Math 481: Course Notes (2025–2026)

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Chapter 1

Real Numbers

Supremum and Infimum

Bounded Sets

A subset $A \subset \mathbb{R}$ is:

- *bounded above* if there exists $K \in \mathbb{R}$ such that $x \leq K$ for all $x \in A$;
- *bounded below* if there exists $K \in \mathbb{R}$ such that $x \geq K$ for all $x \in A$;
- *bounded* if it is both bounded above and bounded below.

Remarks.

1. A is bounded if and only if there exists $M \geq 0$ such that $|x| \leq M$ for all $x \in A$.
2. A sequence is bounded above (resp. below) if and only if the set of its values is bounded above (resp. below).

Supremum and Infimum

Let $A \subset \mathbb{R}$.

- The *supremum* $\sup(A)$ is the *least upper bound* of A:
    1. $\sup(A)$ is an upper bound of A;
    2. if K' is any other upper bound of A, then $\sup(A) \leq K'$.
- The *infimum* $\inf(A)$ is the *greatest lower bound*, defined analogously.

If they exist, $\sup(A)$ and $\inf(A)$ are unique.

Examples

1. Closed interval: $\sup([a,b]) = b$, $\inf([a,b]) = a$.
2. Open interval: $\sup((a,b)) = b$, $\inf((a,b)) = a$. Proof: b is an upper bound. If K is any upper bound, take $x_n = b - 2^{-n}(b-a) \in (a,b)$. Then $x_n \le K$ and $x_n \to b$, so $b \le K$.
3. $A = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$: $\sup(A) = 1$.
4. $A = \left\{\frac{n^2}{2^n} : n \in \mathbb{N}\right\}$, $\sup(A) = \frac{9}{8}$ because $\frac{n^2}{2^n} \le 1 < \frac{9}{8}$ for $n \ne 3$, and $\frac{3^2}{2^3} = \frac{9}{8} \in A$.

Theorem 1 (ε-characterizations) *Let $A \subset \mathbb{R}$ be nonempty and bounded above/below.*

a) $\alpha = \sup A$ *iff*

$$\textit{(i) } x \le \alpha \ \forall x \in A \quad \textit{and} \quad \textit{(ii) } \forall \varepsilon > 0 \ \exists x_\varepsilon \in A : \ \alpha - \varepsilon < x_\varepsilon \le \alpha.$$

b) $\beta = \inf A$ *iff*

$$\textit{(i) } \beta \le x \ \forall x \in A \quad \textit{and} \quad \textit{(ii) } \forall \varepsilon > 0 \ \exists y_\varepsilon \in A : \ \beta \le y_\varepsilon < \beta + \varepsilon.$$

Proof. We prove (a); (b) is analogous. ($\Rightarrow$) If $\alpha = \sup A$, then (i) holds by definition. If (ii) failed, there would be $\varepsilon_0 > 0$ such that $x \le \alpha - \varepsilon_0$ for all $x \in A$, so $\alpha - \varepsilon_0$ would be an upper bound—contradiction to leastness. ($\Leftarrow$) Assume (i)–(ii). Let U be any upper bound of A. From (ii), $\alpha - \varepsilon < x_\varepsilon \le U$ for all $\varepsilon > 0$, hence $\alpha \le U$. Thus α is the least upper bound. ∎

Theorem 2 (Completeness Axiom) *Every nonempty subset $A \subset \mathbb{R}$ that is bounded above has a least upper bound* $\sup A \in \mathbb{R}$*. Equivalently, every nonempty subset that is bounded below has a greatest lower bound* $\inf A \in \mathbb{R}$*.*

Remarks on the Completeness Axiom

- **Failure in $\mathbb{Q}$.** The set $A = \{q \in \mathbb{Q} : q^2 < 2\}$ is nonempty and bounded above in $\mathbb{Q}$, but it has no supremum in $\mathbb{Q}$ (its least upper bound in $\mathbb{R}$ is $\sqrt{2} \notin \mathbb{Q}$).
- **Approximating $\sup A$ and $\inf A$ by points of A.** If $A \ne \varnothing$ is bounded above, there exists an increasing sequence $(a_n) \subset A$ with $a_n \uparrow \sup A$. *Construction:* choose $a_n \in A$ with $\sup A - \frac{1}{n} < a_n \le \sup A$ and set $s_n = \max\{a_1, \dots, a_n\}$. Then $s_n \in A$, s_n is increasing, and $s_n \to \sup A$. Similarly, if A is bounded below, there exists a decreasing $(b_n) \subset A$ with $b_n \downarrow \inf A$.
- **Unbounded sets and extended reals.** If A is unbounded above (resp. below), we set $\sup A = +\infty$ (resp. $\inf A = -\infty$) in the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.
- **Maxima and minima.** If $\sup A \in A$, then $\max A := \sup A$ is the (unique) *maximum* of A. If $\inf A \in A$, then $\min A := \inf A$ is the (unique) *minimum* of A.

Basic consequences.

- **Monotone Convergence for sequences.** If (a_n) is monotone and bounded, then (a_n) converges in $\mathbb{R}$. *Sketch:* If a_n is increasing and bounded above, set $L = \sup\{a_n : n \in \mathbb{N}\}$. Then for every $\varepsilon > 0$ there exists N with $a_N > L - \varepsilon$, hence $L - \varepsilon < a_n \le L$ for all $n \ge N$, so $a_n \to L$.

- **Riemann integral (existence of best bounds).** For a bounded function f on $[a, b]$, define the lower and upper sums over a partition P by $L(f, P)$ and $U(f, P)$. Completeness guarantees the numbers

$$L^*(f) = \sup_P L(f, P), \qquad U_*(f) = \inf_P U(f, P)$$

exist in $\mathbb{R}$. We call f Riemann integrable when $L^*(f) = U_*(f)$, and this common value is $\int_a^b f$. Without completeness, these sup / inf might not exist, and the definition would not be rigorous.

- **Measure and Lebesgue integral (built from inf / sup).** For an arbitrary set $E \subset \mathbb{R}$,

$$m^*(E) = \inf\Big\{ \sum_{k=1}^{\infty} |I_k| : \ E \subset \bigcup_{k=1}^{\infty} I_k, \ I_k \text{ intervals} \Big\}$$

exists by completeness. For a nonnegative measurable f,

$$\int f \, d\mu = \sup\Big\{ \int \varphi \, d\mu : \ 0 \le \varphi \le f, \ \varphi \text{ simple} \Big\}$$

also exists for the same reason. Thus both measure and integral rely essentially on the least-upper-bound and greatest-lower-bound properties of $\mathbb{R}$.

Conclusion. Completeness is not a cosmetic axiom: it is the structural feature of $\mathbb{R}$ that ensures limits, integrals, and measure-theoretic constructions are well defined.

Limsup and Liminf of a Sequence

Limit points (subsequential limits) and Bolzano–Weierstrass

Definition 3 *Let (a_n) be a real sequence and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. A number $L \in \overline{\mathbb{R}}$ is a* limit point *(or* subsequential limit*) of (a_n) if some subsequence (a_{n_k}) satisfies $a_{n_k} \to L$.*

Theorem 4 (Bolzano–Weierstrass) *Every bounded sequence $(a_n) \subset \mathbb{R}$ has a convergent subsequence. Equivalently, every bounded sequence has at least one* real *limit point.*

Corollary 5 *For a real sequence (a_n):*

1. *(a_n) is bounded $\iff$ all its limit points lie in $\mathbb{R}$ (i.e. no $\pm\infty$).*
2. *(a_n) converges in $\mathbb{R}$ to L $\iff$ it has exactly one limit point in $\mathbb{R}$ (and none at $\pm\infty$).*

3. *If (a_n) is bounded, then the set of real limit points is nonempty and compact; moreover*

$$\limsup_{n\to\infty} a_n = \sup \operatorname{Lim}(a_n), \qquad \liminf_{n\to\infty} a_n = \inf \operatorname{Lim}(a_n).$$

Proof sketch of Bolzano–Weierstrass. By boundedness, (a_n) lies in a compact interval $[m, M]$. The nested-interval (or bisection) argument produces a subsequence contained in successively smaller closed intervals with lengths tending to 0; the unique point in the intersection is the subsequential limit. ■

Definition 6 *For a real sequence (a_n) define the tail sup/inf*

$$b_n := \sup_{k\geq n} a_k, \qquad c_n := \inf_{k\geq n} a_k \qquad (n \in \mathbb{N}).$$

Then (b_n) is nonincreasing (possibly $+\infty$) and (c_n) is nondecreasing (possibly $-\infty$), so the limits

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} b_n, \qquad \liminf_{n\to\infty} a_n := \lim_{n\to\infty} c_n$$

always exist in $\overline{\mathbb{R}}$.

Equivalently, if $\operatorname{Lim}(a_n)$ denotes the set of all (finite) limit points of (a_n) in $\mathbb{R}$, then

$$\limsup_{n\to\infty} a_n = \sup \operatorname{Lim}(a_n), \qquad \liminf_{n\to\infty} a_n = \inf \operatorname{Lim}(a_n),$$

with the convention that $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$ if there are no finite limit points.

Basic properties and characterizations

Theorem 7 *Let (a_n) be a real sequence and set $A := \liminf a_n$, $B := \limsup a_n$ (in $\overline{\mathbb{R}}$).*

1. *$A \leq B$.*

2. *(a_n) converges in $\mathbb{R}$ to L iff $A = B = L \in \mathbb{R}$.*

3. *(Quantified bounds) For $x \in \mathbb{R}$:*
   - *If $x > B$, then $\exists N\ \forall n \geq N:\ a_n \leq x$.*
   - *If $x < B$, then $\forall N\ \exists n \geq N:\ a_n > x$.*
   - *If $x < A$, then $\exists N\ \forall n \geq N:\ a_n \geq x$.*
   - *If $x > A$, then $\forall N\ \exists n \geq N:\ a_n < x$.*

4. *(Epsilon–test for $\limsup$) If $B \in \mathbb{R}$ then $\limsup a_n = B$ iff*

$$\forall \varepsilon > 0: \quad \Big(\exists N\ \forall n \geq N:\ a_n < B + \varepsilon\Big) \text{ and } \Big(\forall N\ \exists n \geq N:\ a_n > B - \varepsilon\Big).$$

   (The analogous statement holds for $\liminf$.)

Examples

1. $a_n = (-1)^n$: $\limsup a_n = 1$, $\liminf a_n = -1$ (two limit points).
2. $a_n = n$: $\limsup a_n = \liminf a_n = +\infty$ (diverges to $+\infty$).
3. $a_n = \frac{1}{n}$: $\limsup a_n = \liminf a_n = 0$ (converges to 0).
4. $a_n = (-1)^n\left(1 + \frac{1}{n}\right)$: $\limsup a_n = 1$, $\liminf a_n = -1$.

Useful consequences

Proposition 8 *If $(b_n) \to b > 0$, then*

$$\liminf_{n\to\infty}(a_n b_n) = b \liminf_{n\to\infty} a_n, \qquad \limsup_{n\to\infty}(a_n b_n) = b \limsup_{n\to\infty} a_n.$$

For $b < 0$ the equalities hold with liminf/limsup interchanged.

Exercise

1. Show that (a_n) is bounded above iff $+\infty$ is not a (extended) limit point.
2. Prove Bolzano–Weierstrass: every bounded real sequence has a real limit point.
3. Decide whether the identities

   $$\limsup(a_n + b_n) = \limsup a_n + \limsup b_n, \qquad \limsup(a_n b_n) = (\limsup a_n)(\limsup b_n)$$

   hold in general; provide counterexamples if not.

Proposition 9 (Ratio vs. Root for positive sequences) *Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers. Then*

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \;\leq\; \liminf_{n\to\infty} a_n^{1/n} \;\leq\; \limsup_{n\to\infty} a_n^{1/n} \;\leq\; \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}. \tag{1.1}$$

Moreover, if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists in $\mathbb{R}$, then $\lim_{n\to\infty} a_n^{1/n}$ also exists in $\mathbb{R}$ and both limits are equal.

Proof. The middle inequality in (**??**) is obvious.

Last inequality. Let $\lambda := \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$. If $\lambda = \infty$ there is nothing to prove. Assume $\lambda < \infty$. Fix $\rho > \lambda$. Then, for n large enough, $\frac{a_{n+1}}{a_n} \leq \rho$, hence

$$a_n \leq a_N \rho^{n-N} \qquad (n \geq N).$$

It follows that

$$a_n^{1/n} \leq (a_N \rho^{-N})^{1/n} \rho \;\longrightarrow\; \rho \quad (n \to \infty).$$

Thus $\limsup_{n\to\infty} a_n^{1/n} \leq \rho$. Since $\rho > \lambda$ was arbitrary, we obtain

$$\limsup_{n\to\infty} a_n^{1/n} \leq \lambda.$$

First inequality. Let $L := \liminf_{n\to\infty} a_n^{1/n}$. Fix $\varepsilon > 0$. For n large enough we have $a_n^{1/n} \geq L - \varepsilon$, i.e.,

$$a_n \geq (L-\varepsilon)^n.$$

Hence

$$\frac{a_{n+1}}{a_n} \;\geq\; \frac{(L-\varepsilon)^{n+1}}{(L-\varepsilon)^n} = L - \varepsilon.$$

Taking lim inf on both sides yields

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \;\geq\; L - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \;\geq\; \liminf_{n\to\infty} a_n^{1/n}.$$

Conclusion. If the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L \in \mathbb{R}$ exists, then (**??**) forces $\liminf a_n^{1/n} = \limsup a_n^{1/n} = L$, hence $\lim_{n\to\infty} a_n^{1/n} = L$. ∎

Corollary 10 *From* (**??**) *it follows that if*

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1,$$

then

$$\liminf_{n\to\infty} a_n^{1/n} > 1.$$

That is, the ratio divergence criterion *implies the* root divergence criterion.

Remark 11 *By* (**??**)*, whenever* $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$*, we also have* $\limsup_{n\to\infty} a_n^{1/n} < 1$*. Hence there is no example of a sequence for which the ratio convergence criterion holds but the root test does not.*

Real Series

Definition 12 (Convergent Series) *A real series $\sum_{j=1}^{\infty} a_j$ is called a convergent series if the sequence of its partial sums (s_n) converges. We define the value of the series as the limit of its partial sums. In other words, if*

$$\lim_{n\to\infty} s_n = s,$$

we assign the value s to the series:

$$\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} \sum_{j=1}^{n} a_j = \lim_{n\to\infty} s_n = s.$$

Otherwise, if the sequence of the partial sums (s_n) diverges, the series $\sum_{n=1}^{\infty} a_n$ is called a divergent series.

Consider the series $\sum_{j=1}^{\infty} \frac{1}{j(j+1)}$. The terms of this series can be rewritten as:

$$\frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1}, \quad \text{for every } j \in \mathbb{N}.$$

Thus, the series can be expressed as:

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)} = \sum_{j=1}^{\infty} \left(\frac{1}{j} - \frac{1}{j+1} \right).$$

Now, considering the sequence of partial sums (s_n) for this series, many terms cancel out due to the nature of the expression. Specifically, we have:

$$s_n = \sum_{j=1}^{n} \left(\frac{1}{j} - \frac{1}{j+1} \right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

Applying the limit to the sequence of partial sums, we obtain:

$$\lim_{n\to\infty} s_n = 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1.$$

Thus, the sequence of partial sums converges, and we conclude:

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)} = \lim_{n\to\infty} s_n = 1.$$

In general, a real series that can be written in the form $\sum_{j=1}^{\infty}(f(j) - f(j+1))$ for some function $f : \mathbb{N} \to \mathbb{R}$ is known as a *telescoping series.* The partial sums of such a series simplify to:

$$s_n = f(1) - f(n+1), \quad \text{for all } n \in \mathbb{N},$$

which makes the series easier to analyze due to the cancellation of terms.

Proposition 13 *If the real series $\sum_{j=1}^{\infty} a_j$ converges, then $\lim_{j\to\infty} a_j = 0$.*

Proof. Since the series converges, by definition, the sequence of partial sums $s_n = \sum_{j=1}^{n} a_j$ also converges, say $s_n \to s \in \mathbb{R}$. Note that $a_n = s_n - s_{n-1}$. Taking the limit as $n \to \infty$ on both sides and applying the algebra of limits, we get:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

Thus, we have shown that $\lim_{n\to\infty} a_n = 0$, completing the proof. ■

Proposition 14 *Let $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ be convergent real series. Then:*

1. *For any $\lambda \in \mathbb{R}$, the series $\sum_{j=1}^{\infty} \lambda a_j$ converges, and its sum is equal to $\lambda \sum_{j=1}^{\infty} a_j$.*
2. *The series $\sum_{j=1}^{\infty}(a_j + b_j)$ converges, and its sum is equal to $\sum_{j=1}^{\infty} a_j + \sum_{j=1}^{\infty} b_j$.*

Since the convergence of a series is determined by its limiting behavior, we can safely ignore or add any finite number of terms at the beginning of the series without affecting its convergence. This leads to the following proposition:

Proposition 15 *Let $\sum_{j=1}^{\infty} a_j$ be a real series.*

1. *If there exists $N \in \mathbb{N}$ such that the series $\sum_{j=N}^{\infty} a_j$ converges, then the series $\sum_{j=1}^{\infty} a_j$ also converges, and its sum is given by*

$$\sum_{j=1}^{\infty} a_j = \sum_{j=1}^{N-1} a_j + \sum_{j=N}^{\infty} a_j.$$

2. *If the series $\sum_{j=1}^{\infty} a_j$ converges, then for any $N \in \mathbb{N}$, the series $\sum_{j=N}^{\infty} a_j$ also converges.*

Proof. We prove each assertion separately.

1. For $n \geq N$, consider the sequence of partial sums (t_n) where $t_n = \sum_{j=N}^{n} a_j$. Let (s_n) be the sequence of partial sums where $s_n = \sum_{j=1}^{n} a_j$. For $n \geq N$, we have

$$s_n = \sum_{j=1}^{N-1} a_j + t_n = K + t_n,$$

where $K = \sum_{j=1}^{N-1} a_j \in \mathbb{R}$ is a real constant. Since (t_n) converges as $n \to \infty$, by the algebra of limits, we conclude that (s_n) also converges. Moreover, we have

$$\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(K + t_n) = K + \lim_{n\to\infty} t_n = \sum_{j=1}^{N-1} a_j + \sum_{j=N}^{\infty} a_j.$$

2. Fix $N \in \mathbb{N}$ and for $n \geq N$, consider the sequence of partial sums (t_n) where $t_n = \sum_{j=N}^{n} a_j$. Let (s_n) be the sequence of partial sums for the series $\sum_{j=1}^{\infty} a_j$ where $s_n = \sum_{j=1}^{n} a_j$. Then, for any $n \geq N$, we have

$$t_n = s_n - \sum_{j=1}^{N-1} a_j = s_n - K,$$

where $K = \sum_{j=1}^{N-1} a_j \in \mathbb{R}$ is a real constant. Since (s_n) converges, by the algebra of limits, we conclude that the sequence (t_n) also converges.

■

Absolute and Conditional Convergence

Proposition 16 (Cauchy Criterion for Convergence of a Series) *The real series $\sum_{j=1}^{\infty} a_j$ converges if and only if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n > m \geq N$, we have:*

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Definition 17 (Absolute Convergence) *A real series $\sum_{j=1}^{\infty} a_j$ is said to be absolutely convergent if the corresponding series of absolute values, $\sum_{j=1}^{\infty} |a_j|$, converges.*

Definition 18 (Conditional Convergence) *A real series $\sum_{j=1}^{\infty} a_j$ is called conditionally convergent if $\sum_{j=1}^{\infty} a_j$ converges but $\sum_{j=1}^{\infty} |a_j|$ diverges to infinity.*

An important distinction between absolutely convergent and conditionally convergent series is that the terms of an absolutely convergent series can be rearranged without changing the value of the series. However, in the case of a conditionally convergent series, the terms can be rearranged in such a way that the rearranged series converges to any real number in $\mathbb{R}$ or even diverges to $\pm\infty$. This result is known as the *Riemann rearrangement theorem.*

Alternating Series

To illustrate an example of a conditionally convergent series, we define *alternating series.* As the name suggests, an alternating series is a real series where the terms alternate in sign.

Definition 19 (Alternating Series) *A real series is called alternating if it takes one of the following forms:*

$$\sum_{j=1}^{\infty} (-1)^j b_j \quad or \quad \sum_{j=1}^{\infty} (-1)^{j-1} b_j,$$

where $b_j > 0$ for all $j \in \mathbb{N}$.

Theorem 20 (Alternating Series Test) *An alternating series of the form $\sum_{j=1}^{\infty}(-1)^j b_j$ or $\sum_{j=1}^{\infty}(-1)^{j-1} b_j$, with $b_j > 0$, converges if the terms (b_j) are decreasing and $b_j \to 0$.*

Proof. Without loss of generality (WLOG), consider the alternating series of the form $\sum_{j=1}^{\infty}(-1)^{j-1} b_j$, where the first term in the series is positive. Let (s_n) denote the sequence of partial sums. We analyze the subsequences of even-indexed and odd-indexed partial sums, namely (s_{2n}) and (s_{2n-1}).

For the subsequence of even-indexed partial sums, by grouping some consecutive terms together, we have:

$$s_{2n} = b_1 - b_2 + b_3 - b_4 + \cdots + b_{2n-1} - b_{2n} = b_1 - (b_2 - b_3) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n} \leq b_1,$$

since $b_j \geq b_{j+1}$ for all $j \in \mathbb{N}$. Additionally, we observe that:

$$s_{2(n+1)} - s_{2n} = -b_{2n+2} + b_{2n+1} \geq 0,$$

which implies that $s_{2(n+1)} \geq s_{2n}$ for all $n \in \mathbb{N}$. Thus, the subsequence of even-indexed partial sums (s_{2n}) is increasing and bounded above. By the monotone convergence theorem, the subsequence (s_{2n}) converges.

Using similar arguments, we can show that the subsequence of odd-indexed partial sums (s_{2n-1}) is bounded below and decreasing. Therefore, by the monotone convergence theorem, the subsequence (s_{2n-1}) also converges.

Furthermore, since $-b_{2n} = s_{2n} - s_{2n-1}$, taking the limit as $n \to \infty$ and applying the algebra of limits, we obtain:

$$0 = -\lim_{n\to\infty} b_{2n} = \lim_{n\to\infty}(s_{2n} - s_{2n-1}) = \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} s_{2n-1}.$$

Thus, $\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n-1}$, say s. Finally,it is the entire sequence of partial sums (s_n) converges to the same limit s. Hence, the series converges. ■

Comparison Tests

For real sequences, we have seen that limits preserve weak inequalities, as demonstrated by the sandwich lemma. These results can help us compare or bound the limits of a sequence with those of commonly known sequences. We now extend this idea to series. By comparing series that converge or diverge, we can apply these standard examples to test the behavior of other series.

Direct Comparison Test

The first convergence test is the *direct comparison test* for series. The idea is simple and intuitive: suppose we have two series with non-negative terms such that one series is termwise larger than the other. If the series with the larger terms converges, then the series with the smaller terms must also converge. Similarly, if the series with the smaller terms diverges, the series with the larger terms must diverge as well. We state the following proposition:

Proposition 21 (Direct Comparison Test) *Let $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ be two real series such that $0 \leq a_j \leq b_j$ for all $j \in \mathbb{N}$.*

1. *If the series $\sum_{j=1}^{\infty} b_j$ converges, then the series $\sum_{j=1}^{\infty} a_j$ also converges.*
2. *If the series $\sum_{j=1}^{\infty} a_j$ diverges to ∞, then the series $\sum_{j=1}^{\infty} b_j$ diverges to ∞ as well.*

Proof. Let $s_n = \sum_{j=1}^{n} a_j$ and $t_n = \sum_{j=1}^{n} b_j$ be the sequences of partial sums for the series $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$, respectively. Since $a_j, b_j \geq 0$, both sequences (s_n) and (t_n) are increasing. Moreover, the condition $0 \leq a_j \leq b_j$ implies that $0 \leq s_n \leq t_n$ for all $n \in \mathbb{N}$.

We now prove the two assertions separately.

1. Since the series $\sum_{j=1}^{\infty} b_j$ converges, the sequence (t_n) is bounded, say $t_n \leq M$ for all $n \in \mathbb{N}$ and some $M > 0$. Thus, $s_n \leq t_n \leq M$ for every $n \in \mathbb{N}$. By the boundedness of (s_n), the sequence (s_n) must also converge, and therefore the series $\sum_{j=1}^{\infty} a_j$ converges.

2. Since $\sum_{j=1}^{\infty} a_j$ diverges to ∞, the sequence (s_n) diverges to ∞. As $s_n \le t_n$ for all $n \in \mathbb{N}$, the sequence (t_n) must also diverge to ∞. Therefore, the series $\sum_{j=1}^{\infty} b_j$ diverges to ∞.

■

Proposition 22 (Limit Comparison Test) *Let $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ be two real series such that $a_j \ge 0$ and $b_j > 0$ for all $j \in \mathbb{N}$. Suppose that*

$$\lim_{j\to\infty} \frac{a_j}{b_j} = L$$

for some $0 < L < \infty$. Then, either both series converge or both series diverge. In other words:

$$\sum_{j=1}^{\infty} a_j \text{ converges} \quad \Leftrightarrow \quad \sum_{j=1}^{\infty} b_j \text{ converges.}$$

Proof. Since $\lim_{j\to\infty} \frac{a_j}{b_j} = L$, for $\epsilon = \frac{L}{2} > 0$, there exists $N \in \mathbb{N}$ such that

$$\left|\frac{a_j}{b_j} - L\right| < \frac{L}{2} \quad \text{for all } j \ge N.$$

Equivalently, for any $j \ge N$, we have:

$$\frac{L}{2} b_j < a_j < \frac{3L}{2} b_j.$$

We now prove the two implications separately:

- $(\Rightarrow)$:
  Assume that the series $\sum_{j=1}^{\infty} a_j$ converges. Therefore, the series $\frac{2}{L}\sum_{j=1}^{\infty} a_j = \sum_{j=1}^{\infty} \frac{2}{L} a_j$ also converges by Proposition 7.2.8. Since $0 < b_j < \frac{2}{L} a_j$ for all $j \ge N$, by the direct comparison test, the series $\sum_{j=N}^{\infty} b_j$ converges. Finally, by Proposition 7.2.9, the full series $\sum_{j=1}^{\infty} b_j$ converges.

- $(\Leftarrow)$:
  Similarly, suppose that the series $\sum_{j=1}^{\infty} b_j$ converges. Since $0 \le a_j < \frac{3L}{2} b_j$ for all $j \ge N$, and the series $\sum_{j=N}^{\infty} \frac{3L}{2} b_j$ converges, the series $\sum_{j=N}^{\infty} a_j$ also converges by the direct comparison test. Therefore, the full series $\sum_{j=1}^{\infty} a_j$ converges as well.

■

Theorem 23 (Ratio Test) *Let $\sum_{j=1}^{\infty} a_j$ be a real series such that $a_j \ne 0$ for all $j \in \mathbb{N}$. Let $L = \lim_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| \ge 0$.*

1. *If $L < 1$, then the series converges absolutely.*

2. *If $L > 1$, then the series diverges.*

Proof. We prove the assertions separately.

1. Suppose that $\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| = L < 1$. Then, for $\epsilon = \frac{1-L}{2} > 0$, there exists an $N \in \mathbb{N}$ such that
$$\left|\frac{a_{n+1}}{a_n} - L\right| < \frac{1-L}{2} \quad \text{for all } n \geq N.$$
This implies that
$$\left|\frac{a_{n+1}}{a_n}\right| < \frac{1+L}{2} \quad \text{for all } n \geq N.$$
Denote $r = \frac{1+L}{2} < 1$, so that $|a_{n+1}| < r\,|a_n|$ for all $n \geq N$. By induction, we can show that $|a_{k+N}| < r^k\,|a_N|$ for all $k \in \mathbb{N}$.

   Let us compare the tail of the series $\sum_{j=N+1}^{\infty}|a_j| = \sum_{k=1}^{\infty}|a_{k+N}|$ with the geometric series $\sum_{k=1}^{\infty} r^k\,|a_N|$. Clearly, the geometric series converges since $r < 1$. By the direct comparison test, since $|a_{k+N}| < r^k\,|a_N|$ for all $k \in \mathbb{N}$, the tail of the series $\sum_{j=N+1}^{\infty}|a_j|$ also converges. Proposition 7.2.9 then implies that the entire series $\sum_{j=1}^{\infty}|a_j|$ converges, meaning the series converges absolutely.

2. Suppose that $\lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| = L > 1$. By a similar argument as in the previous case, if we choose $\epsilon = \frac{L-1}{2} > 0$, we can show that there exists $N \in \mathbb{N}$ such that
$$\frac{1+L}{2} < \left|\frac{a_{n+1}}{a_n}\right| \quad \text{for all } n \geq N.$$
Denote $r = \frac{1+L}{2} > 1$, so that $0 < |a_N| < r^k\,|a_N| < |a_{k+N}|$ for all $k \in \mathbb{N}$. Since $r^k\,|a_N| \to \infty$, we have $|a_{k+N}| \to \infty$ as well. This implies that $\lim_{j\to\infty}|a_j| \neq 0$ and, by Lemma 5.9.3, $\lim_{j\to\infty} a_j \neq 0$. Thus, the series $\sum_{j=1}^{\infty} a_j$ cannot converge by Proposition 7.2.5.

■

Chapter 2

The Riemann Integral

1. Partition and Refinement of an Interval

Let $[a, b]$ be a closed and bounded interval with $a < b$. A **partition** P of $[a, b]$ is a finite ordered set of points

$$P = \{x_0, x_1, \ldots, x_n\}, \qquad a = x_0 < x_1 < \cdots < x_n = b,$$

which subdivides $[a, b]$ into the n subintervals

$$[x_{k-1}, x_k], \qquad k = 1, 2, \ldots, n.$$

These subintervals are pairwise disjoint in their interiors and their union is $[a, b]$.

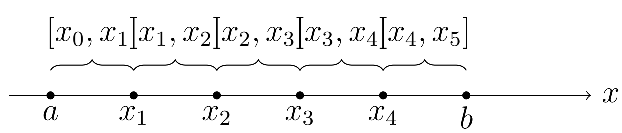


Figure 2.1: Partition P of $[a, b]$ into subintervals.

Let

$$P = \{x_0, \ldots, x_n\} \quad \text{with} \quad a = x_0 < \cdots < x_n = b.$$

A partition Q of $[a, b]$ is called a **refinement** of P if $P \subseteq Q$; that is, every point of P also appears in Q, and Q may contain additional points inside the subintervals determined by P.

Example

Suppose

$$P = \{a, x_1, x_2, b\}, \qquad a < x_1 < x_2 < b,$$

and we insert three additional points

$$q_1 \in (a, x_1), \qquad q_2 \in (x_1, x_2), \qquad q_3 \in (x_2, b).$$

Then the refinement Q is

$$Q = P \cup \{q_1, q_2, q_3\} = \{a, q_1, x_1, q_2, x_2, q_3, b\},$$

listed in strictly increasing order.

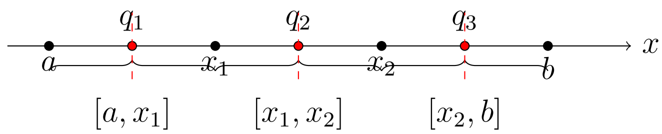


Figure 2.2: Refinement Q of P by inserting q_1, q_2, and q_3.

2. Lower and Upper Sums

Definition 24 (Lower and Upper Sums) *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, \dots, x_n\}$ a partition of $[a, b]$. For each subinterval $[x_{k-1}, x_k]$, define:*

$$m_k := \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}, \quad M_k := \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}.$$

*Then the **lower sum** of f with respect to P is:*

$$L(f, P) = \sum_{k=1}^{n} m_k \cdot (x_k - x_{k-1}),$$

*and the **upper sum** is:*

$$U(f, P) = \sum_{k=1}^{n} M_k \cdot (x_k - x_{k-1}).$$

3. Properties of Riemann Sums

Lemma 25 (Properties of Lower and Upper Sums) *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then:*

1. *For every partition P,*

$$L(f, P) \leq U(f, P).$$

2. *If Q is a refinement of P, then*

$$L(f, P) \leq L(f, Q) \quad \text{and} \quad U(f, P) \geq U(f, Q).$$

3. *For any two partitions P_1, P_2,*

$$L(f, P_1) \leq U(f, P_2).$$

Proof.

1. **Lower sum is always less than or equal to upper sum.**
   For each subinterval $[x_{k-1}, x_k]$, we define:

$$m_k := \inf\{f(x) : x \in [x_{k-1}, x_k]\}, \quad M_k := \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$

   Since $m_k \leq M_k$ for all k, it follows that:

$$L(f, P) = \sum_{k=1}^{n} m_k \Delta x_k \leq \sum_{k=1}^{n} M_k \Delta x_k = U(f, P).$$

Example: Let $f(x) = x^2$ on $[0,1]$, and let $P = \{0, 0.5, 1\}$. Then:

$L(f,P) = 0^2 \cdot 0.5 + (0.5)^2 \cdot 0.5 = 0 + 0.125 = 0.125, U(f,P) = (0.5)^2 \cdot 0.5 + (1)^2 \cdot 0.5 = 0.125 + 0.5 = 0.625.$

So $L(f,P) < U(f,P)$.

2. **Refining increases lower sum and decreases upper sum.**
   A refinement Q of P adds points to subdivide the interval more finely. The infimum over a smaller subinterval is at least as large as over the larger one (because we're minimizing over fewer values), and similarly, the supremum over a smaller subinterval is at most as large.

   Hence:
   $$L(f,Q) \geq L(f,P), \qquad U(f,Q) \leq U(f,P).$$

   Example: Use the same $f(x) = x^2$ on $[0,1]$, but refine $P = \{0, 0.5, 1\}$ to $Q = \{0, 0.25, 0.5, 0.75, 1\}$. You will find:
   $$L(f,Q) > L(f,P), \quad U(f,Q) < U(f,P).$$

3. **Lower sum of one partition is less than upper sum of another.**
   Let $R = P_1 \cup P_2$, which is a common refinement of both P_1 and P_2. Then, by part (2):
   $$L(f,P_1) \leq L(f,R) \leq U(f,R) \leq U(f,P_2),$$
   so:
   $$L(f,P_1) \leq U(f,P_2).$$

   Example: Let $P_1 = \{0, 0.5, 1\}$, $P_2 = \{0, 0.25, 1\}$. Their union is $R = \{0, 0.25, 0.5, 1\}$. Again using $f(x) = x^2$, you can compute and verify the inequality numerically:
   $$L(f,P_1) \leq L(f,R) \leq U(f,R) \leq U(f,P_2).$$

■

Definition 26 *Let $f : [a,b] \to \mathbb{R}$ be bounded.*
1) Lower/upper integrals. *Define*

$$L(f) := \sup\{\, L(f,P) : P \text{ a partition of } [a,b] \,\},$$
$$U(f) := \inf\{\, U(f,P) : P \text{ a partition of } [a,b] \,\}.$$

Because f is bounded, the sets inside the sup *and* inf *are nonempty and bounded, so by completeness of $\mathbb{R}$ the numbers $L(f)$ and $U(f)$ exist. Always $L(f) \leq U(f)$.*
2) Riemann integrability. *We say f is* Riemann integrable *on $[a,b]$ if $L(f) = U(f)$. In that case, the common value is the* ***Riemann integral*** *of f:*

$$\int_a^b f(x)\,dx \quad (\text{also written } \int_a^b f).$$

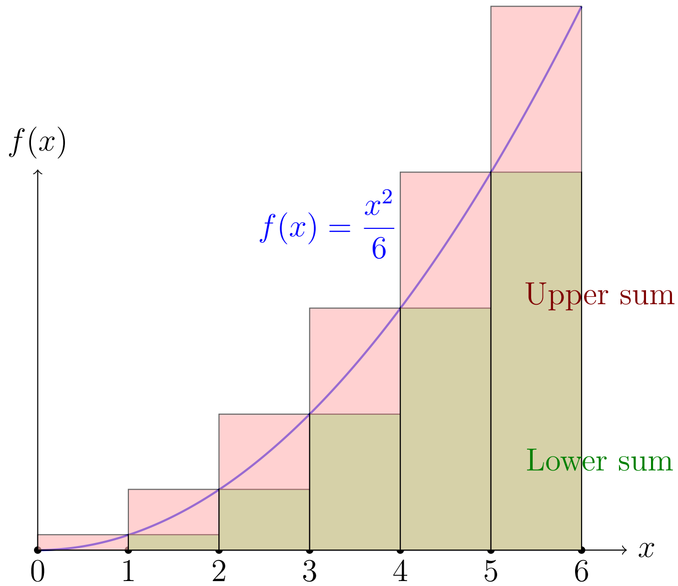


Figure 2.3: Lower and upper sums for the function $f(x) = \frac{x^2}{6}$ on $[0, 6]$.

Intuitively, a bounded function f is Riemann integrable if we can approximate the area under its graph from below (using lower sums) and from above (using upper sums) in such a way that both approximations can be made arbitrarily close to each other by refining the partition.

In the figure above:

- The **green rectangles** represent the *lower sum* $L(f, P)$, constructed using the minimum value of f on each subinterval.
- The **red translucent rectangles** represent the *upper sum* $U(f, P)$, constructed using the maximum value of f on each subinterval.
- The **blue curve** shows the graph of the function $f(x) = \frac{x^2}{6}$.

As the partition becomes finer (i.e., we divide $[a, b]$ into smaller subintervals), the lower and upper rectangles better approximate the area under the curve. The difference between the total areas of the upper and lower sums decreases.

This leads to the following fundamental characterization of Riemann integrability.

Theorem 27 (ε–criterion) *A bounded f is Riemann integrable on $[a, b]$ iff for every $\varepsilon > 0$ there exists a partition P such that*

$$U(f, P) - L(f, P) < \varepsilon.$$

Equivalently, we can make lower and upper sums as close as we wish by choosing P fine enough.

Proof sketch. ($\Rightarrow$) Assume f is Riemann integrable, and let $I = \int_a^b f(x)\,dx = L(f) = U(f)$. Fix $\varepsilon > 0$. By the definitions of sup and inf, choose partitions P_1, P_2 such that

$$L(f, P_1) > I - \frac{\varepsilon}{2} \quad \text{and} \quad U(f, P_2) < I + \frac{\varepsilon}{2}.$$

Let P be any common refinement of P_1 and P_2 (e.g. take the union of their points). By refinement monotonicity, $L(f, P) \geq L(f, P_1)$ and $U(f, P) \leq U(f, P_2)$, hence

$$U(f, P) - L(f, P) \leq U(f, P_2) - L(f, P_1) < \varepsilon.$$

($\Leftarrow$) Conversely, suppose that for every $\varepsilon > 0$ there exists a partition P with $U(f, P) - L(f, P) < \varepsilon$. Since $L(f) = \sup_Q L(f, Q) \geq L(f, P)$ and $U(f) = \inf_Q U(f, Q) \leq U(f, P)$, we have

$$0 \leq U(f) - L(f) \leq U(f, P) - L(f, P) < \varepsilon.$$

Letting $\varepsilon \to 0$ gives $U(f) = L(f)$, so f is Riemann integrable. ∎

Constant function. Let $f : [a, b] \to \mathbb{R}$ be the constant function $f(x) \equiv c$ with $c \in \mathbb{R}$. For any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, write $\Delta x_k := x_k - x_{k-1}$ and $I_k := [x_{k-1}, x_k]$. Then

$$m_k := \inf_{I_k} f = M_k := \sup_{I_k} f = c \qquad (k = 1, \ldots, n).$$

Thus the Riemann sums coincide:

$$L(f, P) = \sum_{k=1}^{n} m_k\, \Delta x_k = c \sum_{k=1}^{n} \Delta x_k = c(b - a), \qquad U(f, P) = \sum_{k=1}^{n} M_k\, \Delta x_k = c(b - a).$$

Hence $U(f, P) - L(f, P) = 0$ for every partition P, so the ε–criterion is trivially satisfied and f is Riemann integrable. Since the common value is independent of P, we conclude

$$\int_a^b f(x)\,dx = c(b - a).$$

Corollary 28 *Let $f : [a, b] \to \mathbb{R}$ be bounded. For a partition P write $U(f, P)$ and $L(f, P)$ for the upper and lower Riemann sums. The following are equivalent:*

(i) f is Riemann integrable on $[a, b]$;

(ii) there exists a sequence of partitions (P_n) such that $U(f, P_n) - L(f, P_n) \to 0$.

