)|.$$

Since $|f_n(x)| \geq 0$, the sequence of partial sums is increasing. By the Monotone Convergence Theorem:

$$\int_E G\, dm = \sum_{n=1}^{\infty} \int_E |f_n|\, dm.$$

(ii) If G is Lebesgue integrable, then $G(x) < \infty$ almost everywhere, so the series $\sum_{n=1}^{\infty} f_n(x)$ converges almost everywhere. Let $F(x) = \sum_{n=1}^{\infty} f_n(x)$ denote the pointwise sum, and define the partial sums

$$F_n(x) = \sum_{k=1}^{n} f_k(x).$$

Then $F_n(x) \to F(x)$ almost everywhere and

$$|F_n(x)| \leq \sum_{k=1}^{n} |f_k(x)| \leq G(x).$$

Thus, by the Dominated Convergence Theorem:

$$\int_E F\, dm = \lim_{n\to\infty} \int_E F_n\, dm = \lim_{n\to\infty} \sum_{k=1}^{n} \int_E f_k\, dm = \sum_{k=1}^{\infty} \int_E f_k\, dm.$$

■

This example illustrates that **the class of Lebesgue integrable functions is strictly larger** than the class of Riemann integrable functions:

$$R(a,b) \subsetneq L^1(a,b).$$

Riemann and Lebesgue Integrals

We now explore an important question: When is a function Riemann integrable? And how does this relate to Lebesgue integrability?

Theorem 84 (Characterization of Riemann Integrability) *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if it is continuous almost everywhere on $[a,b]$; that is,*

$$f \in R[a,b] \quad \Longleftrightarrow \quad m\big(\{x \in [a,b] : f \text{ is not continuous at } x\}\big) = 0.$$

Intuition: A bounded function can be Riemann integrated as long as its discontinuities are "rare"—specifically, they must form a set of measure zero. If the function is continuous almost everywhere, the Riemann integral exists.

Key Fact: If f is Riemann integrable on $[a,b]$, then:

- f is measurable,
- f is Lebesgue integrable,
- and the integrals are equal:

$$\int_a^b f(x)\,dx = \int_{[a,b]} f\,dm.$$

We previously defined the notation $\int_a^b f$ to mean the Riemann integral of f. However, since the Riemann and Lebesgue integrals agree for Riemann integrable functions , we now redefine the expression

$$\int_a^b f(x)dx$$

to denote the Lebesgue integral.

Definition 85 (Lebesgue Integral Notation) *Let $f : (a,b) \to \mathbb{R}$ be a Lebesgue measurable function. Then:*

- *$\int_a^b f(x)\,dx$ or $\int_a^b f$ denotes the Lebesgue integral over (a,b),*

$$\int_a^b f(x)dx := \int_{(a,b)} f\,dm,$$

  where m is the Lebesgue measure.

- *If $a > b$, we define the integral as*

$$\int_a^b f := -\int_b^a f,$$

  so that useful properties like

$$\int_a^b f = \int_a^c f + \int_c^b f$$

  hold for any $a < c < b$.

Conclusion: Riemann integrable functions are also Lebesgue integrable. But the opposite is not always true — some functions can be Lebesgue integrable but not Riemann integrable.

Example 86 (Lebesgue Integrable, Not Riemann Integrable) *Define:*

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 0, & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

This function is not Riemann integrable because it is discontinuous at every point in $[0,1]$. *But it is Lebesgue integrable, and we have:*

$$\int_0^1 f(x)\,dx = 0.$$

Improper Integrals

One advantage of the Lebesgue integral over the Riemann integral is that it can be defined over unbounded domains, as long as the function is measurable.

However, this does not guarantee that the integral is finite. For example, consider the constant function:

$$f(x) = 1 \quad \text{for all } x \in \mathbb{R}.$$

This function is measurable, but its Lebesgue integral over $\mathbb{R}$ diverges:

$$\int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} 1\,dx = \infty.$$

Thus, f is not Lebesgue integrable on $\mathbb{R}$.