Moreover, whenever (ii) *holds,*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n).$$

Proof. Define the lower and upper integrals of f by

$$L(f) := \sup\{L(f,P) : P \text{ a partition of } [a,b]\},$$
$$U(f) := \inf\{U(f,P) : P \text{ a partition of } [a,b]\}.$$

Recall that f is Riemann integrable iff $L(f) = U(f)$, in which case the common value equals $\int_a^b f$.

(i) ⇒ (ii). Assume f is Riemann integrable. Then for every $\varepsilon > 0$ there exists a partition P with $U(f,P) - L(f,P) < \varepsilon$ (ε-criterion). Choose P_n so that $U(f,P_n) - L(f,P_n) < 1/n$. Then $U(f,P_n) - L(f,P_n) \to 0$.

(ii) ⇒ (i). Assume there exists (P_n) with $U(f,P_n) - L(f,P_n) \to 0$. For every partition P we have $L(f,P) \le L(f) \le U(f) \le U(f,P)$, hence

$$L(f,P_n) \le L(f) \le U(f) \le U(f,P_n) \qquad \text{for all } n.$$

Therefore

$$0 \le U(f) - L(f) \le U(f,P_n) - L(f,P_n) \xrightarrow[n\to\infty]{} 0,$$

so $U(f) = L(f)$. Thus f is Riemann integrable and

$$\int_a^b f = L(f) = U(f).$$

Moreover, $L(f,P_n) \le \int_a^b f \le U(f,P_n)$ and $U(f,P_n) - L(f,P_n) \to 0$ imply, by the squeeze theorem,

$$\lim_{n\to\infty} L(f,P_n) = \int_a^b f = \lim_{n\to\infty} U(f,P_n).$$

Remark. If f is integrable and $\|P_n\| \to 0$, then necessarily $U(f,P_n) - L(f,P_n) \to 0$ and the same conclusions hold. ■

We illustrate the above Corollary with the explicit example.

A partition P of $[a,b]$ is *uniform* if all its subintervals have the same length. For $n \in \mathbb{N}$ set

$$P_n = \Big\{a,\ a + \tfrac{b-a}{n},\ a + \tfrac{2(b-a)}{n},\ \dots,\ a + \tfrac{n(b-a)}{n} = b\Big\}, \qquad \|P_n\| = \tfrac{b-a}{n}.$$

Let $f : [0,1] \to \mathbb{R}$, $f(x) = x^2$. For the uniform partitions

$$P_n = \Big\{0, \tfrac{1}{n}, \tfrac{2}{n}, \dots, \tfrac{n-1}{n}, 1\Big\}, \quad \Delta x = \tfrac{1}{n},$$

f is increasing on each $I_k = [\frac{k-1}{n}, \frac{k}{n}]$, so

$$m_k = \Big(\tfrac{k-1}{n}\Big)^2, \qquad M_k = \Big(\tfrac{k}{n}\Big)^2.$$

Hence the Riemann sums are

$$L(f, P_n) = \frac{1}{n^3} \sum_{k=1}^{n} (k-1)^2 = \frac{(n-1)n(2n-1)}{6n^3} = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2},$$

$$U(f, P_n) = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

Therefore

$$U(f, P_n) - L(f, P_n) = \frac{1}{n} \xrightarrow[n\to\infty]{} 0,$$

and, by the ε-criterion,

$$\int_0^1 x^2\, dx = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = \frac{1}{3}.$$

Theorem 29 *Every monotone function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.*

Proof. Suppose f is monotone increasing on $[a, b]$. Then f is bounded, since

$$f(a) \leq f(x) \leq f(b) \quad \text{for all } x \in [a, b].$$

Let $\varepsilon > 0$ be given. We want to find a partition P such that $U(f, P) - L(f, P) < \varepsilon$. Choose $\delta > 0$ such that

$$\delta(f(b) - f(a)) < \varepsilon.$$

Now select a partition $P = \{x_0, x_1, \ldots, x_n\}$ such that the width of every subinterval satisfies:

$$x_k - x_{k-1} < \delta \quad \text{for all } k = 1, \ldots, n.$$

Since f is increasing, on each subinterval $[x_{k-1}, x_k]$ we have:

$$m_k = f(x_{k-1}), \quad M_k = f(x_k),$$

so the difference between the upper and lower sums becomes:

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} (f(x_k) - f(x_{k-1}))(x_k - x_{k-1}).$$

Using the fact that $x_k - x_{k-1} < \delta$, we estimate:

$$U(f, P) - L(f, P) \leq \delta \sum_{k=1}^{n} (f(x_k) - f(x_{k-1})) = \delta(f(b) - f(a)) < \varepsilon.$$

Hence, by the integrability criterion (Lemma), f is Riemann integrable. ■

Theorem 30 *Every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.*

Proof. Since f is continuous on the closed interval $[a,b]$, which is compact, the **Extreme Value Theorem** guarantees that f is bounded and attains its maximum and minimum on each subinterval of any partition. Furthermore, by the **Uniform Continuity Theorem**, f is uniformly continuous on $[a,b]$. Therefore, for any $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$|x-y| < \delta \quad \Rightarrow \quad |f(x)-f(y)| < \frac{\varepsilon}{b-a}.$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ such that:

$$x_k - x_{k-1} < \delta \quad \text{for all } k = 1, \ldots, n.$$

On each subinterval $[x_{k-1}, x_k]$, the function f attains both its maximum M_k and minimum m_k (by continuity), and we have:

$$M_k - m_k < \frac{\varepsilon}{b-a}.$$

Thus,

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) < \frac{\varepsilon}{b-a} \sum_{k=1}^{n} (x_k - x_{k-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

Hence, by the ε-criterion f is Riemann integrable. ■

Generalization: Even though continuity guarantees integrability, the converse is not true. A function can be Riemann integrable without being continuous everywhere.

Theorem 31 (Generalization) *Let $f : [a,b] \to \mathbb{R}$ be bounded and have only **finitely many points of discontinuity**. Then f is Riemann integrable.*

Sketch of proof. Let $D = \{c_1, c_2, \ldots, c_m\} \subset [a,b]$ be the (finite) set of discontinuities of f. Around each c_i, construct an interval of length less than δ/m such that the total contribution to the upper-lower sum difference over these intervals is less than $\varepsilon/2$. On the complement of these intervals, f is continuous, so we apply the previous theorem to choose a partition on that region giving error less than $\varepsilon/2$. Combining both partitions yields a global partition P such that $U(f,P) - L(f,P) < \varepsilon$. ■

Example 32 (Discontinuous but integrable vs non-integrable) *This example illustrates how the nature and number of discontinuities affect integrability.*

(a) ***Integrable with one discontinuity:***

Exercise 33 *Define $f : [-1,1] \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

Show that f is Riemann integrable on $[-1,1]$ and compute $\int_{-1}^{1} f(x)\,dx$.

Solution 34 *For each $n \in \mathbb{N}$, consider the partition*

$$P_n = \Big\{-1,\ -\tfrac{1}{2n},\ \tfrac{1}{2n},\ 1\Big\},$$

which produces the subintervals

$$I_1 = \Big[-1,\ -\tfrac{1}{2n}\Big], \qquad I_2 = \Big[-\tfrac{1}{2n},\ \tfrac{1}{2n}\Big], \qquad I_3 = \Big[\tfrac{1}{2n},\ 1\Big].$$

Since $f \equiv 0$ on $[-1,1] \setminus \{0\}$ and $f(0) = 1$, we have

$$\sup_{I_1} f = \sup_{I_3} f = 0, \quad \inf_{I_1} f = \inf_{I_3} f = 0, \qquad \sup_{I_2} f = 1, \quad \inf_{I_2} f = 0.$$

The lengths are

$$|I_1| = 1 - \tfrac{1}{2n}, \qquad |I_2| = \tfrac{1}{n}, \qquad |I_3| = 1 - \tfrac{1}{2n}.$$

Therefore

$$L(f, P_n) = \sum (\inf f)\,|I| = 0, \qquad U(f, P_n) = \sum (\sup f)\,|I| = 1 \cdot |I_2| = \frac{1}{n}.$$

Integrability. *Note that $\|P_n\| = \max\{|I_1|, |I_2|, |I_3|\} = 1 - \frac{1}{2n} \not\to 0$, but this is irrelevant: the Darboux criterion only requires, for each $\varepsilon > 0$,* some *partition P with $U(f,P) - L(f,P) < \varepsilon$. Given $\varepsilon > 0$, by the Archimedean property choose $N \in \mathbb{N}$ with $1/N < \varepsilon$. Then*

$$U(f, P_N) - L(f, P_N) = \frac{1}{N} < \varepsilon,$$

so f is Riemann integrable on $[-1,1]$.

Value of the integral. *For every n,*

$$0 = L(f, P_n) \ \le\ \int_{-1}^{1} f(x)\,dx \ \le\ U(f, P_n) = \frac{1}{n}.$$

Letting $n \to \infty$ and using the squeeze theorem gives

$$\int_{-1}^{1} f(x)\,dx = 0.$$

(b) ***Not integrable:*** *Define $f : [0,1] \to \mathbb{R}$ by:*

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

This function is known as the Dirichlet function and is discontinuous at ***every point*** *in $[0,1]$. On every subinterval of any partition:*

$$\inf f = 0, \quad \sup f = 1,$$

so:

$$L(f, P) = 0, \quad U(f, P) = 1 \quad \textit{for all } P.$$

Therefore,

$$U(f, P) - L(f, P) = 1 \not\to 0,$$

and f is not Riemann integrable.

Theorem 35 *Let $f : [a, b] \to \mathbb{R}$ be bounded and let $c \in (a, b)$. Then f is integrable on $[a, b]$ if and only if f is integrable on both $[a, c]$ and $[c, b]$. In that case:*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Remark 36 *If f is integrable on $[a, b]$, we define:*

$$\int_a^b f = -\int_b^a f.$$

Also, for any $c \in [a, b]$, we define:

$$\int_c^c f = 0.$$

Then, for any three points $a, b, c \in I$, where $I \subseteq \mathbb{R}$ is a compact interval and $f : I \to \mathbb{R}$ is integrable, we have:

$$\int_a^b f + \int_b^c f = \int_a^c f.$$

We leave the verification as an exercise.

Theorem 37 *Suppose f and g are Riemann integrable on $[a, b]$, and let $k \in \mathbb{R}$. Then:*

1. *The function $f + g$ is integrable, and*
$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

2. *The function kf is integrable, and*
$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

3. *If $f(x) \leq g(x)$ for all $x \in [a, b]$, then*
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

4. *The function $|f|$ is integrable, and*
$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Proof. We prove parts (1) and (4). Parts (2) and (3) follow from similar arguments and are left as exercises.

(1) Linearity of the integral. Let f and g be integrable on $[a, b]$, and let P be any partition of $[a, b]$ into subintervals $[x_{k-1}, x_k]$, $k = 1, \ldots, n$.

Define:
$$m_k^f = \inf_{x \in [x_{k-1}, x_k]} f(x), \quad M_k^f = \sup_{x \in [x_{k-1}, x_k]} f(x),$$
and similarly for g, and for $f + g$:
$$m_k^{f+g} = \inf_{x \in [x_{k-1}, x_k]} (f(x) + g(x)), \quad M_k^{f+g} = \sup_{x \in [x_{k-1}, x_k]} (f(x) + g(x)).$$

From basic properties of infima and suprema over sets:
$$m_k^f + m_k^g \leq m_k^{f+g}, \quad M_k^{f+g} \leq M_k^f + M_k^g.$$

Multiplying by the subinterval length $\Delta x_k = x_k - x_{k-1}$, and summing over all k, we obtain:
$$L(f, P) + L(g, P) \leq L(f + g, P), \quad U(f + g, P) \leq U(f, P) + U(g, P).$$

Let $\varepsilon > 0$. Since f and g are integrable, there exist partitions P_1 and P_2 such that:
$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2}, \quad U(g, P_2) - L(g, P_2) < \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$, a common refinement. Then using monotonicity of upper and lower sums under refinement:

$$U(f, P) \leq U(f, P_1), \quad L(f, P) \geq L(f, P_1), \quad \text{and similarly for } g.$$

Then:

$$U(f+g, P) \leq U(f, P) + U(g, P) \leq U(f, P_1) + U(g, P_2) < U(f) + U(g) + \varepsilon,$$
$$L(f+g, P) \geq L(f, P) + L(g, P) \geq L(f, P_1) + L(g, P_2) > L(f) + L(g) - \varepsilon.$$

Thus:

$$U(f+g) \leq U(f) + U(g), \quad L(f+g) \geq L(f) + L(g),$$

and since:

$$L(f+g) \leq U(f+g),$$

we conclude that:

$$L(f+g) = U(f+g) = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

So $f + g$ is integrable and its integral is the sum of the integrals.

(4) Integrability of $|f|$ and inequality.

First, note that since f is integrable, it is bounded, say $|f(x)| \leq M$ for all $x \in [a, b]$. Let P be a partition of $[a, b]$. Define:

$$m_k^{|f|} = \inf_{x \in [x_{k-1}, x_k]} |f(x)|, \quad M_k^{|f|} = \sup_{x \in [x_{k-1}, x_k]} |f(x)|.$$

Since $|f(x)|$ is Lipschitz continuous with respect to $f(x)$ (triangle inequality), we have:

$$M_k^{|f|} - m_k^{|f|} \leq M_k^f - m_k^f.$$

Summing over all subintervals gives:

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P).$$

Now, since f is integrable, for any $\varepsilon > 0$, there exists a partition P such that:

$$U(f, P) - L(f, P) < \varepsilon \quad \Rightarrow \quad U(|f|, P) - L(|f|, P) < \varepsilon.$$

So $|f|$ is also integrable.

To prove the inequality:

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx,$$

observe that for all $x \in [a, b]$:

$$-|f(x)| \leq f(x) \leq |f(x)|.$$

Integrating all parts and using the order property (proved in part 3), we get:

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx,$$

which implies:

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

■

Riemann Sums

Let $f : [a, b] \to \mathbb{R}$ be a function, and let

$$P = \{x_0,\, x_1,\, \ldots,\, x_n\}$$

be a partition of the interval $[a, b]$. For each $k \in \{1, \ldots, n\}$, choose a point $\xi_k \in [x_{k-1}, x_k]$, $\xi = (\xi_1, \ldots, \xi_2)$ called a *mark* (or *tag*).

Definition 38 (Riemann Sum) *The* Riemann sum *of f with respect to P and ξ is defined as:*

$$S(f, P, \xi) := \sum_{k=1}^{n} f(\xi_k)\,(x_k - x_{k-1}).$$

Definition 39 (Norm of a Partition) *The* Norm *of a partition $P = \{x_0,\, x_1,\, \ldots,\, x_n\}$ is defined by:*

$$\|P\| := \max_{1 \le k \le n} (x_k - x_{k-1}).$$

Theorem 40 (Convergence of tagged Riemann sums) *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. For a partition*

$$P :\ a = x_0 < x_1 < \cdots < x_n = b, \qquad \|P\| := \max_{1 \le k \le n} (x_k - x_{k-1}),$$

and any choice of tags $\xi = (\xi_1, \ldots, \xi_n)$ with $\xi_k \in [x_{k-1}, x_k]$, define the Riemann sum

$$S(f, P, \xi) := \sum_{k=1}^{n} f(\xi_k)\,(x_k - x_{k-1}).$$

Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all partitions P with $\|P\| \le \delta$ and for all choices of tags ξ,

$$\left| \int_a^b f(x)\,dx - S(f, P, \xi) \right| < \varepsilon.$$

Equivalently,

$$\lim_{\|P\| \to 0} S(f, P, \xi) = \int_a^b f(x)\,dx \qquad (\textit{uniformly in the choice of tags } \xi).$$

Proof. Write $U(f, P) = \sum_{k=1}^{n} (\sup_{[x_{k-1}, x_k]} f)\,(x_k - x_{k-1})$ and $L(f, P) = \sum_{k=1}^{n} (\inf_{[x_{k-1}, x_k]} f)\,(x_k - x_{k-1})$. For any tags ξ one has

$$L(f, P) \ \le\ S(f, P, \xi) \ \le\ U(f, P).$$

Since f is Riemann integrable, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P\| < \delta \ \Rightarrow\ U(f, P) - L(f, P) < \varepsilon$. Combining the two displays gives

$$\left| S(f, P, \xi) - \textstyle\int_a^b f \right| \le U(f, P) - L(f, P) < \varepsilon,$$

as required. ■

Example

Fix $a > 0$ and for $n \in \mathbb{N}$ take the uniform partition $x_k = \frac{ka}{n}$, $k = 0, \ldots, n$, with right-endpoint tags $\xi_k = x_k$. Then

$$S_n = \sum_{k=1}^{n} f(\xi_k)\,(x_k - x_{k-1}) = \sum_{k=1}^{n} \left(\frac{ka}{n}\right)\left(\frac{a}{n}\right) = \frac{a^2}{n^2}\sum_{k=1}^{n} k = \frac{a^2}{2}\left(1 + \frac{1}{n}\right) \xrightarrow[n\to\infty]{} \frac{a^2}{2}.$$

Hence $\displaystyle\int_0^a x\,dx = \frac{a^2}{2}$.

Exercise 41 *Let $f \in \mathcal{R}[a, b]$. Show that*

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right) = \int_a^b f(x)\,dx.$$

Proof. Set $\Delta_n = \frac{b-a}{n}$ and $x_k = a + k\Delta_n$. Then $\|P_n\| = \Delta_n \to 0$ and

$$\frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right) = \sum_{k=1}^{n} f(x_k)\,\Delta_n = S\Big(f, P_n, \xi^{\text{right}}\Big),$$

the right-endpoint Riemann sum ($\xi_k = x_k$). By the theorem (with arbitrary tags), $S(f, P_n, \xi^{\text{right}}) \to \int_a^b f$. ■

For the *left*-endpoint sums define ($\xi_k = x_{k-1}$)

$$L_n := \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(a + \frac{k(b-a)}{n}\right) = \sum_{k=1}^{n} f(x_{k-1})\,\Delta_n, \qquad x_k = a + k\Delta_n.$$

Then $L_n \to \displaystyle\int_a^b f$ as $n \to \infty$ as well. *Illustration ($f(x) = x$ on $[0, 1]$).*

$$L_n = \frac{1}{n}\sum_{k=0}^{n-1} \frac{k}{n} = \frac{n-1}{2n} \to \frac{1}{2} = \int_0^1 x\,dx, \qquad R_n = \frac{1}{n}\sum_{k=1}^{n} \frac{k}{n} = \frac{n+1}{2n} \to \frac{1}{2}.$$

Applications: write each limit as an integral (and evaluate).

1.
$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\,\frac{1}{1 + k/n} = \int_0^1 \frac{dx}{1+x} = \ln 2.$$

2.
$$\begin{aligned}\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2} &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\,\frac{(k/n)}{1+(k/n)^2} \\ &= \int_0^1 \frac{x}{1+x^2}\,dx \\ &= \frac{1}{2}\ln(1+x^2)\Big|_0^1 \\ &= \frac{1}{2}\ln 2.\end{aligned}$$

4.The Fundamental Theorem of Calculus

This central theorem states that the operations of differentiation and integration are, in a sense, inverses of each other.

Theorem 42 (Fundamental Theorem of Calculus)

1. *Let $f : [a,b] \to \mathbb{R}$ be integrable and let $F : [a,b] \to \mathbb{R}$ be differentiable with $F'(x) = f(x)$ for all $x \in [a,b]$. Then*
$$\int_a^b f = F(b) - F(a).$$
2. *Let $g : [a,b] \to \mathbb{R}$ be integrable and define*
$$G(x) := \int_a^x g(t)\,dt, \quad x \in [a,b].$$
Then G is continuous on $[a,b]$. Moreover, if g is continuous at $c \in [a,b]$, then G is differentiable at c, and
$$G'(c) = g(c).$$

In part (1), the function F is called an **antiderivative** of f.
Proof of Theorem ??. **(1)** Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. By the Mean Value Theorem, for each interval $[x_{k-1}, x_k]$, there exists $t_k \in (x_{k-1}, x_k)$ such that

$$F(x_k) - F(x_{k-1}) = F'(t_k)(x_k - x_{k-1}) = f(t_k)(x_k - x_{k-1}).$$

Since $m_k \le f(t_k) \le M_k$, we get

$$L(f,P) \le \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \le U(f,P).$$

The sum $\sum_{k=1}^{n} f(t_k)(x_k - x_{k-1})$ is a telescoping sum equal to $F(b) - F(a)$, hence

$$\int_a^b f = F(b) - F(a).$$

(2) Suppose $|g(x)| \le M$ on $[a,b]$. For any $x, y \in [a,b]$,

$$|G(x) - G(y)| = \left|\int_a^x g - \int_a^y g\right| = \left|\int_y^x g\right| \le \left|\int_y^x |g|\right| \le M|x-y|.$$

This shows that G is uniformly continuous.

Now suppose g is continuous at $c \in [a,b]$. Then for $x \ne c$:

$$\frac{G(x) - G(c)}{x - c} = \frac{1}{x-c}\int_c^x g(t)\,dt.$$

Given $\varepsilon > 0$, by continuity of g at c, there exists $\delta > 0$ such that $|g(t) - g(c)| < \varepsilon$ whenever $|t - c| < \delta$. Then for $0 < |x - c| < \delta$:

$$\left|\frac{G(x) - G(c)}{x - c} - g(c)\right| = \left|\frac{1}{x-c}\int_c^x (g(t) - g(c))\,dt\right| \le \varepsilon.$$

Hence $G'(c) = g(c)$. ■

Usual Antiderivatives

All antiderivatives are up to an additive constant $+C$.

Integrand $f(x)$	An antiderivative $F(x)$	Conditions / Notes
Powers & logs		
x^n	$\dfrac{x^{n+1}}{n+1}$	$n \neq -1.$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$	$x \neq 0.$
$(ax+b)^n$	$\dfrac{(ax+b)^{n+1}}{a(n+1)}$	$a \neq 0,\ n \neq -1.$
$\dfrac{1}{ax+b}$	$\dfrac{1}{a}\ln\lvert ax+b\rvert$	$a \neq 0.$
$\ln x$	$x\ln x - x$	$x > 0.$
Exponentials		
e^x	e^x	
e^{ax}	$\dfrac{1}{a}e^{ax}$	$a \neq 0.$
a^x	$\dfrac{a^x}{\ln a}$	$a > 0,\ a \neq 1.$
Trigonometric		
$\sin x$	$-\cos x$	
$\cos x$	$\sin x$	
$\tan x$	$-\ln\lvert\cos x\rvert$	also $\ln\lvert\sec x\rvert$.
$\cot x$	$\ln\lvert\sin x\rvert$	also $-\ln\lvert\csc x\rvert$.
$\sec^2 x$	$\tan x$	
$\csc^2 x$	$-\cot x$	
$\sec x\ \tan x$	$\sec x$	
$\csc x\ \cot x$	$-\csc x$	
$\sec x$	$\ln\lvert\sec x + \tan x\rvert$	
$\csc x$	$\ln\lvert\csc x - \cot x\rvert$	
Inverse trig (standard forms)		
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$	Also $\int -\dfrac{dx}{\sqrt{1-x^2}} = \arccos x.$
$\dfrac{1}{1+x^2}$	$\arctan x$	
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\arctan\dfrac{x}{a}$	$a > 0.$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\arcsin\dfrac{x}{a}$	$a > 0,\ \lvert x\rvert < a.$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$	$a > 0,\ x \neq \pm a.$
Hyperbolic		
$\sinh x$	$\cosh x$	
$\cosh x$	$\sinh x$	
$\tanh x$	$\ln(\cosh x)$	
$\operatorname{sech}^2 x$	$\tanh x$	
$\operatorname{csch}^2 x$	$-\coth x$	
$\operatorname{sech} x\ \tanh x$	$-\operatorname{sech} x$	
$\operatorname{csch} x\ \coth x$	$-\operatorname{csch} x$	
Inverse hyperbolic (log forms)		
$\dfrac{1}{\sqrt{x^2+1}}$	$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$	
$\dfrac{1}{\sqrt{x^2-1}}$	$\operatorname{arcosh}\lvert x\rvert = \ln\left(\lvert x\rvert + \sqrt{x^2-1}\right)$	$\lvert x\rvert > 1.$
1	$1+x$	

Theorem 43 (Integration by Parts) *Let $f, g : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. If $f', g' \in \mathcal{R}([a, b])$ (Riemann integrable), then*

$$\int_a^b f(x)\, g'(x)\, dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)\, g(x)\, dx. \tag{8.20}$$

Proof. Let $h = fg$. By the product rule, $h' = f'g + fg'$. By the Fundamental Theorem of Calculus and the assumed integrability of f', g', we obtain

$$h(b) - h(a) = \int_a^b h'(x)\, dx = \int_a^b \left(f'(x)g(x) + f(x)g'(x)\right) dx.$$

Rearranging gives (**??**). ■

Example 44 *Evaluate the integral*

$$I = \int_0^1 xe^x\, dx.$$

Solution. *We apply integration by parts with*

$$f(x) = x, \quad g'(x) = e^x \quad \Rightarrow \quad f'(x) = 1,\ g(x) = e^x.$$

By Theorem **??**,

$$I = \Big[xe^x\Big]_0^1 - \int_0^1 1 \cdot e^x\, dx = (1 \cdot e^1 - 0) - (e^1 - e^0).$$

Thus

$$I = e - (e - 1) = 1.$$

Theorem 45 (First Substitution Rule) *Suppose φ is differentiable on $[a, b]$ and its derivative $\varphi'(t)$ is continuous. If f is continuous on the range of φ, then*

$$\int_a^b f(\varphi(t))\, \varphi'(t)\, dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\, dx.$$

Example 46 *Evaluate*

$$I = \int_0^{\pi/2} \sin^2(t) \cos(t)\, dt.$$

Solution. *Let $\varphi(t) = \sin(t)$. Then $\varphi'(t) = \cos(t)$, and when $t = 0$, $\varphi(0) = 0$; when $t = \frac{\pi}{2}$, $\varphi(\frac{\pi}{2}) = 1$. Thus, by Theorem* **??**,

$$I = \int_0^{\pi/2} \sin^2(t) \cos(t)\, dt = \int_0^1 x^2\, dx = \left[\tfrac{x^3}{3}\right]_0^1 = \tfrac{1}{3}.$$

Theorem 47 (Second Substitution Rule) *Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $\varphi : [\alpha, \beta] \to [a, b]$ be differentiable with continuous derivative. Then*

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f(\varphi(t))\, \varphi'(t)\, dt.$$

Example 48 *Evaluate*

$$I = \int_0^4 \sqrt{x}\,dx.$$

Solution. *Let* $\varphi(t) = t^2$*, so that* $\varphi'(t) = 2t$*. When* $t = 0$*,* $\varphi(0) = 0$*; when* $t = 2$*,* $\varphi(2) = 4$*. Thus, by Theorem* **??***,*

$$I = \int_0^4 \sqrt{x}\,dx = \int_0^2 \sqrt{t^2} \cdot 2t\,dt = \int_0^2 2t^2\,dt.$$

Since $\sqrt{t^2} = t$ *for* $t \geq 0$*, we compute*

$$I = \left[\tfrac{2}{3}t^3\right]_0^2 = \tfrac{16}{3}.$$

Theorem 49 (Mean Value Theorem for Integrals) *If* $g : [a, b] \to \mathbb{R}$ *is continuous, then there exists* $c \in (a, b)$ *such that*

$$\int_a^b g = (b - a)g(c).$$

Proof. Apply the Mean Value Theorem to the function $x \mapsto \int_a^x g$, which by the Fundamental Theorem of Calculus is an antiderivative of g. ∎

Exercise 50 *Suppose* f *is continuous on* $[0, \infty)$ *and* $\lim_{x\to\infty} f(x) = a$*. Show that*

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(t)\,dt \; = \; a.$$

Hint. *Fix* N *large and use the mean value theorem for integrals on* $[N, x]$*.*

Proof. Fix $\varepsilon > 0$. Since $\lim_{x\to\infty} f(x) = a$, choose N so large that $|f(t) - a| < \varepsilon$ for all $t \geq N$. For $x > N$, write

$$\frac{1}{x}\int_0^x f(t)\,dt = \frac{N}{x}\,\frac{1}{N}\int_0^N f(t)\,dt \; + \; \frac{x-N}{x}\,\frac{1}{x-N}\int_N^x f(t)\,dt.$$

By the mean value theorem for integrals (continuity of f), there exists $\xi_x \in (N, x)$ such that

$$\frac{1}{x-N}\int_N^x f(t)\,dt = f(\xi_x).$$

Hence

$$\frac{1}{x}\int_0^x f(t)\,dt = \frac{N}{x}\,m_N \; + \; \frac{x-N}{x}\,f(\xi_x), \qquad \text{where } m_N := \frac{1}{N}\int_0^N f(t)\,dt.$$

Subtract a and estimate:

$$\left|\frac{1}{x}\int_0^x f(t)\,dt - a\right| \leq \frac{N}{x}\,|m_N| \; + \; \frac{x-N}{x}\,|f(\xi_x) - a| \; + \; \frac{N}{x}\,|a| \leq \frac{N}{x}(|m_N| + |a|) + \varepsilon,$$

because $\xi_x \in (N, x)$ implies $|f(\xi_x) - a| < \varepsilon$. Letting $x \to \infty$, the first term tends to 0, so the whole expression is $\leq \varepsilon$ in the limit. Since $\varepsilon > 0$ is arbitrary, the limit equals a. ∎

Improper Integrals

In this section, we study improper integrals, which arise in two main situations:

- One of the integration limits is infinite,
- The function becomes unbounded (e.g., has a vertical asymptote) at a boundary point.

We will consider these two cases in detail.

Case 1: Integration over an Infinite Interval

Definition 51 *Let $f : [a, \infty) \to \mathbb{R}$ be a function that is Riemann integrable over every finite interval $[a, R]$, for $a < R < \infty$. If the limit*

$$\lim_{R \to \infty} \int_a^R f(x)\, dx$$

exists and is finite, then the improper integral is said to converge, and we define

$$\int_a^\infty f(x)\, dx := \lim_{R \to \infty} \int_a^R f(x)\, dx.$$

Similarly, for a function $f : (-\infty, a] \to \mathbb{R}$, we define

$$\int_{-\infty}^a f(x)\, dx := \lim_{R \to -\infty} \int_R^a f(x)\, dx,$$

provided the limit exists.

Example

Consider the integral

$$\int_1^\infty \frac{1}{x^s}\, dx.$$

We compute:

$$\int_1^R \frac{1}{x^s}\, dx = \begin{cases} \frac{1}{s-1}\left(1 - \frac{1}{R^{s-1}}\right), & s \neq 1, \\ \log R, & s = 1. \end{cases}$$

Taking the limit as $R \to \infty$, we get:

$$\int_1^\infty \frac{1}{x^s}\, dx = \begin{cases} \frac{1}{s-1}, & \text{if } s > 1, \\ \text{diverges}, & \text{if } s \leq 1. \end{cases}$$

Case 2: The Function is Unbounded at an Endpoint

Definition 52 *Let $f : (a, b] \to \mathbb{R}$ be a function that is Riemann integrable over every interval $[a + \varepsilon, b]$, for $0 < \varepsilon < b - a$. If the limit*

$$\lim_{\varepsilon \searrow 0} \int_{a+\varepsilon}^b f(x)\, dx$$

exists and is finite, then the improper integral is said to converge, and we define

$$\int_a^b f(x)\, dx := \lim_{\varepsilon \searrow 0} \int_{a+\varepsilon}^b f(x)\, dx.$$

Example

Let us evaluate

$$\int_0^1 \frac{1}{x^s}\,dx.$$

For $s \neq 1$, we compute:

$$\int_\varepsilon^1 \frac{1}{x^s}\,dx = \frac{1}{1-s}\left(1-\varepsilon^{1-s}\right).$$

Now take the limit as $\varepsilon \to 0^+$:

$$\lim_{\varepsilon\to 0^+} \varepsilon^{1-s} = \begin{cases} 0, & s<1, \\ \infty, & s>1. \end{cases}$$

Hence,

$$\int_0^1 \frac{1}{x^s}\,dx = \begin{cases} \dfrac{1}{1-s}, & \text{if } s<1, \\ \text{diverges}, & \text{if } s\geq 1. \end{cases}$$

We now consider the general case of improper integrals over open intervals.

Definition 53 *Let $f:(a,b)\to\mathbb{R}$, where $a\in\mathbb{R}\cup\{-\infty\}$ and $b\in\mathbb{R}\cup\{\infty\}$, be a function that is Riemann integrable over every compact subinterval $[\alpha,\beta]\subset(a,b)$. Let $c\in(a,b)$ be arbitrary. If both of the improper integrals*

$$\int_a^c f(x)\,dx := \lim_{\alpha\searrow a}\int_\alpha^c f(x)\,dx, \quad \int_c^b f(x)\,dx := \lim_{\beta\nearrow b}\int_c^\beta f(x)\,dx$$

converge, then the integral over the full interval is called convergent, and we define:

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Note that this definition is independent of the choice of the intermediate point $c\in(a,b)$.

Examples

Example 1

According to previous examples, the integral

$$\int_0^\infty \frac{1}{x^s}\,dx$$

diverges for all $s\in\mathbb{R}$.

Example 2

The integral

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

converges. We compute:

$$\begin{aligned}\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx &= \lim_{\varepsilon\searrow 0}\int_{-1+\varepsilon}^0 \frac{1}{\sqrt{1-x^2}}\,dx + \lim_{\varepsilon\searrow 0}\int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx \\ &= -\lim_{\varepsilon\searrow 0}\sin^{-1}(-1+\varepsilon) + \lim_{\varepsilon\searrow 0}\sin^{-1}(1-\varepsilon) \\ &= -(-\tfrac{\pi}{2}) + \tfrac{\pi}{2} = \pi.\end{aligned}$$

Example 3

The integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

also converges:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{R\to\infty}\int_{-R}^{0} \frac{1}{1+x^2}\,dx + \lim_{R\to\infty}\int_{0}^{R} \frac{1}{1+x^2}\,dx\\ &= -\lim_{R\to\infty}\tan^{-1}(-R) + \lim_{R\to\infty}\tan^{-1}(R)\\ &= -(-\tfrac{\pi}{2}) + \tfrac{\pi}{2} = \pi.\end{aligned}$$

Comparison Test for Improper Riemann Integrals.

Let $I = [a,\infty)$. Suppose that $f, g : I \to \mathbb{R}$ are continuous non-negative functions such that $0 \le f(x) \le g(x)$ for all $x \in I$.

1. If $\int_a^\infty g(x)\,dx$ exists, then $\int_a^\infty f(x)\,dx$ also exists.
2. If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ also diverges.

A similar result can be proven for $I = (-\infty, a]$ and improper integrals over this domain.

Proof. We prove each assertion separately. Since f and g are continuous over $[a,\infty)$, these functions are Riemann integrable over the interval $[a,t]$ for any finite $t > a$.

1. Since f and g are non-negative, by the ordering property and additivity of integrals, for any $t \ge a$ we have

$$\int_a^t f(x)\,dx \le \int_a^t g(x)\,dx \le \lim_{t\to\infty}\int_a^t g(x)\,dx = \int_a^\infty g(x)\,dx.$$

Moreover, the integral function $F(t) = \int_a^t f(x)\,dx$ on $[a,\infty)$ is an increasing function. Thus, the limit of $F(t)$ as $t \to \infty$ exists since $F(t)$ is bounded by the finite number $\int_a^\infty g(x)\,dx$.

2. For any $t \ge a$, we have the ordering

$$\int_a^t f(x)\,dx \le \int_a^t g(x)\,dx.$$

Taking the limit $t \to \infty$ on both sides, since $\int_a^\infty f(x)\,dx$ diverges, it must approach ∞. Thus, we conclude that $\lim_{t\to\infty}\int_a^t g(x)\,dx = \infty$, implying that $\int_a^\infty g(x)\,dx$ also diverges. ■

Limit Comparison Test for Improper Riemann Integrals

Let $I = [a,\infty)$. Suppose that $f, g : I \to \mathbb{R}$ are continuous positive functions. Suppose further that

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$$

for some $0 < L < \infty$. Then either both improper Riemann integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ exist, or both diverge. In other words,

$$\int_a^\infty f(x)\,dx \text{ exists} \iff \int_a^\infty g(x)\,dx \text{ exists}.$$

Theorem 54 (Integral Test) *Suppose $f : [1, \infty) \to [0, \infty)$ is decreasing and Riemann integrable on $[1, b]$ for all $b > 1$.*

(i) The series $\sum_{n=1}^{\infty} f(n)$ is convergent if and only if the improper integral $\int_1^{\infty} f(t)\,dt$ is convergent.

(ii) If the series is convergent, then for every $n \in \mathbb{N}$,

$$\int_{n+1}^{\infty} f(t)\,dt \;\le\; \sum_{k=n+1}^{\infty} f(k) \;\le\; \int_{n}^{\infty} f(t)\,dt.$$

Proof. (i) Since f is decreasing and integrable over $[k-1, k]$, we have

$$f(k) \;\le\; \int_{k-1}^{k} f(t)\,dt \;\le\; f(k-1) \qquad (k \ge 2). \tag{2.1}$$

Summing (**??**) from $k = 2$ to $k = n$ gives

$$\sum_{k=2}^{n} f(k) \;\le\; \int_1^n f(t)\,dt \;\le\; \sum_{k=1}^{n-1} f(k).$$

If $\sum_{k=1}^{\infty} f(k)$ converges, then the rightmost bound shows $\int_1^n f$ is bounded above by the series' sum; hence $\int_1^{\infty} f$ converges. Conversely, if $\int_1^{\infty} f$ converges, then the leftmost bound shows the partial sums $\sum_{k=2}^{n} f(k)$ are bounded, and since $f \ge 0$ they form an increasing sequence; thus $\sum_{k=1}^{\infty} f(k)$ converges.

(ii) Summing (**??**) from $k = n+1$ to $k = m$ $(m > n)$ yields

$$\sum_{k=n+1}^{m} f(k) \;\le\; \int_n^m f(t)\,dt \;\le\; \sum_{k=n}^{m-1} f(k).$$

Rewriting the middle term by shifting limits,

$$\int_{n+1}^{m+1} f(t)\,dt \;\le\; \sum_{k=n+1}^{m} f(k) \;\le\; \int_n^m f(t)\,dt.$$

Letting $m \to \infty$ (and using (i) for existence of the improper integral) gives

$$\int_{n+1}^{\infty} f(t)\,dt \;\le\; \sum_{k=n+1}^{\infty} f(k) \;\le\; \int_{n}^{\infty} f(t)\,dt,$$

which proves (ii). $\square$

Exercises: Improper Integrals

1. Evaluate the following improper integrals:

(a) $\displaystyle\int_0^1 \log x\,dx$

Solution. This is improper at $x = 0$. Integrate by parts (or use a known primitive):

$$\int \log x\,dx = x\log x - x + C.$$

Hence

$$\int_0^1 \log x\,dx = \lim_{\varepsilon\to 0^+}\Big[x\log x - x\Big]_\varepsilon^1 = \Big(1\cdot 0 - 1\Big) - \lim_{\varepsilon\to 0^+}(\varepsilon\log\varepsilon - \varepsilon) = -1,$$

since $\varepsilon\log\varepsilon \to 0$ as $\varepsilon \to 0^+$.