In contrast, the Riemann integral is only defined on bounded intervals. To handle unbounded domains or functions, it must be extended using limits, leading to the concept of **improper integrals**.

On bounded intervals, if a function is Riemann integrable, then its Riemann and Lebesgue integrals agree. But on unbounded, the agreement may break down. In many cases, a function that is improperly Riemann integrable is also Lebesgue integrable. However, this is not always true. Below, we present examples that illustrate when the two approaches agree and when they differ.

(i) Measurability of Improperly Riemann Integrable Functions

Suppose the improper integral

$$\int_a^\infty f(x)\,dx$$

converges in the Riemann sense. Then f is Riemann integrable on every finite interval $[a,b]$ for all $b > a$. Since Riemann integrability implies Lebesgue integrability on compact intervals, it follows that $f \in L^1([a,b])$ for all $b > a$.

Moreover, we can express f as the pointwise limit:

$$f(x) = \lim_{n\to\infty} f(x) \cdot \chi_{[a,n]}(x),$$

where $\chi_{[a,n]}$ is the indicator function of the interval $[a,n]$. Each function $f \cdot \chi_{[a,n]}$ is measurable, and the pointwise limit of measurable functions is measurable. Hence, f is measurable on $[a,\infty)$.

(ii) Nonnegative Functions and the Monotone Convergence Theorem

Suppose $f \geq 0$ on $[a,\infty)$, and the improper Riemann integral

$$\int_a^\infty f(x)\,dx$$

converges. Define the sequence of functions $f_n = f \cdot \chi_{[a,n]}$. Then $f_n \nearrow f$ pointwise, and by the Monotone Convergence Theorem:

$$\int_{[a,\infty)} f\,dm = \lim_{n\to\infty} \int_{[a,n]} f\,dm = \lim_{n\to\infty} \int_a^n f(x)\,dx = \int_a^\infty f(x)\,dx. \tag{11.27}$$

(iii) Lebesgue Integrability Implies Convergence of the Improper Riemann Integral

Assume $f \in R(a,b)$ for all $b > a$, and that $f \in L^1([a,\infty))$. Then the improper Riemann integral $\int_a^\infty f(x)\,dx$ converges, and:

$$\int_a^\infty f(x)\,dx = \int_{[a,\infty)} f\,dm.$$

Proof. Let (b_n) be a sequence such that $b_n \to \infty$, and define $f_n = f \cdot \chi_{[a,b_n]}$. Then $f_n \to f$ pointwise, and $|f_n| \le |f| \in L^1([a,\infty))$. By the Dominated Convergence Theorem:

$$\int_{[a,\infty)} f\,dm = \lim_{n\to\infty} \int_{[a,b_n]} f\,dm = \lim_{n\to\infty} \int_a^{b_n} f(x)\,dx = \int_a^\infty f(x)\,dx.$$

(iv) Absolute Convergence Implies Agreement of Integrals

Suppose $f \in R(a,b)$ for all $b > a$, and:

$$\int_a^\infty |f(x)|\,dx < \infty.$$

Then $f \in L^1([a,\infty))$, and by part (iii), the improper Riemann integral exists and:

$$\int_a^\infty f(x)\,dx = \int_{[a,\infty)} f\,dm.$$

Conclusion

If both of the following improper integrals exist:

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty |f(x)|\,dx,$$

then the Lebesgue integral $\int_{[a,\infty)} f\,dm$ also exists and agrees with the improper Riemann integral.

- The convergence of $\int_a^\infty f(x)\,dx$ implies $f \in R(a,b)$ for all $b > a$.
- The convergence of $\int_a^\infty |f(x)|\,dx$ implies $f \in L^1([a,\infty))$.