(b) $\displaystyle\int_1^2 \frac{1}{x\log x}\,dx$

Solution. Improper at $x = 1^+$. With $u = \log x$, $du = dx/x$, so

$$\int \frac{1}{x\log x}\,dx = \int \frac{1}{u}\,du = \log|u| + C = \log(\log x) + C.$$

Thus

$$\int_1^2 \frac{1}{x\log x}\,dx = \lim_{\varepsilon\to 0^+}\Big[\log(\log x)\Big]_{1+\varepsilon}^2 = \log(\log 2) - \lim_{\varepsilon\to 0^+}\log(\log(1+\varepsilon)).$$

Since $\log(1+\varepsilon) \sim \varepsilon \to 0^+$, we have $\log(\log(1+\varepsilon)) \to -\infty$, hence the integral

$$\int_1^2 \frac{1}{x\log x}\,dx = +\infty \quad \text{(diverges)}.$$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

Solution. Use $\arctan x$:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \Big[\arctan x\Big]_{-\infty}^{\infty} = \frac{\pi}{2} - \Big(-\frac{\pi}{2}\Big) = \pi.$$

(d) $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}\,(x+1)}$

Solution. Let $x = t^2$ $(t \ge 0)$. Then $dx = 2t\,dt$ and $\sqrt{x} = t$:

$$\int_0^{\infty} \frac{dx}{\sqrt{x}\,(x+1)} = \int_0^{\infty} \frac{2t\,dt}{t\,(t^2+1)} = 2\int_0^{\infty} \frac{dt}{t^2+1} = 2\Big[\arctan t\Big]_0^{\infty} = 2\cdot\frac{\pi}{2} = \pi.$$

(e) $\displaystyle\int_{-2}^{2} \frac{dx}{\sqrt{4-x^2}}$

Solution. Set $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$, and $\sqrt{4-x^2} = 2\cos\theta$ with $\theta \in [-\pi/2, \pi/2]$:

$$\int_{-2}^{2} \frac{dx}{\sqrt{4-x^2}} = \int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\,d\theta}{2\cos\theta} = \int_{-\pi/2}^{\pi/2} d\theta = \pi.$$

Exercise 55 *Suppose $f \in \mathcal{R}(a,c)$ for all $c > a$ (i.e. f is Riemann integrable on every finite interval $[a,c]$). Prove the equivalence of the following statements:*

(a) f has a convergent improper integral on $[a,\infty)$, i.e.

$$\int_a^\infty f(x)\,dx := \lim_{t\to\infty} \int_a^t f(x)\,dx \quad \textit{exists (as a finite real number).}$$

(b) For every $\varepsilon > 0$ there exists $N > a$ such that for all $b, c > N$,

$$\left|\int_b^c f(x)\,dx\right| < \varepsilon.$$

Proof. Define the function of the upper limit

$$F(t) := \int_a^t f(x)\,dx, \qquad t > a.$$

By assumption $F(t)$ is well-defined for all $t > a$.

(a)⇒(b). Assume $\lim_{t\to\infty} F(t) = L \in \mathbb{R}$. Let $\varepsilon > 0$. Then there exists $N > a$ such that for all $u > N$,

$$|F(u) - L| < \varepsilon/2.$$

Hence for any $b, c > N$,

$$\left|\int_b^c f(x)\,dx\right| = |F(c) - F(b)| \le |F(c) - L| + |F(b) - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus (b) holds.

(b)⇒(a). Assume (b). We show that $\left(F(t)\right)_{t>a}$ is a Cauchy net (equivalently, the limit $\lim_{t\to\infty} F(t)$ exists). Fix $\varepsilon > 0$ and choose N as in (b). Then for all $b, c > N$,

$$|F(c) - F(b)| = \left|\int_b^c f(x)\,dx\right| < \varepsilon.$$

Hence the values $F(t)$ form a Cauchy family for large t, and since $\mathbb{R}$ is complete, there exists $L \in \mathbb{R}$ with

$$\lim_{t\to\infty} F(t) = L.$$

By definition, this means the improper integral $\int_a^\infty f(x)\,dx$ converges (to L). Thus (a) holds.

We have proved (a) $\iff$ (b). ∎

Exercise 56 *Suppose $f, g \in \mathcal{R}(a,c)$ for all $c > a$, and $|f(x)| \le g(x)$ for all $x \in [a,\infty)$. If $\int_a^\infty g(x)\,dx$ is convergent, prove that $\int_a^\infty f(x)\,dx$ is also convergent. Use this to prove the existence of the integral $\int_0^\infty \frac{dx}{1+x^4}$.*

Proof. Assume $|f| \le g$ on $[a, \infty)$ and $\int_a^\infty g < \infty$. By the Cauchy criterion for improper integrals, for every $\varepsilon > 0$ there exists $N > a$ such that for all $b, c > N$,

$$\left| \int_b^c g(x)\,dx \right| = \int_b^c g(x)\,dx < \varepsilon.$$

Then, for all $b, c > N$,

$$\left| \int_b^c f(x)\,dx \right| \;\le\; \int_b^c |f(x)|\,dx \;\le\; \int_b^c g(x)\,dx \;<\; \varepsilon.$$

Hence $\int_a^\infty f(x)\,dx$ satisfies the Cauchy criterion and therefore converges. Moreover, $\int_a^\infty |f(x)|\,dx \le \int_a^\infty g(x)\,dx < \infty$, so the convergence is absolute.

(Application to $\displaystyle\int_0^\infty \frac{dx}{1+x^4}$). Let $f(x) = \frac{1}{1+x^4}$ on $(0, \infty)$.
Near 0*:* f is continuous on $[0, 1]$, hence Riemann integrable there.
On the tail $[1, \infty)$*:* For $x \ge 1$,

$$0 \le \frac{1}{1+x^4} \;\le\; \frac{1}{x^4} =: g(x).$$

Since

$$\int_1^\infty \frac{dx}{x^4} = \left[-\frac{1}{3x^3} \right]_1^\infty = \frac{1}{3},$$

the comparison above shows $\int_1^\infty \frac{dx}{1+x^4}$ converges. Combining with integrability on $[0, 1]$, we conclude

$$\int_0^\infty \frac{dx}{1+x^4} \quad \text{exists (is finite).}$$

■

Example 4: Evaluation of the Dirichlet Integral

We evaluate the improper integral:

$$\int_0^\infty \frac{\sin x}{x}\,dx.$$

Although the integrand is undefined at $x = 0$, we extend it continuously by defining:

$$\frac{\sin x}{x}\Big|_{x=0} := \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

This makes the function continuous on $[0, \infty)$. We define the sine integral function:

$$\mathrm{Si}(x) := \int_0^x \frac{\sin t}{t}\,dt.$$

The function $\mathrm{Si}(x)$ is continuous for all $x \ge 0$, although it cannot be written using elementary functions.

The integrand $\frac{\sin x}{x}$ changes sign on each interval $[n\pi, (n+1)\pi]$, and we define:

$$a_n := \left| \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x}\,dx \right|.$$

Then (a_n) is a decreasing sequence with $a_n \to 0$, and:

$$\mathrm{Si}(n\pi) = \sum_{k=0}^{n-1} (-1)^k a_k.$$

By the Leibniz criterion (alternating series test), this sum converges, so:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{n\to\infty} \mathrm{Si}(n\pi)$$

exists.

To evaluate the limit, we consider:

$$\mathrm{Si}\left(\frac{\lambda\pi}{2}\right) = \int_0^{\pi/2} \frac{\sin(\lambda x)}{x}\,dx,$$

by the substitution $t = \lambda x$.

Define the auxiliary function:

$$g(x) := \begin{cases} \frac{1}{x} - \frac{1}{\sin x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then g is continuous on $[0, \pi/2]$, and we decompose the integrand:

$$\frac{\sin(\lambda x)}{x} = \frac{\sin(\lambda x)}{\sin x} + \sin(\lambda x) \cdot g(x).$$

We now use the following key result:

Theorem 57 (Riemann's Lemma) *Let $f \in C^1([a, b])$. Then:*

$$\lim_{|k|\to\infty} \int_a^b f(x) \sin(kx)\,dx = 0.$$

Proof. Let $F(k) := \int_a^b f(x) \sin(kx)\,dx$. For $k \neq 0$, we integrate by parts:

$$F(k) = -\frac{f(x)\cos(kx)}{k}\bigg|_a^b + \frac{1}{k}\int_a^b f'(x)\cos(kx)\,dx.$$

If $|f(x)| \leq M$ and $|f'(x)| \leq M$, then:

$$|F(k)| \leq \frac{2M}{|k|} + \frac{M(b-a)}{|k|} = \frac{2M + M(b-a)}{|k|} \to 0 \quad \text{as } |k| \to \infty.$$

■

We apply this lemma with $f(x) = g(x) \in C^1([0, \pi/2])$, which gives:

$$\lim_{\lambda\to\infty} \int_0^{\pi/2} \sin(\lambda x) \cdot g(x)\,dx = 0.$$

Hence,

$$\lim_{\lambda\to\infty} \int_0^{\pi/2} \frac{\sin(\lambda x)}{x}\,dx = \lim_{\lambda\to\infty} \int_0^{\pi/2} \frac{\sin(\lambda x)}{\sin x}\,dx.$$

We now evaluate the remaining limit. For every integer $n \geq 1$, the following identity holds:

$$\frac{\sin((2n+1)x)}{\sin x} = 1 + 2\sum_{k=1}^{n} \cos(2kx).$$

Integrating term-by-term over $[0, \pi/2]$, and noting that each $\cos(2kx)$ integrates to zero, we get:

$$\int_0^{\pi/2} \frac{\sin((2n+1)x)}{\sin x}\, dx = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}.$$

Taking the limit $n \to \infty$, we conclude:

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}. \quad \blacksquare$$

The Gamma Function

Definition 58 (Euler's Integral Representation of the Gamma Function) *For $x > 0$, the Gamma function is defined by*

$$\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t}\, dt.$$

Convergence of the improper integral $\Gamma(x) = \displaystyle\int_0^{\infty} t^{x-1}e^{-t}\, dt$] We show carefully that the integral converges for all $x > 0$ by splitting it at a convenient point (say 1):

$$\int_0^{\infty} t^{x-1}e^{-t}\, dt = \underbrace{\int_0^{1} t^{x-1}e^{-t}\, dt}_{(*)} + \underbrace{\int_1^{\infty} t^{x-1}e^{-t}\, dt}_{(**)}.$$

The integral $(*)$**.** Since $e^{-t} \leq 1$ for all $t \geq 0$,

$$0 \leq t^{x-1}e^{-t} \leq t^{x-1} \quad (0 < t \leq 1).$$

Hence, by comparison,

$$\int_0^{1} t^{x-1}e^{-t}\, dt \;\leq\; \int_0^{1} t^{x-1}\, dt = \left[\frac{t^x}{x}\right]_0^1 = \frac{1}{x} < \infty \quad \text{iff } x > 0.$$

This also shows the *necessity* of $x > 0$: for $x \leq 0$, the model integral $\int_0^1 t^{x-1}\, dt$ diverges (the exponent at $t = 0$ is ≤ -1).

The integral $(**)$**.** There are two standard (equivalent) ways to bound the tail.

Method A: Limit/comparison. Using the well-known limit

$$\lim_{t\to\infty} t^{x+1}e^{-t} = 0,$$

by the definition of a limit there exists $T \geq 1$ such that

$$t^{x+1}e^{-t} \leq 1 \quad \text{for all } t \geq T.$$

Equivalently, for $t \geq T$,

$$e^{-t} \leq t^{-(x+1)} \quad \Rightarrow \quad t^{x-1}e^{-t} \leq t^{x-1} \cdot t^{-(x+1)} = t^{-2}.$$

Therefore

$$\int_T^\infty t^{x-1}e^{-t}\, dt \;\leq\; \int_T^\infty t^{-2}\, dt = \frac{1}{T} < \infty.$$

The finite piece $\int_1^T t^{x-1}e^{-t}\, dt$ is also finite because the integrand is continuous on $[1, T]$.

Method B: For any integer $m \geq 1$, the series $e^t = \sum_{k=0}^\infty \frac{t^k}{k!}$ implies

$$e^t \;\geq\; \frac{t^m}{m!} \quad \text{for all } t \geq 0 \quad \Longrightarrow \quad e^{-t} \;\leq\; m!\, t^{-m}.$$

Choose an integer $m \geq x + 1$. Then for $t \geq 1$,

$$t^{x-1}e^{-t} \;\leq\; m!\, t^{x-1-m} \;\leq\; m!\, t^{-2},$$

since $x - 1 - m \leq -2$. Hence

$$\int_1^\infty t^{x-1}e^{-t}\, dt \;\leq\; m! \int_1^\infty t^{-2}\, dt = m! < \infty.$$

Both $(*)$ and $(**)$ converge for $x > 0$, so the improper integral $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt$ is finite exactly for $x > 0$.

Theorem 59 (Functional Equation and Factorial Formula) *For all $x > 0$,*

$$\Gamma(x + 1) = x\, \Gamma(x).$$

In particular, for all $n \in \mathbb{N}$,

$$\Gamma(n + 1) = n!.$$

Proof. Consider

$$\int_\varepsilon^R t^x e^{-t}\, dt \quad \text{for } 0 < \varepsilon < R.$$

Integration by parts with $u = t^x$, $dv = e^{-t}dt$ (hence $du = xt^{x-1}dt$, $v = -e^{-t}$) yields

$$\int_\varepsilon^R t^x e^{-t}\, dt = -t^x e^{-t}\Big|_{t=\varepsilon}^{t=R} + x \int_\varepsilon^R t^{x-1}e^{-t}\, dt.$$

Letting $\varepsilon \to 0$ and $R \to \infty$, the boundary terms vanish because $\lim_{R\to\infty} R^x e^{-R} = 0$ and $\lim_{\varepsilon\to 0} \varepsilon^x e^{-\varepsilon} = 0$. Thus,

$$\Gamma(x + 1) = x\Gamma(x).$$

Moreover,

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1,$$

so by iteration we obtain $\Gamma(n + 1) = n!$. ∎

Remark The Gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ extends the factorial function to the positive reals in the sense that $\Gamma(n+1) = n!$ for integers n. The functional equation alone does not determine Γ uniquely; an additional property such as *logarithmic convexity* is needed for full characterization.

Exercises

Exercise 1. Let

$$f(x) = \begin{cases} 1, & x \in [0, \frac{1}{2}), \\ 0, & x \in [\frac{1}{2}, 1]. \end{cases}$$

Show that f is Riemann integrable on $[0, 1]$ and compute $\int_0^1 f(x)\,dx$.

Solution.

Fix $\varepsilon > 0$ and consider the partition

$$P = \Big\{0,\ \tfrac{1}{2} - \tfrac{\varepsilon}{2},\ \tfrac{1}{2},\ \tfrac{1}{2} + \tfrac{\varepsilon}{2},\ 1\Big\}.$$

On this partition, the only discrepancy between upper and lower sums occurs in the small interval $[\frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2}]$, where the function jumps from 1 to 0. Thus

$$L(P, f) = \tfrac{1}{2} - \tfrac{\varepsilon}{2}, \qquad U(P, f) = \tfrac{1}{2}.$$

Hence

$$0 \le U(P, f) - L(P, f) = \tfrac{\varepsilon}{2}.$$

Therefore f is Riemann integrable.

Finally, by the squeeze theorem,

$$\tfrac{1}{2} - \tfrac{\varepsilon}{2} \ \le\ \int_0^1 f(x)\,dx \ \le\ \tfrac{1}{2} \quad \text{for all } \varepsilon > 0.$$

Letting $\varepsilon \to 0$, we conclude

$$\int_0^1 f(x)\,dx = \tfrac{1}{2}.$$

Exercise 2. Determine whether

$$f(x) = \begin{cases} x, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases} \qquad x \in [0, 1],$$

is Riemann integrable on $[0, 1]$.

Solution. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[0, 1]$. Since each interval $[x_{k-1}, x_k]$ contains irrational points, we always have

$$\inf_{[x_{k-1}, x_k]} f = 0,$$

so that

$$L(f, P) = \sum_{k=1}^{n} \inf_{[x_{k-1}, x_k]} f\,(x_k - x_{k-1}) = 0 \quad \text{for all } P,$$

and therefore $\sup_P L(f, P) = 0$.

On the other hand, since rationals are dense, the supremum of f on $[x_{k-1}, x_k]$ is attained arbitrarily close to x_k, hence

$$\sup_{[x_{k-1}, x_k]} f = x_k,$$

and thus

$$U(f,P)=\sum_{k=1}^{n} x_k(x_k-x_{k-1}).$$

A direct computation shows

$$x_k(x_k-x_{k-1})=\tfrac{1}{2}(x_k^2-x_{k-1}^2)+\tfrac{1}{2}(x_k-x_{k-1})^2,$$

and summing over all k yields

$$U(f,P)=\tfrac{1}{2}+\tfrac{1}{2}\sum_{k=1}^{n}(x_k-x_{k-1})^2.$$

If we denote $\Delta_k=x_k-x_{k-1}$ and $\|P\|=\max_k \Delta_k$, then

$$0\le U(f,P)-\tfrac{1}{2}=\tfrac{1}{2}\sum_{k=1}^{n}\Delta_k^2\le \tfrac{1}{2}\,\|P\|\sum_{k=1}^{n}\Delta_k=\tfrac{1}{2}\,\|P\|.$$

Thus, whenever $\|P\|<2\varepsilon$, we obtain

$$\tfrac{1}{2}\le U(f,P)\le \tfrac{1}{2}+\varepsilon.$$

It follows that

$$\inf_P U(f,P)=\tfrac{1}{2}.$$

Collecting the results,

$$U(f)=\tfrac{1}{2},\qquad L(f)=0.$$

Since the upper and lower Riemann integrals differ, the function f is not Riemann integrable on $[0,1]$.

Remark. Because $f(x)=0$ on all irrationals and the rationals form a set of measure zero, the Lebesgue integral exists and equals

$$\int_0^1 f(x)\,dx=0.$$

Exercise 3. Decide whether

$$f(x)=\begin{cases}2x, & x\in[0,\frac{1}{2}),\\ x-2, & x\in[\frac{1}{2},1],\end{cases}$$

is Riemann integrable on $[0,1]$ using an explicit sequence of partitions P_n. Then compute $\displaystyle\int_0^1 f(x)\,dx$.

For $n\in\mathbb{N}$, set $h=\frac{1}{2n}$ and take

$$P_n=\{0,h,2h,\ldots,\tfrac{1}{2},\tfrac{1}{2}+h,\ldots,1\},\qquad \|P_n\|=h\to 0.$$

No subinterval crosses the jump at $x=\frac{1}{2}$.

On $[0,\frac{1}{2}]$, $f(x)=2x$ is increasing, so on $I_k=[(k-1)h,kh]$ the minimum is $2(k-1)h$:

$$L(f,[0,\tfrac{1}{2}],P_n)=\sum_{k=1}^{n}(2(k-1)h)\cdot h=2h^2\sum_{k=1}^{n}(k-1)=h^2\,\frac{n(n-1)}{1}=\frac{n-1}{4n}.$$

On $[\frac{1}{2}, 1]$, $f(x) = x - 2$ is increasing, so on $J_j = [\frac{1}{2} + (j-1)h, \frac{1}{2} + jh]$ the minimum is $-\frac{3}{2} + (j-1)h$:

$$L(f, [\tfrac{1}{2}, 1], P_n) = \sum_{j=1}^{n} (-\tfrac{3}{2} + (j-1)h) \cdot h = -\frac{3n}{2}h + h^2 \sum_{j=1}^{n} (j-1) = -\frac{3}{4} + \frac{n-1}{8n}.$$

Hence

$$L(f, P_n) = \frac{n-1}{4n} - \frac{3}{4} + \frac{n-1}{8n} = \frac{3(n-1)}{8n} - \frac{3}{4}.$$

On $[0, \frac{1}{2}]$, the maximum on I_k is $2kh$:

$$U(f, [0, \tfrac{1}{2}], P_n) = \sum_{k=1}^{n} (2kh) \cdot h = 2h^2 \sum_{k=1}^{n} k = h^2 n(n+1) = \frac{n+1}{4n}.$$

On $[\frac{1}{2}, 1]$, the maximum on J_j is $-\frac{3}{2} + jh$:

$$U(f, [\tfrac{1}{2}, 1], P_n) = \sum_{j=1}^{n} (-\tfrac{3}{2} + jh) \cdot h = -\frac{3n}{2}h + h^2 \sum_{j=1}^{n} j = -\frac{3}{4} + \frac{n+1}{8n}.$$

Thus

$$U(f, P_n) = \frac{n+1}{4n} - \frac{3}{4} + \frac{n+1}{8n} = \frac{3(n+1)}{8n} - \frac{3}{4}.$$

Note

$$U(f, P_n) - L(f, P_n) = \frac{3(n+1)}{8n} - \frac{3}{4} - \Big(\frac{3(n-1)}{8n} - \frac{3}{4}\Big) = \frac{3}{4n} \xrightarrow[n\to\infty]{} 0,$$

and

$$\lim_{n\to\infty} L(f, P_n) = \frac{3}{8} - \frac{3}{4} = -\frac{3}{8} = \lim_{n\to\infty} U(f, P_n).$$

Since for all n,

$$L(f, P_n) \le \int_0^1 f(x)\,dx \le U(f, P_n),$$

the squeeze theorem yields

$$\int_0^1 f(x)\,dx = -\frac{3}{8}.$$

Exercise 4. Suppose $f, g : [a, b] \to \mathbb{R}$ and $f(x) = g(x)$ for all $x \in [a, b)$. If $f \in \mathcal{R}[a, b]$, prove that $g \in \mathcal{R}[a, b]$ and

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx.$$

Solution. Both f and g are bounded, since they differ only at the single point b. Let $M > 0$ be such that $|f(x)|, |g(x)| \le M$ on $[a, b]$.

Because $f \in \mathcal{R}[a, b]$, for any $\varepsilon > 0$ there exists a partition Q of $[a, b]$ with

$$U(f, Q) - L(f, Q) < \tfrac{\varepsilon}{2}.$$

Now refine Q by inserting $b - \delta$ for some $\delta > 0$ to obtain a partition P. Refinement cannot increase the gap, so

$$U(f, P) - L(f, P) < \tfrac{\varepsilon}{2}.$$

The only difference between f and g occurs on the final subinterval $[b-\delta, b]$. On that interval, the upper (resp. lower) sum of a bounded function h contributes at most $M\delta$ in magnitude. Hence

$$|U(f,P)-U(g,P)| \le 2M\delta, \qquad |L(f,P)-L(g,P)| \le 2M\delta.$$

Choosing $\delta < \frac{\varepsilon}{4M}$ ensures that each difference is $< \frac{\varepsilon}{2}$. Therefore

$$U(g,P)-L(g,P) \;\le\; (U(f,P)-L(f,P)) + |U(g,P)-U(f,P)| + |L(f,P)-L(g,P)| < \varepsilon.$$

Thus $g \in \mathcal{R}[a,b]$.

Finally, since integrals are trapped between upper and lower sums,

$$\Big|\int_a^b f(x)\,dx - \int_a^b g(x)\,dx\Big| \le \max\{|U(f,P)-U(g,P)|,\ |L(f,P)-L(g,P)|\} < \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we conclude

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

Remark. In general, if we modify the value of a Riemann integrable function at a finite set of points, the function remains Riemann integrable and the value of the integral does not change. This is because upper and lower sums depend only on suprema and infima over intervals, and changing finitely many points cannot affect these values.

Example. Let

$$g(x) = \begin{cases} 1, & x = 0, \\ \dfrac{\sin x}{x}, & x \in (0,1], \end{cases} \qquad f(x) = \begin{cases} A, & x = 0 \quad (A \in \mathbb{R}), \\ \dfrac{\sin x}{x}, & x \in (0,1]. \end{cases}$$

The function g is continuous on $[0,1]$, since

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1 = g(0),$$

and therefore $g \in \mathcal{R}[0,1]$. Since f differs from g only at the single point $x = 0$, the remark applies, giving

$$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx.$$

Thus the value of the integral is independent of the choice of A.

Exercise 5. Let $f : [a,b] \to \mathbb{R}$ be continuous and $f(x) \ge 0$ on $[a,b]$. If $\displaystyle\int_a^b f(x)\,dx = 0$, prove that $f(x) = 0$ for all $x \in [a,b]$.

Solution. Suppose, for contradiction, that there exists $x_0 \in [a,b]$ with $f(x_0) > 0$. Set

$$m = \tfrac{1}{2} f(x_0) > 0.$$

By continuity of f at x_0, for $\varepsilon = m$ there exists $\delta > 0$ such that

$$|x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - f(x_0)| < m.$$

From this inequality we obtain

$$f(x) > f(x_0) - m = m,$$

so every point sufficiently close to x_0 has $f(x) \geq m$.

Now define

$$\delta' = \min\{\delta,\ x_0 - a,\ b - x_0\} > 0,$$

and set

$$I = [x_0 - \delta',\ x_0 + \delta'].$$

Then $I \subseteq [a, b]$ and $f(x) \geq m$ for all $x \in I$.

Next, introduce the auxiliary function

$$g(x) = \begin{cases} f(x), & x \in I, \\ 0, & x \notin I. \end{cases}$$

Clearly $0 \leq g(x) \leq f(x)$ on $[a, b]$, so

$$\int_a^b f(x)\,dx \;\geq\; \int_a^b g(x)\,dx = \int_I f(x)\,dx.$$

But since $f(x) \geq m$ on I and $|I| = 2\delta'$, we obtain

$$\int_I f(x)\,dx \;\geq\; \int_I m\,dx = m \cdot |I| = 2m\delta' > 0.$$

Thus

$$\int_a^b f(x)\,dx > 0,$$

contradicting the hypothesis $\int_a^b f(x)\,dx = 0$.

Therefore no such x_0 can exist, and the only possibility is that

$$f(x) = 0 \quad \text{for all } x \in [a, b].$$

Counterexample (necessity of continuity). Consider

$$f(x) = \begin{cases} 1, & x = 0, \\ 0, & x \in (0, 1], \end{cases}$$

on the interval $[0, 1]$. Here $f(x) \geq 0$ everywhere, but

$$\int_0^1 f(x)\,dx = 0,$$

since changing the value of a function at a single point does not affect the Riemann integral. Nevertheless, $f(0) = 1 \neq 0$, so the conclusion fails if continuity is dropped.

Counterexample (necessity of nonnegativity). Consider the continuous function

$$f(x) = \sin(2\pi x), \qquad x \in [0, 1].$$

It is continuous but changes sign: $f(x) > 0$ on $(0, \frac{1}{2})$ and $f(x) < 0$ on $(\frac{1}{2}, 1)$. Computing the integral,

$$\int_0^1 \sin(2\pi x)\,dx = \Big[-\tfrac{1}{2\pi}\cos(2\pi x)\Big]_0^1 = -\tfrac{1}{2\pi}(\cos(2\pi) - \cos(0)) = 0.$$

Hence $\int_0^1 f(x)\,dx = 0$ but $f \not\equiv 0$.

Exercise 6. Let $f : [a, b] \to \mathbb{R}$ be continuous. Assume that

$$\int_a^b f(x)\,g(x)\,dx = 0 \quad \text{for every } g \in \mathcal{R}(a, b).$$

Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution. Since f is continuous, we know $f \in \mathcal{R}(a, b)$. Taking $g = f$ in the hypothesis gives

$$\int_a^b f(x)^2\,dx = 0.$$

The function f^2 is continuous and satisfies $f^2(x) \geq 0$ for all $x \in [a, b]$. Applying the result of Exercise 5 to f^2, we conclude that

$$f(x)^2 = 0 \quad \text{for all } x \in [a, b].$$

Hence $f(x) = 0$ identically on $[a, b]$.

Remark (Hilbert space perspective). Consider the space $\mathcal{R}([0, 1])$ of Riemann integrable functions on $[0, 1]$, equipped with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

Exercise 6 shows that if $f \in C([0, 1])$ satisfies

$$\langle f, g\rangle = 0 \quad \text{for all } g \in \mathcal{R}([0, 1]),$$

then necessarily $f \equiv 0$. In other words, the orthogonal complement of the set of continuous functions $C([0, 1])$ inside the inner product space $\mathcal{R}([0, 1])$ is trivial:

$$C([0, 1])^{\perp} = \{0\}.$$

This means that continuous functions are "dense in the sense of orthogonality" within $\mathcal{R}([0, 1])$: no nonzero function can be orthogonal to all continuous test functions.

Definition 60 (Metric and Metric Space) *Let X be a set. A function $\rho : X \times X \to [0, \infty)$ is called a* metric *on X if, for all $x, y, z \in X$,*

(M1) ***Nonnegativity:*** $\rho(x, y) \geq 0$.

(M2) ***Identity of indiscernibles:*** $\rho(x, y) = 0 \iff x = y$.

(M3) ***Symmetry:*** $\rho(x, y) = \rho(y, x)$.

(M4) ***Triangle inequality:*** $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

The pair (X, ρ) is then called a metric space.

Exercise 7. Let $C([a,b])$ denote the space of real-valued continuous functions on $[a,b]$. For $f,g \in C([a,b])$, define

$$\rho(f,g) := \int_a^b |f(x) - g(x)|\,dx.$$

(i) ρ is a metric on $C([a,b])$ in the sense of (M1)–(M4).

(ii) Writing $\|h\|_{L^1} := \int_a^b |h(x)|\,dx$, we have $\rho(f,g) = \|f-g\|_{L^1}$; i.e. ρ is the distance associated with the L^1–norm.

Proof. *(M1)* For each x, $|f(x)-g(x)| \geq 0$, hence $\rho(f,g) = \int_a^b |f-g|\,dx \geq 0$.

(M2) If $f = g$, then $|f-g| \equiv 0$ and $\rho(f,g) = 0$. Conversely, if $\rho(f,g) = 0$ then $\phi := |f-g| \in C([a,b])$, $\phi \geq 0$ and $\int_a^b \phi = 0$. If $\phi(x_0) > 0$ for some x_0, continuity gives $\varepsilon, \delta > 0$ with $\phi(x) \geq \varepsilon$ on $(x_0 - \delta, x_0 + \delta) \cap [a,b]$, whence

$$\int_a^b \phi \;\geq\; \varepsilon \cdot \text{length}\big((x_0-\delta, x_0+\delta) \cap [a,b]\big) \;>\; 0,$$

a contradiction. Thus $\phi \equiv 0$ and $f = g$.

(M3) $|f-g| = |g-f|$ pointwise, so $\rho(f,g) = \rho(g,f)$.

(M4) The pointwise triangle inequality gives $|f-h| \leq |f-g| + |g-h|$ on $[a,b]$. Integrating and using linearity/monotonicity of the integral,

$$\rho(f,h) = \int_a^b |f-h| \;\leq\; \int_a^b |f-g| + \int_a^b |g-h| \;=\; \rho(f,g) + \rho(g,h).$$

This proves (i). For (ii), the equality $\rho(f,g) = \|f-g\|_{L^1}$ is immediate from the definitions. ■

Exercise 8.

For $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0, & x \leq \frac{1}{2}, \\ n\Big(x - \frac{1}{2}\Big), & \frac{1}{2} < x \leq \frac{1}{2} + \frac{1}{n}, \\ 1, & x > \frac{1}{2} + \frac{1}{n}. \end{cases}$$

Show that for all m, n,

$$\rho(f_n, f_m) = \int_0^1 |f_n - f_m|\,dx = \Big|\frac{1}{2m} - \frac{1}{2n}\Big| \xrightarrow[n,m\to\infty]{} 0.$$

Solution Assume $n \geq m$ and set $t = x - \frac{1}{2}$. Then f_n and f_m can differ only for $t \in (0, \frac{1}{m}]$. Split into two regions.

(1) $t \in (0, \frac{1}{n}]$: $f_n = nt$, $f_m = mt$, hence $|f_n - f_m| = (n-m)t$ and

$$\int_0^{1/n} (n-m)t\,dt \;=\; \frac{n-m}{2n^2}.$$

(2) $t \in (\frac{1}{n}, \frac{1}{m}]$: $f_n = 1$, $f_m = mt$, so $|f_n - f_m| = 1 - mt$ and

$$\int_{1/n}^{1/m} (1-mt)\,dt = \Big[t - \tfrac{m}{2}t^2\Big]_{1/n}^{1/m} = \frac{1}{2m} - \frac{1}{n} + \frac{m}{2n^2}.$$

Adding the two contributions gives

$$\rho(f_n, f_m) = \frac{n-m}{2n^2} + \Big(\frac{1}{2m} - \frac{1}{n} + \frac{m}{2n^2}\Big) = \frac{1}{2m} - \frac{1}{2n} = \Big|\frac{1}{2m} - \frac{1}{2n}\Big|.$$

Hence $\rho(f_n, f_m) \to 0$ as $n, m \to \infty$, so (f_n) is Cauchy in the L^1 metric.

Remark With the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$ on $\mathcal{R}([0,1])$, Exercise from above shows that the orthogonal complement of $C([0,1])$ is trivial: $C([0,1])^{\perp} = \{0\}$. On the other hand, the metric space $(C([0,1]), \rho)$ is *not* complete: in Exercise from above, (f_n) is Cauchy but converges pointwise (and in L^1) to the discontinuous step function $\mathbf{1}_{(1/2,1]}$, which is not in $C([0,1])$. The completion of $(C([0,1]), \rho)$ is the Banach space $L^1([0,1])$ (functions modulo equality a.e.).

Exercise 9.

$$\text{(a)} \quad \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\cos\Big(\frac{k\pi}{n}\Big) = \int_0^1 \cos(\pi x)\,dx = \frac{1}{\pi}\sin(\pi x)\Big|_0^1 = 0.$$

$$\text{(b)} \quad \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\,\frac{1}{1+(k/n)^2} = \int_0^1 \frac{dx}{1+x^2} = \arctan x\Big|_0^1 = \frac{\pi}{4}.$$

$$\text{(c)} \quad \lim_{n\to\infty} \frac{\sum_{k=1}^{2n} k^3}{24\,n^4} = \lim_{n\to\infty} \frac{\big(\frac{(2n)(2n+1)}{2}\big)^2}{24\,n^4} = \lim_{n\to\infty} \frac{n^2(2n+1)^2}{24\,n^4} = \lim_{n\to\infty} \frac{4+\frac{4}{n}+\frac{1}{n^2}}{24} = \frac{1}{6}.$$

Exercise 10. Suppose $f \in \mathcal{R}(a,b)$ and $c \in \mathbb{R}$. Define $g : [a+c, b+c] \to \mathbb{R}$ by

$$g(x) = f(x-c) \quad (x \in [a+c, b+c]).$$

Show that $g \in \mathcal{R}(a+c, b+c)$ and

$$\int_{a+c}^{b+c} g(x)\,dx = \int_a^b f(x)\,dx.$$

Solution Let $P = \{x_0 < \cdots < x_n\}$ be a partition of $[a,b]$ and set $Q = P + c = \{y_i := x_i + c\}_{i=0}^n$, a partition of $[a+c, b+c]$. For each i,

$$[y_{i-1}, y_i] = [x_{i-1}+c, x_i+c], \qquad y_i - y_{i-1} = x_i - x_{i-1},$$

and since $g(t) = f(t-c)$ the ranges on corresponding subintervals agree:

$$\{g(t) : t \in [y_{i-1}, y_i]\} = \{f(s) : s \in [x_{i-1}, x_i]\}.$$

Hence $m_i^g = m_i^f$ and $M_i^g = M_i^f$, so

$$L(g,Q) = \sum_{i=1}^{n} m_i^g\,(y_i - y_{i-1}) = \sum_{i=1}^{n} m_i^f\,(x_i - x_{i-1}) = L(f,P), \qquad U(g,Q) = U(f,P).$$

Because $f \in \mathcal{R}(a,b)$, for every $\varepsilon > 0$ there is P with $U(f,P) - L(f,P) < \varepsilon$; the corresponding Q then satisfies $U(g,Q) - L(g,Q) < \varepsilon$, so $g \in \mathcal{R}(a+c, b+c)$. Taking the supremum of lower sums and infimum of upper sums over all partitions (equivalently, translating partitions back and forth) yields

$$\int_{a+c}^{b+c} g(x)\,dx = \int_a^b f(x)\,dx.$$

Chapter 3

Uniform Convergence of Sequences of Functions

In the mathematical literature, there are several different notions of convergence for sequences of functions. In this chapter, we describe the two most important ones: *pointwise convergence* and *uniform convergence*. (Other types also exist, such as L^1-convergence, L^2-convergence, etc.)

Uniform convergence

Definition 61 *Let $D \subset \mathbb{R}$, and let $f_n : D \to \mathbb{R}$ be a sequence of functions.*

(a) *We say that (f_n) **converges pointwise** to a function $f : D \to \mathbb{R}$ if, for every $x \in D$, the sequence of real numbers $f_n(x)$ converges to $f(x)$:*

$$\forall x \in D,\ \forall \varepsilon > 0,\ \exists N = N(x, \varepsilon) \text{ such that } |f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N.$$

(b) *We say that (f_n) **converges uniformly** to f on D if:*

$$\forall \varepsilon > 0,\ \exists N = N(\varepsilon) \text{ such that } |f_n(x) - f(x)| < \varepsilon \quad \forall x \in D,\ \forall n \geq N.$$

In logical quantifiers:

Pointwise convergence:

$$\forall x \in D\ \forall \varepsilon > 0\ \exists N\ \forall n \geq N :\ |f(x) - f_n(x)| < \varepsilon,$$

or equivalently

$$\forall \varepsilon > 0\ \forall x \in D\ \exists N\ \forall n \geq N :\ |f(x) - f_n(x)| < \varepsilon.$$

Uniform convergence:

$$\forall \varepsilon > 0\ \exists N\ \forall x \in D\ \forall n \geq N :\ |f(x) - f_n(x)| < \varepsilon,$$

or equivalently

$$\forall \varepsilon > 0\ \exists N\ \forall n \geq N :\ \sup_{x \in D} |f(x) - f_n(x)| \leq \varepsilon.$$

Note the dependence $N = N(\varepsilon, x)$ in the pointwise case vs. $N = N(\varepsilon)$ in the uniform case. Uniform convergence always implies pointwise convergence; this is just the logical implication

$$(\exists N\ \forall x\, B(x, N)) \ \Rightarrow\ (\forall x\ \exists N\, B(x, N)).$$

Supremum Criterion

Theorem 62 (Sup–norm characterization of uniform convergence) *Let $(f_n)_{n\in\mathbb{N}}$ be functions $f_n : D \to \mathbb{R}$ and let $f : D \to \mathbb{R}$. Then $f_n \to f$ uniformly on D if and only if*

$$\sup_{x\in D} |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0.$$

Proof. ($\Rightarrow$) Assume $f_n \to f$ uniformly on D. By definition, for every $\varepsilon > 0$ there exists N such that for all $n \geq N$ and all $x \in D$, $|f_n(x) - f(x)| < \varepsilon$. Taking the supremum over $x \in D$ gives

$$\sup_{x\in D} |f_n(x) - f(x)| \leq \varepsilon \quad \text{for all } n \geq N,$$

hence $\sup_{x\in D} |f_n - f| \to 0$.

($\Leftarrow$) Conversely, suppose $\sup_{x\in D} |f_n(x) - f(x)| \to 0$. Let $\varepsilon > 0$. Then there exists N such that for all $n \geq N$,

$$\sup_{x\in D} |f_n(x) - f(x)| < \varepsilon.$$

In particular, for every $x \in D$ we have $|f_n(x) - f(x)| \leq \sup_{y\in D} |f_n(y) - f(y)| < \varepsilon$. This is precisely the definition of uniform convergence. ∎

Remark 63 *Equivalently, writing $\|g\|_{\infty,D} := \sup_{x\in D} |g(x)|$ (the sup–norm on D), the theorem states:*

$$f_n \xrightarrow{\text{unif. on } D} f \quad \Longleftrightarrow \quad \|f_n - f\|_{\infty,D} \to 0.$$

Examples

Example 1 (powers on different domains). Let $D_1 = [0, \frac{1}{2}]$, $D_2 = [0, 1)$, $D_3 = [0, 1]$, and $f_n(x) = x^n$. Set $f(x) := \lim_{n\to\infty} f_n(x)$ pointwise.

- On D_1 and D_2, $f(x) = 0$ for all x (since $|x| < 1$). Define

  $$M_n(D) := \sup_{x\in D} |f(x) - f_n(x)|.$$

  On D_1,

  $$M_n(D_1) = \sup_{0\leq x\leq 1/2} x^n = (1/2)^n \longrightarrow 0,$$

  so $f_n \to f$ uniformly on D_1.

- On D_2,

  $$M_n(D_2) = \sup_{0\leq x<1} x^n = 1,$$

  because $x^n \to 1$ as $x \uparrow 1$ (the supremum is not attained). Hence no uniform convergence, although $f_n \to f$ pointwise.

- On D_3, the pointwise limit is

  $$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1, \end{cases}$$

  which is discontinuous. Uniform convergence fails again; in fact we have a quantitative obstruction: taking $x_n := 1 - \frac{1}{n}$,

  $$|f(x_n) - f_n(x_n)| = (1 - \tfrac{1}{n})^n \longrightarrow e^{-1} > 0,$$

  so $M_n(D_3) \geq (1 - \frac{1}{n})^n \not\to 0$.

Example 2 . On $D = [0,1]$ define

$$f_n(x) = \begin{cases} 1 - nx, & 0 \le x < \frac{1}{n}, \\ 0, & \frac{1}{n} \le x \le 1, \end{cases} \qquad f(x) = \begin{cases} 1, & x = 0, \\ 0, & x > 0. \end{cases}$$

Then $f_n \to f$ pointwise on $[0,1]$. Moreover,

$$|f(x) - f_n(x)| = \begin{cases} 0, & x = 0 \text{ or } x \ge \frac{1}{n}, \\ 1 - nx, & 0 < x < \frac{1}{n}, \end{cases}$$

so

$$M_n([0,1]) = \sup_{0<x<1/n} (1 - nx) = 1,$$

(the supremum is approached as $x \downarrow 0$ but not attained). Hence convergence is not uniform, and the limit f is discontinuous despite each f_n being continuous.

Exercise 64 *Suppose $f_n \to f$ uniformly on D and $g_n \to g$ uniformly on D. Show (or give counterexamples, where appropriate):*

1. *$(f_n + g_n) \to f + g$ uniformly on D.*
2. *$(f_n g_n) \to fg$ uniformly on D (hint: assume boundedness or supply a counterexample).*

Exercise 65 *If f_n converges uniformly to f on D and also uniformly to f on E, then it converges uniformly to f on $D \cup E$.*

If $f_n : [a,b] \to \mathbb{R}$ converges uniformly on (a,b) and the sequences $(f_n(a))$ and $(f_n(b))$ converge, show that f_n converges uniformly on $[a,b]$.

Geometric Picture

For $D \subset \mathbb{R}$ and real f, the ε–band around the graph of f is

$$\mathcal{G}_\varepsilon(f) := \{(x,y) \in D \times \mathbb{R} : |y - f(x)| < \varepsilon\}.$$

Then $f_n \to f$ uniformly iff for each $\varepsilon > 0$ there exists N such that for every $n \ge N$, the graph of f_n is contained in $\mathcal{G}_\varepsilon(f)$.

A Majorant Test for Uniform Convergence

Exercise 66 (Majorant criterion) *If there exist numbers $M_n \ge 0$ with $M_n \to 0$ and*

$$|f(x) - f_n(x)| \le M_n \quad \textit{for all } x \in D,\ n \in \mathbb{N},$$

then $f_n \to f$ uniformly on D.

Exercise 67 *Show that $\lim_{n\to\infty} \dfrac{\sin(nx)}{n^2 + x^2} = 0$ uniformly for $x \in [0,\infty)$.*

Exercise 68 (Majorant criterion) *If there exist numbers $M_n \ge 0$ with $M_n \to 0$ and*

$$|f(x) - f_n(x)| \le M_n \quad \textit{for all } x \in D,\ n \in \mathbb{N},$$

then $f_n \to f$ uniformly on D.

Proof. Taking suprema over $x \in D$ gives

$$\sup_{x\in D} |f(x) - f_n(x)| \le M_n \qquad (n \in \mathbb{N}).$$

Since $M_n \to 0$, it follows that $\sup_{x\in D} |f(x) - f_n(x)| \to 0$. By the supremum criterion for uniform convergence, $f_n \to f$ uniformly on D. ■

Exercise 69 *Show that* $\lim_{n\to\infty} \dfrac{\sin(nx)}{n^2 + x^2} = 0$ *uniformly for* $x \in [0, \infty)$.

Proof. For every $n \in \mathbb{N}$ and $x \ge 0$,

$$\left|\frac{\sin(nx)}{n^2 + x^2}\right| \le \frac{1}{n^2 + x^2} \le \frac{1}{n^2}.$$

Hence

$$\sup_{x\in[0,\infty)} \left|\frac{\sin(nx)}{n^2 + x^2}\right| \le \frac{1}{n^2} \xrightarrow[n\to\infty]{} 0,$$

so the convergence to 0 is uniform on $[0, \infty)$. ■

Detecting Failure of Uniform Convergence

Exercise 70 *The following are equivalent for a sequence* (f_n) *and a function* f *on* D*:*

1. $f_n \not\to f$ *uniformly on* D.
2. *There exist* $\varepsilon > 0$*, a sequence* (x_k) *in* D*, and integers* $n_1 < n_2 < \cdots$ *such that* $|f(x_k) - f_{n_k}(x_k)| \ge \varepsilon$ *for all* k.
3. *There is a countable subset* $D_1 \subset D$ *such that* $f_n \not\to f$ *uniformly on* D_1.

Proof. (1) ⇒ (2): Negating uniform convergence yields

$$\exists\, \varepsilon > 0\ \forall N\ \exists n \ge N\ \exists x \in D : \ |f(x) - f_n(x)| \ge \varepsilon.$$

Choose n_k inductively with $n_k \ge k$ and corresponding $x_k \in D$ so that $|f(x_k) - f_{n_k}(x_k)| \ge \varepsilon$. This gives (2).

(2) ⇒ (3): Let $D_1 := \{x_k : k \in \mathbb{N}\}$ (countable). If the convergence were uniform on D_1, then $\sup_{x\in D_1} |f(x) - f_n(x)| \to 0$, contradicting $|f(x_k) - f_{n_k}(x_k)| \ge \varepsilon$.

(3) ⇒ (1): If convergence were uniform on D, then *a fortiori* it would be uniform on every subset, contradicting (3). ■

Exercise 71 *Show that* $\lim_{n\to\infty} \dfrac{x\cos(nx)}{x + n} = 0$ *pointwise on* $[0, \infty)$ *but not uniformly.*

Proof. *Pointwise:* Fix $x \ge 0$. Then

$$\left|\frac{x\cos(nx)}{x + n}\right| \le \frac{x}{x + n} \xrightarrow[n\to\infty]{} 0,$$

so the limit is 0 for each x (and equals 0 trivially at $x = 0$).

Not uniform: Take $x_n := n\pi$. Then $\cos(nx_n) = \cos(n^2\pi) = 1$, hence

$$\left|\frac{x_n \cos(nx_n)}{x_n + n}\right| = \frac{n\pi}{n\pi + n} = \frac{\pi}{\pi + 1}.$$

Thus

$$\sup_{x\in[0,\infty)} \left|\frac{x\cos(nx)}{x+n}\right| \geq \frac{\pi}{\pi+1}$$

for all n, so the suprema do not tend to 0; the convergence is not uniform. ■

Exercise 72 *If there exists a sequence* (x_n) *with* $\lim_{n\to\infty} |f(x_n) - f_n(x_n)| = c \neq 0$*, then* $f_n \not\to f$ *uniformly.*

Proof. Uniform convergence implies $\sup_{x\in D} |f(x)-f_n(x)| \to 0$, hence $|f(x_n)-f_n(x_n)| \leq \sup_{x\in D} |f(x)-f_n(x)| \to 0$. This contradicts $\lim_{n\to\infty} |f(x_n) - f_n(x_n)| = c \neq 0$. Therefore the convergence cannot be uniform. ■

Uniform Cauchy Criterion

Definition 73 *A sequence* (f_n) *on* D *satisfies the* uniform Cauchy condition *if for every* $\varepsilon > 0$ *there exists* N *such that for all* $n, m \geq N$,

$$|f_n(x) - f_m(x)| < \varepsilon \quad \textit{for every } x \in D.$$

Equivalently,

$$\forall\varepsilon > 0\ \exists N\ \forall n, m \geq N: \ \sup_{x\in D} |f_n(x) - f_m(x)| \leq \varepsilon.$$

Theorem 74 (Uniform Cauchy criterion) *A sequence* (f_n) *converges uniformly on* D *to some* f *if and only if it satisfies the uniform Cauchy condition.*

Limitations of Pointwise Convergence

Pointwise convergence alone does not preserve important properties:

- It does not preserve continuity.
- It does not preserve limits.
- It does not preserve integrals.
- It does not preserve differentiability.

We illustrate this with detailed examples.

(a) Continuity is not preserved.

Consider the sequence $f_n(x) = x^n$ on $[0, 1]$. Each f_n is continuous, since it is a polynomial. The pointwise limit is

$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1. \end{cases}$$

At $x = 1$, the limit jumps from 0 (approached from the left) to 1 at the endpoint. Hence f is not continuous, even though every f_n is continuous.

This shows that pointwise convergence does not guarantee preservation of continuity.

(b) Limits are not preserved.

Again, let $f_n(x) = x^n$ on $(0,1)$.

- For fixed $x \in (0,1)$, we have $\lim_{n\to\infty} f_n(x) = 0$. - But if we first let $x \to 1^-$, then $\lim_{x\to 1^-} f_n(x) = 1$ for every n.

Now compare the two iterated limits:

$$\lim_{n\to\infty} \big(\lim_{x\to 1^-} f_n(x) \big) = 1, \qquad \lim_{x\to 1^-} \big(\lim_{n\to\infty} f_n(x) \big) = \lim_{x\to 1^-} 0 = 0.$$

Since these two results differ, the order of limits cannot be interchanged under pointwise convergence.

(c) Integrals are not preserved.

Define

$$f_n(x) = \begin{cases} n, & 0 \le x \le \frac{1}{n}, \\ 0, & \frac{1}{n} < x \le 1. \end{cases}$$

Then $f_n(x) \to 0$ for every fixed $x \in [0,1]$. Thus the pointwise limit is the zero function.

However, compute the integral:

$$\int_0^1 f_n(x)\,dx = \int_0^{1/n} n\,dx = n \cdot \frac{1}{n} = 1.$$

Therefore:

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1, \qquad \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0.$$

Since these are different, we see that pointwise convergence does not justify interchanging limit and integration.

(d) Differentiability is not preserved.

Consider

$$f_n(x) = \sqrt{x^2 + \tfrac{1}{n}}, \qquad x \in \mathbb{R}.$$

Each f_n is differentiable everywhere (being a smooth function).

The pointwise limit is

$$\lim_{n\to\infty} f_n(x) = |x|.$$

But $|x|$ is not differentiable at $x = 0$, because:

$$f'_-(0) = \lim_{h\to 0^-} \frac{|h| - 0}{h} = -1, \qquad f'_+(0) = \lim_{h\to 0^+} \frac{|h| - 0}{h} = 1.$$

Since the left and right derivatives differ, the derivative does not exist at 0.

Thus, differentiability is not preserved under pointwise convergence.

Hereditary Theorems

We call a statement a *hereditary theorem* if a certain property possessed by all f_n in the sequence is "inherited" by the limit function f. Such properties include continuity, Riemann integrability, differentiability, etc. Pointwise convergence is typically too weak to ensure inheritance (see section above); uniform convergence has much better behavior. We begin with the prototype: continuity. For analogous results concerning other properties, an additional issue appears, namely whether an operation (integration or differentiation) can be interchanged with the limit.

Uniform Convergence and Continuity

Theorem 75 *Let $D \subset \mathbb{R}$ and let $f_n : D \to \mathbb{R}$ be a sequence of continuous functions that converges uniformly to a function $f : D \to \mathbb{R}$. Then f is also continuous.*

In other words: the limit of a uniformly convergent sequence of continuous functions is itself continuous.

Proof. Let $x \in D$. We aim to show that f is continuous at x, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(x')| < \varepsilon \quad \text{for all } x' \in D \text{ with } |x - x'| < \delta.$$

Since (f_n) converges uniformly to f, there exists $N \in \mathbb{N}$ such that

$$|f_N(\xi) - f(\xi)| < \frac{\varepsilon}{3} \quad \text{for all } \xi \in D.$$

Because f_N is continuous at x, there exists $\delta > 0$ such that

$$|f_N(x) - f_N(x')| < \frac{\varepsilon}{3} \quad \text{whenever } |x - x'| < \delta.$$

Then for such $x' \in K$, we estimate:

$$\begin{aligned} |f(x) - f(x')| &\le |f(x) - f_N(x)| + |f_N(x) - f_N(x')| + |f_N(x') - f(x')| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This proves that f is continuous at x. Since x was arbitrary, f is continuous on D. ■

Let D be any set of $\mathbb{R}$. For a bounded function $h : D \to \mathbb{R}$, write

$$\|h\|_\infty := \sup_{x \in D} |h(x)|.$$

This gives a natural "distance" between two bounded functions:

$$\rho(f, g) := \|f - g\|_\infty = \sup_{x \in D} |f(x) - g(x)|.$$

Uniform convergence of f_n to f is exactly saying $\rho(f_n, f) \to 0$.

Let $\mathcal{B}(D)$ be all bounded functions on D, and let $\mathcal{C}(D)$ be those that are continuous .

Theorem 76 (Continuous limits stay continuous)

1. ***(Closedness)*** *If $f_n \in \mathcal{C}(D)$ and $f_n \to f$* uniformly *on D, then f is continuous. Equivalently: $C(D)$ contains all its uniform limits.*

2. ***(Completeness)*** *If* (f_n) *is* uniformly Cauchy *(i.e. for every* $\varepsilon > 0$ *there is* N *so that* $\sup_x |f_n(x) - f_m(x)| < \varepsilon$ *for all* $m, n \geq N$*) and each* f_n *is continuous, then* f_n *converges uniformly to some continuous* f*. In short: every uniformly Cauchy sequence of continuous functions has a continuous uniform limit.*

Part (1) is the standard fact "uniform limit of continuous functions is continuous." Part (2) just says you don't leave the world of continuous functions when taking uniform limits of good approximations: uniform Cauchy $\Rightarrow$ uniform convergence *to a continuous function.*

Uniform Convergence and Integrability

We recall some Riemann–integration notions for a bounded function $f : [a, b] \to \mathbb{R}$. A *partition* is $P = \{x_0, \dots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_n = b$, and we write $\Delta x_i := x_i - x_{i-1}$. Set

$$m_i := \inf_{x \in [x_{i-1}, x_i]} f(x), \qquad M_i := \sup_{x \in [x_{i-1}, x_i]} f(x), \qquad \omega_i := M_i - m_i = \sup_{x, y \in [x_{i-1}, x_i]} |f(x) - f(y)|.$$

The *lower* and *upper* sums are

$$L(f, P) := \sum_{i=1}^{n} m_i \, \Delta x_i, \qquad U(f, P) := \sum_{i=1}^{n} M_i \, \Delta x_i,$$

and they satisfy

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} \omega_i \, \Delta x_i, \qquad \sup_P L(f, P) \ \leq \ \inf_P U(f, P).$$

The function f is *Riemann integrable* on $[a, b]$ iff equality holds:

$$\int_a^b f(x) \, dx = \sup_P L(f, P) = \inf_P U(f, P).$$

Riemann's criterion. f is integrable iff for every $\varepsilon > 0$ there exists a partition P with $U(f, P) - L(f, P) < \varepsilon$.

Every continuous function on $[a, b]$ is Riemann integrable. Moreover, if f is integrable then so is $|f|$, and

$$\Big| \int_a^b f(x) \, dx \Big| \ \leq \ \int_a^b |f(x)| \, dx \ \leq \ (b - a) \sup_{x \in [a, b]} |f(x)|.$$

Theorem 77 (Hereditary theorem for integrability) *Let* $[a, b]$ *be a bounded interval. If* (f_n) *is a sequence of Riemann integrable functions on* $[a, b]$ *that converges uniformly to* f*, then* f *is Riemann integrable and*

$$\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx.$$

Proof. First, if each f_n is continuous, then f is continuous and hence integrable. For the identity, fix $\varepsilon > 0$. Uniform convergence gives N with $|f - f_n| \leq \varepsilon$ on $[a, b]$ for all $n \geq N$, hence

$$\Big| \int_a^b f - \int_a^b f_n \Big| \leq \int_a^b |f - f_n| \leq \varepsilon(b - a).$$

For general (not necessarily continuous) f_n, let $\omega_i(g)$ be the oscillation of g on $[x_{i-1}, x_i]$. For any partition P and any n,

$$\omega_i(f) \le 2 \sup_{[a,b]} |f - f_n| + \omega_i(f_n).$$

Multiply by Δx_i and sum. Choose n so that $\sup |f - f_n| < \varepsilon$, and then a partition P with $\sum_i \omega_i(f_n)\Delta x_i < \varepsilon$. Then $\sum_i \omega_i(f)\Delta x_i < \varepsilon(2(b-a)+1)$, which ensures integrability of f by Riemann's criterion. ■

Complex-valued case. For $f : [a,b] \to \mathbb{C}$, define integrability componentwise via $\int_a^b f = \int_a^b \Re f + i \int_a^b \Im f$. The theorem remains valid.

Function space formulation. Let $\mathcal{B}([a,b])$ be the (real or complex) vector space of all bounded functions on $[a,b]$, equipped with the sup norm

$$\|f\|_\infty := \sup_{x\in[a,b]} |f(x)|,$$

and the associated (sup) metric

$$\rho(f,g) := \|f - g\|_\infty = \sup_{x\in[a,b]} |f(x) - g(x)|.$$

Let $\mathcal{R}([a,b]) \subset \mathcal{B}([a,b])$ denote the subspace of (Riemann) integrable functions.

Theorem 78 *With respect to the metric ρ:*

1. *$\mathcal{R}([a,b])$ is closed in $\mathcal{B}([a,b])$.*
2. *$(\mathcal{R}([a,b]), \rho)$ is complete.*
3. *The linear functional $I : \mathcal{R}([a,b]) \to \mathbb{K}$ (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) given by*

$$I(f) := \int_a^b f(x)\,dx$$

is continuous (indeed, Lipschitz), with

$$|I(f) - I(g)| = \Big| \int_a^b (f-g) \Big| \le \int_a^b |f-g| \le (b-a)\,\|f-g\|_\infty = (b-a)\,\rho(f,g).$$

Proof sketch. (1) If $f_n \in \mathcal{R}([a,b])$ and $f_n \to f$ uniformly, then f is Riemann integrable and $\int f_n \to \int f$. This is the standard "uniform limit preserves Riemann integrability" fact (prove via upper/lower sums or via oscillations on a common partition). Hence $\mathcal{R}([a,b])$ is closed in $\mathcal{B}([a,b])$.

(2) $\mathcal{B}([a,b])$ is complete under ρ (a ρ–Cauchy sequence converges uniformly to a bounded limit). A closed subspace of a complete metric space is complete, so (1) implies $(\mathcal{R}([a,b]), \rho)$ is complete.

(3) The displayed inequality shows I is $(b-a)$–Lipschitz, hence continuous (and bounded) on $(\mathcal{R}([a,b]), \rho)$. ■

Remark 79

- ***Density of continuous functions.*** *Continuous functions are Riemann integrable, and are dense in $\mathcal{R}([a,b])$ under ρ (e.g. by approximating Riemann integrable f with step functions and then smoothing). Thus $\overline{\mathcal{C}([a,b])}^{\|\cdot\|_\infty} = \mathcal{R}([a,b])$.*

- ***Characterization of Riemann integrability.*** *A bounded f is Riemann integrable iff the set of its discontinuities has Lebesgue measure zero (Lebesgue's criterion). This gives a practical test and clarifies why uniform limits of Riemann integrable functions stay integrable.*
- ***Completeness comparison.*** $(\mathcal{C}([a,b]), \|\cdot\|_\infty)$ *is also complete (a Banach space). We have* $\mathcal{C}([a,b]) \subset \mathcal{R}([a,b]) \subset \mathcal{B}([a,b])$*, with* $\mathcal{C}([a,b])$ *and* $\mathcal{R}([a,b])$ *both closed in* $\mathcal{B}([a,b])$*.*
- ***Operator norm of I.*** *From the estimate above,* $\|I\| := \sup_{\|f\|_\infty \le 1} |\int_a^b f| \le b-a$*. In fact,* $\|I\| = b - a$*, attained by* $f \equiv 1$*.*
- ***Non-example (bounded but not Riemann integrable).*** *The Dirichlet function* $f = \mathbf{1}_{\mathbb{Q}\cap[a,b]}$ *is bounded but not in* $\mathcal{R}([a,b])$*. It also shows* $\mathcal{R}([a,b]) \subsetneq \mathcal{B}([a,b])$*.*

Three possibilities under nonuniform convergence. If f_n are Riemann integrable and $f_n \to f$ pointwise but not uniformly, then any of the following can happen:

1. f is not Riemann integrable.
2. f is Riemann integrable but $\int f_n \not\to \int f$.
3. f is Riemann integrable and $\int f_n \to \int f$.

Example 80 (Case 1) *Enumerate rationals in* $[0,1]$ *as* $(r_k)_{k\ge1}$ *and set* $f_n = \mathbf{1}_{\{r_1,\dots,r_n\}}$*. Then* $f_n \to f$ *pointwise where* $f = \mathbf{1}_{\mathbb{Q}\cap[0,1]}$*, but f is not Riemann integrable (every upper sum is 1, every lower sum is 0). Each f_n is Riemann integrable.*

Example 81 (Case 2) *Let* $f_n(x) = n$ *for* $0 < x \le 1/n$ *and* $f_n(0) = f_n(x) = 0$ *for* $x > 1/n$*. Then* $f_n \to 0$ *pointwise, each f_n and $f \equiv 0$ are Riemann integrable, but* $\int_0^1 f_n = 1 \not\to 0$*.*

Example 82 (Case 3) *Let* $f_n = \mathbf{1}_{(0,1/n]}$ *on* $[0,1]$*. Then* $f_n \to 0$ *pointwise, each f_n and $f \equiv 0$ are Riemann integrable, and* $\int_0^1 f_n = 1/n \to 0$*.*

The dominated convergence theorem (Lebesgue) vastly strengthens the result above:

Uniform convergence and differentiability

For differentiability the hypotheses are subtler: we require uniform convergence of derivatives, plus convergence at a single base point.

Theorem 83 (Hereditary theorem for differentiability) *Let (f_n) be differentiable on a bounded interval $[a,b]$ (right-derivative at a, left-derivative at b). Suppose (f_n') converges uniformly on $[a,b]$, and for some $x_0 \in [a,b]$ the sequence $(f_n(x_0))$ converges. Then (f_n) converges uniformly on $[a,b]$ to a differentiable f and* $f' = \lim_{n\to\infty} f_n'$ *on $[a,b]$. If each f_n' is continuous, then f' is continuous.*

Sketch under $f_n \in C^1$. Let $g = \lim f_n'$ (uniform limit, hence continuous). By the fundamental theorem of calculus,

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)\,dt.$$

The right-hand side converges as $n \to \infty$ by uniform convergence of f_n' (plus convergence of $f_n(x_0)$). Define $f(x) := \lim f_n(x)$. Then

$$f(x) = f(x_0) + \int_{x_0}^{x} g(t)\,dt,$$

so f is differentiable with $f' = g$. Using the mean value estimate

$$|(f - f_n)(x) - (f - f_n)(a)| \leq (b - a) \sup_{t \in [a,b]} |f'(t) - f_n'(t)|,$$

and $f_n(a) \to f(a)$, we get $\sup_x |f(x) - f_n(x)| \to 0$. ■

General proof idea. Define

$$\phi_{y,n}(x) = \begin{cases} \dfrac{f_n(x) - f_n(y)}{x - y}, & x \neq y, \\ f_n'(y), & x = y. \end{cases}$$

Each $\phi_{y,n}$ is continuous on $[a, b]$. By the mean value theorem and uniform convergence of f_n', the sequence $(\phi_{y,n})$ is uniformly Cauchy, hence converges uniformly to a continuous ϕ_y. Using $f_n(x) = (x - x_0)\phi_{x_0,n}(x) + f_n(x_0)$ and convergence of $f_n(x_0)$, one gets uniform convergence $f_n \to f$. Finally $\phi_y(y) = \lim_{x \to y} \phi_y(x) = \lim f_n'(y)$, so f is differentiable with $f' = \lim f_n'$; continuity of f' follows if f_n' are continuous and converge uniformly. ■

Remark 84 *Without convergence of $(f_n(x_0))$ at some point, neither uniform nor pointwise convergence of (f_n) is guaranteed. Indeed, if (c_n) is a divergent scalar sequence and $g_n = f_n + c_n$, then $(g_n') = (f_n')$ and the other hypotheses remain true, but (g_n) cannot converge pointwise on $[a, b]$ while (f_n) does.*

Example 85 *Let*

$$f_n(x) := x - \frac{\sin(nx)}{n^2}, \qquad x \in [0, 1].$$

- *Each f_n is differentiable on $[0, 1]$. At $x_0 = 0$, we have*

  $$f_n(0) = 0 - \frac{\sin(0)}{n^2} = 0,$$

  so (f_n) converges at the point $x_0 = 0$.

- *Differentiating gives*

  $$f_n'(x) = 1 - \frac{\cos(nx)}{n}.$$

- *As $n \to \infty$, for each fixed $x \in [0, 1]$,*

  $$f_n'(x) \longrightarrow 1.$$

  To check uniform convergence, compute the uniform error:

  $$\sup_{x \in [0,1]} \left| f_n'(x) - 1 \right| = \sup_{x \in [0,1]} \frac{|\cos(nx)|}{n}.$$

  Since $|\cos(nx)| \leq 1$ for all x, this gives

  $$\sup_{x \in [0,1]} \left| f_n'(x) - 1 \right| \leq \frac{1}{n} \longrightarrow 0.$$

  Hence (f_n') converges uniformly on $[0, 1]$ to the constant function 1.

- *By the theorem, the sequence* (f_n) *converges uniformly to a function* f*. Let us determine* f*. For fixed* x*,*

$$f_n(x) = x - \frac{\sin(nx)}{n^2}.$$

Since $|\sin(nx)| \leq 1$*, we have*

$$\left|\frac{\sin(nx)}{n^2}\right| \leq \frac{1}{n^2} \to 0.$$

Thus

$$\lim_{n\to\infty} f_n(x) = x,$$

so the limit function is

$$f(x) = x.$$

The convergence is uniform because the error term is bounded uniformly in x *by* $1/n^2$*.*

Theorem 86 *Let* $D \subseteq \mathbb{R}$ *and* (f_n) *be a sequence of functions* $f_n : D \to \mathbb{R}$ *that converges uniformly over* D *to a function* $f : X \to \mathbb{R}$*. Assume that for* $x_0 \in X$*, both* $\lim_{x\to x_0} f(x)$ *and* $\lim_{x\to x_0} f_n(x)$ *for all* $n \in \mathbb{N}$ *exist. Then:*

$$\lim_{x\to x_0}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right).$$

Proof. First, note that since the sequence (f_n) converges uniformly to f, this convergence is also pointwise, meaning $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in D$. Therefore, we want to prove the equality:

$$\lim_{x\to x_0} f(x) = \lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right).$$

Let $p_n = \lim_{x\to x_0} f_n(x)$ for each $n \in \mathbb{N}$, and let $p = \lim_{x\to x_0} f(x)$. Proving this equation is equivalent to showing the convergence of the real sequence $p_n \to p$.

Fix $\varepsilon > 0$. Since (f_n) converges uniformly to f, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have:

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \text{for any } x \in D.$$

Now, take the limit as $x \to x_0$ on both sides. Since limits preserve weak inequalities (as seen in Exercise 9.10), we get:

$$\lim_{x\to x_0} |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} < \varepsilon.$$

By applying the algebra of limits, we then have:

$$|p_n - p| = \left|\lim_{x\to x_0} f_n(x) - \lim_{x\to x_0} f(x)\right| = \left|\lim_{x\to x_0} (f_n(x) - f(x))\right| = \lim_{x\to x_0} |f_n(x) - f(x)| < \varepsilon.$$

Thus, for all $n \geq N$, we have $|p_n - p| < \varepsilon$, which is what we wanted to prove. Therefore

$$\lim_{x\to x_0}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right).$$

■ As a consequence, knowing that the functions in (f_n) are continuous everywhere guarantees that their uniform limit is also continuous everywhere.

Example 87 *Let*

$$f_n(x) = \frac{x}{1+nx}, \qquad x \in [0,1].$$

Then $f_n \to f$ *uniformly on* $[0,1]$*, where* $f(x) \equiv 0$*. At* $x_0 = 0$ *we have*

$$\lim_{x\to 0}\Big(\lim_{n\to\infty} f_n(x)\Big) = \lim_{x\to 0} f(x) = 0, \qquad \lim_{n\to\infty}\left(\lim_{x\to 0} f_n(x)\right) = \lim_{n\to\infty} 0 = 0.$$

Thus both iterated limits coincide, illustrating the theorem.

Exercises

Practical guide to study uniform convergence (sequences (f_n)).

1. **Pointwise first.** Compute $f(x) := \lim_{n\to\infty} f_n(x)$. If this fails anywhere, uniform convergence is impossible.

2. **A Majorant Test for Uniform Convergence.** Define

$$M_n := \sup_{x\in X} |f_n(x) - f(x)|.$$

   Then $f_n \to f$ uniformly $\iff$ $M_n \to 0$.

3. **Easy domination.** If you can find numbers $A_n \downarrow 0$ with

$$|f_n(x) - f(x)| \le A_n \quad (\forall x \in X),$$

   uniform convergence follows immediately.

4. **Uniform Cauchy (when f is unknown).** (f_n) is uniformly convergent on X $\iff$

$$\forall \varepsilon > 0\ \exists N\ \forall m, n \ge N: \ \sup_{x\in X} |f_n(x) - f_m(x)| < \varepsilon.$$

5. **Test on a smaller set.** If $E \subset D$ and $\sup_{x\in E} |f_n(x) - f(x)| \not\to 0$, then uniform convergence fails on D.

6. **Witness to failure.** To disprove uniform convergence, find $\varepsilon > 0$ and sequences $(x_k) \subset D$, $n_k \uparrow \infty$ with

$$|f_{n_k}(x_k) - f(x_k)| \ge \varepsilon \quad (\forall k).$$

7. **Monotone + compact $\Rightarrow$ Dini.** On compact D, if each f_n is continuous, $f_n \to f$ pointwise with f continuous, and (f_n) is pointwise monotone in n, then $f_n \to f$ uniformly (Dini).

8. **Uniform equicontinuity + pointwise $\Rightarrow$ uniform (on compact D).** If (f_n) is equicontinuous and uniformly bounded on compact D and $f_n \to f$ pointwise, then $f_n \to f$ uniformly (Arzelà–Ascoli corollary).

9. **Common bounding patterns.**

   - *Factor out n–only pieces:* $|f_n(x) - f(x)| \le a_n\, g(x)$ with $a_n \downarrow 0$ and g bounded on $D \Rightarrow$ uniform.

- *Squeeze by maxima:* If $|f_n - f| \le h_n$ and $\|h_n\|_\infty \to 0$, done.
- *Localize oscillations:* If oscillations live on shrinking sets E_n with $\sup_{x\in D\setminus E_n} |f_n - f| \to 0$ and also $\sup_{x\in E_n} |f_n - f| \to 0$, combine.

10. **Uniform limits preserve structure.** If f_n are continuous (resp. uniformly continuous) and $f_n \to f$ uniformly on a metric space, then f is continuous (resp. uniformly continuous).

11. **Interchanging limits (use only with uniformity).**

    - *Integration:* If f_n are Riemann integrable on $[a, b]$ and $f_n \to f$ uniformly, then $\int_a^b f_n \to \int_a^b f$.
    - *Differentiation:* If f_n' converge uniformly and $f_n(x_0)$ converges for some x_0, then $f_n \to f$ uniformly and $f' = \lim f_n'$.

Exercises

Ex 1.1 For $f_n(x) = x^n$ on $\mathbb{R}$: find the pointwise limit and decide on uniform convergence on relevant domains.

Ex 1.2 For $f_n(x) = \dfrac{\sin(nx)}{nx}$ on $(0, 1)$: find the pointwise limit and decide on uniform convergence (also on $[a, 1]$, $0 < a < 1$).

Ex 1.3 For $f_n(x) = \dfrac{1}{nx+1}$ on $(0, 1)$: find the pointwise limit and decide on uniform convergence (also on $[a, 1]$, $0 < a < 1$).

Ex 1.4 For $f_n(x) = \dfrac{x}{nx+1}$ on $[0, 1]$: find the pointwise limit and decide on uniform convergence.

Ex 1.5 For $f_n(x) = \dfrac{nx^3}{1+nx}$ on $[0, 1]$: show $f_n \to f$ pointwise and decide on uniform convergence.

Ex 1.6 For $f_n(x) = x^n(1 - x)$ on $[0, 1]$: pointwise limit and uniform convergence.

Ex 1.7 For $f_n(x) = x^n(1 - x^n)$ on $[0, 1]$: pointwise limit and uniform convergence.

Ex 1.8 For

$$f_n(x) = \begin{cases} nx, & x \in [0, 1/n], \\ 0, & x \in (1/n, 1], \end{cases} \quad \text{on } [0, 1]:$$

pointwise limit and uniform convergence.

Ex 1.9 For

$$f_n(x) = \begin{cases} \sqrt{nx}, & x \in [0, 1/n], \\ 0, & x \in (1/n, 1], \end{cases} \quad \text{on } [0, 1]:$$

pointwise limit and uniform convergence.

Ex 1.10 If $f_n \to f$ uniformly on D and on E, prove $f_n \to f$ uniformly on $D \cup E$.

Ex 1.11 If $f_n \to f$ and $g_n \to g$ uniformly on D, prove $\alpha f_n + \beta g_n \to \alpha f + \beta g$ uniformly for all $\alpha, \beta \in \mathbb{R}$.

Ex 1.12 If $f_n \to f$ and $g_n \to g$ uniformly on D, and each f_n, g_n is bounded on D, prove $f_n g_n \to fg$ uniformly. Give a counterexample if boundedness fails.

Ex 1.13 (Dini) On compact $D \subset \mathbb{R}$, if (f_n) is continuous, decreases pointwise to a continuous f, prove $f_n \to f$ uniformly. Show compactness is needed with $f_n(x) = x^n$ on $(0, 1)$.

Ex 1.14 For $f_n(x) = \dfrac{x^n}{1 + x^n}$ on $[0, 2]$: pointwise limit and (non)uniform convergence.

Ex 1.15 Construct $f_n : [0, 1] \to \mathbb{R}$, each discontinuous everywhere, with $f_n \to f$ uniformly for a continuous f.

Ex 1.16 Let $\varphi \in C[0, 1]$ and $f_n(x) = \varphi(x)x^n$. Show f_n converges uniformly on $[0, 1]$ iff $\varphi(1) = 0$. Deduce $g_n(x) = nx(1 - x)^n \to 0$ pointwise but not uniformly.

Ex 1.17 Define

$$f_n(x) = \begin{cases} n^2 x, & x \in [0, 1/n], \\ -n^2(x - 2/n), & x \in (1/n, 2/n], \\ 0, & x \in (2/n, 1], \end{cases}$$

on $[0, 1]$. Find the pointwise limit; decide uniform convergence; compare $\int f_n$ vs. $\int \lim f_n$.

Ex 1.18 For $p > 0$, $f_n(x) = \dfrac{nx}{1 + n^2 x^p}$ on $[0, 1]$: (a) for which p does $f_n \to f$ uniformly? (b) For $p = 2$, compute $\int_0^1 f_n \to ?$ and compare with $\int_0^1 f$.

Quick Checks

Ex 1.1 *Pointwise:* $\lim x^n = 0$ for $|x| < 1$; $= 1$ at $x = 1$; diverges for $|x| > 1$ or $x = -1$. *Uniform:* Not uniform on $(-1, 1)$ (since $\sup_{(-1,1)} |x^n| = 1$) nor on $(-1, 1]$ (limit discontinuous at 1). Uniform on any $[-a, a]$ with $a < 1$ since $\sup |x^n| = a^n \to 0$. Trivially uniform on $\{1\}$.

Ex 1.2 *Pointwise:* $|\sin(nx)/(nx)| \le 1/(nx) \to 0$, so $f_n \to 0$. *Uniform:* Not on $(0, 1)$ (take $x_n = \pi/(2n)$ gives $|f_n(x_n)| = 2/\pi$). Uniform on $[a, 1]$ for any $a > 0$ since $\sup_{[a,1]} |f_n| \le 1/(na) \to 0$.

Ex 1.3 *Pointwise:* $1/(nx + 1) \to 0$. *Uniform:* Not on $(0, 1)$ (at $x_n = 1/n$, value $= 1/2$). Uniform on $[a, 1]$ for $a > 0$ since $\sup \le 1/(na + 1) \to 0$.

Ex 1.4 *Pointwise:* $x/(nx + 1) \to 0$ on $[0, 1]$. *Uniform:* Yes. $\sup_{[0,1]} x/(nx + 1) = 1/(n + 1) \to 0$.

Ex 1.5 *Pointwise:* $f_n(x) = \dfrac{nx^3}{1 + nx} \to x^2$; at $x = 0$, 0. *Uniform:* Yes. $|f_n - x^2| = \dfrac{x^2}{1 + nx} \le \dfrac{1}{n + 1} \to 0$.

Ex 1.6 *Pointwise:* $x^n(1-x) \to 0$ on $[0, 1]$. *Uniform:* Yes. $\max_{[0,1]} x^n(1-x) = \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} \to e^{-1} \cdot 0 = 0$.

Ex 1.7 *Pointwise:* $x^n(1 - x^n) \to 0$ on $[0, 1]$. *Uniform:* No. $\max f_n = 1/4$ (at $x^n = 1/2$), so $\sup |f_n - 0| = 1/4 \not\to 0$.

Ex 1.8 *Pointwise:* $\to 0$ on $[0, 1]$. *Uniform:* No. $\sup f_n = f_n(1/n) = 1$ for all n.

Ex 1.9 *Pointwise:* $\to 0$ on $[0, 1]$. *Uniform:* No. $\sup f_n = f_n(1/n) = 1$ for all n.

Ex 1.10 For $\varepsilon > 0$, take $N = \max(N_D, N_E)$ from the two uniform convergences. Then $\sup_{D\cup E} |f_n - f| < \varepsilon$ for $n \geq N$.

Ex 1.11 For $\varepsilon > 0$, choose $N = \max(N_f, N_g)$ with $\sup_D |f_n - f| < \varepsilon/(2(|\alpha| + |\beta| + 1))$ and similarly for g_n. Then $\sup_D |\alpha f_n + \beta g_n - (\alpha f + \beta g)| < \varepsilon$.

Ex 1.12 If $\sup_D |f_n| \leq M$ and $\sup_D |g_n| \leq K$, then

$$\sup_D |f_n g_n - fg| \leq K \sup_D |f_n - f| + \|f\|_\infty \sup_D |g_n - g| \to 0.$$

Counterexample: On $D = \mathbb{R}$, $f_n = g_n = x + 1/n$ converge uniformly to x, but $|f_n g_n - x^2| = |2x/n + 1/n^2|$ has unbounded supremum $\Rightarrow$ not uniform.

Ex 1.13 (Dini) Compactness $\Rightarrow$ $f_n \downarrow f$ with f continuous gives uniform convergence. Counterexample: x^n on $(0, 1)$ is not uniform.

Ex 1.14 $f_n(x) = \dfrac{x^n}{1 + x^n} \to 0$ for $x < 1$, $= 1/2$ at $x = 1$, and $\to 1$ for $x > 1$. Limit is discontinuous at 1; hence not uniform on $[0, 2]$.

Ex 1.15 Let $f_n = \mathbf{1}_{\mathbb{Q}\cap[0,1]}/n$. Each f_n is everywhere discontinuous, while $\sup |f_n| = 1/n \to 0$; thus $f_n \to 0$ uniformly (continuous limit).

Ex 1.16 $f_n(x) = \varphi(x)x^n$. *Uniform iff* $\varphi(1) = 0$*:* If uniform, limit must be continuous at 1, forcing $\varphi(1) = 0$. Conversely, if $\varphi(1) = 0$, use continuity near 1 and geometric decay away from 1 to get $\sup |f_n| \to 0$. For $g_n(x) = nx(1 - x)^n$, $\sup g_n = \frac{n}{n+1}\left(\frac{n}{n+1}\right)^n \to 1/e \neq 0 \Rightarrow$ not uniform (though pointwise $\to 0$).

Ex 1.17 Pointwise $f_n \to 0$ on $[0, 1]$. Not uniform: $\sup |f_n| = n \to \infty$. Moreover, $\int_0^1 f_n = 1 \not\to 0 = \int_0^1 \lim f_n$ (no interchange without uniform integrable control).

Ex 1.18 $f_n(x) = \dfrac{nx}{1 + n^2 x^p}$ on $[0, 1]$. Pointwise $f_n \to 0$ for all $p > 0$. Uniform:

- $0 < p < 2$: uniform $\to 0$ (max $\to 0$).
- $p = 2$: not uniform; $\max f_n = f_n(1/n) = 1/2$.
- $p > 2$: not uniform; $\sup f_n \to \infty$.