Therefore,

$$\int_a^\infty f(x)\,dx = \int_{[a,\infty)} f\,dm.$$

Example 87 *Consider some improper Riemann integrals and investigate whether they agree with the Lebesgue integral.*

1. Let consider the function $f : (0,1] \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{\sqrt{x}}.$$

This function is improperly Riemann integrable over $(0,1]$ with a value of 2. We now demonstrate that it is also Lebesgue integrable over this region with the same value. Since f is continuous, it is measurable, so asking if it is Lebesgue integrable is valid.

To compute the Lebesgue integral, let us define a sequence of functions f_n where $f_n : (0,1] \to \mathbb{R}$ is given by

$$f_n = f \cdot 1_{[\frac{1}{n},1]}.$$

Here, f_n is a pointwise increasing sequence whose limit is f. By the Monotone Convergence Theorem (MCT), we have:

$$\int_{(0,1]} f\,d\mu = \int_{(0,1]} \lim_{n\to\infty} f_n\,d\mu = \lim_{n\to\infty} \int_{(0,1]} f_n\,d\mu = \lim_{n\to\infty} \int_{[\frac{1}{n},1]} f\,d\mu.$$

For any $n \in \mathbb{N}$, f is continuous on the compact domain $\left[\frac{1}{n}, 1\right]$, so the Lebesgue integral over this interval equals the Riemann integral. Using the Fundamental Theorem of Calculus, we find

$$\int_{\left[\frac{1}{n},1\right]} f\,d\mu = \int_{1/n}^{1} \frac{1}{\sqrt{x}}\,dx = 2 - \frac{2}{\sqrt{n}}.$$

Substituting into the earlier limit, we obtain

$$\int_{(0,1]} f\,d\mu = \lim_{n\to\infty}\left(2 - \frac{2}{\sqrt{n}}\right) = 2 = \int_0^1 f(x)\,dx.$$

Thus, for this function, the improper Riemann integral agrees with the Lebesgue integral.

2. Consider the function $f : [0,\infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{\sin(x)}{x} \quad \textit{for } x \neq 0, \quad \textit{and} \quad f(0) = 1.$$

Since f is continuous, it is measurable. To compute its Lebesgue integral, we decompose it into positive and negative parts. Define sets E and F where f is non-negative and non-positive, respectively:

$$E = \bigcup_{\substack{n\in\mathbb{N}\\ n\ odd}} [(n-1)\pi, n\pi], \quad F = \bigcup_{\substack{n\in\mathbb{N}\\ n\ even}} [(n-1)\pi, n\pi].$$

The positive and negative parts of f are then given by:

$$f^+(x) = \frac{\sin(x)}{x}\cdot 1_E(x), \quad f^-(x) = -\frac{\sin(x)}{x}\cdot 1_F(x).$$

To evaluate the entire Lebesgue integral, we separately integrate these parts. Focusing on the positive part f^+, we split the domain into smaller compact intervals. Since f is continuous over each compact interval in E (and hence Riemann integrable there), we can compute the Lebesgue integral as a Riemann integral. Since $\sin(x)$ is non-negative over each interval in E, we have:

$$\int_{[0,\infty)} f^+\,d\mu = \int_E \frac{\sin(x)}{x}\,dx = \sum_{\substack{n\in\mathbb{N}\\ n\ odd}} \int_{(n-1)\pi}^{n\pi} \frac{\sin(x)}{x}\,dx.$$

By approximating the lower bound of $\sin(x)$ over each interval, we find

$$\int_{[0,\infty)} f^+\,d\mu \geq \sum_{\substack{n\in\mathbb{N}\\ n\ odd}} \int_{(n-1)\pi}^{n\pi} \frac{\sin(x)}{n\pi}\,dx = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}.$$

However, this sum diverges to ∞ by comparison with the harmonic series. Similarly, the integral of the negative part of f, namely $\int_{[0,\infty)} f^-\,d\mu$, also diverges to ∞. Therefore, we encounter an indeterminate $\infty - \infty$ case for the Lebesgue integral, implying that this function is not Lebesgue integrable.