For $p = 2$, $\displaystyle\int_0^1 f_n = \frac{1}{2n^2}\ln(1 + n^2) \to 0 = \int_0^1 0.$

Detailed Answers

Exercise 1. For each $n \in \mathbb{N}$, let $f_n(x) = x^n$ on $\mathbb{R}$. Determine the pointwise limit of (f_n) and decide whether the convergence is uniform on the given domain.

Solution

Pointwise convergence

For fixed $x \in \mathbb{R}$,

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & |x| < 1, \\ 1, & x = 1, \\ \text{does not exist}, & x = -1 \text{ or } |x| > 1. \end{cases}$$

Thus (f_n) converges pointwise to 0 on $(-1,1)$, to 1 at $x=1$, and fails to converge at $x=-1$ or any $|x|>1$.

Uniform convergence. Uniform convergence on a set E *implies* pointwise convergence on E. Hence any set on which we can even hope for uniform convergence must be contained in the pointwise–convergence set. Therefore, it suffices to study uniform convergence only on subsets of $(-1,1]$ (in particular, sets containing -1 or any $|x|>1$ are automatically excluded).

- *On* $(-1,1]$ *the convergence is not uniform.* The pointwise limit on $(-1,1]$ is

$$f(x)=\begin{cases}0, & x\in[0,1),\\ 1, & x=1,\end{cases}$$

  which is discontinuous at $x=1$, whereas each f_n is continuous. A uniform limit of continuous functions must be continuous, so uniform convergence to f on $(-1,1]$ is impossible.

- *On* $(-1,1)$ *the convergence is not uniform.* We have

$$\sup_{x\in(-1,1)}|x^n|=1 \quad \text{for all } n$$

  (since values arbitrarily close to 1 occur as $x\to 1^-$), so the suprema do not tend to 0.

- For any $0<a<1$, the convergence is uniform on $[-a,a]$

$$\sup_{x\in[-a,a]}|x^n|=a^n \xrightarrow[n\to\infty]{} 0,$$

  so $f_n\to 0$ uniformly on $[-a,a]$. More generally, if $E\subset(-1,1)$ satisfies $\sup_{x\in E}|x|\le a<1$, then $\sup_{x\in E}|x^n|\le a^n\to 0$ and the convergence is uniform on E.

- *On the singleton* $\{1\}$*, the convergence is trivially uniform* (to the constant 1).

Exercise 2. For each $n\in\mathbb{N}$, let $f_n(x)=\dfrac{\sin(nx)}{nx}$ on $(0,1)$. Determine the pointwise limit of (f_n) and decide whether the convergence is uniform on the given domain.

Solution

Pointwise limit. For each fixed $x\in(0,1)$,

$$\left|\frac{\sin(nx)}{nx}\right|\le\frac{1}{nx}\xrightarrow[n\to\infty]{} 0,$$

so $f_n(x)\to 0$ pointwise on $(0,1)$.

Uniform convergence.

The convergence is *not* uniform on $(0,1)$. Take $x_n=\frac{\pi}{2n}\in(0,1)$ for all large n. Then

$$|f_n(x_n)|=\frac{|\sin(nx_n)|}{nx_n}=\frac{1}{n\cdot\pi/(2n)}=\frac{2}{\pi}\not\to 0,$$

so $\sup_{x\in(0,1)}|f_n(x)|\ge\frac{2}{\pi}$ for all n.

Uniform convergence away from 0. Let $0 < a < 1$. Then for every $x \in [a, 1]$,

$$\left|\frac{\sin(nx)}{nx}\right| \leq \frac{1}{na}.$$

Hence $\sup_{x\in[a,1]} |f_n(x)| \leq \frac{1}{na} \to 0$, so $f_n \to 0$ uniformly on $[a, 1]$. More generally, if $E \subset [a, 1]$ for some $0 < a < 1$, then

$$\sup_{x\in E}\left|\frac{\sin(nx)}{nx}\right| \leq \frac{1}{na} \xrightarrow[n\to\infty]{} 0,$$

so the convergence is *uniform on every subset* $E \subset [a,])$ *with* $0 < a < 1$.

Exercise 4. For each $n \in \mathbb{N}$, let $f_n(x) = \dfrac{1}{nx+1}$ on $(0, 1)$. Determine the pointwise limit of (f_n) and decide whether the convergence is uniform on the given domain.

Pointwise limit.

For every fixed $x \in (0, 1)$,

$$f_n(x) = \frac{1}{nx+1} \xrightarrow[n\to\infty]{} 0,$$

so $f_n \to 0$ pointwise on $(0, 1)$.

Non–uniform convergence on $(0, 1)$. To show the convergence is not uniform, take the sequence $x_n = \frac{1}{n} \in (0, 1)$. Then

$$f_n(x_n) = \frac{1}{n \cdot (1/n) + 1} = \frac{1}{2}.$$

Hence for every n,

$$\sup_{x\in(0,1)} |f_n(x) - 0| \;\geq\; |f_n(x_n)| = \frac{1}{2} \not\longrightarrow 0,$$

so $f_n \nrightarrow 0$ uniformly on $(0, 1)$.

Uniform convergence away from 0. If $a \in (0, 1)$, then for $x \in [a, 1]$,

$$0 \leq f_n(x) = \frac{1}{nx+1} \leq \frac{1}{na+1} \xrightarrow[n\to\infty]{} 0,$$

so $f_n \to 0$ uniformly on $[a, 1]$. More generally, for any $E \subset [a, 1]$ the convergence is uniform on E with the same bound.

Exercise 5. For each $n \in \mathbb{N}$, let $f_n(x) = \dfrac{x}{nx+1}$ on $[0, 1]$. Determine the pointwise limit of (f_n) and decide whether the convergence is uniform on the given domain.

For every $x \in [0, 1]$,

$$f_n(x) = \frac{x}{nx+1} \xrightarrow[n\to\infty]{} 0,$$

so $f_n \to 0$ pointwise on $[0, 1]$.

Moreover, the convergence is *uniform* on $[0, 1]$. Indeed, for fixed n, the function $x \mapsto \dfrac{x}{nx+1}$ is increasing on $[0, 1]$ (its derivative is $\dfrac{1}{(nx+1)^2} > 0$), hence

$$\sup_{x\in[0,1]}\left|\frac{x}{nx+1}\right| = \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0.$$

Thus $\sup_{[0,1]} |f_n - 0| \to 0$, proving uniform convergence.

Exercise 6.

$$f_n(x) = \frac{nx^3}{1+nx}, \quad x \in [0,1].$$

Pointwise limit: For $x = 0$, $f_n(0) = 0$. For $x > 0$,

$$f_n(x) = \frac{x^2}{x + \frac{1}{n}} \xrightarrow[n\to\infty]{} x^2,$$

so

$$f(x) = x^2.$$

Uniform convergence: We must compute

$$\sup_{x\in[0,1]} |f_n(x) - x^2|.$$

Since $f_n(x) \le x^2$ for all x, we have

$$|f_n(x) - x^2| = x^2 - f_n(x).$$

Thus

$$x^2 - f_n(x) = x^2 - \frac{nx^3}{1+nx} = \frac{x^2(1+nx) - nx^3}{1+nx} = \frac{x^2}{1+nx}.$$

Hence

$$\sup_{x\in[0,1]} |f_n(x) - x^2| = \sup_{x\in[0,1]} \frac{x^2}{1+nx}.$$

The function $\frac{x^2}{1+nx}$ is increasing in $x \in [0,1]$, so the maximum is at $x = 1$:

$$\sup_{x\in[0,1]} |f_n(x) - x^2| = \frac{1}{1+n}.$$

Conclusion: Since

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - x^2| = \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

the convergence $f_n \to f$ is **uniform on** $[0,1]$.

Exercise 7.

$$f_n(x) = x^n(1-x), \quad x \in [0,1].$$

Pointwise limit: For $0 \le x < 1$, we have $\lim_{n\to\infty} x^n = 0$, hence

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n(1-x) = 0.$$

At $x = 1$, $f_n(1) = 1^n(0) = 0$. Thus the pointwise limit function is

$$f(x) \equiv 0 \quad \text{on } [0,1].$$

Uniform convergence: We compute

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} f_n(x).$$

Since $f_n(0) = f_n(1) = 0$ and $f_n(x) \geq 0$ for all x, the maximum occurs at a critical point. Differentiate:

$$f_n'(x) = nx^{n-1}(1-x) - x^n = x^{n-1}(n-(n+1)x).$$

Setting $f_n'(x) = 0$ gives the unique critical point

$$x_n = \frac{n}{n+1}.$$

At this point,

$$f_n(x_n) = \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}.$$

Hence

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}.$$

Asymptotics: Note that

$$\left(\frac{n}{n+1}\right)^n = \left(1 - \frac{1}{n+1}\right)^n \xrightarrow[n\to\infty]{} e^{-1}.$$

Therefore

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0.$$

Since the supremum of the error tends to zero, the convergence is in fact **uniform on** $[0,1]$.
Exercise 8.

$$f_n(x) = x^n(1-x^n), \quad x \in [0,1].$$

Pointwise limit: For $0 \leq x < 1$, $x^n \to 0$, hence $f_n(x) \to 0$. At $x = 1$, $f_n(1) = 0$. Thus,

$$f(x) \equiv 0 \quad \text{on } [0,1].$$

Uniform convergence: We have

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} f_n(x) = \tfrac{1}{4}, \quad \forall n.$$

Since this supremum does not tend to 0, the convergence is **not uniform**. **Exercise 8.**

$$f_n(x) = x^n(1-x^n), \quad x \in [0,1].$$

Pointwise limit: For $0 \leq x < 1$, we know $x^n \to 0$ as $n \to \infty$. Therefore

$$f_n(x) = x^n - x^{2n} \longrightarrow 0.$$

At $x = 1$, we have

$$f_n(1) = 1 - 1 = 0.$$

Thus, the pointwise limit function is

$$f(x) \equiv 0 \quad \text{on } [0,1].$$

Uniform convergence: We compute

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} f_n(x),$$

since $f(x) = 0$ and $f_n(x) \geq 0$.

Now

$$f_n(x) = x^n - x^{2n}.$$

The derivative is

$$f_n'(x) = nx^{n-1}(1 - 2x^n).$$

Hence $f_n'(x) = 0$ if and only if $x^n = \frac{1}{2}$, which gives

$$x_n = \left(\tfrac{1}{2}\right)^{1/n}.$$

At this point,

$$f_n(x_n) = \tfrac{1}{2}\left(1 - \tfrac{1}{2}\right) = \tfrac{1}{4}.$$

Therefore,

$$\sup_{x \in [0,1]} f_n(x) = \frac{1}{4}, \quad \forall n \in \mathbb{N}.$$

Since

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = \frac{1}{4} \not\longrightarrow 0,$$

the sequence (f_n) converges pointwise to $f(x) = 0$ but the convergence is **not uniform**.

Exercise 9.

$$f_n(x) = \begin{cases} nx, & x \in [0, 1/n], \\ 0, & x \in (1/n, 1]. \end{cases}$$

Pointwise limit: - If $x = 0$, then $f_n(0) = 0$ for all n. - If $x > 0$, then for sufficiently large n we have $x > 1/n$, which implies $f_n(x) = 0$. Hence

$$\lim_{n \to \infty} f_n(x) = 0, \quad \forall x \in [0, 1].$$

Thus the pointwise limit function is

$$f(x) \equiv 0.$$

Uniform convergence: We compute

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = \sup_{x \in [0,1]} f_n(x).$$

On $[0, 1/n]$,

$$f_n(x) = nx,$$

which is increasing in x. Its maximum is therefore attained at $x = 1/n$, giving

$$f_n(1/n) = n \cdot \frac{1}{n} = 1.$$

On $(1/n, 1]$, $f_n(x) = 0$. Therefore

$$\sup_{x \in [0,1]} f_n(x) = 1 \quad \text{for all } n.$$

Conclusion: Since

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = 1 \not\to 0,$$

the convergence is **not uniform**.

Exercise 10.

$$f_n(x) = \begin{cases} \sqrt{nx}, & x \in [0, 1/n], \\ 0, & x \in (1/n, 1]. \end{cases}$$

Pointwise limit: At $x = 0$, clearly $f_n(0) = \sqrt{n \cdot 0} = 0$ for all n. If $x > 0$, then for sufficiently large n we have $x > 1/n$, which implies

$$f_n(x) = 0.$$

Therefore

$$\lim_{n\to\infty} f_n(x) = 0, \quad \forall x \in [0, 1].$$

Thus, the pointwise limit function is

$$f(x) \equiv 0.$$

Uniform convergence: We compute

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} f_n(x),$$

since $f(x) = 0$ and $f_n(x) \geq 0$.

On the interval $[0, 1/n]$, the function $f_n(x) = \sqrt{nx}$ is increasing in x. Hence the maximum is attained at $x = 1/n$, giving

$$f_n(1/n) = \sqrt{n \cdot \tfrac{1}{n}} = 1.$$

On $(1/n, 1]$, we have $f_n(x) = 0$. Therefore

$$\sup_{x\in[0,1]} f_n(x) = 1, \quad \forall n \in \mathbb{N}.$$

Since

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1 \not\longrightarrow 0,$$

the convergence is **not uniform**.

Exercise 11. Suppose that (f_n) converges uniformly to f on D and also converges uniformly to f on E. Prove that (f_n) converges uniformly to f on $D \cup E$.

Solution. By definition of uniform convergence, for every $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \quad \Longrightarrow \quad \sup_{x\in D} |f_n(x) - f(x)| < \varepsilon.$$

Similarly, since $f_n \to f$ uniformly on E, there exists $N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \quad \Longrightarrow \quad \sup_{x\in E} |f_n(x) - f(x)| < \varepsilon.$$

Now set

$$N = \max(N_1, N_2).$$

Then for every $n \geq N$, we have simultaneously

$$\sup_{x\in D} |f_n(x) - f(x)| < \varepsilon \quad \text{and} \quad \sup_{x\in E} |f_n(x) - f(x)| < \varepsilon.$$

Therefore, for every $x \in D \cup E$,

$$|f_n(x) - f(x)| < \varepsilon.$$

This implies

$$\sup_{x \in D \cup E} |f_n(x) - f(x)| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $f_n \to f$ uniformly on $D \cup E$.

Exercise 12. Suppose that (f_n) converges uniformly to f on D and (g_n) converges uniformly to g on D. Prove that for all $\alpha, \beta \in \mathbb{R}$, we have

$$\alpha f_n + \beta g_n \xrightarrow{u} \alpha f + \beta g \quad \text{on } D.$$

Solution. Fix $\varepsilon > 0$.

Since $f_n \xrightarrow{u} f$ on D, there exists $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \quad \Longrightarrow \quad \sup_{x \in D} |f_n(x) - f(x)| < \frac{\varepsilon}{2(|\alpha| + |\beta| + 1)}.$$

Similarly, since $g_n \xrightarrow{u} g$ on D, there exists $N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \quad \Longrightarrow \quad \sup_{x \in D} |g_n(x) - g(x)| < \frac{\varepsilon}{2(|\alpha| + |\beta| + 1)}.$$

Let

$$N = \max(N_1, N_2).$$

Then for all $n \geq N$ and all $x \in D$,

$$\begin{aligned} |(\alpha f_n(x) + \beta g_n(x)) - (\alpha f(x) + \beta g(x))| &= |\alpha(f_n(x) - f(x)) + \beta(g_n(x) - g(x))| \\ &\leq |\alpha|\,|f_n(x) - f(x)| + |\beta|\,|g_n(x) - g(x)| \\ &< (|\alpha| + |\beta|) \cdot \frac{\varepsilon}{2(|\alpha| + |\beta| + 1)} \\ &< \varepsilon. \end{aligned}$$

Taking the supremum over $x \in D$ gives

$$\sup_{x \in D} |(\alpha f_n(x) + \beta g_n(x)) - (\alpha f(x) + \beta g(x))| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\alpha f_n + \beta g_n \xrightarrow{u} \alpha f + \beta g \quad \text{on } D.$$

Exercise 13. Suppose $f_n \xrightarrow{u} f$ and $g_n \xrightarrow{u} g$ on D. If each f_n and each g_n is bounded on D, prove that $(f_n g_n) \xrightarrow{u} fg$ on D. Give an example where $f_n \xrightarrow{u} f$ and $g_n \xrightarrow{u} g$ on D but $(f_n g_n)$ does not converge uniformly.

Solution.

<u>Proof under boundedness.</u> Since each f_n is bounded on D, the family $\{f_n\}$ is uniformly bounded. Thus, there exists a constant $M > 0$ such that

$$|f_n(x)| \leq M, \quad \forall x \in D, \ \forall n \in \mathbb{N}.$$

Similarly, since each g_n is bounded, there exists $K > 0$ such that

$$|g_n(x)| \le K, \quad \forall x \in D,\ \forall n \in \mathbb{N}.$$

For $x \in D$,

$$f_n(x)g_n(x) - f(x)g(x) = (f_n(x) - f(x))g_n(x) + f(x)(g_n(x) - g(x)).$$

Taking absolute values,

$$|f_n(x)g_n(x) - f(x)g(x)| \le |f_n(x) - f(x)| \cdot |g_n(x)| + |f(x)| \cdot |g_n(x) - g(x)|.$$

Now take the supremum over $x \in D$:

$$\sup_{x \in D} |f_n(x)g_n(x) - f(x)g(x)| \le K \cdot \sup_{x \in D} |f_n(x) - f(x)| + \|f\|_\infty \cdot \sup_{x \in D} |g_n(x) - g(x)|.$$

Since $f_n \xrightarrow{u} f$ and $g_n \xrightarrow{u} g$, both suprema tend to 0. Therefore

$$f_n g_n \xrightarrow{u} fg \quad \text{on } D.$$

<u>Counterexample when boundedness is missing.</u>
Let $D = \mathbb{R}$, define

$$f_n(x) = g_n(x) = x + \frac{1}{n}.$$

Then: - For each fixed $x \in \mathbb{R}$, $f_n(x) \to f(x) = x$ and $g_n(x) \to g(x) = x$, uniformly (because $|f_n(x) - f(x)| = 1/n$ is independent of x). So $f_n \xrightarrow{u} f$ and $g_n \xrightarrow{u} g$ on $\mathbb{R}$.

But their product is

$$f_n(x)g_n(x) = \left(x + \tfrac{1}{n}\right)^2, \qquad fg(x) = x^2.$$

Thus

$$|f_n(x)g_n(x) - f(x)g(x)| = \left|\left(x + \tfrac{1}{n}\right)^2 - x^2\right| = \left|\tfrac{2x}{n} + \tfrac{1}{n^2}\right|.$$

For each fixed x, this tends to 0, so $(f_n g_n)(x) \to fg(x)$ pointwise. But

$$\sup_{x \in \mathbb{R}} \left|\tfrac{2x}{n} + \tfrac{1}{n^2}\right| = \infty,$$

since x can be arbitrarily large. Hence convergence is not uniform.

Conclusion. If $f_n \to f$ and $g_n \to g$ uniformly and the sequences are uniformly bounded, then $f_n g_n \to fg$ uniformly. If boundedness is absent, uniform convergence of the product can fail, as shown by the example above.

Exercise 15 (Dini's Theorem). Let D be a compact set in $\mathbb{R}$. Suppose (f_n) is a sequence of continuous functions on D such that

- $f_{n+1}(x) \le f_n(x)$ for all $x \in D$ and $n \in \mathbb{N}$ (monotone decreasing), - $f_n(x) \to f(x)$ pointwise for all $x \in D$, - f is continuous on D.

Prove that $f_n \to f$ uniformly on D. Show by the example $f_n(x) = x^n$ on $(0, 1)$ that compactness is necessary.

Solution. Let $\varepsilon > 0$. For each $c \in D$, since $f_n(c) \to f(c)$, there exists $N_c \in \mathbb{N}$ with

$$n \ge N_c \ \Rightarrow\ f_n(c) - f(c) < \tfrac{\varepsilon}{2}.$$

Because f_n and f are continuous, the difference $f_n - f$ is continuous. Hence there exists $\delta_c > 0$ such that

$$|x-c| < \delta_c,\ x \in D \quad \Rightarrow \quad |(f_n(x) - f(x)) - (f_n(c) - f(c))| < \tfrac{\varepsilon}{2}.$$

Thus for $n \geq N_c$ and $|x - c| < \delta_c$,

$$f_n(x) - f(x) \leq (f_n(c) - f(c)) + \tfrac{\varepsilon}{2} < \varepsilon.$$

The sets $I_c = (c - \delta_c, c + \delta_c) \cap D$ form an open cover of D. Since D is compact, we can select a finite subcover $I_{c_1}, \dots, I_{c_m}$. Let

$$N = \max(N_{c_1}, \dots, N_{c_m}).$$

Then for all $n \geq N$ and all $x \in D$, we have $f_n(x) - f(x) < \varepsilon$. Because the sequence is decreasing, $f_n(x) \geq f(x)$, so

$$0 \leq f_n(x) - f(x) < \varepsilon.$$

Taking the supremum,

$$\sup_{x \in D} |f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq N,$$

which proves $f_n \to f$ uniformly on D.

Counterexample. Take $f_n(x) = x^n$ on $(0,1)$. For each $x \in (0,1)$, $f_n(x) \to 0$, so the limit function is $f \equiv 0$. But

$$\sup_{x \in (0,1)} |f_n(x) - 0| = \sup_{x \in (0,1)} x^n = 1,$$

since $x^n \to 1$ as $x \to 1^-$ for all n. Thus the convergence is not uniform.

This shows that the compactness of D is essential.

Exercise 16. Find the pointwise limit of $f_n(x) = \dfrac{x^n}{1+x^n}$ on $[0,2]$, and determine whether the convergence is uniform.

Pointwise limit.

- If $0 \leq x < 1$: then $x^n \to 0$, so

$$f_n(x) = \frac{x^n}{1+x^n} \longrightarrow 0.$$

- If $x = 1$:

$$f_n(1) = \frac{1}{2} \quad \forall n.$$

- If $1 < x \leq 2$: then $x^n \to \infty$, so

$$f_n(x) = \frac{x^n}{1+x^n} \longrightarrow 1.$$

Thus the pointwise limit function is

$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ \dfrac{1}{2}, & x = 1, \\ 1, & 1 < x \leq 2. \end{cases}$$

Uniform convergence.

We check whether $\sup_{x\in[0,2]} |f_n(x) - f(x)| \to 0$. Note that f is discontinuous at $x = 1$, while each f_n is continuous. But a sequence of continuous functions converging uniformly must converge to a continuous limit. Therefore, uniform convergence on $[0, 2]$ is impossible.

More explicitly: near $x = 1$, if $x = 1 - \frac{1}{n}$ then $f_n(x)$ is close to 0, while if $x = 1 + \frac{1}{n}$ then $f_n(x)$ is close to 1. So $\sup_{x\in[0,2]} |f_n(x) - f(x)| \geq \frac{1}{2}$ for all n, which shows the error does not vanish.

The pointwise limit is

$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ \frac{1}{2}, & x = 1, \\ 1, & 1 < x \leq 2, \end{cases}$$

and the convergence is **not uniform**.

Exercise 18. Define a sequence $f_n : [0, 1] \to \mathbb{R}$ such that each f_n is discontinuous at every point of $[0, 1]$, and yet (f_n) converges uniformly to a continuous function on $[0, 1]$.

Solution. For $x \in [0, 1]$, let

$$f_n(x) = \begin{cases} \frac{1}{n}, & \text{if } x \text{ is rational}, \\ 0, & \text{if } x \text{ is irrational}. \end{cases}$$

- Each f_n is discontinuous at every point of $[0, 1]$, since rationals and irrationals are dense in $[0, 1]$ and the left/right limits oscillate between 0 and $\frac{1}{n}$.
- For every $x \in [0, 1]$, we have

$$\lim_{n\to\infty} f_n(x) = 0.$$

Thus the pointwise limit function is

$$f(x) \equiv 0,$$

which is continuous on $[0, 1]$.
- To check uniform convergence:

$$\sup_{x\in[0,1]} |f_n(x) - 0| = \sup_{x\in[0,1]} f_n(x) = \frac{1}{n}.$$

Since $\frac{1}{n} \to 0$, the convergence is uniform.

The sequence

$$f_n(x) = \begin{cases} \frac{1}{n}, & x \in \mathbb{Q} \cap [0, 1], \\ 0, & x \in [0, 1] \setminus \mathbb{Q} \end{cases}$$

is discontinuous everywhere for each n, but converges uniformly to the continuous function $f(x) \equiv 0$.

Exercise 19. Let φ be a continuous function on $[0, 1]$, and define

$$f_n(x) = \varphi(x)\, x^n, \quad x \in [0, 1].$$

Show that (f_n) converges uniformly if and only if $\varphi(1) = 0$. Deduce from this that

$$g_n(x) = nx(1 - x)^n \to 0 \quad \text{pointwise on } [0, 1],$$

but not uniformly.

Solution.

Step 1. Pointwise limit. For $0 \le x < 1$, $x^n \to 0$ as $n \to \infty$, hence $f_n(x) \to 0$. At $x = 1$, $f_n(1) = \varphi(1) \cdot 1^n = \varphi(1)$. Thus the pointwise limit is

$$f(x) = \begin{cases} 0, & 0 \le x < 1, \\ \varphi(1), & x = 1. \end{cases}$$

Step 2. Necessity of $\varphi(1) = 0$. If (f_n) converges uniformly, then the limit function f must be continuous on $[0, 1]$ (as a uniform limit of continuous functions). But the function f above is continuous at $x = 1$ if and only if $\varphi(1) = 0$. Therefore, uniform convergence implies $\varphi(1) = 0$.

Step 3. Sufficiency of $\varphi(1) = 0$. Assume $\varphi(1) = 0$. Since φ is continuous, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - 1| < \delta \quad \Longrightarrow \quad |\varphi(x)| < \varepsilon.$$

Now for $x \in [0, 1 - \delta]$, we have

$$|f_n(x)| \le \|\varphi\|_\infty (1 - \delta)^n \to 0.$$

For $x \in (1 - \delta, 1]$, we have

$$|f_n(x)| = |\varphi(x)|\, x^n \le \varepsilon \cdot 1 = \varepsilon.$$

Thus

$$\sup_{x \in [0,1]} |f_n(x)| \to 0,$$

which proves uniform convergence to 0.

Step 4. Deduction for $g_n(x) = nx(1 - x)^n$.

We first note pointwise convergence: - If $x = 0$, then $g_n(0) = 0$. - If $0 < x < 1$, then $(1-x)^n \to 0$ exponentially fast, so $nx(1-x)^n \to 0$. - If $x = 1$, then $g_n(1) = 0$. Hence $g_n(x) \to 0$ for all $x \in [0, 1]$.

To test uniform convergence, compute the maximum. Differentiate:

$$g_n'(x) = n(1 - x)^n - n^2 x(1 - x)^{n-1} = n(1 - x)^{n-1}\big((1 - x) - nx\big).$$

Setting $g_n'(x) = 0$ gives $1 - x = nx$, i.e.

$$x = \frac{1}{n+1}.$$

At this point,

$$g_n\!\left(\tfrac{1}{n+1}\right) = \frac{n}{n+1} \left(\frac{n}{n+1}\right)^n.$$

As $n \to \infty$,

$$\left(\frac{n}{n+1}\right)^n \to e^{-1}, \qquad \frac{n}{n+1} \to 1,$$

so

$$\sup_{x \in [0,1]} g_n(x) \to \frac{1}{e}.$$

Thus

$$\sup_{x \in [0,1]} |g_n(x) - 0| \not\to 0,$$

and the convergence is not uniform.

- (f_n) converges uniformly on $[0,1]$ if and only if $\varphi(1)=0$. - The sequence $g_n(x)=nx(1-x)^n$ converges pointwise to 0 on $[0,1]$, but not uniformly, since $\sup g_n = \frac{n}{n+1}\left(\frac{n}{n+1}\right)^n \to \frac{1}{e} \neq 0$.

Exercise 20. Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n^2 x, & x \in [0,1/n], \\ -n^2(x-2/n), & x \in (1/n, 2/n], \\ 0, & x \in (2/n, 1]. \end{cases}$$

Pointwise limit. If $x > 0$, then for large n we have $x > 2/n$, hence $f_n(x) = 0$. At $x = 0$, $f_n(0) = 0$. Thus $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$. Hence $f = 0$.

Uniform convergence. For each n, the maximum of f_n occurs at $x = 1/n$:

$$f_n(1/n) = n^2 \cdot \frac{1}{n} = n.$$

Thus

$$\sup_{x\in[0,1]} |f_n(x)| = n \to \infty.$$

So (f_n) does *not* converge uniformly to 0, even though the pointwise limit is 0.

Integral comparison. Compute

$$\int_0^1 f_n(x)\,dx = \int_0^{1/n} n^2 x\,dx + \int_{1/n}^{2/n} -n^2(x-2/n)\,dx.$$

First term:

$$\int_0^{1/n} n^2 x\,dx = n^2 \cdot \frac{1}{2} \cdot \frac{1}{n^2} = \frac{1}{2}.$$

Second term:

$$\int_{1/n}^{2/n} -n^2(x-2/n)\,dx = -n^2 \cdot \left[\frac{1}{2}(x-2/n)^2\right]_{1/n}^{2/n} = -n^2 \cdot \left(0 - \frac{1}{2}\cdot\frac{1}{n^2}\right) = \frac{1}{2}.$$

So

$$\int_0^1 f_n(x)\,dx = \frac{1}{2} + \frac{1}{2} = 1.$$

But

$$\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0.$$

We have $\lim f_n = f = 0$ pointwise, but

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1 \neq \int_0^1 f(x)\,dx.$$

This shows how pointwise convergence (without uniform convergence) does not allow us to interchange limit and integral.

Exercise 21. Let $p > 0$ and define

$$f_n(x) = \frac{nx}{1+n^2x^p}, \quad x \in [0,1].$$

(a) For what values of p does (f_n) converge uniformly on $[0,1]$ to a limit f? (b) If $p=2$, does

$$\int_0^1 f_n(x)\,dx \;\longrightarrow\; \int_0^1 f(x)\,dx\,?$$

Pointwise convergence.
Fix $x \in [0,1]$.
- If $x=0$, then $f_n(0)=0$ for all n.
- If $0 < x \le 1$, then for large n the denominator behaves like n^2x^p, so

$$f_n(x) = \frac{nx}{1+n^2x^p} \sim \frac{nx}{n^2x^p} = \frac{1}{nx^{p-1}}.$$

If $p>1$, the denominator has power n^2, so the whole fraction goes to 0. If $p=1$, then

$$f_n(x) = \frac{nx}{1+n^2x} \le \frac{n}{n^2} = \frac{1}{n} \to 0.$$

If $0<p<1$, then $n^2x^p \to \infty$ as $n\to\infty$ for each fixed $x>0$, so again $f_n(x)\to 0$.

Therefore in every case

$$\lim_{n\to\infty} f_n(x) = 0, \quad \forall x \in [0,1].$$

So the pointwise limit is the zero function

$$f(x) \equiv 0 \quad \text{on } [0,1].$$

Uniform convergence .
We test whether

$$\sup_{x\in[0,1]} |f_n(x)-f(x)| = \sup_{x\in[0,1]} f_n(x) \to 0.$$

Thus the behavior of the maximum of f_n determines uniform convergence.

Maximization of f_n.
For $x>0$, compute the derivative:

$$f_n'(x) = \frac{n(1+n^2x^p)-(nx)(n^2px^{p-1})}{(1+n^2x^p)^2} = \frac{n+n^3x^p-n^3px^p}{(1+n^2x^p)^2} = \frac{n-(p-1)n^3x^p}{(1+n^2x^p)^2}.$$

So critical points satisfy

$$n-(p-1)n^3x^p = 0.$$

- If $p=1$, the numerator is always $n>0$, so f_n is increasing and maximum at $x=1$.
- If $p\neq 1$, we get

$$x^p = \frac{1}{(p-1)n^2}.$$

Case 1: $0<p<1$. Here $(p-1)<0$, so the numerator $n-(p-1)n^3x^p$ is always positive, hence f_n increases on $[0,1]$. Maximum at $x=1$:

$$\sup_{x\in[0,1]} f_n(x) = f_n(1) = \frac{n}{1+n^2} \sim \frac{1}{n} \to 0.$$

So convergence is uniform.

Case 2: $p=1$. Then

$$f_n(x) = \frac{nx}{1+n^2x}.$$

Maximum at $x = 1$, so

$$\sup_{x\in[0,1]} f_n(x) = \frac{n}{1+n^2} \sim \frac{1}{n} \to 0.$$

So convergence is uniform.

Case 3: $1 < p < 2$. Critical point at

$$x_n = \left(\tfrac{1}{(p-1)n^2}\right)^{1/p}.$$

Evaluate:

$$f_n(x_n) = \frac{nx_n}{1+n^2x_n^p}.$$

But $n^2x_n^p = \frac{1}{p-1}$, so

$$f_n(x_n) = \frac{nx_n}{1+1/(p-1)} = \frac{nx_n}{p/(p-1)}.$$

Now

$$nx_n = n \cdot \left((p-1)^{-1/p}n^{-2/p}\right) = (p-1)^{-1/p}n^{1-2/p}.$$

Since $1 < p < 2$, we have $1 - 2/p < 0$, hence $n^{1-2/p} \to 0$. Thus $f_n(x_n) \to 0$. Therefore $\sup f_n(x) \to 0$, so convergence is uniform.

Case 4: $p = 2$. Critical point $x_n = 1/n$. Then

$$f_n(1/n) = \frac{n\cdot(1/n)}{1+n^2(1/n)^2} = \frac{1}{2}.$$

So

$$\sup f_n(x) \geq \tfrac{1}{2}, \quad \forall n.$$

Therefore $\sup f_n(x) \not\to 0$; no uniform convergence.

Case 5: $p > 2$. At the maximizing point, $nx_n = (p-1)^{-1/p}n^{1-2/p}$, and now $1 - 2/p > 0$, so this tends to infinity. Thus $f_n(x_n) \to \infty$, so $\sup f_n(x) \not\to 0$; no uniform convergence.

- For $0 < p < 2$, we have uniform convergence to $f = 0$.
- For $p \geq 2$, convergence is pointwise to 0 but not uniform.

Integral when $p = 2$.

We compute

$$f_n(x) = \frac{nx}{1+n^2x^2}.$$

Substitute $u = nx$, $dx = du/n$, limits $x = 0 \to 1$ give $u = 0 \to n$:

$$\int_0^1 f_n(x)\,dx = \int_0^1 \frac{nx}{1+n^2x^2}\,dx = \frac{1}{n^2}\int_0^n \frac{u}{1+u^2}\,du.$$

Now

$$\int_0^n \frac{u}{1+u^2}\,du = \frac{1}{2}\ln(1+n^2).$$

So

$$\int_0^1 f_n(x)\,dx = \frac{1}{2n^2}\ln(1+n^2).$$

As $n \to \infty$,

$$\frac{\ln(1+n^2)}{2n^2} \to 0.$$

Meanwhile the limit function $f \equiv 0$, so

$$\int_0^1 f(x)\,dx = 0.$$

Thus

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

indicate any domain restrictions explicitly.

Uniform Convergence for Series of Functions

Let D be a set and, for each $n \in \mathbb{N}$, let $f_n : X \to \mathbb{R}$ be a function. As with numerical series, a series of functions

$$\sum_{k=1}^{\infty} f_k$$

has two related meanings:

(a) The sequence of partial sums $(s_n)_{n\geq 1}$ on D, where

$$s_n(x) := \sum_{k=1}^{n} f_k(x).$$

(b) The sum function $f : D \to \mathbb{R}$ defined by

$$f(x) := \lim_{n\to\infty} s_n(x),$$

whenever the limit exists (pointwise or uniformly) on D.

Definition 88 *With the notation above:*

1. *The series* $\sum_{k=1}^{\infty} f_k$ converges pointwise *on D with sum f if the sequence (s_n) converges pointwise to f on D.*

2. *The series* converges uniformly *on X with sum f if (s_n) converges uniformly to f on D.*

When the sum function is clear from context, we simply say "$\sum f_k$ converges uniformly on D."

Supremum Criterion for Series

Proposition 89 (Supremum criterion for uniform convergence of series) *The series* $\sum_{k=1}^{\infty} f_k$ *converges uniformly on D if and only if*

1. *it converges pointwise on D, and*

2. $$\lim_{n\to\infty} \sup_{x\in D} \Big| \sum_{k=n+1}^{\infty} f_k(x) \Big| = 0.$$

Idea. Let $s_n = \sum_{k=1}^{n} f_k$ and $f = \lim s_n$ pointwise. Then $f(x) - s_n(x) = \sum_{k=n+1}^{\infty} f_k(x)$. The stated limit is exactly the uniform version of $\|f - s_n\|_\infty \to 0$. ■

It is often convenient to write the tail

$$R_n(x) := \sum_{k=n+1}^{\infty} f_k(x), \qquad M_n := \sup_{x\in X} |R_n(x)| \in [0, \infty],$$

so that uniform convergence $\iff M_n \to 0$.

Example 90 *Let $D = [0, \frac{1}{2}]$ or $D = [0, 1)$ and $f_k(x) = x^k$. Then $\sum_{k=1}^{\infty} f_k(x) = \frac{x}{1-x}$ pointwise for $x \in [0, 1)$. The tail is $R_n(x) = \dfrac{x^{n+1}}{1-x}$, which is increasing in x on $[0, 1)$. Hence $M_n = 2^{-(n+1)} \to 0$ on $[0, \frac{1}{2}]$ (uniform convergence), while $M_n = \infty$ on $[0, 1)$ (no uniform convergence).*

Uniform Cauchy Criterion for Series

Theorem 91 (Uniform Cauchy criterion) *The series $\sum_{k=1}^{\infty} f_k$ converges uniformly on D if and only if for every $\varepsilon > 0$ there exists N such that for all $m > n \geq N$,*

$$\Big|\sum_{k=n+1}^{m} f_k(x)\Big| < \varepsilon \quad \text{for all } x \in D.$$

Weierstrass M-Test

Theorem 92 *Let $\{f_n\}$ be a sequence of functions $f_n : D \to \mathbb{R}$. Suppose there exist constants $M_n \geq 0$ such that:*

1. *$\forall n,\ \sup_{x\in D} |f_n(x)| \leq M_n$, and*
2. *The series $\sum_{n=1}^{\infty} M_n$ converges.*

Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on D, and:

$$\sup_{x\in D} \left|\sum_{n=1}^{\infty} f_n(x)\right| \leq \sum_{n=1}^{\infty} M_n.$$

Example 93 *Consider the series:*

$$\sum_{n=1}^{\infty} \frac{n\sin(nx)}{e^n}, \qquad x \in \mathbb{R}.$$

We note:

$$\sup_{x\in\mathbb{R}} \left|\frac{n\sin(nx)}{e^n}\right| \leq \frac{n}{e^n}.$$

Since $\sum_{n=1}^{\infty} \frac{n}{e^n}$ converges, the Weierstrass M-test implies that the function series converges uniformly on $\mathbb{R}$.

Dirichlet's Test for Uniform Convergence

Theorem 94 (Dirichlet's Test) *Let $\{f_n\}$ and $\{g_n\}$ be sequences of real-valued functions defined on a common domain $D \subseteq \mathbb{R}$, with:*

$$f_n, g_n : D \to \mathbb{R}.$$

Suppose:

1. *The partial sums $S_n(x) = \sum_{k=1}^{n} f_k(x)$ are uniformly bounded on D, i.e., there exists $M > 0$ such that:*

$$|S_n(x)| \leq M \quad \text{for all } x \in D \text{ and all } n \in \mathbb{N}.$$

2. *The sequence $\{g_n(x)\}$ is monotonic in n for every $x \in D$.*

3. *$\{g_n(x)\}$ converges pointwise to 0 on D.*

Then the series $\sum_{n=1}^{\infty} f_n(x)g_n(x)$ converges uniformly on D.

Example 95 *Consider the function series:*

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{3n+1}, \qquad x \in [0,1].$$

We define:

$$f_n(x) = (-1)^n, \qquad g_n(x) = \frac{x^{3n+1}}{3n+1}.$$

Then:

- *The partial sums $\sum_{k=0}^{n}(-1)^k$ are bounded by 1.*
- *$\{g_n(x)\}$ is decreasing for each fixed $x \in [0,1]$.*
- *$g_n(x) \to 0$ uniformly on $[0,1]$ since:*

$$\sup_{x\in[0,1]} |g_n(x)| = \frac{1}{3n+1} \to 0.$$

Therefore, by Dirichlet's Test, the series converges uniformly on $[0,1]$.

Abel's Test for Uniform Convergence

Theorem 96 (Abel's Test) *Let $\{f_n\}$ and $\{g_n\}$ be sequences of real-valued functions defined on D, with:*

$$f_n, g_n : D \to \mathbb{R}.$$

Suppose:

1. *The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on D.*

2. *The sequence $\{g_n(x)\}$ is uniformly bounded and monotonic in n for each $x \in D$, i.e., there exists $M > 0$ such that:*

$$|g_n(x)| \leq M \quad \text{for all } x \in D \text{ and all } n \in \mathbb{N}.$$

Then the series $\sum_{n=1}^{\infty} f_n(x)g_n(x)$ converges uniformly on D.

Example 97 *Let:*

$$f_n(x) = \frac{x^n}{n}, \qquad g_n(x) = (-1)^n, \quad \text{for } x \in [0,1].$$

- $\sum_{n=1}^{\infty} \frac{x^n}{n}$ *converges uniformly on* $[0,1]$ *(this is the Taylor series of* $-\log(1-x)$*, and it converges uniformly on* $[0,a]$ *for any* $a < 1$*).*
- *The sequence* $g_n(x) = (-1)^n$ *is bounded and monotonic (since it oscillates and is fixed in absolute value).*

Hence, by Abel's Test, the series $\sum_{n=1}^{\infty} \frac{x^n}{n}(-1)^n$ *converges uniformly on* $[0,a]$ *for any* $a < 1$.

Corollary 98 (Alternating uniform convergence) *Suppose* $f_n : D \to [0,\infty)$ *satisfy* $f_{n+1}(x) \leq f_n(x)$ *for all* n *and* x*, and* $f_n \to 0$ *uniformly on* D*. Then the alternating series* $\sum_{k=1}^{\infty}(-1)^k f_k$ *converges uniformly on* D.

Idea. For $m > n$, alternating partial sums are bounded by the next term: $\left|\sum_{k=n+1}^{m}(-1)^k f_k(x)\right| \leq f_{n+1}(x)$. Taking suprema and using uniform $f_{n+1} \to 0$ gives the uniform Cauchy property. ■

Exercises

Practical guide to study uniform convergence of series $\sum_{k=1}^{\infty} f_k$

1. **Pointwise first.** For each $x \in D$, check $\sum_k f_k(x)$ converges and identify the sum $f(x)$ (or detect divergence).
2. **Tail–sup test.** Define the tail $R_n(x) := \sum_{k>n} f_k(x)$ and
$$M_n := \sup_{x \in D} |R_n(x)|.$$
Then $\sum f_k$ converges uniformly on $D \iff M_n \to 0$.
3. **Weierstrass M–test (fast sufficient).** If $\exists\, M_k \geq 0$ with $|f_k(x)| \leq M_k$ on D and $\sum_k M_k < \infty$, then $\sum f_k$ converges uniformly on D.
4. **Alternating (uniform) test.** If $f_k \geq 0$, $f_{k+1}(x) \leq f_k(x)$ for all $x \in D$, and $f_k \to 0$ *uniformly* on D, then $\sum(-1)^k f_k$ converges uniformly.
5. **Uniform Cauchy (when the sum is unknown).** Uniform convergence $\iff$
$$\forall \varepsilon > 0\ \exists N\ \forall m > n \geq N :\ \sup_{x \in D}\Big|\sum_{k=n+1}^{m} f_k(x)\Big| < \varepsilon.$$
6. **Domination patterns (quick wins).**
   - If $|f_k(x)| \leq a_k\, g(x)$ with $\sum a_k$ converges and g bounded on D, apply the M–test with $M_k = a_k \|g\|_\infty$.
   - If $|R_n(x)| \leq A_n$ on D and $A_n \to 0$, then uniform convergence follows (tail–sup test).

7. **Compactness.** On compact D, if (f_k) are continuous and the tails R_n are *equicontinuous* with $R_n \to 0$ pointwise, Dini-type arguments (monotone tails) or Arzelà–Ascoli corollaries may upgrade to uniform convergence.

8. **Interchange rules (need uniformity).** If $\sum f_k$ converges uniformly and each f_k is Riemann integrable on $[a, b]$, then

$$\int_a^b \Big(\sum f_k\Big) = \sum \int_a^b f_k.$$

   If $\sum f_k'$ converges uniformly on $[a, b]$ and $\sum f_k(x_0)$ converges at some x_0, then $\sum f_k$ converges uniformly and $\Big(\sum f_k\Big)' = \sum f_k'$.

9. **When M–test fails.** Try Dirichlet/Abel–type criteria on x–dependent signs:

   - **Dirichlet (uniform flavor):** if partial sums $A_n(x) = \sum_{k=1}^n a_k(x)$ are uniformly bounded in x, and $b_k(x) \downarrow 0$ uniformly in x, then $\sum a_k(x)b_k(x)$ converges uniformly.
   - **Abel:** combine bounded partial sums with uniformly bounded variation in the coefficient sequence.

10. **Subset test (to disprove).** If uniform convergence fails on some $E \subset D$, it fails on D. So it suffices to find a "bad" subset where tails stay large.

Problems

Ex 2.1 Prove that $\displaystyle\lim_{x\to 0} \sum_{n=0}^{\infty} \frac{e^{-n^3x^2}}{2^n} = 2.$

Ex 2.2 Show that

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}\,dx = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^3}.$$

Ex 2.3 Compute:

(a) $\displaystyle\int_1^2 \sum_{n=1}^{\infty} \frac{1}{(n+x)^2}\,dx;$

(b) $\displaystyle\int_0^{\pi} \sum_{n=1}^{\infty} \frac{n\sin(nx)}{e^n}\,dx.$

Ex 2.4 Let $f_n(x) = x^n(1-x)$ for $x \in [0, 1]$.

(a) Show that $\sum_{n=1}^{\infty} f_n$ does not converge uniformly on $[0, 1]$.

(b) Decide whether $\displaystyle\int_0^1 \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_0^1 f_n(x)\,dx.$

Ex 2.5

Ex 2.6 Let $f(x) := \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$ on $\mathbb{R}$. Show that

$$f'(x) = -2x \sum_{n=1}^{\infty} \frac{1}{(n + n^2 x^2)^2}.$$

Ex 2.7 Let $f_n(x) := 2^n x e^{-nx^2}$ for $x \in \mathbb{R}$.

(a) For each $A > \delta > 0$, prove that $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[\delta, A]$.

(b) Compute $\sum_{n=1}^{\infty} f_n(x)$ for $x > \sqrt{\ln 2}$.

Ex 2.8 Let $f_n(x) := (-1)^n \frac{1}{nx}$ on $(0, \infty)$. Show that $\sum_{n=1}^{\infty} f_n$ is continuous and differentiable on $(0, \infty)$.

Ex 2.9 Let $f_n(x) := \frac{nx^2}{n^3 + x^3}$.

(a) Show that $\sum_{n=1}^{\infty} f_n$ does not converge uniformly on $[0, \infty)$.

(b) Show that $\sum_{n=1}^{\infty} f_n$ is continuous on $[0, \infty)$.

Ex 2.10 Same as Ex 2.9 with $f_n(x) := \frac{nx}{n^3 + x^3}$.

Ex 2.11 Define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) := (n + x)^{-2} \sin(nx)$.

(a) Prove that $\sum_{n=1}^{\infty} f_n$ converges pointwise on $[0, \infty)$ and write the sum as $f(x) := \sum_{n=1}^{\infty} f_n(x)$.

(b) Prove that $f : [0, \infty) \to \mathbb{R}$ is continuous.

Answers

Ex 2.1 Since $0 \le e^{-n^3 x^2} \le 1$ and $\sum_{n=0}^{\infty} 2^{-n} < \infty$, by dominated convergence

$$\lim_{x \to 0} \sum_{n=0}^{\infty} \frac{e^{-n^3 x^2}}{2^n} = \sum_{n=0}^{\infty} \lim_{x \to 0} \frac{e^{-n^3 x^2}}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

Ex 2.2 Termwise integration is justified by Weierstrass M–test (since $|\sin(nx)|/n^2 \le 1/n^2$):

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}\, dx = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{\pi} \sin(nx)\, dx = \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} = \sum_{k=1}^{\infty} \frac{2}{(2k-1)^3}.$$

Ex 2.3 (a) Since $\sup_{x \in [1,2]} \frac{1}{(n+x)^2}) \le \frac{1}{n^2+1})$, we gey convergence uniform and we can interchange sum and the integral:

$$\int_1^2 \sum_{n=1}^{\infty} \frac{1}{(n+x)^2}\, dx = \sum_{n=1}^{\infty} \int_1^2 \frac{dx}{(n+x)^2} = \sum_{n=1}^{\infty} \Big[-\frac{1}{n+x}\Big]_1^2 = \sum_{n=1}^{\infty} \Big(\frac{1}{n+1} - \frac{1}{n+2}\Big) = \frac{1}{2}.$$

(b)

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{n \sin(nx)}{e^n}\, dx = \sum_{n=1}^{\infty} \frac{n}{e^n} \frac{1 - (-1)^n}{n} = \sum_{k=0}^{\infty} \frac{2}{e^{2k+1}} = \frac{2e^{-1}}{1 - e^{-2}} = \frac{2e}{e^2 - 1}.$$

Ex 2.4 $f_n(x) = x^n(1-x)$ on $[0,1]$. For $x \in [0,1)$,

$$\sum_{n=1}^{\infty} f_n(x) = (1-x)\sum_{n=1}^{\infty} x^n = x.$$

At $x=1$, all terms are 0, so the pointwise sum is $f(x)=x$ on $[0,1)$ and $f(1)=0$ (discontinuous). Uniform convergence would preserve continuity, hence it fails on $[0,1]$. Moreover,

$$\sum_{n=1}^{\infty}\int_0^1 f_n(x)\,dx = \sum_{n=1}^{\infty}\Big(\frac{1}{n+1} - \frac{1}{n+2}\Big) = \frac{1}{2} = \int_0^1 x\,dx = \int_0^1 \Big(\sum_{n=1}^{\infty} f_n(x)\Big)\,dx.$$

Ex 2.5

Ex 2.6 For each n,

$$\frac{d}{dx}\frac{1}{n^3+n^4x^2} = \frac{1}{n^3}\cdot\frac{d}{dx}\frac{1}{1+nx^2} = -\frac{2x}{n^2(1+nx^2)^2} = -2x\,\frac{1}{(n+n^2x^2)^2}.$$

Uniform convergence of the derivative series on compact sets (compare with $2|x|/n^2$) justifies termwise differentiation:

$$f'(x) = -2x\sum_{n=1}^{\infty}\frac{1}{(n+n^2x^2)^2}.$$

Ex 2.7 Let $f_n(x) = 2^n x e^{-nx^2}$.

(a) For $A \ge x \ge \delta > 0$,

$$f_n(x) = x\,(2e^{-x^2})^n \le A\,(2e^{-\delta^2})^n.$$

Thus $\sum f_n$ converges uniformly on $[\delta, A]$ provided $2e^{-\delta^2} < 1$, i.e. $\delta > \sqrt{\ln 2}$. (For $\delta \le \sqrt{\ln 2}$ the series diverges since the ratio ≥ 1.)

(b) For $x > \sqrt{\ln 2}$,

$$\sum_{n=1}^{\infty} f_n(x) = x\sum_{n=1}^{\infty}(2e^{-x^2})^n = \frac{2x\,e^{-x^2}}{1-2e^{-x^2}}.$$

For $x = \sqrt{\ln 2}$ the terms are constant in n and the series diverges; for $0 < x < \sqrt{\ln 2}$ it diverges geometrically; at $x=0$ the sum is 0.

Ex 2.8 Fix $a > 0$ and write

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{nx} = \sum_{n=1}^{\infty} a_n\,b_n(x), \qquad a_n := (-1)^n,\quad b_n(x) := \frac{1}{nx},\quad x \in [a,\infty).$$

The partial sums $A_N := \sum_{n=1}^{N} a_n$ are uniformly bounded: $|A_N| \le 1$ for all N. For each $x \ge a$, the sequence $b_n(x)$ is nonincreasing in n, and $b_n(x) \to 0$ as $n \to \infty$. Moreover this tends to 0 *uniformly* on $[a,\infty)$, since

$$\sup_{x\ge a} b_n(x) = \frac{1}{na} \xrightarrow[n\to\infty]{} 0.$$

We now invoke the *uniform Dirichlet test*: if (A_N) is bounded and $b_n(\cdot)$ is nonincreasing in n with $b_n \to 0$ uniformly on a set D, then $\sum a_n b_n(\cdot)$ converges uniformly on D. For completeness, we sketch the proof. For $p < q$ and $x \in D$, partial summation gives

$$\sum_{n=p}^{q} a_n b_n(x) = A_q b_{q+1}(x) - A_{p-1} b_p(x) + \sum_{n=p}^{q} A_n\big(b_n(x) - b_{n+1}(x)\big).$$

Taking absolute values and using $|A_n| \le M$ (here $M = 1$) and the monotonicity of b_n,

$$\Big|\sum_{n=p}^{q} a_n b_n(x)\Big| \le M\, b_{q+1}(x) + M\, b_p(x) + M \sum_{n=p}^{q} \big(b_n(x) - b_{n+1}(x)\big) \le 2M\, b_p(x).$$

Hence

$$\sup_{x\in[a,\infty)} \Big|\sum_{n=p}^{q} a_n b_n(x)\Big| \le 2M \sup_{x\ge a} b_p(x) = \frac{2M}{pa} \xrightarrow[p\to\infty]{} 0,$$

which is precisely the Cauchy criterion for uniform convergence on $[a, \infty)$.

Therefore $\sum_{n=1}^{\infty}(-1)^n/(nx)$ converges uniformly on every $[a, \infty)$. In particular, the sum $f(x)$ is continuous on each $[a, \infty)$; since $a > 0$ was arbitrary, f is continuous on $(0, \infty)$.

Ex 2.9 $f_n(x) = \dfrac{nx^2}{n^3 + x^3} \ge 0.$

(a) Let $y = x/n$. Then

$$\sup_{x\ge 0} f_n(x) = \sup_{y\ge 0} \frac{y^2}{1 + y^3} = \frac{2^{2/3}}{3} > 0$$

(attained at $y = \sqrt[3]{2}$). Since $\sup_x f_n(x)$ does not tend to 0, the series cannot converge uniformly on $[0, \infty)$. (Indeed, tails are bounded below by a fixed positive amount because the terms are nonnegative.)

(b) On each $[0, A]$ and for $n > A$,

$$0 \le f_n(x) \le \frac{nA^2}{n^3} = \frac{A^2}{n^2},$$

so $\sum f_n$ converges uniformly on $[0, A]$ by the M–test, hence the sum is continuous on $[0, \infty)$.

Ex 2.10 $f_n(x) = \dfrac{nx}{n^3 + x^3}.$

(a) With $y = x/n$,

$$\sup_{x\ge 0} f_n(x) = \sup_{y\ge 0} \frac{y}{1 + y^3} = \frac{1}{\sqrt[3]{4}} > 0,$$

so no uniform convergence on $[0, \infty)$ (same positivity argument).

(b) On $[0, A]$ and $n > A$,

$$0 \le f_n(x) \le \frac{nA}{n^3} = \frac{A}{n^2},$$

hence uniform convergence on $[0, A]$ and continuity of the sum on $[0, \infty)$.

Ex 2.11 $f_n(x) = (n + x)^{-2} \sin(nx)$ on $[0, \infty)$.

(a) **Pointwise convergence for each fixed $x \geq 0$.** For $x = 0$ one has $\sin(n \cdot 0) = 0$, so the series is identically 0. For $x > 0$ set $a_n(x) := (n+x)^{-2}$ and $b_n(x) := \sin(nx)$. Then $a_n(x) \downarrow 0$ in n. The partial sums of $b_n(x)$ satisfy the classical bound (for $x \notin 2\pi\mathbb{Z}$)

$$\sum_{k=1}^{N} \sin(kx) = \frac{\sin\left(\frac{Nx}{2}\right)\,\sin\left(\frac{(N+1)x}{2}\right)}{\sin(x/2)}, \qquad \Big|\sum_{k=1}^{N} \sin(kx)\Big| \leq \frac{1}{2\,|\sin(x/2)|},$$

and if $x \in 2\pi\mathbb{Z}$ then the partial sums are 0. Thus for each fixed $x \geq 0$ the sequence $\left(\sum_{k=1}^{N} \sin(kx)\right)_N$ is bounded. By Dirichlet's test, $\sum_{n=1}^{\infty} a_n(x)\sin(nx)$ converges for every $x \geq 0$.

(b) **Uniform (absolute) convergence and continuity on $[0, \infty)$.** For all $x \geq 0$ and all $n \in \mathbb{N}$,

$$|f_n(x)| = \frac{|\sin(nx)|}{(n+x)^2} \leq \frac{1}{(n+x)^2} \leq \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, the Weierstrass M–test implies that $\sum_{n=1}^{\infty} f_n(x)$ converges *uniformly and absolutely* on the whole $[0, \infty)$. Each f_n is continuous on $[0, \infty)$, hence the uniform limit

$$f(x) := \sum_{n=1}^{\infty} \frac{\sin(nx)}{(n+x)^2}$$

is continuous on $[0, \infty)$.

Power Series

Uniform (Absolute) Convergence for Series of Functions

Let $\sum_{n=1}^{\infty} f_n$ be a series of functions on a set X. The lower summation index is immaterial: for any $n_0 \in \mathbb{Z}$ we say that

$$\sum_{n=n_0}^{\infty} f_n \quad \text{converges uniformly on } D$$

if the sequence of partial sums $s_N(x) := \sum_{k=n_0}^{n_0+N-1} f_k(x)$ converges uniformly on D.

We say that $\sum_{n=n_0}^{\infty} f_n$ *converges uniformly absolutely* on D if the series of nonnegative functions $\sum_{n=n_0}^{\infty} |f_n|$ converges uniformly on D. By the Weierstrass M–test, the usual hypotheses imply uniform absolute convergence, and uniform absolute convergence implies uniform convergence.

Power Series

A *power series* is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n \qquad (x \in \mathbb{R}), \tag{3.1}$$

where $(c_n)_{n \geq 0} \subset \mathbb{R}$ are the *coefficients*. For each fixed $x \in \mathbb{R}$ the series in (**??**) may converge or diverge. The set $E \subset \mathbb{R}$ of all x for which it converges is the *domain of convergence*; on E the sum defines a function. Conversely, for a given function f on a set $V \subset \mathbb{R}$ one may ask whether f can be represented as the sum of a power series whose domain of convergence contains V.

Radius of convergence (Cauchy–Hadamard)

Theorem 99 *For the series (**??**) set*

$$d := \limsup_{n\to\infty} |c_n|^{1/n}, \qquad R := d^{-1} \quad (R := 0 \text{ if } d = \infty,\ R := \infty \text{ if } d = 0).$$

Then:

1. *If $|x| < R$, the series converges absolutely.*
2. *If $|x| > R$, the series diverges.*
3. *At the boundary points $x = \pm R$ (when $0 < R < \infty$), either convergence or divergence may occur; no general conclusion is possible.*

Proof sketch. For $x = 0$ the series converges. For $x \neq 0$, apply the root test to $a_n := c_n x^n$:

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} \left(|c_n|^{1/n}\,|x|\right) = |x|\,d = \begin{cases} 0, & d = 0, \\ \infty, & d = \infty, \\ |x|/R, & d \in (0,\infty). \end{cases}$$

The conclusions follow from the root test. ■

The number $R \in [0,\infty]$ is uniquely determined and is called the *radius of convergence*. The series converges for $|x| < R$ and diverges for $|x| > R$.

Exercise 100 *Assume only finitely many c_n are nonzero for large n, and that the limit $\lambda := \lim_{n\to\infty} |c_{n+1}|/|c_n|$ exists in $\mathbb{R}$. Define R from this d as above. Show that the conclusions of Theorem **??** still hold (ratio-test version).*

Examples

1. $c_0 := 0$, $c_n := n^n$ for $n \geq 1$. Then $d = \infty$, hence $R = 0$. So $\sum_{n=1}^{\infty} n^n x^n$ converges only at $x = 0$.
2. $c_n := 1/n!$. Then $\lim_{n\to\infty} |c_{n+1}|/|c_n| = 0$, so $R = \infty$. The sum defines the entire function

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (x \in \mathbb{R}), \tag{3.2}$$

   with $\exp(0) = 1$ and $\exp(1) = e$.
3. $c_n := 1$ (the geometric series). Then $R = 1$. For $|x| = 1$ the series $\sum_{n=0}^{\infty} x^n$ diverges because $x^n \nrightarrow 0$ as $n \to \infty$.
4. $c_n := 1/n$ $(n \geq 1)$. Then $R = 1$. At $x = 1$ the series diverges (harmonic series); at $x = -1$ it converges conditionally (alternating harmonic series). For real $|x| = 1$, these are the only boundary points.
5. $c_n := 1/n^2$ $(n \geq 1)$. Then $R = 1$ and the series converges absolutely at $x = \pm 1$.
6. If $c_n = 0$ for all $n > N$, the power series (**??**) *terminates* and is a polynomial $\sum_{n=0}^{N} c_n x^n$; the radius is $R = \infty$.

Termwise differentiation on $(-R, R)$

Restrict to real $x \in (-R, R)$ and define

$$f(x) := \sum_{n=0}^{\infty} c_n x^n. \tag{3.3}$$

Formally differentiating term by term suggests

$$g(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n.$$

The next lemma ensures the same radius for the derived series.

Lemma 101 *The power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$ have the same radius of convergence.*

Proof. By Cauchy–Hadamard, multiplying coefficients by n does not change the radius since $\lim_{n\to\infty} n^{1/n} = 1$. An index shift also does not affect the radius. ■

Theorem 102 *Let $R > 0$ be the radius of convergence of (**??**). Define f by (**??**) on $(-R, R)$ and $g(x) := \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$ on $(-R, R)$. Then f is differentiable on $(-R, R)$ and $f'(x) = g(x)$ for all $x \in (-R, R)$. In particular, f is continuous on $(-R, R)$.*

Idea of proof. Fix $x \in (-R, R)$ and choose ρ with $|x| < \rho < R$. For small $|h|$ estimate

$$\left| \frac{f(x+h) - f(x)}{h} - g(x) \right|$$

by splitting into finitely many terms (whose difference quotients $\to 0$) and a tail, bounded using $\sum n |c_n| \rho^{n-1} < \infty$. Use Lemma **??** and the mean value estimate to conclude. ■

Corollary 103 *On $(-R, R)$ the function f is C^∞ and, for $p \in \mathbb{N}$,*

$$f^{(p)}(x) = \sum_{n=0}^{\infty} (n+1)(n+2)\cdots(n+p)\, c_{n+p}\, x^n. \tag{3.4}$$

In particular,

$$f^{(p)}(0) = p!\, c_p. \tag{3.5}$$

Exercise 104 (Real Taylor formula on $(-R, R)$) *Assume all $c_n \in \mathbb{R}$. Show that for each $n \geq 0$ and $x \in (-R, R)$ there exists $\theta_x \in (0, 1)$ such that*

$$f(x) = \sum_{k=0}^{n} c_k x^k + \frac{f^{(n+1)}(\theta_x x)}{(n+1)!}\, x^{n+1}.$$

Remark 105 (Analytic vs. C^∞) *Equation (**??**) shows that a function given by a power series on $(-R, R)$ is completely determined by all of its derivatives at 0, and in fact by its values on any smaller open interval $(-\delta, \delta)$. There exist C^∞ functions f with $f^{(p)}(0) = 0$ for all p but $f \not\equiv 0$; such functions are not of the form (**??**). Functions representable by (**??**) are called* analytic.

Cauchy product and products of power series

Definition 106 (Cauchy product) *Given series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ in $\mathbb{R}$, their* Cauchy product *is the series $\sum_{n=0}^{\infty} c_n$ with*

$$c_n := \sum_{k=0}^{n} a_k\, b_{n-k} \qquad (n = 0, 1, 2, \dots). \tag{3.6}$$

Theorem 107 (Mertens, absolutely convergent case) *If $\sum |a_n| < \infty$ and $\sum |b_n| < \infty$ with sums A and B, then the Cauchy product $\sum c_n$ converges absolutely and $\sum c_n = AB$.*

Theorem 108 *Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be power series with radii of convergence $R_1, R_2 > 0$. Put $R := \min\{R_1, R_2\}$ and define c_n by (??). Then for $|x| < R$,*

$$\Big(\sum_{n=0}^{\infty} a_n x^n\Big)\Big(\sum_{n=0}^{\infty} b_n x^n\Big) = \sum_{n=0}^{\infty} c_n x^n,$$

and the product series has radius of convergence at least R.

Example 109 *Multiplying the geometric series with itself yields*

$$\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2} \qquad (|x| < 1),$$

consistent with differentiating $(1-x)^{-1}$.

Exercises

Ex 3.1. Determine the radius of convergence of $\sum a_n x^n$, where

$$\text{(a) } a_n = \frac{1}{n^n}, \quad \text{(b) } a_n = \frac{n^n}{n!}, \text{(c) } a_n = \frac{(n!)^2}{(2n)!}, \quad \text{(d) } a_n = n - \sqrt{n}, \quad \text{(e) } a_n = \begin{cases} 1, & n = k^2 \\ 0, & \text{otherwise.} \end{cases}$$

Ex 3.2. Suppose $0 < b \le |a_n| < c$ for all n. Find the radius of convergence of $\sum a_n x^n$.

Ex 3.3. Give a power series that converges on $(-1, 1]$ and diverges at -1.

Ex 3.4. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is even on $(-R, R)$, prove $a_n = 0$ for all odd n; if f is odd, prove $a_n = 0$ for all even n.

Ex 3.5. For which $c > 0$ is each series uniformly convergent on $[-c, c]$?

$$\text{(a) } \sum \frac{x^n}{n^{1/n}}, \quad \text{(b) } \sum n x^n, \quad \text{(c) } \sum \frac{2^n}{\sqrt{n!}}\, x^n, \quad \text{(d) } \sum \frac{(-1)^n}{5^n(n+1)}\, x^n, \text{(e) } \sum \frac{x^2}{\sqrt{n}\,(1+nx^2)}, \quad \text{(f) } \sum x^n(1-$$

Ex 3.6. Let $0 < R < \infty$ be the radius of convergence of $\sum a_n x^n$ and $k \in \mathbb{N}$. Find the radius of convergence of

$$\text{(a) } \sum (a_n)^k x^n, \qquad \text{(b) } \sum a_n x^{kn}, \qquad \text{(c) } \sum a_n x^{nk}.$$

Ex 3.7. Suppose $\sum a_n x^n$ and $\sum b_n x^n$ have radii R_1 and R_2, respectively.

(a) If $R_1 \neq R_2$, show $\sum(a_n+b_n)x^n$ has radius $\min\{R_1, R_2\}$. What can happen if $R_1 = R_2$?

(b) Show the radius of $\sum(a_n b_n)x^n$ is at least $R_1 R_2$.

Ex 3.8. (a) If $\sum a_n$ converges, prove $\sum \dfrac{a_n}{n+1}$ converges and

$$\int_0^1 \Big(\sum_{n=0}^{\infty} a_n x^n\Big)\, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}.$$

(b) If $\sum \dfrac{a_n}{n+1}$ converges, show the same identity holds (the integral may be improper).

Ex 3.9. Prove $(1+x)^\alpha$ is analytic on $(-1,1)$ for all $\alpha \in \mathbb{R}$ with

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \binom{\alpha}{0} = 1, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

Deduce x^α is analytic on $(0,\infty)$ by writing $x^\alpha = c^\alpha\,[1+(x/c-1)]^\alpha$.

Ex 3.10. From the Taylor series of $\dfrac{1}{1-x}$ show

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad |x| < 1,$$

and deduce $\log 2 = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$.

Ex 3.11. Differentiate $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ about $x=0$ to prove

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}, \qquad |x| < 1.$$

Ex 3.12. Find the Taylor expansions of $\dfrac{1}{2-x}$ about $x=0$ and $x=1$, and give each radius of convergence.

Ex 3.13. Find the Taylor expansion of $\dfrac{1}{x(2-x)}$ about $x=1$ and its radius of convergence.

Ex 3.14. Let $f(x) = \displaystyle\int_0^x e^{-t^2}\, dt$. Find its power series and evaluate $\lim_{x\to\infty} f(x)$.

Ex 3.15. Let R be the radius of $\sum a_n x^n$.

(a) If $f(x) = \sum_{n=0}^{\infty} a_n x^n \equiv 0$ on $(-R,R)$, prove $a_n = 0$ for all n.

(b) If $f(c_k) = 0$ for a sequence (c_k) of distinct points with $c_k \to c \in (-R,R)$, prove $a_n = 0$ for all n.

Answers

Ans 3.1. Use Cauchy–Hadamard: $R = (\limsup |a_n|^{1/n})^{-1}$.

$$
\begin{aligned}
&\text{(a) } |a_n|^{1/n} = n^{-1} \to 0 \;\Rightarrow\; R = \infty.\\
&\text{(b) } \frac{a_{n+1}}{a_n} = \left(1 + \tfrac{1}{n}\right)^n \to e \;\Rightarrow\; R = \tfrac{1}{e}.\\
&\text{(c) } a_n \sim \frac{\sqrt{\pi n}}{4^n} \;\Rightarrow\; |a_n|^{1/n} \to \tfrac{1}{4} \;\Rightarrow\; R = 4.\\
&\text{(d) } |a_n|^{1/n} \to 1 \;\Rightarrow\; R = 1.\\
&\text{(e) } \limsup |a_n|^{1/n} = 1 \;\Rightarrow\; R = 1.
\end{aligned}
$$

Ans 3.2. $b^{1/n} \le |a_n|^{1/n} \le c^{1/n} \to 1$, hence $\limsup |a_n|^{1/n} = 1$ and $R = 1$.

Ans 3.3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$: at $x = 1$ the alternating harmonic converges; at $x = -1$ it becomes $\sum \frac{1}{n}$ (diverges).

Ans 3.4. $f(-x) = \sum a_n (-1)^n x^n$. If f is even, $f(-x) = f(x)$, hence $a_n(1 - (-1)^n) = 0$, so $a_n = 0$ for odd n. The odd case is analogous.

Ans 3.5.

(a) Uniform on $[-c, c]$ $\Leftrightarrow$ $c < 1$ (Weierstrass; at $x = \pm 1$ no conv.).

(b) Uniform on $[-c, c]$ $\Leftrightarrow$ $c < 1$ ($\sum n|x|^n$ uniform for $|x| \le c < 1$).

(c) $R = \infty$; uniform on every $[-c, c]$ by $\sum \frac{2^n}{\sqrt{n!}} c^n < \infty$.

(d) Uniform on $[-c, c]$ for $c < 5$ (majorant $\sum (c/5)^n/(n+1)$). Not uniform at $c = 5$.

(e) $\sup_{|x| \le c} \frac{x^2}{\sqrt{n}(1 + nx^2)} \le \frac{1}{2n^{3/2}}$ for large n; $\sum n^{-3/2} < \infty$

$\Rightarrow$ uniform on all $[-c, c]$.

(f) $\sum x^n(1 - x) = 1$ for $|x| < 1$; diverges at $x = -1$. Uniform on $[-c, c]$ $\Leftrightarrow$ $c < 1$.

Ans 3.6. Let $d = \limsup |a_n|^{1/n} = 1/R$.

(a) $\limsup |(a_n)^k|^{1/n} = d^k \;\Rightarrow\; R_{\text{new}} = R^k$; (b),(c) $\sum a_n (x^k)^n \Rightarrow |x| < R^{1/k}$.

Ans 3.7.

(a) If $R_1 \ne R_2$, then $|a_n + b_n|^{1/n} \to \max\{1/R_1, 1/R_2\}$, hence radius $= \min\{R_1, R_2\}$. If $R_1 = R_2$, cancellation can increase the radius (e.g. $a_n = 1$, $b_n = -1$ gives the zero series).

(b) $\limsup |a_n b_n|^{1/n} \le (\limsup |a_n|^{1/n})(\limsup |b_n|^{1/n}) = (R_1 R_2)^{-1}$, so the radius $\ge R_1 R_2$.

Ans 3.8.

(a) For $0 \le r < 1$, $\int_0^1 \sum a_n (rx)^n dx = \sum a_n \frac{r^n}{n+1}$. Let $r \uparrow 1$ and use Abel/Dirichlet to pass to the limit; also $\sum \frac{a_n}{n+1}$ converges by Dirichlet.

(b) Same identity holds: define $F(r) = \sum \frac{a_n r^n}{n+1}$, note $F(r) = \int_0^1 \sum a_n (rx)^n dx$, then let $r \uparrow 1$; the LHS is an improper integral if necessary.

Ans 3.9. For $|x| < 1$, $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ by binomial series (ratio/root test). For x^α, fix $c > 0$ and write $x^\alpha = c^\alpha[1 + (x/c - 1)]^\alpha$; this gives a power series in $(x-c)$ on $(c - \mathrm{rad}, c + \mathrm{rad}) \subset (0, \infty)$, hence x^α is analytic on $(0, \infty)$.

Ans 3.10. Since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, integrate from 0 to x:

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}, \quad |x| < 1.$$

Set $x = \frac{1}{2}$ to get $\log 2 = \sum_{n=1}^{\infty} \dfrac{1}{n2^n}$.

Ans 3.11. Differentiate $\sum_{n=0}^{\infty} x^n$ termwise: $\dfrac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$, $|x| < 1$.

Ans 3.12. About 0: $\dfrac{1}{2-x} = \dfrac{1}{2} \cdot \dfrac{1}{1 - \frac{x}{2}} = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$, radius 2. About 1: let $h = x - 1$, then $\dfrac{1}{2-x} = \dfrac{1}{1-h} = \sum_{n=0}^{\infty} h^n$, radius 1.

Ans 3.13. Partial fractions: $\dfrac{1}{x(2-x)} = \dfrac{1}{2}\Big(\dfrac{1}{x} + \dfrac{1}{2-x}\Big)$. About $x = 1$ with $h = x - 1$:

$$\frac{1}{x} = \frac{1}{1+h} = \sum_{n=0}^{\infty} (-1)^n h^n, \quad \frac{1}{2-x} = \frac{1}{1-h} = \sum_{n=0}^{\infty} h^n.$$

Thus $\dfrac{1}{x(2-x)} = \sum_{m=0}^{\infty} h^{2m}$ (all odd terms cancel). Radius is $\min\{1, 1\} = 1$ (distance to nearest pole 0 or 2).

Ans 3.14. $e^{-t^2} = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{n!} t^{2n}$, so

$$f(x) = \int_0^x e^{-t^2} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} \quad \text{(entire power series)}.$$

Moreover $\lim_{x\to\infty} f(x) = \dfrac{\sqrt{\pi}}{2}$ (Gaussian integral).

Ans 3.15.

(a) An analytic function with all values 0 on an interval is identically 0; hence all Taylor coefficients a_n vanish.

(b) Zeros with an accumulation point inside the disc of convergence force the analytic function to be identically 0; hence $a_n = 0$ for all n.

Chapter 4

Lebesuge Integral

Introduction to the Lebesgue Integral

This section presents a short but rigorous introduction to the Lebesgue integral on a bounded interval in $\mathbb{R}$, in comparison with the classical Riemann integral.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. We begin with the Riemann integral, which is defined via partitions of the *domain*.

A partition of $[a, b]$ is a finite collection of points

$$a = a_0 < a_1 < \cdots < a_n = b.$$

Given such a partition, we define the following sums:

- The *lower Riemann sum*:

$$L(f, P) := \sum_{i=1}^{n} (a_i - a_{i-1}) \inf_{x \in [a_{i-1}, a_i]} f(x).$$

- The *upper Riemann sum*:

$$U(f, P) := \sum_{i=1}^{n} (a_i - a_{i-1}) \sup_{x \in [a_{i-1}, a_i]} f(x).$$

The function f is called *Riemann integrable* on $[a, b]$ if

$$\sup_P L(f, P) = \inf_P U(f, P).$$

In this case, the common value is denoted by

$$\int_a^b f(x)\, dx$$

and called the *Riemann integral* of f.

The Lebesgue integral takes a completely different approach: instead of partitioning the *domain* of f, we partition its *range*.

Let $f : [a, b] \to [c, d)$ be bounded. Choose a partition Q of the range:

$$c = c_0 < c_1 < \cdots < c_n = d.$$

Define

$$L^*(f, Q) := \sum_{i=1}^{n} c_{i-1}\, m(f^{-1}([c_{i-1}, c_i))),$$

$$U^*(f, Q) := \sum_{i=1}^{n} c_i\, m(f^{-1}([c_{i-1}, c_i))),$$

where $m(\cdot)$ denotes Lebesgue measure.

The function f is called *Lebesgue integrable* if

$$\sup_Q L^*(f, Q) = \inf_Q U^*(f, Q).$$

The common value is the *Lebesgue integral* of f and is written

$$\int_{[a,b]} f\, dm.$$

If f is Riemann integrable on $[a,b]$, then f is also Lebesgue integrable, and

$$\int_a^b f(x)\, dx = \int_{[a,b]} f\, dm.$$

The integral relies on the Lebesgue measure $m(\cdot)$, which we now define step by step.

1. If $E \subseteq [a,b]$ is an interval, then $m(E)$ is the length of the interval.
2. If $E \subseteq [a,b]$ is open, then E can be written as a countable disjoint union of intervals:

$$E = \bigsqcup_{k=1}^{\infty} E_k, \qquad E_k \subseteq [a,b].$$

   We define

$$m(E) := \sum_{k=1}^{\infty} m(E_k).$$

3. For an arbitrary set $E \subseteq [a,b]$, the *outer measure* is defined as

$$m^e(E) := \inf\{m(U) : E \subseteq U,\ U \text{ open}\}.$$

4. A set $E \subseteq [a,b]$ is called *Lebesgue measurable* if

$$m^e(E) + m^e([a,b] \setminus E) = b - a.$$

   In this case, we define $m(E) := m^e(E)$.

Remark 110 *For a function f, the Lebesgue integral is well-defined whenever $f^{-1}([c', d'])$ is Lebesgue measurable for every subinterval $[c', d'] \subseteq [c, d)$.*

If $\{E_k\}$ is a countable collection of pairwise disjoint Lebesgue-measurable sets, then

$$m\Big(\bigcup_k E_k\Big) = \sum_k m(E_k).$$

The Lebesgue integral has several fundamental advantages over the Riemann integral:

1. **More functions are integrable.** Every Riemann integrable function is Lebesgue integrable, but not conversely.
2. **Handles unbounded cases.** The definition extends naturally to unbounded functions and unbounded domains.
3. **Limit theorems.** Under very general conditions (Dominated Convergence Theorem, Monotone Convergence Theorem), one can interchange limit and integral:

$$\lim_{n\to\infty} \int f_n\, dm = \int \lim_{n\to\infty} f_n\, dm.$$

4. **Generalization.** Lebesgue integration applies to functions defined on arbitrary measure spaces, not just subsets of $\mathbb{R}$.

5. **Normed spaces.** The Lebesgue integral allows us to define the norms

$$\|f\|_1 := \int_a^b |f(x)|\, dm, \qquad \|f\|_2 := \left(\int_a^b |f(x)|^2\, dm \right)^{1/2},$$

   which are fundamental for L^p spaces and Hilbert space theory.

6. **Probability and statistics.** Lebesgue measure and integration provide the natural framework for probability theory, where measurable sets correspond to events and integrals correspond to expectations.

Lecture 1: Outer measure

The length $\ell(I)$ of an open interval $I \subset \mathbb{R}$ is defined as:

$$\ell(I) = \begin{cases} b-a & \text{if } I=(a,b) \text{ for some } a<b\in\mathbb{R}, \\ 0 & \text{if } I=\emptyset, \\ \infty & \text{if } I=(-\infty,a) \text{ or } I=(a,\infty), \\ \infty & \text{if } I=(-\infty,\infty). \end{cases}$$

This notion of length can be extended to a finite or infinite disjoint union of open intervals. Suppose

$$A = \bigcup_n I_n, \quad \text{with } I_n \cap I_m = \emptyset \text{ for } n \neq m,$$

then the total length of A is defined as:

$$\ell(A) = \sum_n \ell(I_n),$$

where $\ell(A) = \infty$ if the series diverges—this includes the case where at least one I_n is unbounded.