On the other hand, if we use the Riemann integral, the unbounded domain requires us to apply the concept of the improper Riemann integral. This improper integral can be defined as the limit of the integral function $I : [0,\infty) \to \mathbb{R}$, given by

$$I(t) = \int_0^t f(x)\,dx \quad \textit{as} \quad t \to \infty.$$

For $t > 1$, we can apply integration by parts to compute $I(t)$:

$$I(t) = \int_0^1 f(x)\,dx + \int_1^t \frac{\sin(x)}{x}\,dx.$$

Applying integration by parts to the second integral, we get:

$$I(t) = \int_0^1 f(x)\,dx + \left[-\frac{\cos(x)}{x}\right]_1^t - \int_1^t \frac{\cos(x)}{x^2}\,dx.$$

This simplifies to

$$I(t) = \int_0^1 f(x)\,dx - \frac{\cos(t)}{t} + \cos(1) - \int_1^t \frac{\cos(x)}{x^2}\,dx.$$

We observe that the Riemann integral $\int_0^1 f(x)\,dx$ *exists because the integrand* f *is continuous over the compact interval* $[0,1]$. *Furthermore, the limits*

$$\lim_{t\to\infty} \frac{\cos(t)}{t} = 0 \quad \textit{and} \quad \lim_{t\to\infty} \int_1^t \frac{\cos(x)}{x^2}\,dx$$

both exist. To check the improper Riemann integrability of the function $f : [1,\infty) \to \mathbb{R}$ *defined by*

$$f(x) = \frac{\cos(x)}{x^2}$$

over $[1,\infty)$, *we note that this function has mixed signs, meaning we cannot directly apply the comparison test. To proceed, we split* f *into its positive and negative parts. Specifically, we define* f^+ *and* $f^- : [1,\infty) \to \mathbb{R}$ *as follows:*

$$f^+ = \max(f,0), \quad f^- = -\min(f,0),$$

so that $f = f^+ - f^-$.

Both f^+ *and* f^- *are non-negative and continuous. We can therefore apply the direct comparison test to* f^+. *Observe that*

$$0 \le f^+(x) \le \frac{|\cos(x)|}{x^2} \le \frac{1}{x^2}.$$

Since $\frac{1}{x^2}$ *is improperly Riemann integrable over* $[1,\infty)$, *as shown in Example 16.4.4(3), it follows by direct comparison that the improper integral*

$$\int_1^\infty f^+(x)\,dx$$

exists. By a similar argument, the improper Riemann integral

$$\int_1^\infty f^-(x)\,dx$$

also exists.

For any finite $t > 1$, *we have*

$$\int_1^t f(x)\,dx = \int_1^t f^+(x)\,dx - \int_1^t f^-(x)\,dx.$$

Applying the algebra of limits, we get:

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t f(x)\,dx = \lim_{t\to\infty} \left(\int_1^t f^+(x)\,dx - \int_1^t f^-(x)\,dx\right).$$

This simplifies to

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t f^+(x)\,dx - \lim_{t\to\infty} \int_1^t f^-(x)\,dx,$$

which exists because both limits on the right-hand side exist.

Thus, we conclude that the function f *is improperly Riemann integrable over* $[1,\infty)$.

Therefore, by the algebra of limits, the improper integral

$$\lim_{t\to\infty} I(t) = \int_0^\infty f(x)\,dx$$

exists. Hence, the function f *is Riemann integrable over* $\mathbb{R}$ *in the improper sense, even though it is not Lebesgue integrable.*

The next step is to recall the Direct Comparison Test for Improper Riemann Integrals. Let $I = [a, \infty)$. Suppose that $f, g : I \to \mathbb{R}$ are continuous non-negative functions such that $0 \le f(x) \le g(x)$ for all $x \in I$.

1. If $\int_a^\infty g(x)\,dx$ exists, then $\int_a^\infty f(x)\,dx$ also exists.

2. If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ also diverges.

A similar result can be proven for $I = (-\infty, a]$ and improper integrals over this domain.

Proof. We prove each assertion separately. Since f and g are continuous over $[a, \infty)$, these functions are Riemann integrable over the interval $[a, t]$ for any finite $t > a$.

1. Since f and g are non-negative, by the ordering property and additivity of integrals, for any $t \ge a$ we have

$$\int_a^t f(x)\,dx \le \int_a^t g(x)\,dx \le \lim_{t\to\infty} \int_a^t g(x)\,dx = \int_a^\infty g(x)\,dx.$$

Moreover, the integral function $F(t) = \int_a^t f(x)\,dx$ on $[a, \infty)$ is an increasing function. Thus, the limit of $F(t)$ as $t \to \infty$ exists since $F(t)$ is bounded by the finite number $\int_a^\infty g(x)\,dx$.

2. For any $t \ge a$, we have the ordering

$$\int_a^t f(x)\,dx \le \int_a^t g(x)\,dx.$$

Taking the limit $t \to \infty$ on both sides, since $\int_a^\infty f(x)\,dx$ diverges, it must approach ∞. Thus, we conclude that $\lim_{t\to\infty} \int_a^t g(x)\,dx = \infty$, implying that $\int_a^\infty g(x)\,dx$ also diverges.

Limit Comparison Test for Improper Riemann Integrals:
Let $I = [a, \infty)$. Suppose that $f, g : I \to \mathbb{R}$ are continuous positive functions. Suppose further that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

for some $0 < L < \infty$. Then either both improper Riemann integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ exist, or both diverge. In other words,

$$\int_a^\infty f(x)\,dx \text{ exists} \iff \int_a^\infty g(x)\,dx \text{ exists.}$$

■

The idea behind the Limit Comparison Test is similar to the comparison test for series. If f and g exhibit similar behavior asymptotically (up to a constant scale L) as $x \to \infty$, then the convergence or divergence of $\int_a^\infty f(x)\,dx$ is equivalent to that of $\int_a^\infty g(x)\,dx$.

Lecture 9: Examples

Example 88 *Evaluate*

$$\lim_{n\to\infty}\int_0^1 \frac{nx}{1+n^2x^2}\,dx.$$

Solution. *For each $n \in \mathbb{N}$, define the sequence of functions*

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad x \in [0,1].$$

Observe that as $n \to \infty$, $f_n(x)$ converges pointwise to 0 for all $x \in (0,1]$, since the term n^2x^2 in the denominator dominates as n becomes large, driving $f_n(x)$ towards 0. At $x = 0$, the function value is also 0 for all n, so $f_n(x)$ converges pointwise to 0.

To understand the behavior of $f_n(x)$ on $[0,1]$, we find the maximum value of $f_n(x)$. Taking the derivative, we see that $f_n(x)$ attains its maximum at $x = \frac{1}{n}$. Evaluating f_n at this point, we get

$$f_n\left(\frac{1}{n}\right) = \frac{n\cdot\frac{1}{n}}{1+n^2\cdot\left(\frac{1}{n}\right)^2} = \frac{1}{2}.$$

Thus,

$$\sup_{x\in[0,1]}|f_n(x)| = \frac{1}{2},$$

showing that the convergence $f_n \to 0$ is not uniform on $[0,1]$.

Since the convergence is not uniform, we cannot interchange the limit and the integral directly using properties of the Riemann integral. However, we can consider this as a Lebesgue integral and apply the Bounded Convergence Theorem. Each $f_n(x)$ is bounded and measurable, and $f_n(x) \to 0$ pointwise. Thus, by the Bounded Convergence Theorem,

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0.$$

Example 89 *In many problems, one often needs to use the following upper bound:*

$$\left(1+\frac{x}{n}\right)^n \le e^x$$

which holds for all

$$n \ge 1 \quad \text{and} \quad x > -n.$$

This bound must be proved; one cannot simply refer to a calculus or advanced calculus text where this fact may have been mentioned.