Definition 111 *The* ***outer measure*** *of a set $A \subset \mathbb{R}$, denoted $m^*(A)$, is defined by:*

$$m^*(A) = \inf\left\{ \sum_{k=1}^{\infty} \ell(I_k) \ : \ A \subset \bigcup_{k=1}^{\infty} I_k,\ I_k \textit{ are open intervals in } \mathbb{R} \right\}.$$

This means:

- We look at **all possible countable collections** of open intervals $I_1, I_2, I_3, \ldots$ that cover the set A.
- For each such collection, we calculate the total length:

$$\sum_{k=1}^{\infty} \ell(I_k).$$

- The outer measure $m^*(A)$ is the **smallest possible total length** (i.e., the infimum over all such sums).

Let's calculate the outer measure of the closed interval $A = [0, 1]$.

- To do this, we cover $[0,1]$ using open intervals. One simple choice is to take a slightly larger open interval that contains all of $[0,1]$. For any small $\varepsilon > 0$, let:

$$I_1 = (-\varepsilon, 1+\varepsilon), \quad \text{and set } I_2 = I_3 = \cdots = \emptyset.$$

- The total length of this cover is:

$$\sum_{k=1}^{\infty} \ell(I_k) = \ell(I_1) = (1+\varepsilon) - (-\varepsilon) = 1 + 2\varepsilon.$$

- Since ε can be made arbitrarily small, we take the infimum over all such covers:

$$m^*([0,1]) \leq \inf\{1+2\varepsilon : \varepsilon > 0\} = 1.$$

To prove the opposite inequality, let $I_1, I_2, \ldots$ be a countable collection of open intervals such that:

$$[0,1] \subset \bigcup_{k=1}^{\infty} I_k.$$

By the Heine–Borel Theorem, there exists a finite subcover; that is, there exists $n \in \mathbb{N}$ such that:

$$[0,1] \subset I_1 \cup \cdots \cup I_n.$$

We will now show by induction on n that this implies:

$$\sum_{k=1}^{n} \ell(I_k) \geq 1.$$

Since this finite sum is a lower bound for the total infinite sum, it follows that:

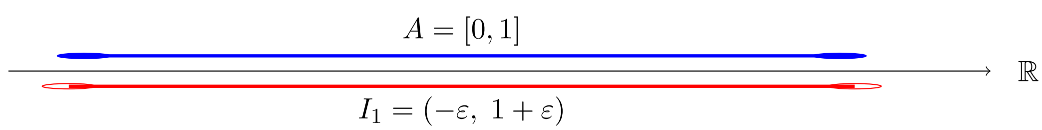
$$\sum_{k=1}^{\infty} \ell(I_k) \geq \sum_{k=1}^{n} \ell(I_k) \geq 1.$$

Thus, for every such cover:

$$m^*([0,1]) \geq 1.$$

Combining both inequalities, we conclude:

$$m^*([0,1]) = 1.$$



This example shows how outer measure works: we cover the set with open intervals and try to minimize the total length. Since every subset $A \subset \mathbb{R}$ can be covered by a countable union of bounded open intervals, and since all interval lengths are nonnegative (or infinite), the outer measure $m^*(A)$ is always well-defined. If every covering gives an infinite total length, then $m^*(A) = \infty$.

Properties of Outer Measure

- **Countable Sets Have Zero Measure:**

  If $A \subset \mathbb{R}$ is countable (finite or infinite), then:

$$m^*(A) = 0.$$

Why? Let $A = \{a_1, a_2, a_3, \ldots\}$. For any $\varepsilon > 0$, surround each point a_n with an open interval:

$$I_n = \left(a_n - \frac{\varepsilon}{2^{n+1}},\, a_n + \frac{\varepsilon}{2^{n+1}}\right), \quad \text{so} \quad \ell(I_n) = \frac{\varepsilon}{2^n}.$$

These intervals cover A, and the total length is:

$$\sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

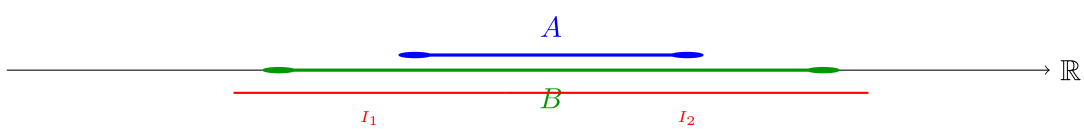
Since ε can be made arbitrarily small, the outer measure must be zero:

$$m^*(A) = 0.$$

Examples: Finite sets and $\mathbb{Q} \cap [0,1]$ are countable, so they have outer measure zero.

- **Monotonicity:** If $A \subset B$, then:
$$m^*(A) \leq m^*(B).$$

  Why? Any collection of open intervals that covers B also covers A. Since outer measure is defined as the smallest such total length, the measure of A can't exceed that of B.



  Interpretation:

  – The red intervals cover B (green), so they also cover A (blue).
  – The total length needed to cover A is at most the length needed to cover B.

- **Countable Subadditivity:**

  For any sequence of sets $E_1, E_2, E_3, \dots \subset \mathbb{R}$:

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

  Example 112 *Let $\mathbb{Q} \cap [0,1] = \{q_1, q_2, q_3, \dots\}$, and set $E_k = \{q_k\}$. Then:*

  – *$m^*(E_k) = 0$ for all k,*
  – *$\sum m^*(E_k) = 0$,*
  – *$\bigcup E_k = \mathbb{Q} \cap [0,1]$, so:*
$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) = 0.$$

  Here, we get equality.

  Now consider $A = \mathbb{Q} \cap [0,1]$ and $B = [0,1] \setminus \mathbb{Q}$. Then:

  – *$A \cup B = [0,1]$, and $A \cap B = \emptyset$,*

- $m^*(A) = 0$, $m^*(B) = 1$,
- *So:*

$$m^*(A \cup B) = 1 = m^*(A) + m^*(B).$$

Again, we have equality, but this is not always true.

Important: *Outer measure is* not *always additive! Even for disjoint sets A and B, it can happen that:*

$$m^*(A \cup B) \neq m^*(A) + m^*(B).$$

*So while outer measure is always **countably subadditive**, it is not generally **countably additive**.*

Lecture 2: σ-algebra

Definition 113 (Sigma-Algebra and Measurable Space) *Let X be a set, and let $\mathcal{S}$ be a collection of subsets of X.*

*We say that $\mathcal{S}$ is a **σ-algebra** on X if it satisfies the following:*

- $\emptyset \in \mathcal{S}$ *(the empty set is included),*
- *If $E \in \mathcal{S}$, then the complement $X \setminus E \in \mathcal{S}$,*
- *If $E_1, E_2, E_3, \dots \in \mathcal{S}$, then the union*

$$\bigcup_{k=1}^{\infty} E_k \in \mathcal{S}$$

  (closed under countable unions).

*If $\mathcal{S}$ is a σ-algebra on X, then the pair $(X, \mathcal{S})$ is called a **measurable space**.*

Example 114

- $\{\emptyset, X\}$*: the smallest possible σ-algebra on X,*
- $\mathcal{P}(X)$*: the power set of X, containing all subsets — the largest possible σ-algebra,*
- *The collection of all subsets $E \subseteq X$ such that either E is countable or $X \setminus E$ is countable.*

Proposition 115 *Let $\mathcal{S}$ be a σ-algebra on a set X. Then:*

(a) $X \in \mathcal{S}$

(b) *If $D, E \in \mathcal{S}$, then:*

$$D \cup E \in \mathcal{S}, \quad D \cap E \in \mathcal{S}, \quad D \setminus E \in \mathcal{S}$$

(c) *If $E_1, E_2, E_3, \dots \in \mathcal{S}$, then:*

$$\bigcap_{k=1}^{\infty} E_k \in \mathcal{S}$$

Proof. **(a)** Since $\emptyset \in \mathcal{S}$ (by definition), and $X = X \setminus \emptyset$, closure under complements gives $X \in \mathcal{S}$.

(b) Suppose $D, E \in \mathcal{S}$. Then:

- $D \cup E \in \mathcal{S}$ because $\mathcal{S}$ is closed under countable unions.
- For $D \cap E$, use De Morgan's law:

$$X \setminus (D \cap E) = (X \setminus D) \cup (X \setminus E)$$

  The right-hand side is in $\mathcal{S}$, so the left-hand side is too. Taking its complement shows $D \cap E \in \mathcal{S}$.

- For $D \setminus E$, note that:

$$D \setminus E = D \cap (X \setminus E)$$

  Both sets on the right are in $\mathcal{S}$, so their intersection is too.

(c) Let $E_1, E_2, \dots \in \mathcal{S}$. Then by De Morgan's law:

$$X \setminus \left(\bigcap_{k=1}^{\infty} E_k \right) = \bigcup_{k=1}^{\infty} (X \setminus E_k)$$

Since each $X \setminus E_k \in \mathcal{S}$ and $\mathcal{S}$ is closed under countable unions, the right-hand side is in $\mathcal{S}$. Taking the complement, we conclude:

$$\bigcap_{k=1}^{\infty} E_k \in \mathcal{S}$$

■

Borel σ-Algebra on $\mathbb{R}$

Definition 116 *The* ***Borel σ-algebra*** *on $\mathbb{R}$, denoted by $\mathcal{B}(\mathbb{R})$, is the smallest σ-algebra that contains all open intervals (a, b), where $a, b \in \mathbb{R}$.*

- It includes many familiar sets in real analysis: open, closed, half-open intervals, countable sets, and more.
- It is the foundation for defining measures (like Lebesgue measure) on subsets of $\mathbb{R}$.
- Any set in $\mathcal{B}(\mathbb{R})$ is called a **Borel set**.

Examples of Borel Sets:

- **Open intervals:** $(a, b) \in \mathcal{B}(\mathbb{R})$ by definition.
- **Half-open intervals:**

  $$[a, b) = \bigcap_{k=1}^{\infty} \left(a - \frac{1}{k}, b \right).$$

  Since each interval on the right is open, and Borel sets are closed under countable intersections, $[a, b) \in \mathcal{B}(\mathbb{R})$.
- **Unbounded intervals:**

  $$(a, \infty) = \bigcup_{k=1}^{\infty} (a + k, a + k + 1).$$

- **Closed intervals:**

  $$[a, b] = \mathbb{R} \setminus ((-\infty, a) \cup (b, \infty)).$$

  Since open sets are Borel, so are their complements.
- **Countable sets:** Any countable set, like the rationals in $[0, 1]$, is Borel. For example:

  $$B = \{x_1, x_2, x_3, \dots\}, \quad B = \bigcup_{k=1}^{\infty} \{x_k\},$$

  where each $\{x_k\}$ is a closed set.

- **Continuity sets of functions:** If $f : \mathbb{R} \to \mathbb{R}$, then the set where f is continuous is a Borel set, because it can be written as a countable intersection of open sets.

How is the Borel σ-algebra built?

- Start with all open intervals (a, b),
- Add all their complements to get closed sets,
- Then include all countable unions and intersections of those sets.

The Borel σ-algebra is large enough to cover most useful sets in analysis, but not all subsets of $\mathbb{R}$. Some sets are too "wild" to be Borel and require Lebesgue theory to handle.

Measure

Definition 117 *Let $\mathcal{S}$ be a σ-algebra on a set X. A function*

$$\mu : \mathcal{S} \to [0, \infty]$$

is called a ***measure*** *if it satisfies the following properties:*

1. ***Empty Set Has Zero Measure:***
$$\mu(\emptyset) = 0.$$

2. ***Countable Additivity (or σ-Additivity):***
If $\{E_n\}_{n=1}^{\infty}$ is a countable collection of pairwise disjoint sets in $\mathcal{S}$ (i.e., $E_i \cap E_j = \emptyset$ for $i \neq j$), then:
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

A triple $(X, \mathcal{S}, \mu)$ is called a ***measure space****.*

Let's explore several types of measures to better understand what a measure is and what properties it must satisfy.

(i) Counting Measure (Finite Case):

Define a function μ on all subsets of $\mathbb{R}$ by:

$$\mu(E) = \begin{cases} \text{Number of elements in } E, & \text{if } E \text{ is finite,} \\ \infty, & \text{if } E \text{ is infinite.} \end{cases}$$

- This measure simply counts how many elements are in a set.
- If the set is infinite (for example, the set of all natural numbers), we define its measure to be ∞.
- For instance:
$$\mu(\{1, 2, 4\}) = 3, \qquad \mu(\mathbb{N}) = \infty.$$

Why this is a measure:

- $\mu(\emptyset) = 0$, which satisfies the null empty set property.
- For any countable collection of disjoint finite sets $E_1, E_2, \ldots$, we have:

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n),$$

because union just adds up all the elements with no overlap.

(ii) Dirac Measure at a Point $c \in \mathbb{R}$:

Define:

$$\mu_c(E) = \begin{cases} 1, & \text{if } c \in E, \\ 0, & \text{if } c \notin E. \end{cases}$$

Explanation:

- This measure concentrates all the "mass" at a single point c.
- Think of placing a unit of "weight" only at point c. Any set containing c will have measure 1; otherwise, 0.
- For example:

$$\mu_5([4,6]) = 1, \quad \mu_5((0,4)) = 0.$$

Why this is a measure:

- $\mu_c(\emptyset) = 0$ since $c \notin \emptyset$.
- For disjoint sets $E_1, E_2, \ldots$, only one of them (at most) can contain c, so:

$$\mu_c\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu_c(E_n),$$

which is either 1 or 0 depending on whether $c \in \bigcup E_n$.

(iii) Weighted Dirac Measures (Discrete Probability Model):

Let $c_1, c_2, \dots \in \mathbb{R}$ be points, and $p_1, p_2, \dots \geq 0$ be corresponding weights (think of probabilities or masses). Define:

$$\mu(E) = \sum_{\{i : c_i \in E\}} p_i.$$

Explanation:

- Each point c_i has a fixed weight $p_i \geq 0$.
- To measure a set E, we sum up all the weights of those c_i that lie in E.
- Example:

$$\text{If } c_1 = 1,\ p_1 = 0.3;\ c_2 = 2,\ p_2 = 0.7;\ \text{then } \mu(\{1,2\}) = 1.$$

Why this is a measure:

- $\mu(\emptyset) = 0$, because none of the c_i are in $\emptyset$.
- Countable additivity holds: if $E_1, E_2, \ldots$ are disjoint, the weights of points in each are disjoint too, so:

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Note: This kind of measure is used in probability theory to model discrete random variables with weighted outcomes.

(iv) Define a set function μ by:

$$\mu(E) = \begin{cases} 0, & \text{if } E \text{ is finite,} \\ \infty, & \text{if } E \text{ is infinite.} \end{cases}$$

Why this fails to be a measure:

- It satisfies $\mu(\emptyset) = 0$, and is **finitely additive**, meaning:

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2),$$

  when E_1, E_2 are disjoint and finite.

- However, it is **not countably additive**. For example, take the disjoint sets:

$$E_n = \{n\}, \quad n = 1, 2, 3, \ldots$$

  Then each $\mu(E_n) = 0$, so:

$$\sum_{n=1}^{\infty} \mu(E_n) = 0.$$

  But their union is the infinite set $\mathbb{N}$, so:

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu(\mathbb{N}) = \infty.$$

  This contradicts countable additivity.

Lecture 3: Lebesgue measure

Lebesgue Measurable Sets

Definition 118 *A set $E \subset \mathbb{R}$ is called **Lebesgue measurable** if, for every subset $A \subset \mathbb{R}$, the following equality holds:*

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c),$$

where:

- *$m^*(\cdot)$ denotes the **outer measure**, and*
- *$E^c = \mathbb{R} \setminus E$ is the **complement** of E.*

*This condition is known as the **Carathéodory criterion**.*

The intuition behind this definition is that a Lebesgue measurable set E splits any other set $A \subset \mathbb{R}$ into two disjoint parts—$A \cap E$ and $A \cap E^c$—in a way that preserves the total outer measure. That is, measuring the parts separately and adding the results gives exactly the same outer measure as measuring the whole set A directly.

From the properties of outer measure, we always have the inequality:

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c),$$

since $A \subset (A \cap E) \cup (A \cap E^c)$ and outer measure is **countably subadditive**.

Therefore, to verify that E is measurable, we only need to check the **reverse inequality**:

$$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A) \quad \text{for all } A \subset \mathbb{R}.$$

If this inequality holds, then equality follows automatically from the previous inequality, and E is Lebesgue measurable.

Summary: A set $E \subset \mathbb{R}$ is Lebesgue measurable if splitting any set A using E and its complement does not increase the outer measure. This ensures that E behaves well with respect to measure and integration.

Properties of Measurable Sets

- **The empty set $\emptyset$ and the real line $\mathbb{R}$ are measurable.**

  Why? For any set $A \subset \mathbb{R}$:

  $$A \cap \emptyset = \emptyset, \quad A \cap \emptyset^c = A \quad \Rightarrow \quad m^*(A) = 0 + m^*(A).$$

  Similarly, for $E = \mathbb{R}$:

  $$A \cap \mathbb{R} = A, \quad A \cap \mathbb{R}^c = \emptyset \quad \Rightarrow \quad m^*(A) = m^*(A) + 0.$$

- **A set is measurable if and only if its complement is measurable.**

  Why? If E is measurable, then for all $A \subset \mathbb{R}$:

  $$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

  This expression is symmetric in E and E^c, so E^c is also measurable.

- **Every set of outer measure zero is measurable.**

  Why? If $m^*(E) = 0$, then for any $A \subset \mathbb{R}$:
$$m^*(A \cap E) \leq m^*(E) = 0 \quad \Rightarrow \quad m^*(A \cap E) = 0.$$
  Hence,
$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$
  So E satisfies the measurability condition.

- **The union of two measurable sets is measurable.**

  Why? Let $E, F \in \mathcal{M}$, and let $A \subset \mathbb{R}$. Define:
$$A_1 = A \cap E, \quad A_2 = A \cap E^c \cap F, \quad A_3 = A \cap E^c \cap F^c.$$
  These three parts are disjoint and cover A, and since E and F are measurable:
$$m^*(A) = m^*(A_1) + m^*(A_2) + m^*(A_3).$$
  Notice:
$$A \cap (E \cup F) = A_1 \cup A_2, \quad A \cap (E \cup F)^c = A_3,$$
  so:
$$m^*(A) = m^*(A \cap (E \cup F)) + m^*(A \cap (E \cup F)^c),$$
  and thus $E \cup F$ is measurable.

- **The interval (a, ∞) is measurable for any $a \in \mathbb{R}$.**

  Why? Let $A \subset \mathbb{R}$. Define:
$$A_1 = A \cap (a, \infty), \quad A_2 = A \cap (-\infty, a].$$
  These cover A, and are disjoint:
$$A = A_1 \cup A_2, \quad A_1 \cap A_2 = \emptyset.$$
  If $m^*(A) = \infty$, then the inequality
$$m^*(A_1) + m^*(A_2) \leq m^*(A)$$
  holds trivially. Otherwise, for any $\varepsilon > 0$, choose an open cover $\{I_n\}$ of A such that:
$$\sum \ell(I_n) \leq m^*(A) + \varepsilon.$$
  Define:
$$J_n = I_n \cap (a, \infty), \quad K_n = I_n \cap (-\infty, a].$$
  Then $A_1 \subset \bigcup J_n$, $A_2 \subset \bigcup K_n$, and:
$$\ell(I_n) = \ell(J_n) + \ell(K_n) \Rightarrow \sum \ell(J_n) + \sum \ell(K_n) = \sum \ell(I_n).$$
  Therefore:
$$m^*(A_1) + m^*(A_2) \leq m^*(A) + \varepsilon.$$
  Letting $\varepsilon \to 0$, we conclude:
$$m^*(A) = m^*(A_1) + m^*(A_2),$$
  proving that (a, ∞) is measurable.

Theorem 119 *The collection of Lebesgue measurable sets $\mathcal{M}$ is a σ-**algebra**.*

Proof. We have already established that:

- $\mathcal{M}$ contains $\emptyset$ and $\mathbb{R}$,
- $\mathcal{M}$ is closed under complements,
- $\mathcal{M}$ is closed under finite unions.

Now let $E_1, E_2, \cdots \in \mathcal{M}$ be a countable collection of pairwise disjoint measurable sets, and define:

$$F_n = \bigcup_{i=1}^{n} E_i, \quad F = \bigcup_{i=1}^{\infty} E_i.$$

Since each F_n is a finite union of measurable sets, $F_n \in \mathcal{M}$ for all n. For any $A \subset \mathbb{R}$, we have:

$$m^*(A) = \lim_{n\to\infty} \left[m^*(A \cap F_n) + m^*(A \cap F_n^c)\right].$$

As $F_n \uparrow F$, we get:

$$\lim_{n\to\infty} m^*(A \cap F_n) = m^*(A \cap F),$$

and since $A \cap F_n^c \downarrow A \cap F^c$, we also have:

$$\lim_{n\to\infty} m^*(A \cap F_n^c) = m^*(A \cap F^c).$$

Thus:

$$m^*(A) = m^*(A \cap F) + m^*(A \cap F^c),$$

which shows $F \in \mathcal{M}$. Therefore, $\mathcal{M}$ is closed under countable unions, and hence is a σ-algebra. ∎

Theorem 120 *The Borel σ-algebra $\mathcal{B}$ is contained in the collection of Lebesgue measurable sets $\mathcal{M}$, i.e., $\mathcal{B} \subset \mathcal{M}$.*

Proof. We previously showed that every interval of the form (a, ∞) is measurable.

Now consider an open interval of the form $(-\infty, b)$. Observe that:

$$(-\infty, b) = \bigcup_{n=1}^{\infty} \left(-\infty, b - \frac{1}{n}\right),$$

and since each $(b - \frac{1}{n}, \infty)$ is measurable, their complements $(-\infty, b - \frac{1}{n})$ are also measurable. Therefore, $(-\infty, b)$ is measurable as a countable union of measurable sets.

Consequently, any open interval (a, b) can be written as:

$$(a, b) = (a, \infty) \cap (-\infty, b),$$

which is an intersection of two measurable sets, and hence also measurable.

Since any open set in $\mathbb{R}$ can be written as a countable union of open intervals, and $\mathcal{M}$ is a σ-algebra, it follows that all open sets are measurable.

Therefore, the Borel σ-algebra $\mathcal{B}$, which is generated by open intervals, is a subset of $\mathcal{M}$. ∎

Theorem 121 *The restriction of the outer measure m^* to the collection $\mathcal{M}$ of Lebesgue measurable sets defines a measure. That is,*

$$m := m^*|_{\mathcal{M}}$$

is a measure on the measurable space $(\mathbb{R}, \mathcal{M})$.

$$(\mathbb{R}, \mathcal{M}, m)$$

is called the ***Lebesgue measure space****.*

Idea of the Proof. To prove that m is a measure, we must verify the two key properties of a measure:

1. $m(\emptyset) = 0$,
2. Countable additivity: For any disjoint collection $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$,

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

Since $m = m^*$ on $\mathcal{M}$, and $m^*(\emptyset) = 0$, the first property is automatically satisfied.
Now we focus on countable additivity.
Let $\{E_i\}_{i=1}^{\infty}$ be a disjoint family of measurable sets. We want to show:

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i).$$

Step 1: Lower bound (using monotonicity and finite additivity).
For any $n \in \mathbb{N}$, we have

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \geq m^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(E_i),$$

because the sets E_i are disjoint and measurable.

Taking the limit as $n \to \infty$ gives:

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} m^*(E_i).$$

Step 2: Upper bound (using subadditivity of m^*).
By the definition of outer measure and countable subadditivity:

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i).$$

Conclusion: Since we have both inequalities ($\leq$ and $\geq$, equality holds:

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i).$$

Therefore, m is countably additive on $\mathcal{M}$, so it is indeed a measure. ∎

Key Remark: Although the outer measure m^* is defined on all subsets of $\mathbb{R}$, it is not countably additive in general. However, when restricted to the collection $\mathcal{M}$ of Lebesgue measurable sets, it becomes a proper measure. Importantly, $\mathcal{M}$ contains all Borel sets, which are sufficient for most practical applications in analysis .

Theorem 122 *Then the following properties hold:*

(a) Monotonicity: *If $E, F \in \mathcal{M}$ with $E \subset F$, then*

$$m(E) \leq m(F).$$

(b) Countable Sub-additivity: *For any countable collection $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$,*

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m(E_n).$$

(c) Continuity from Below: *If $E_1 \subset E_2 \subset \cdots$ (increasing sequence), then*

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n).$$

(d) Continuity from Above: *If $E_1 \supset E_2 \supset \cdots$ (decreasing sequence) and $m(E_k) < \infty$ for some k, then*

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n).$$

We prove each property individually.

(a) Monotonicity: Suppose $E \subset F$. Then the set difference $F \setminus E \in \mathcal{M}$, and the sets E and $F \setminus E$ are disjoint. Since $E \cup (F \setminus E) = F$, we get:

$$m(F) = m(E) + m(F \setminus E) \geq m(E).$$

(b) Countable Sub-additivity: Let $\{E_n\}$ be any sequence of measurable sets. Define:

$$F_1 = E_1, \quad F_n = E_n \setminus \bigcup_{k=1}^{n-1} E_k \quad (n \geq 2).$$

Then the F_n are disjoint, and:

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n.$$

Thus:

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(F_n) \leq \sum_{n=1}^{\infty} m(E_n).$$

(c) Continuity from Below: Let $E_1 \subset E_2 \subset \cdots$, and set:

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Define $A_1 = E_1$, and $A_n = E_n \setminus E_{n-1}$ for $n \geq 2$. Then $E = \bigsqcup_{n=1}^{\infty} A_n$, so:

$$m(E) = \sum_{n=1}^{\infty} m(A_n), \quad \text{and} \quad m(E_k) = \sum_{n=1}^{k} m(A_n).$$

Hence,

$$\lim_{k\to\infty} m(E_k) = m(E).$$

(d) Continuity from Above: Let $E_1 \supset E_2 \supset \cdots$, and assume $m(E_k) < \infty$ for some k. Let:

$$E = \bigcap_{n=1}^{\infty} E_n = \bigcap_{n=k}^{\infty} E_n,$$

and set $A_n = E_k \setminus E_n$. Then $A_n \subset A_{n+1}$ and:

$$E_k \setminus E = \bigcup_{n=k}^{\infty} A_n.$$

By continuity from below:

$$m(E_k \setminus E) = \lim_{n\to\infty} m(E_k \setminus E_n).$$

Therefore:

$$\lim_{n\to\infty} m(E_n) = m(E_k) - \lim_{n\to\infty} m(E_k \setminus E_n) = m(E).$$

Lecture 4: Lebesgue Measurable Function

Definition 123 *Let $f : E \to \mathbb{R}$ be a function, where $E \subseteq \mathbb{R}$ is a measurable set.*

*We say that f is **Lebesgue measurable** (or simply **measurable**) if for every real number $\alpha \in \mathbb{R}$, the set*

$$\{x \in E : f(x) > \alpha\}$$

belongs to $\mathcal{M}$; that is, it is a measurable set.

We now present several examples to illustrate the concept of measurable functions. In each case, we examine whether the set $\{x \in \mathbb{R} : f(x) > \alpha\}$ belongs to $\mathcal{M}$ (e.g., Lebesgue measurable set). If this condition holds for every $\alpha \in \mathbb{R}$, then f is measurable.

1. **Constant function:** Let $f(x) \equiv c$, a constant function for some $c \in \mathbb{R}$. Consider the set

   $$\{x \in \mathbb{R} : f(x) > \alpha\}.$$

   - If $\alpha \geq c$, then $f(x) > \alpha$ is never true, so the set is empty: $\emptyset$. - If $\alpha < c$, then $f(x) > \alpha$ for all $x \in \mathbb{R}$, so the set is $\mathbb{R}$.

   Since both $\emptyset$ and $\mathbb{R}$ are elements of $\mathcal{M}$, this shows that constant functions are always measurable .

2. **Continuous functions:** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. For any $\alpha \in \mathbb{R}$, the set

   $$\{x \in \mathbb{R} : f(x) > \alpha\}$$

   is an open set, because the preimage of an open interval (α, ∞) under a continuous function is open. Since every open set is a Borel set, it follows that every continuous function is Borel measurable.

3. **Characteristic function of a measurable set:** If $E, F \subset \mathbb{R}$ are two measurable sets, then the indicator function $\chi_F : E \to \mathbb{R}$, defined by

   $$\chi_F(x) = \begin{cases} 1, & x \in F, \\ 0, & x \notin F, \end{cases}$$

   is measurable.

   This can be verified by direct computation. For any $\alpha \in \mathbb{R}$, the preimage $\chi_F^{-1}((\alpha, \infty])$ is given by

   $$\{x \in \mathbb{R} : \chi_F(x) > \alpha\} = \begin{cases} \emptyset, & \alpha > 1, \\ E \cap F, & 0 \leq \alpha < 1, \\ E, & \alpha < 0. \end{cases}$$

   Since E and F are measurable, each of these preimages is measurable, thus making χ_F measurable.

4. **Monotone functions:** Let $f : \mathbb{R} \to \mathbb{R}$ be any monotone increasing function, and let $\alpha \in \mathbb{R}$.

   Then the set

   $$\{x \in \mathbb{R} : f(x) > \alpha\}$$

   is one of the following:

- a right-open half-line of the form $\{x \in \mathbb{R} : x > \gamma\}$,
- a right-closed half-line $\{x \in \mathbb{R} : x \geq \gamma\}$,
- the entire real line $\mathbb{R}$, or
- the empty set $\emptyset$,

Conclusion. These examples illustrate that the class of measurable functions includes:

- all continuous functions,
- all characteristic functions of measurable sets,
- and all monotone functions.

Remark. The collection of measurable functions is closed under arithmetic operations (addition, subtraction, scalar multiplication, etc.), pointwise limits, and taking absolute values. This makes them very useful in integration theory and probability.

Theorem 124 *Let $E \subset \mathbb{R}$ be measurable, and suppose $f, g : E \to \mathbb{R}$ are two measurable functions, and let $c \in \mathbb{R}$ be a constant. Then the following functions are also measurable:*

$$cf, \quad f^2, \quad f + g, \quad f \cdot g, \quad |f|.$$

Proof. We verify measurability for each case:

1. **Scalar multiplication:** Assume $c > 0$ (the case $c < 0$ is similar and $c = 0$ is trivial). For any $\alpha \in \mathbb{R}$, we have:

   $$\{x \in \mathbb{R} : cf(x) > \alpha\} = \{x \in \mathbb{R} : f(x) > \alpha/c\}.$$

   Since f is measurable, the right-hand side is in $\mathcal{M}$, hence cf is measurable.

2. **Square function:** Assume $\alpha > 0$ (for $\alpha \leq 0$, the set $\{f^2 > \alpha\}$ is either $\mathbb{R}$ or empty, and thus measurable). Then:

   $$\{x \in \mathbb{R} : f^2(x) > \alpha\} = \{x \in \mathbb{R} : f(x) > \sqrt{\alpha}\} \cup \{x \in \mathbb{R} : f(x) < -\sqrt{\alpha}\}.$$

   Both sets on the right are measurable since f is measurable. Therefore, f^2 is measurable.

3. **Sum $f + g$:** Fix $\alpha \in \mathbb{R}$. For each rational number $r \in \mathbb{Q}$, define:

   $$S_r = \{x \in \mathbb{R} : f(x) > r\} \cap \{x \in \mathbb{R} : g(x) > \alpha - r\}.$$

   Each set $S_r \in \mathcal{M}$, since f and g are measurable. Moreover,

   $$\{x \in \mathbb{R} : f(x) + g(x) > \alpha\} = \bigcup_{r \in \mathbb{Q}} S_r,$$

   which is a countable union of measurable sets, hence measurable. Thus $f + g$ is measurable.

4. **Product $f \cdot g$:** Using the identity:

   $$f \cdot g = \frac{1}{4}\left[(f + g)^2 - (f - g)^2\right],$$

   and since sums, differences, and squares of measurable functions are measurable (as shown above), it follows that $f \cdot g$ is measurable.

5. **Absolute value:** For $\alpha > 0$, we write:

$$\{x \in \mathbb{R} : |f(x)| > \alpha\} = \{x \in \mathbb{R} : f(x) > \alpha\} \cup \{x \in \mathbb{R} : f(x) < -\alpha\}.$$

   Each set on the right is measurable, hence $|f|$ is measurable.

■

Suppose f is a function. We define the positive part f^+ and the negative part f^- of f as functions from Ω to $[0, \infty]$ as follows:

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{if } f(x) < 0, \end{cases}$$

and

$$f^-(x) = \begin{cases} 0 & \text{if } f(x) \geq 0, \\ -f(x) & \text{if } f(x) < 0. \end{cases}$$

Note that

$$f = f^+ - f^-, \quad |f| = f^+ + f^-.$$

Theorem 125 *The function f is measurable if and only if f^+ and f^- are both measurable.*

In dealing with sequences of measurable functions, it is often convenient to consider operations such as suprema, infima, lim sup, lim inf, and pointwise limits. These operations naturally lead us to consider functions that may take infinite values. Therefore, it is useful—and often necessary—to allow functions to take values in the *extended real line*, that is, to take the values $+\infty$ and $-\infty$ in addition to the usual real values.

We denote the set of **extended real numbers** by:

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}.$$

Definition 126 (Measurable Extended Real-Valued Function) *Let $f : E \to \overline{\mathbb{R}}$, where $E \subseteq \mathbb{R}$ is a measurable set and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ denotes the extended real line.*

*We say that f is **measurable** (with respect to a σ-algebra $\mathcal{M}$) if the following conditions are satisfied:*

- *For every $\alpha \in \mathbb{R}$, the set*
$$\{x \in E : f(x) > \alpha\} \in \mathcal{M}.$$
- *The sets*
$$\{x \in E : f(x) = +\infty\} \quad \textit{and} \quad \{x \in E : f(x) = -\infty\}$$
  also belong to $\mathcal{M}$.

Definition 127 *Let $E \subset \mathbb{R}$ be a measurable set. A statement $P(x)$ is said to hold **almost everywhere (a.e.)** on E if*

$$m(\{x \in E : P(x) \text{ does not hold}\}) = 0.$$

In other words, the set where $P(x)$ does not hold has measure zero. Note that any set with outer measure zero also has measure zero, so using m^* instead of m in this definition would yield the same statement.

Theorem 128 *If two functions $f, g : E \to [-\infty, \infty]$ satisfy $f = g$ almost everywhere on E, and f is measurable, then g is also measurable.*

In other words, modifying a measurable function on a set of measure zero does not affect its measurability.

Proof. Let $N = \{x \in E : f(x) \neq g(x)\}$. By assumption, N has outer measure zero, so $m(N) = 0$. For any $\alpha \in \mathbb{R}$, define

$$N_\alpha = \{x \in N : g(x) > \alpha\} \subset N,$$

which also has measure zero since $m^*(N_\alpha) \leq m^*(N) = 0$.

Now, for each $\alpha \in \mathbb{R}$, we can express the preimage $g^{-1}((\alpha, \infty])$ as

$$g^{-1}((\alpha, \infty]) = \left(f^{-1}((\alpha, \infty]) \setminus N\right) \cup N_\alpha.$$

Since f is measurable, $f^{-1}((\alpha, \infty])$ is measurable, and both N and N_α have measure zero. Thus, $g^{-1}((\alpha, \infty])$ is a union of measurable sets, making it measurable as well. This proves that g is measurable. ■

Corollary 129 *If f and g are measurable, then the sets $\{x : f(x) < g(x)\}$, $\{x : f(x) \leq g(x)\}$, and $\{x : f(x) = g(x)\}$ are also measurable.*

Theorem 130 *Let $\{f_n(x)\}$ be a sequence of measurable functions. Then the functions*

$$\inf_n f_n(x), \quad \sup_n f_n(x), \quad \liminf_{n\to\infty} f_n(x), \quad \text{and} \quad \limsup_{n\to\infty} f_n(x)$$

are all measurable.

Proof. Define $g(x) = \sup_n f_n(x)$ and let $a \in \mathbb{R}$. Then we can express the set $\{x : g(x) \leq a\}$ as

$$\{x : g(x) \leq a\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) \leq a\}.$$

This set is measurable, as it is the countable intersection of measurable sets, each $\{x : f_n(x) \leq a\}$ being measurable by the measurability of f_n.

Now, let $h(x) = \limsup_{n\to\infty} f_n(x)$. For $h(x) \leq a$ (where $a \in \mathbb{R}$), it is true if and only if for every $n \in \mathbb{N}$, there exists $m \geq n$ such that $f_m(x) \leq a$. This can be written as

$$\{x : h(x) \leq a\} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{x : f_m(x) \leq a\},$$

which is measurable as it is a countable intersection of countable unions of measurable sets.

The arguments for $\inf_n f_n$ and $\liminf_{n\to\infty} f_n$ follow similarly and are left as an exercise. ■

A function is called a **simple function** if it takes only a finite number of values and can be written as a finite linear combination of characteristic functions of measurable sets:

$$f(x) = \sum_{i=1}^{N} a_i \chi_{A_i}(x), \quad \text{where } A_i \in \mathcal{M}.$$

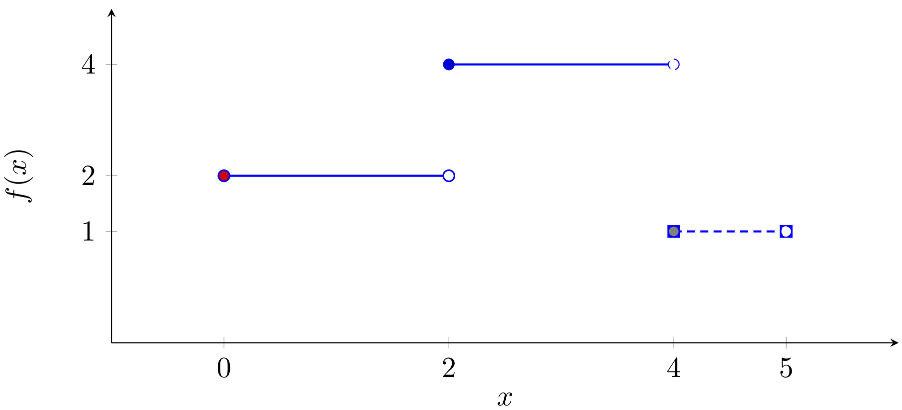
Here, $\chi_A(x)$ is the characteristic function of the set A, defined by:

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Example: Let

$$f(x) = 2\chi_{[0,2)}(x) + 4\chi_{[2,4)}(x) + 1\chi_{[4,5)}(x).$$

Then f is a simple function defined on $[0, 5]$, taking the values 2, 4, and 1 over disjoint intervals.



Theorem 131 *If $f : \Omega \to [0, \infty]$ is a Lebesgue measurable function, then there exists a sequence of non-negative simple functions (φ_n) such that:*

(i) $\varphi_{n+1}(x) \geq \varphi_n(x)$ *for all* $n \in \mathbb{N}$ *and* $x \in \Omega$*,*

(ii) $\lim_{n\to\infty} \varphi_n(x) = f(x)$ *for all* $x \in \Omega$*.*

We write $\varphi_n \uparrow f$ *to denote that* φ_n *increases to* f*.*

For each $n \in \mathbb{N}$, define the sets:

$$F_{n,i} = f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right), \quad i \in \{1, 2, \ldots, n2^n\},$$

$$F_{n,\infty} = f^{-1}([n, \infty]) \cup f^{-1}(\{\infty\}),$$

and the simple function

$$\varphi_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{F_{n,i}} + n\chi_{F_{n,\infty}}.$$

Each φ_n is measurable because each interval $\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)$ and $[n, \infty)$ is a Borel set.

(i) For any $x \in F_{n,i}$, we have $\frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}$. Then either: - $\frac{i-1}{2^n} \leq f(x) < \frac{2i-1}{2^{n+1}}$, so $x \in F_{n+1,2i-1}$ and $\varphi_{n+1}(x) = \frac{i-1}{2^n} = \varphi_n(x)$, - $\frac{2i-1}{2^{n+1}} \leq f(x) < \frac{i}{2^n}$, so $x \in F_{n+1,2i}$ and $\varphi_{n+1}(x) > \varphi_n(x)$.