To prove it, we take the logarithm of both sides in the inequality and convert it as follows:

$$n\ln\left(1+\frac{x}{n}\right) \le x.$$

Define $t = \frac{x}{n} + 1$, then $t > 0$ due to the condition $x > -n$. The inequality becomes

$$\ln t \le t - 1, \quad \forall t > 0,$$

which is a well-known inequality that can be used here.

Additionally, with a bit more effort, we can show that the sequence

$$a_n = \left(1+\frac{x}{n}\right)^n$$

is monotonically increasing in n for all n satisfying $x > -n$. To prove this, we treat n as a continuous variable and take the derivative:

$$\frac{da_n}{dn} = a_n\left(\ln\left(1+\frac{x}{n}\right) - \frac{x}{x+n}\right).$$

Since $a_n > 0$, we have $\frac{da_n}{dn} \geq 0$ if and only if

$$\ln\left(1+\frac{x}{n}\right) \geq \frac{x}{x+n},$$

or equivalently,

$$\frac{n}{x+n}\ln\left(1+\frac{x}{n}\right) \leq \frac{x}{x+n}.$$

Simplifying further, we find that

$$\ln\left(1+\frac{x}{n}\right) \leq \frac{x}{x+n},$$

which implies the desired result for the monotonicity of a_n.

Evaluate

$$\lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n e^{-2x}\,dx.$$

Proof. *To express this limit as a Lebesgue integral, define the sequence of functions*

$$f_n(x) = \chi_{[0,n]}(x)\cdot\left(1-\frac{x}{n}\right)^n e^{-2x},$$

where $\chi_{[0,n]}(x)$ is the characteristic function of the interval $[0,n]$. This gives us

$$\int_0^n \left(1-\frac{x}{n}\right)^n e^{-2x}\,dx = \int_{[0,\infty)} f_n(x)\,d\mu.$$

Let's analyze the behavior of $f_n(x)$ as $n\to\infty$. For a fixed x,

$$\left(1-\frac{x}{n}\right)^n \to e^{-x} \quad \text{as } n\to\infty.$$

Therefore, $f_n(x)\to e^{-x}\cdot e^{-2x} = e^{-3x}$ pointwise on $[0,\infty)$.

Since $f_n(x)$ converges pointwise to e^{-3x}, and $f_n(x)\leq e^{-x}$ for all $x\in[0,\infty)$, we can apply the Dominated Convergence Theorem, using $g(x)=e^{-x}$ as a dominating function, which is integrable over $[0,\infty)$. Thus,

$$\lim_{n\to\infty}\int_{[0,\infty)} f_n(x)\,d\mu = \int_{[0,\infty)}\lim_{n\to\infty} f_n(x)\,d\mu = \int_{[0,\infty)} e^{-3x}\,dx.$$

Evaluating this integral, we have

$$\int_{[0,\infty)} e^{-3x}\,dx = \left[-\frac{e^{-3x}}{3}\right]_0^\infty = \frac{1}{3}.$$

Thus,

$$\lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)^n e^{-2x}\,dx = \frac{1}{3}.$$

■

Example 90 *Prove that*

$$\int_0^1 \left(\frac{\log x}{1-x}\right)^2 dx = 2\sum_{n=1}^\infty \frac{1}{n^2}.$$

Solution. *For every $x\in(-1,1)$, we have the power series representation*

$$\frac{1}{1-x} = \sum_{k=0}^\infty x^n.$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{(1-x)^2} = \sum_{n=1}^\infty n x^{n-1}.$$

We can now express the integral as

$$\int_0^1 \left(\frac{\log x}{1-x}\right)^2 dx = \sum_{n=1}^{\infty} n \int_0^1 x^{n-1} (\log x)^2 \, dx.$$

Let's focus on evaluating each integral on the right-hand side.