(ii) If $f(x) < N$ for some $N \in \mathbb{N}$, then for all $n \geq N$ there is an integer i such that

$$\frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n},$$

which implies $0 \leq f(x) - \varphi_n(x) < \frac{1}{2^n}$. Hence $\varphi_n(x) \to f(x)$.

Lecture 5: Lebesgue Integral of Nonnegative Measurable Functions

We often work with functions that can take the value $+\infty$ and sets with infinite measure. For this reason, we adopt the following conventions:

$$a + \infty = \infty + a = \infty \quad \text{for } a \in [0, \infty],$$

$$a \cdot \infty = \infty \cdot a = \infty \quad \text{for } a \in (0, \infty],$$

$$0 \cdot \infty = \infty \cdot 0 = 0.$$

Integral of a Simple Function

Let $f : E \to [0, \infty]$ be a simple measurable function, which means it takes only finitely many values. Suppose these values are $\alpha_1, \ldots, \alpha_N$. For each $j = 1, \ldots, N$, define:

$$A_j = \{x \in E : f(x) = \alpha_j\}.$$

Then the Lebesgue integral of f over E is:

$$\int_E f\, dm = \sum_{j=1}^{N} \alpha_j\, m(A_j).$$

Alternatively, if f is written as a sum of characteristic functions:

$$f = \sum_{i=1}^{n} a_i \chi_{A_i},$$

then:

$$\int_E f\, dm = \sum_{i=1}^{n} a_i\, m(A_i).$$

Example: Define the function $f : \mathbb{R} \to \mathbb{R}$ by:

$$f(x) = \begin{cases} 2, & \text{if } -1 < x < 1, \\ 3, & \text{if } 3 < x < 7, \\ -1, & \text{if } -4 \le x < -3, \\ 0, & \text{otherwise.} \end{cases}$$

Then:

$$\int_{\mathbb{R}} f(x)\, dm = 2 \cdot 2 + 3 \cdot 4 + (-1) \cdot 1 = 4 + 12 - 1 = \boxed{15}.$$

Integral of a General Nonnegative Function

Let $f : E \to [0, \infty]$ be any nonnegative measurable function. We define:

$$\int_E f\, dm = \sup \left\{ \int_E \varphi\, dm \;\middle|\; \varphi \in S^+(E),\, 0 \le \varphi \le f \right\},$$

where $S^+(E)$ is the set of all nonnegative simple functions on E.

Basic Properties

If $f, g : E \to [0, \infty]$ are measurable, and $\lambda \geq 0$, then:

- If $f \leq g$, then $\int_E f\, dm \leq \int_E g\, dm$.
- $\int_E \lambda f\, dm = \lambda \int_E f\, dm$.
- If $F \subset E$, then $\int_F f\, dm = \int_E f \chi_F\, dm$.
- If $m(E) = 0$, then $\int_E f\, dm = 0$.

Monotone Convergence Theorem

Theorem 132 *Let $f_n : E \to [0, \infty]$ be an increasing sequence of measurable functions (i.e., $f_1 \leq f_2 \leq \cdots$), and let $f(x) = \lim_{n\to\infty} f_n(x)$. Then:*

$$\lim_{n\to\infty} \int_E f_n\, dm = \int_E f\, dm.$$

Idea of the proof: Since the sequence $\int_E f_n$ increases, its limit exists. Also, for any simple function $\phi \leq f$, eventually $f_n \geq \phi$, so:

$$\int_E \phi \leq \lim_{n\to\infty} \int_E f_n.$$

Taking the supremum over all such ϕ, we get the reverse inequality and conclude:

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

This theorem justifies interchanging limits and integrals for nonnegative functions that grow pointwise.

Note: Additivity of the integral over disjoint measurable subsets is not obvious and will be proved using the Monotone Convergence Theorem.

Theorem 133 (Monotone Convergence Theorem) *Let $\{f_n\}$ be a sequence of nonnegative measurable functions in E such that $f_1 \leq f_2 \leq \ldots$ pointwise on E, and suppose $f_n \to f$ pointwise on E for some f (which will also be a measurable functions is measurable). Then*

$$\lim_{n\to\infty} \int_E f_n\, dm = \int_E f\, dm.$$

Proof. Since $f_1 \leq f_2 \leq \cdots$, it follows that $\int_E f_1 \leq \int_E f_2 \leq \cdots$. Thus, $\int_E f_n$ forms a nonnegative, increasing sequence, which ensures that the limit $\lim_{n\to\infty} \int_E f_n$ exists within the interval $[0, \infty]$. Additionally, because $\lim_{n\to\infty} f_n(x) = f(x)$ for each x, we know $f_n \leq f$ for all n, implying that $\int_E f$ (a finite value in $[0, \infty]$) must satisfy

$$\int_E f_n \leq \int_E f \Rightarrow \lim_{n\to\infty} \int_E f_n \leq \int_E f.$$

To establish the reverse inequality (i.e., $\int_E f \leq \lim_{n\to\infty} \int_E f_n$), we will show that $\int_E \phi \leq \lim_{n\to\infty} \int_E f_n$ for every simple function $\phi \leq f$, noting that eventually, f_n will exceed ϕ.

Let $\epsilon \in (0,1)$ be chosen as a "margin." For any simple function $\phi = \sum_{j=1}^{m} a_j \chi_{A_j}$ with $\phi \le f$, we define the set

$$E_n = \{x \in E : f_n(x) \ge (1-\epsilon)\phi(x)\}.$$

Since $(1-\epsilon)\phi(x) < f(x)$ for all x (strict inequality holds as ϵ is positive) and $\lim_{n\to\infty} f_n(x) = f(x)$, each x must belong to some E_n. Thus, we have

$$\bigcup_{n=1}^{\infty} E_n = E.$$

Moreover, because $f_1 \le f_2 \le \cdots$, it follows that $E_1 \subset E_2 \subset \cdots$, so the sets E_n are nested by inclusion. Now, observe that

$$\int_E f_n \ge \int_{E_n} f_n \ge \int_{E_n} (1-\epsilon)\phi = (1-\epsilon)\int_{E_n} \phi = (1-\epsilon)\sum_{j=1}^{m} a_j m(A_j \cap E_n),$$

since the inequality holds on E_n, and the sets $A_j \cap E_n$ are measurable and disjoint. As E_n increases to E, the sets $E_1 \cap A_j \subset E_2 \cap A_j \subset \cdots$ expand to cover A_j. By the continuity of the Lebesgue measure, we conclude that as $n \to \infty$,

$$m(A_j \cap E_n) \to m(A_j).$$

Taking limits on both sides (noting that we have a finite sum on the right) gives, for all $\epsilon \in (0,1)$,

$$\lim_{n\to\infty} \int_E f_n \ge \lim_{n\to\infty} (1-\epsilon)\sum_{j=1}^{m} a_j m(A_j \cap E_n) = (1-\epsilon)\sum_{j=1}^{m} a_j m(A_j) = (1-\epsilon)\int_E \phi.$$

By letting $\epsilon \to 0$, we obtain the desired inequality $\int_E \phi \le \lim_{n\to\infty} \int_E f_n$. Combining this with the initial inequality completes the proof. ■

Theorem 134 (Fatou's Lemma) *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative measurable functions on a measurable set E. Then:*

$$\int_E \liminf_{n\to\infty} f_n \, dm \le \liminf_{n\to\infty} \int_E f_n \, dm.$$

Proof. We begin by expressing the pointwise $\liminf$ using the identity:

$$\liminf_{n\to\infty} f_n(x) = \sup_{n\ge 1} \left(\inf_{k\ge n} f_k(x) \right).$$

Define:

$$g_n(x) := \inf_{k\ge n} f_k(x).$$

Then $g_n(x)$ is an increasing sequence of measurable functions (since $g_n(x) \le g_{n+1}(x)$) and:

$$\lim_{n\to\infty} g_n(x) = \liminf_{n\to\infty} f_n(x).$$

Now apply the Monotone Convergence Theorem:

$$\int_E \liminf_{n\to\infty} f_n \, dm = \lim_{n\to\infty} \int_E g_n \, dm.$$

For each n, we know $g_n(x) \leq f_k(x)$ for all $k \geq n$, so:

$$\int_E g_n \, dm \leq \int_E f_k \, dm \quad \text{for all } k \geq n.$$

Hence,

$$\int_E g_n \, dm \leq \inf_{k \geq n} \int_E f_k \, dm.$$

Now take the limit as $n \to \infty$ on both sides:

$$\lim_{n \to \infty} \int_E g_n \, dm \leq \lim_{n \to \infty} \inf_{k \geq n} \int_E f_k \, dm = \liminf_{n \to \infty} \int_E f_n \, dm.$$

Putting it all together:

$$\int_E \liminf_{n \to \infty} f_n \, dm \leq \liminf_{n \to \infty} \int_E f_n \, dm.$$

This completes the proof. ■

Lebesgue Integrable Functions

Definition 135 (Lebesgue Integrable Function and Integral) *Let $E \subset \mathbb{R}$ be a measurable set, and let $f : E \to \mathbb{R}$ be a measurable function. Define the positive and negative parts of f as*

$$f^+(x) = \max(f(x), 0), \quad f^-(x) = \max(-f(x), 0).$$

These are nonnegative measurable functions, and satisfy

$$f = f^+ - f^-, \quad |f| = f^+ + f^-.$$

*We say that f is **Lebesgue integrable** over E if*

$$\int_E |f|\, dm = \int_E f^+\, dm + \int_E f^-\, dm < \infty.$$

*In this case, the **Lebesgue integral** of f over E is defined as*

$$\int_E f dm := \int_E f^+\, dm - \int_E f^-\, dm.$$

Proposition 136 *Let $f, g : E \to \mathbb{R}$ be Lebesgue integrable functions. Then:*

1. *For any scalar $c \in \mathbb{R}$, the function cf is integrable, and*

$$\int_E cf\, dm = c \int_E f\, dm.$$

2. *The sum $f + g$ is integrable, and*

$$\int_E (f+g)\, dm = \int_E f\, dm + \int_E g\, dm.$$

3. *If $A, B \subset E$ are disjoint measurable subsets, then*

$$\int_{A \cup B} f\, dm = \int_A f\, dm + \int_B f\, dm.$$

Proof. (1) Since $|cf| = |c| \cdot |f|$ and $f \in L^1(E)$, we know $|cf| \in L^1(E)$, so cf is integrable. Linearity of the integral gives:

$$\int_E cf\, dm = c \int_E f\, dm.$$

(2) By the triangle inequality:

$$|f+g| \leq |f| + |g| \quad \Rightarrow \quad \int_E |f+g|\, dm \leq \int_E |f|\, dm + \int_E |g|\, dm < \infty,$$

so $f + g \in L^1(E)$. Using the decomposition $f = f^+ - f^-$ and similarly for g, we get:

$$f + g = (f^+ + g^+) - (f^- + g^-),$$

and since all terms are nonnegative measurable functions, we apply linearity:

$$\int_E (f+g)\, dm = \int_E f^+\, dm + \int_E g^+\, dm - \int_E f^-\, dm - \int_E g^-\, dm = \int_E f\, dm + \int_E g\, dm.$$

(3) Since A and B are disjoint,

$$\chi_{A\cup B} = \chi_A + \chi_B.$$

Hence,

$$f\chi_{A\cup B} = f\chi_A + f\chi_B,$$

and since the product of a measurable function with an indicator function restricts the domain of integration:

$$\int_{A\cup B} f\,dm = \int_E f\chi_{A\cup B}\,dm = \int_E f\chi_A\,dm + \int_E f\chi_B\,dm = \int_A f\,dm + \int_B f\,dm.$$

■

Proposition 137 *Let $f, g : E \to \mathbb{R}$ be measurable functions. Then:*

1. *If f is Lebesgue integrable, then*

$$\left|\int_E f\,dm\right| \leq \int_E |f|\,dm.$$

2. *If $f = g$ almost everywhere and $g \in L^1(E)$, then $f \in L^1(E)$ and*

$$\int_E f\,dm = \int_E g\,dm.$$

3. *If $f, g \in L^1(E)$ and $f(x) \leq g(x)$ almost everywhere on E, then*

$$\int_E f\,dm \leq \int_E g\,dm.$$

Proof. **(1)** Since $f = f^+ - f^-$, we have

$$\left|\int_E f\,dm\right| = \left|\int_E f^+\,dm - \int_E f^-\,dm\right| \leq \int_E f^+\,dm + \int_E f^-\,dm.$$

Using the identity $|f| = f^+ + f^-$, it follows that

$$\int_E |f|\,dm = \int_E f^+\,dm + \int_E f^-\,dm.$$

(2) Since $f = g$ almost everywhere, we also have $|f| = |g|$ almost everywhere. Thus,

$$\int_E |f|\,dm = \int_E |g|\,dm < \infty,$$

so f is Lebesgue integralble. Also, $f - g = 0$ almost everywhere implies

$$\left|\int_E f\,dm - \int_E g\,dm\right| = \left|\int_E (f-g)\,dm\right| \leq \int_E |f-g|\,dm = 0,$$

which gives $\int_E f\,dm = \int_E g\,dm$.

(3) Define the function

$$h(x) = \begin{cases} g(x) - f(x), & \text{if } g(x) \geq f(x), \\ 0, & \text{otherwise.} \end{cases}$$

Then $h \geq 0$, measurable, and $h = g - f$ almost everywhere. Therefore,

$$\int_E h\,dm = \int_E (g-f)\,dm = \int_E g\,dm - \int_E f\,dm \geq 0,$$

which yields $\int_E f\,dm \leq \int_E g\,dm$. ■

Theorem 138 (Dominated Convergence Theorem) *Let $g : E \to [0, \infty)$ be a Lebesgue integrable function. Suppose $\{f_n\}$ is a sequence of measurable functions $f_n : E \to \mathbb{R}$ such that:*

1. *$|f_n(x)| \leq g(x)$ almost everywhere on E, for all $n \in \mathbb{N}$,*
2. *$f_n(x) \to f(x)$ pointwise almost everywhere on E, for some function $f : E \to \mathbb{R}$.*

Then f is Lebesgue integrable, and

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm.$$

Proof. Since $|f_n| \leq g$ and g is Lebesgue integrable, it follows that each f_n is also Lebesgue integrable. The pointwise limit f is measurable and satisfies $|f| \leq g$, so f is also Lebesgue integrable.

We aim to prove:

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm.$$

Apply Fatou's Lemma to the nonnegative functions $g - f_n$:

$$\int_E \liminf_{n\to\infty} (g - f_n) \, dm \leq \liminf_{n\to\infty} \int_E (g - f_n) \, dm.$$

Since $f_n \to f$ pointwise, the left-hand side becomes $\int_E (g - f) \, dm$, yielding:

$$\int_E (g - f) \, dm \leq \liminf_{n\to\infty} \left(\int_E g \, dm - \int_E f_n \, dm \right).$$

Rewriting this, we obtain:

$$\limsup_{n\to\infty} \int_E f_n \, dm \leq \int_E f \, dm.$$

Similarly, apply Fatou's Lemma to $g + f_n$:

$$\int_E \liminf_{n\to\infty} (g + f_n) \, dm \leq \liminf_{n\to\infty} \int_E (g + f_n) \, dm,$$

which gives:

$$\int_E (g + f) \, dm \leq \liminf_{n\to\infty} \left(\int_E g \, dm + \int_E f_n \, dm \right).$$

Rearranging:

$$\int_E f \, dm \leq \liminf_{n\to\infty} \int_E f_n \, dm.$$

Combining both inequalities:

$$\limsup_{n\to\infty} \int_E f_n \, dm \leq \int_E f \, dm \leq \liminf_{n\to\infty} \int_E f_n \, dm.$$

Since $\liminf \leq \limsup$ always holds, we conclude:

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm.$$

■

Proposition 139 *Let (f_n) be a bounded sequence of measurable functions on a set E with finite measure $m(E) < \infty$. If $f_n \to f$ almost everywhere on E, then the limit function f is Lebesgue integrable and*

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm.$$

Proof. Assume there exists a constant $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in E$ and all n. Define $g(x) = M$, which is clearly Lebesgue integrable on E since $m(E) < \infty$. Then $|f_n(x)| \leq g(x)$ for all n, and $f_n \to f$ almost everywhere. The Dominated Convergence Theorem applies and yields the result. ■ ■

Theorem 140 (Term-by-Term Integration of a Series) *Let $\{f_n\}$ be a sequence of measurable functions on a measurable set E. Then:*

(i) The series of integrals of absolute values satisfies:

$$\int_E \left(\sum_{n=1}^{\infty} |f_n| \right) dm = \sum_{n=1}^{\infty} \int_E |f_n| \, dm.$$

Both sides may be infinite, or both are finite and equal.

(ii) If the right-hand side is finite, then each f_n is Lebesgue integrable, the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges almost everywhere on E, and the sum defines a Lebesgue integrable function F. Moreover,

$$\int_E \left(\sum_{n=1}^{\infty} f_n \right) dm = \sum_{n=1}^{\infty} \int_E f_n \, dm.$$

Proof. (i) Define the function

$$G(x) := \sum_{n=1}^{\infty} |f_n(x)|.$$

Since $|f_n(x)| \geq 0$, the sequence of partial sums is increasing. By the Monotone Convergence Theorem:

$$\int_E G \, dm = \sum_{n=1}^{\infty} \int_E |f_n| \, dm.$$

(ii) If G is Lebesgue integrable, then $G(x) < \infty$ almost everywhere, so the series $\sum_{n=1}^{\infty} f_n(x)$ converges almost everywhere. Let $F(x) = \sum_{n=1}^{\infty} f_n(x)$ denote the pointwise sum, and define the partial sums

$$F_n(x) = \sum_{k=1}^{n} f_k(x).$$

Then $F_n(x) \to F(x)$ almost everywhere and

$$|F_n(x)| \leq \sum_{k=1}^{n} |f_k(x)| \leq G(x).$$

Thus, by the Dominated Convergence Theorem:

$$\int_E F \, dm = \lim_{n\to\infty} \int_E F_n \, dm = \lim_{n\to\infty} \sum_{k=1}^{n} \int_E f_k \, dm = \sum_{k=1}^{\infty} \int_E f_k \, dm.$$

■

Riemann and Lebesgue Integrals

We now explore an important question: When is a function Riemann integrable? And how does this relate to Lebesgue integrability?

Theorem 141 (Characterization of Riemann Integrability) *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if it is continuous almost everywhere on $[a, b]$; that is,*

$$f \in R[a, b] \iff m(\{x \in [a, b] : f \text{ is not continuous at } x\}) = 0.$$

Intuition: A bounded function can be Riemann integrated as long as its discontinuities are "rare"—specifically, they must form a set of measure zero. If the function is continuous almost everywhere, the Riemann integral exists.

Key Fact: If f is Riemann integrable on $[a, b]$, then:

- f is measurable,
- f is Lebesgue integrable,
- and the integrals are equal:

$$\int_a^b f(x)\, dx = \int_{[a,b]} f\, dm.$$

We previously defined the notation $\int_a^b f$ to mean the Riemann integral of f. However, since the Riemann and Lebesgue integrals agree for Riemann integrable functions , we now redefine the expression

$$\int_a^b f(x)dx$$

to denote the Lebesgue integral.

Definition 142 (Lebesgue Integral Notation) *Let $f : (a, b) \to \mathbb{R}$ be a Lebesgue measurable function. Then:*

- *$\int_a^b f(x)\, dx$ or $\int_a^b f$ denotes the Lebesgue integral over (a, b),*

  $$\int_a^b f(x)dx := \int_{(a,b)} f\, dm,$$

  where m is the Lebesgue measure.

- *If $a > b$, we define the integral as*

  $$\int_a^b f := -\int_b^a f,$$

  so that useful properties like

  $$\int_a^b f = \int_a^c f + \int_c^b f$$

  hold for any $a < c < b$.

Conclusion: Riemann integrable functions are also Lebesgue integrable. But the opposite is not always true — some functions can be Lebesgue integrable but not Riemann integrable.

Example 143 (Lebesgue Integrable, Not Riemann Integrable) *Define:*

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 0, & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

This function is not Riemann integrable because it is discontinuous at every point in $[0,1]$*. But it is Lebesgue integrable, and we have:*

$$\int_0^1 f(x)\, dx = 0.$$

Improper Integrals

One advantage of the Lebesgue integral over the Riemann integral is that it can be defined over unbounded domains, as long as the function is measurable.

However, this does not guarantee that the integral is finite. For example, consider the constant function:

$$f(x) = 1 \quad \text{for all } x \in \mathbb{R}.$$

This function is measurable, but its Lebesgue integral over $\mathbb{R}$ diverges:

$$\int_{\mathbb{R}} f(x)\, dx = \int_{\mathbb{R}} 1\, dx = \infty.$$

Thus, f is not Lebesgue integrable on $\mathbb{R}$.

In contrast, the Riemann integral is only defined on bounded intervals. To handle unbounded domains or functions, it must be extended using limits, leading to the concept of **improper integrals**.

On bounded intervals, if a function is Riemann integrable, then its Riemann and Lebesgue integrals agree. But on unbounded, the agreement may break down. In many cases, a function that is improperly Riemann integrable is also Lebesgue integrable. However, this is not always true. Below, we present examples that illustrate when the two approaches agree and when they differ.

(i) Measurability of Improperly Riemann Integrable Functions

Suppose the improper integral

$$\int_a^\infty f(x)\, dx$$

converges in the Riemann sense. Then f is Riemann integrable on every finite interval $[a,b]$ for all $b > a$. Since Riemann integrability implies Lebesgue integrability on compact intervals, it follows that $f \in L^1([a,b])$ for all $b > a$.

Moreover, we can express f as the pointwise limit:

$$f(x) = \lim_{n\to\infty} f(x) \cdot \chi_{[a,n]}(x),$$

where $\chi_{[a,n]}$ is the indicator function of the interval $[a,n]$. Each function $f \cdot \chi_{[a,n]}$ is measurable, and the pointwise limit of measurable functions is measurable. Hence, f is measurable on $[a,\infty)$.

(ii) Nonnegative Functions and the Monotone Convergence Theorem

Suppose $f \geq 0$ on $[a, \infty)$, and the improper Riemann integral

$$\int_a^\infty f(x)\,dx$$

converges. Define the sequence of functions $f_n = f \cdot \chi_{[a,n]}$. Then $f_n \nearrow f$ pointwise, and by the Monotone Convergence Theorem:

$$\int_{[a,\infty)} f\,dm = \lim_{n\to\infty} \int_{[a,n]} f\,dm = \lim_{n\to\infty} \int_a^n f(x)\,dx = \int_a^\infty f(x)\,dx. \tag{11.27}$$

(iii) Lebesgue Integrability Implies Convergence of the Improper Riemann Integral

Assume $f \in R(a,b)$ for all $b > a$, and that $f \in L^1([a,\infty))$. Then the improper Riemann integral $\int_a^\infty f(x)\,dx$ converges, and:

$$\int_a^\infty f(x)\,dx = \int_{[a,\infty)} f\,dm.$$

Proof. Let (b_n) be a sequence such that $b_n \to \infty$, and define $f_n = f \cdot \chi_{[a,b_n]}$. Then $f_n \to f$ pointwise, and $|f_n| \leq |f| \in L^1([a,\infty))$. By the Dominated Convergence Theorem:

$$\int_{[a,\infty)} f\,dm = \lim_{n\to\infty} \int_{[a,b_n]} f\,dm = \lim_{n\to\infty} \int_a^{b_n} f(x)\,dx = \int_a^\infty f(x)\,dx.$$

(iv) Absolute Convergence Implies Agreement of Integrals

Suppose $f \in R(a,b)$ for all $b > a$, and:

$$\int_a^\infty |f(x)|\,dx < \infty.$$

Then $f \in L^1([a,\infty))$, and by part (iii), the improper Riemann integral exists and:

$$\int_a^\infty f(x)\,dx = \int_{[a,\infty)} f\,dm.$$

Conclusion

If both of the following improper integrals exist:

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty |f(x)|\,dx,$$

then the Lebesgue integral $\int_{[a,\infty)} f\,dm$ also exists and agrees with the improper Riemann integral.

- The convergence of $\int_a^\infty f(x)\,dx$ implies $f \in R(a,b)$ for all $b > a$.
- The convergence of $\int_a^\infty |f(x)|\,dx$ implies $f \in L^1([a,\infty))$.

Therefore,

$$\int_a^\infty f(x)\,dx = \int_{[a,\infty)} f\,dm.$$

Example 144 *Consider some improper Riemann integrals and investigate whether they agree with the Lebesgue integral.*

1. Let consider the function $f : (0,1] \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{\sqrt{x}}.$$

This function is improperly Riemann integrable over $(0,1]$ with a value of 2. We now demonstrate that it is also Lebesgue integrable over this region with the same value. Since f is continuous, it is measurable, so asking if it is Lebesgue integrable is valid.

To compute the Lebesgue integral, let us define a sequence of functions f_n where $f_n : (0,1] \to \mathbb{R}$ is given by

$$f_n = f \cdot 1_{\left[\frac{1}{n},1\right]}.$$

Here, f_n is a pointwise increasing sequence whose limit is f. By the Monotone Convergence Theorem (MCT), we have:

$$\int_{(0,1]} f\,d\mu = \int_{(0,1]} \lim_{n\to\infty} f_n\,d\mu = \lim_{n\to\infty} \int_{(0,1]} f_n\,d\mu = \lim_{n\to\infty} \int_{\left[\frac{1}{n},1\right]} f\,d\mu.$$

For any $n \in \mathbb{N}$, f is continuous on the compact domain $\left[\frac{1}{n},1\right]$, so the Lebesgue integral over this interval equals the Riemann integral. Using the Fundamental Theorem of Calculus, we find

$$\int_{\left[\frac{1}{n},1\right]} f\,d\mu = \int_{1/n}^{1} \frac{1}{\sqrt{x}}\,dx = 2 - \frac{2}{\sqrt{n}}.$$

Substituting into the earlier limit, we obtain

$$\int_{(0,1]} f\,d\mu = \lim_{n\to\infty}\left(2 - \frac{2}{\sqrt{n}}\right) = 2 = \int_0^1 f(x)\,dx.$$

Thus, for this function, the improper Riemann integral agrees with the Lebesgue integral.

2. Consider the function $f : [0,\infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{\sin(x)}{x} \quad \text{for } x \neq 0, \quad \text{and} \quad f(0) = 1.$$

Since f is continuous, it is measurable. To compute its Lebesgue integral, we decompose it into positive and negative parts. Define sets E and F where f is non-negative and non-positive, respectively:

$$E = \bigcup_{\substack{n\in\mathbb{N}\\ n \text{ odd}}} [(n-1)\pi, n\pi], \quad F = \bigcup_{\substack{n\in\mathbb{N}\\ n \text{ even}}} [(n-1)\pi, n\pi].$$

The positive and negative parts of f are then given by:

$$f^+(x) = \frac{\sin(x)}{x} \cdot 1_E(x), \quad f^-(x) = -\frac{\sin(x)}{x} \cdot 1_F(x).$$

To evaluate the entire Lebesgue integral, we separately integrate these parts. Focusing on the positive part f^+, we split the domain into smaller compact intervals. Since f is continuous over each compact interval in E (and hence Riemann integrable there), we can compute the Lebesgue integral as a Riemann integral. Since $\sin(x)$ is non-negative over each interval in E, we have:

$$\int_{[0,\infty)} f^+\,d\mu = \int_E \frac{\sin(x)}{x}\,dx = \sum_{\substack{n\in\mathbb{N}\\ n \text{ odd}}} \int_{(n-1)\pi}^{n\pi} \frac{\sin(x)}{x}\,dx.$$

By approximating the lower bound of $\sin(x)$ *over each interval, we find*

$$\int_{[0,\infty)} f^+ \, d\mu \geq \sum_{\substack{n\in\mathbb{N} \\ n \text{ odd}}} \int_{(n-1)\pi}^{n\pi} \frac{\sin(x)}{n\pi} \, dx = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1}.$$

However, this sum diverges to ∞ *by comparison with the harmonic series. Similarly, the integral of the negative part of* f*, namely* $\int_{[0,\infty)} f^- \, d\mu$*, also diverges to* ∞*. Therefore, we encounter an indeterminate* $\infty - \infty$ *case for the Lebesgue integral, implying that this function is not Lebesgue integrable.*

On the other hand, if we use the Riemann integral, the unbounded domain requires us to apply the concept of the improper Riemann integral. This improper integral can be defined as the limit of the integral function $I : [0, \infty) \to \mathbb{R}$*, given by*

$$I(t) = \int_0^t f(x) \, dx \quad \text{as} \quad t \to \infty.$$

For $t > 1$*, we can apply integration by parts to compute* $I(t)$*:*

$$I(t) = \int_0^1 f(x) \, dx + \int_1^t \frac{\sin(x)}{x} \, dx.$$

Applying integration by parts to the second integral, we get:

$$I(t) = \int_0^1 f(x) \, dx + \left[-\frac{\cos(x)}{x} \right]_1^t - \int_1^t \frac{\cos(x)}{x^2} \, dx.$$

This simplifies to

$$I(t) = \int_0^1 f(x) \, dx - \frac{\cos(t)}{t} + \cos(1) - \int_1^t \frac{\cos(x)}{x^2} \, dx.$$

We observe that the Riemann integral $\int_0^1 f(x) \, dx$ *exists because the integrand* f *is continuous over the compact interval* $[0, 1]$*. Furthermore, the limits*

$$\lim_{t\to\infty} \frac{\cos(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \int_1^t \frac{\cos(x)}{x^2} \, dx$$

both exist. To check the improper Riemann integrability of the function $f : [1, \infty) \to \mathbb{R}$ *defined by*

$$f(x) = \frac{\cos(x)}{x^2}$$

over $[1, \infty)$*, we note that this function has mixed signs, meaning we cannot directly apply the comparison test. To proceed, we split* f *into its positive and negative parts. Specifically, we define* f^+ *and* $f^- : [1, \infty) \to \mathbb{R}$ *as follows:*

$$f^+ = \max(f, 0), \quad f^- = -\min(f, 0),$$

so that $f = f^+ - f^-$*.*

Both f^+ *and* f^- *are non-negative and continuous. We can therefore apply the direct comparison test to* f^+*. Observe that*

$$0 \leq f^+(x) \leq \frac{|\cos(x)|}{x^2} \leq \frac{1}{x^2}.$$

Since $\frac{1}{x^2}$ is improperly Riemann integrable over $[1, \infty)$, as shown in Example 16.4.4(3), it follows by direct comparison that the improper integral

$$\int_1^\infty f^+(x)\,dx$$

exists. By a similar argument, the improper Riemann integral

$$\int_1^\infty f^-(x)\,dx$$

also exists.

For any finite $t > 1$, we have

$$\int_1^t f(x)\,dx = \int_1^t f^+(x)\,dx - \int_1^t f^-(x)\,dx.$$

Applying the algebra of limits, we get:

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t f(x)\,dx = \lim_{t\to\infty} \left(\int_1^t f^+(x)\,dx - \int_1^t f^-(x)\,dx \right).$$

This simplifies to

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t f^+(x)\,dx - \lim_{t\to\infty} \int_1^t f^-(x)\,dx,$$

which exists because both limits on the right-hand side exist.

Thus, we conclude that the function f is improperly Riemann integrable over $[1, \infty)$.

Therefore, by the algebra of limits, the improper integral

$$\lim_{t\to\infty} I(t) = \int_0^\infty f(x)\,dx$$

exists. Hence, the function f is Riemann integrable over $\mathbb{R}$ in the improper sense, even though it is not Lebesgue integrable.

Lecture 9: Examples

Example 145 *Evaluate*

$$\lim_{n\to\infty}\int_0^1 \frac{nx}{1+n^2x^2}\,dx.$$

Solution. *For each $n \in \mathbb{N}$, define the sequence of functions*

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad x \in [0,1].$$

Observe that as $n \to \infty$, $f_n(x)$ converges pointwise to 0 for all $x \in (0,1]$, since the term n^2x^2 in the denominator dominates as n becomes large, driving $f_n(x)$ towards 0. At $x = 0$, the function value is also 0 for all n, so $f_n(x)$ converges pointwise to 0.

To understand the behavior of $f_n(x)$ on $[0,1]$, we find the maximum value of $f_n(x)$. Taking the derivative, we see that $f_n(x)$ attains its maximum at $x = \frac{1}{n}$. Evaluating f_n at this point, we get

$$f_n\left(\frac{1}{n}\right) = \frac{n \cdot \frac{1}{n}}{1+n^2\cdot\left(\frac{1}{n}\right)^2} = \frac{1}{2}.$$

Thus,

$$\sup_{x\in[0,1]} |f_n(x)| = \frac{1}{2},$$

showing that the convergence $f_n \to 0$ is not uniform on $[0,1]$.

Since the convergence is not uniform, we cannot interchange the limit and the integral directly using properties of the Riemann integral. However, we can consider this as a Lebesgue integral and apply the Bounded Convergence Theorem. Each $f_n(x)$ is bounded and measurable, and $f_n(x) \to 0$ pointwise. Thus, by the Bounded Convergence Theorem,

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0.$$

Example 146 *In many problems, one often needs to use the following upper bound:*

$$\left(1+\frac{x}{n}\right)^n \le e^x$$

which holds for all

$$n \ge 1 \quad \text{and} \quad x > -n.$$

This bound must be proved; one cannot simply refer to a calculus or advanced calculus text where this fact may have been mentioned.

To prove it, we take the logarithm of both sides in the inequality and convert it as follows:

$$n\ln\left(1+\frac{x}{n}\right) \le x.$$

Define $t = \frac{x}{n} + 1$, then $t > 0$ due to the condition $x > -n$. The inequality becomes

$$\ln t \le t - 1, \quad \forall t > 0,$$

which is a well-known inequality that can be used here.

Additionally, with a bit more effort, we can show that the sequence

$$a_n = \left(1 + \frac{x}{n}\right)^n$$

is monotonically increasing in n for all n satisfying $x > -n$. To prove this, we treat n as a continuous variable and take the derivative:

$$\frac{da_n}{dn} = a_n \left(\ln\left(1 + \frac{x}{n}\right) - \frac{x}{x+n}\right).$$

Since $a_n > 0$, we have $\frac{da_n}{dn} \geq 0$ if and only if

$$\ln\left(1 + \frac{x}{n}\right) \geq \frac{x}{x+n},$$

or equivalently,

$$\frac{n}{x+n} \ln\left(1 + \frac{x}{n}\right) \leq \frac{x}{x+n}.$$

Simplifying further, we find that

$$\ln\left(1 + \frac{x}{n}\right) \leq \frac{x}{x+n},$$

which implies the desired result for the monotonicity of a_n.

Evaluate

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{-2x}\, dx.$$

Proof. *To express this limit as a Lebesgue integral, define the sequence of functions*

$$f_n(x) = \chi_{[0,n]}(x) \cdot \left(1 - \frac{x}{n}\right)^n e^{-2x},$$

where $\chi_{[0,n]}(x)$ is the characteristic function of the interval $[0, n]$. This gives us

$$\int_0^n \left(1 - \frac{x}{n}\right)^n e^{-2x}\, dx = \int_{[0,\infty)} f_n(x)\, d\mu.$$

Let's analyze the behavior of $f_n(x)$ as $n \to \infty$. For a fixed x,

$$\left(1 - \frac{x}{n}\right)^n \to e^{-x} \quad \text{as } n \to \infty.$$

Therefore, $f_n(x) \to e^{-x} \cdot e^{-2x} = e^{-3x}$ pointwise on $[0, \infty)$.

Since $f_n(x)$ converges pointwise to e^{-3x}, and $f_n(x) \leq e^{-x}$ for all $x \in [0, \infty)$, we can apply the Dominated Convergence Theorem, using $g(x) = e^{-x}$ as a dominating function, which is integrable over $[0, \infty)$. Thus,

$$\lim_{n\to\infty} \int_{[0,\infty)} f_n(x)\, d\mu = \int_{[0,\infty)} \lim_{n\to\infty} f_n(x)\, d\mu = \int_{[0,\infty)} e^{-3x}\, dx.$$

Evaluating this integral, we have

$$\int_{[0,\infty)} e^{-3x}\, dx = \left[-\frac{e^{-3x}}{3}\right]_0^\infty = \frac{1}{3}.$$

Thus,

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{-2x}\, dx = \frac{1}{3}.$$

■

Example 147 *Prove that*

$$\int_0^1 \left(\frac{\log x}{1-x}\right)^2 dx = 2\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Solution. *For every $x \in (-1,1)$, we have the power series representation*

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^n.$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}.$$

We can now express the integral as

$$\int_0^1 \left(\frac{\log x}{1-x}\right)^2 dx = \sum_{n=1}^{\infty} n\int_0^1 x^{n-1}(\log x)^2\, dx.$$

Let's focus on evaluating each integral on the right-hand side.

When $n = 1$, the integral $\int_0^1 (\log x)^2\, dx$ is improper, as $\log x$ diverges at $x = 0$. For $n > 1$, however, the integrals $\int_0^1 x^{n-1}(\log x)^2\, dx$ are Riemann integrals. In both cases, we use integration by parts to evaluate the integrals, employing L'Hôpital's rule to handle any indeterminate forms.

Consider

$$\int_0^1 x^{n-1}(\log x)^2\, dx.$$

Using integration by parts, set $u = (\log x)^2$ and $dv = x^{n-1}dx$, giving $du = \frac{2\log x}{x}dx$ and $v = \frac{x^n}{n}$. Then

$$\int_0^1 x^{n-1}(\log x)^2\, dx = \frac{x^n}{n}(\log x)^2\Big|_0^1 - \int_0^1 \frac{2x^n \log x}{n}\, dx.$$

Evaluating the boundary term, we find

$$\frac{x^n}{n}(\log x)^2\Big|_0^1 = 0.$$

This gives

$$\int_0^1 x^{n-1}(\log x)^2\, dx = -\frac{2}{n}\int_0^1 x^n \log x\, dx.$$

Applying integration by parts again to $\int_0^1 x^n \log x\, dx$, with $u = \log x$ and $dv = x^n dx$, we get $du = \frac{1}{x}dx$ and $v = \frac{x^{n+1}}{n+1}$. Thus,

$$\int_0^1 x^n \log x\, dx = \frac{x^{n+1}}{n+1}\log x\Big|_0^1 - \int_0^1 \frac{x^{n+1}}{n+1}\cdot\frac{1}{x}\, dx.$$

The boundary term again vanishes, so we are left with

$$\int_0^1 x^n \log x\, dx = -\frac{1}{(n+1)^2}.$$

Substituting back, we find

$$\int_0^1 x^{n-1}(\log x)^2\,dx = \frac{2}{n^3}.$$

Therefore,

$$\int_0^1 \left(\frac{\log x}{1-x}\right)^2 dx = \sum_{n=1}^{\infty} n \cdot \frac{2}{n^3} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This completes the proof.

Example 148 *Prove that*

$$\int_0^1 \sin x \log x\,dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n(2n)!}.$$

We start by expanding $\sin x$ *as a power series:*

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

Substituting this expansion into the integral, we get:

$$\int_0^1 \sin x \log x\,dx = \int_0^1 \left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}\right) \log x\,dx.$$

Next, we justify the interchange of the sum and integral by checking the absolute convergence:

$$\int_0^1 \sum_{k=0}^{\infty} \left|\frac{(-1)^k x^{2k+1}\log x}{(2k+1)!}\right| dx = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\int_0^1 x^{2k+1}|\log x|\,dx.$$

We compute each integral $\int_0^1 x^{2k+1}\log x\,dx$ *using integration by parts. Setting* $u = \log x$ *and* $dv = x^{2k+1}\,dx$*, we find:*

$$\int_0^1 x^{2k+1}\log x\,dx = -\frac{1}{(2k+2)^2}.$$

Thus

$$\int_0^1 \sum_{k=0}^{\infty} \left|\frac{(-1)^k x^{2k+1}\log x}{(2k+1)!}\right| dx = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!(2k+1)^2} < \infty.$$

By the Theorem of Term-by-Term Integration, we can interchange the sum and the integral:

$$\begin{aligned}
\int_0^1 \sin x \log x\,dx &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\int_0^1 x^{2k+1}\log x\,dx. \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(-\frac{1}{(2k+2)^2}\right).
\end{aligned}$$

Simplifying, we obtain:

$$\int_0^1 \sin x \log x\,dx = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k(2k)!}.$$