When $n = 1$, *the integral* $\int_0^1 (\log x)^2 \, dx$ *is improper, as* $\log x$ *diverges at* $x = 0$. *For* $n > 1$, *however, the integrals* $\int_0^1 x^{n-1} (\log x)^2 \, dx$ *are Riemann integrals. In both cases, we use integration by parts to evaluate the integrals, employing L'Hôpital's rule to handle any indeterminate forms.*

Consider

$$\int_0^1 x^{n-1} (\log x)^2 \, dx.$$

Using integration by parts, set $u = (\log x)^2$ *and* $dv = x^{n-1} dx$, *giving* $du = \frac{2 \log x}{x} dx$ *and* $v = \frac{x^n}{n}$. *Then*

$$\int_0^1 x^{n-1} (\log x)^2 \, dx = \left. \frac{x^n}{n} (\log x)^2 \right|_0^1 - \int_0^1 \frac{2x^n \log x}{n} \, dx.$$

Evaluating the boundary term, we find

$$\left. \frac{x^n}{n} (\log x)^2 \right|_0^1 = 0.$$

This gives

$$\int_0^1 x^{n-1} (\log x)^2 \, dx = -\frac{2}{n} \int_0^1 x^n \log x \, dx.$$

Applying integration by parts again to $\int_0^1 x^n \log x \, dx$, *with* $u = \log x$ *and* $dv = x^n dx$, *we get* $du = \frac{1}{x} dx$ *and* $v = \frac{x^{n+1}}{n+1}$. *Thus,*

$$\int_0^1 x^n \log x \, dx = \left. \frac{x^{n+1}}{n+1} \log x \right|_0^1 - \int_0^1 \frac{x^{n+1}}{n+1} \cdot \frac{1}{x} \, dx.$$

The boundary term again vanishes, so we are left with

$$\int_0^1 x^n \log x \, dx = -\frac{1}{(n+1)^2}.$$

Substituting back, we find

$$\int_0^1 x^{n-1} (\log x)^2 \, dx = \frac{2}{n^3}.$$

Therefore,

$$\int_0^1 \left(\frac{\log x}{1-x}\right)^2 dx = \sum_{n=1}^{\infty} n \cdot \frac{2}{n^3} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This completes the proof.

Example 91 *Prove that*

$$\int_0^1 \sin x \log x \, dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n)!}.$$

We start by expanding $\sin x$ *as a power series:*

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

Substituting this expansion into the integral, we get:

$$\int_0^1 \sin x \log x \, dx = \int_0^1 \left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \right) \log x \, dx.$$

Next, we justify the interchange of the sum and integral by checking the absolute convergence:

$$\int_0^1 \sum_{k=0}^{\infty} \left| \frac{(-1)^k x^{2k+1} \log x}{(2k+1)!} \right| \, dx = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \int_0^1 x^{2k+1} |\log x| \, dx.$$

We compute each integral $\int_0^1 x^{2k+1} \log x \, dx$ *using integration by parts. Setting* $u = \log x$ *and* $dv = x^{2k+1} \, dx$*, we find:*

$$\int_0^1 x^{2k+1} \log x \, dx = -\frac{1}{(2k+2)^2}.$$

Thus

$$\int_0^1 \sum_{k=0}^{\infty} \left| \frac{(-1)^k x^{2k+1} \log x}{(2k+1)!} \right| \, dx = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!(2k+1)^2} < \infty.$$

By the Theorem of Term-by-Term Integration, we can interchange the sum and the integral:

$$\begin{aligned} \int_0^1 \sin x \log x \, dx &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^1 x^{2k+1} \log x \, dx. \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(-\frac{1}{(2k+2)^2} \right). \end{aligned}$$

Simplifying, we obtain:

$$\int_0^1 \sin x \log x \, dx = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k(2k)!}.$$