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Lecture 30

• Bayesian Premium & Bühlmann Premium estimates

Example 17.10 p. 413 * Revise lectures 28 + 29

(EX 17.8 p. 410 continued) Determine the Bayesian premium

Ans: ∵ the amount of a claim has an exponential distⁿ with mean $1/\theta$ (see EX 17.3 p. 403)

$\therefore \mu_4(\theta) = \theta^{-1} = \frac{1}{\theta}$

For Bayesian premium estimate, we can use Eq. 7 in lecture 29.

$$E(X_{n+1} \mid X = x) = \int \mu_{n+1}(\theta)\, \pi_{\Theta|X}(\theta|x)\, d\theta$$

From EX 17.8

$$\pi(\theta \mid 100, 950, 450) = \frac{\theta^6 e^{-2500\theta} (2500)^7}{720}$$

$$\therefore E(X_4 \mid 100, 950, 450) = \int_0^\infty \frac{1}{\theta} \cdot \frac{\theta^6 e^{-2500\theta} (2500)^7}{720}\, d\theta$$

$$= \frac{2500^7}{720} \int_0^\infty \theta^5 e^{-2500\theta}\, d\theta$$

$$= \frac{2500^7}{720} \int_0^\infty \left(\frac{u}{2500}\right)^5 e^{-u} \frac{du}{2500}, \quad u = 2500\theta$$

$$= \frac{2500}{720} \int_0^\infty u^5 e^{-u}\, du$$

$$= \frac{2500}{720} \Gamma(6) = \frac{2500}{720}(5!)$$

$$\therefore E(X_4 \mid 100, 950, 450) = \frac{2500(120)}{720} = 416.67$$

this result could also have been obtained as follows.

From EX 17.8, the predictive density is

$$f(x_4 \mid 100, 950, 450) = \frac{7(2500)^7}{(2500 + x_4)^8}$$

which is a Pareto density with parameters

$$(\alpha, \theta) \Rightarrow (7, 2500)$$

$$\therefore E(X_4 \mid 100, 950, 450) = \frac{\theta}{\alpha - 1} = \frac{2500}{6} = 416.67$$

Example 17.11 p. 413

Generalize the result of Example 17.10 for an arbitrary sample size of n and an arbitrary perior gamma distribution with parameter α and β, where β is the reciprocal of the usual scale parameter.

Ans: The Bayesian premium estimate is given by

$$E(X_{n+1} \mid X = x) = \int \mu_{n+1}(\theta)\, \pi_{\Theta|X}(\theta|x)\, d\theta$$

* From previous Examples (17.3, 17.8 and 17.10), we can determine the posterior dist^{n} as follows.

$$\pi_{\Theta|X}(\theta|x) \propto \underbrace{\left(\prod_{j=1}^{n} \theta e^{-\theta x_j}\right)}_{\text{product of exp. density values}} \underbrace{\frac{\theta^{\alpha-1} e^{-\beta\theta} \beta^{\alpha}}{\Gamma(\alpha)}}_{\text{gamma density fn } (\theta|\alpha,\beta)}$$

i.e. $\pi_{\Theta|X}(\theta|x) \propto \theta^{n+\alpha-1} e^{-(\sum x_j + \beta)\theta}$

• So, we can write the posterior density fn as gamma density fn with parameters $n+\alpha$, $\dfrac{1}{\sum x_j + \beta}$, where $(\sum x_j + \beta)^{-1}$ is the scale parameter.

$$\therefore \pi_{\Theta|X}(\theta|x) = \frac{[(\sum x_j + \beta)\theta]^{n+\alpha} \exp[-(\sum x_j + \beta)\theta]}{\theta\, \Gamma(n+\alpha)}$$

The Bayes estimate of X_{n+1} is the expected value of Θ^{-1} using the posterior distribution. It is

Remember:
$X \sim \text{gamma}(\alpha, \theta)$
$\Rightarrow f(x) = \dfrac{(x/\theta)^{\alpha} e^{-x/\theta}}{x\, \Gamma(\alpha)}$
see p. 497

$\alpha \to n+\alpha$

scale $\theta \to (\sum x_j + \beta)^{-1}$

$$E(X_{n+1} \mid X = x)$$

$$= \int \frac{1}{\theta} \cdot \frac{[(\sum x_j + \beta)\theta]^{n+\alpha} \exp[-(\sum x_j + \beta)\theta]}{\theta\, \Gamma(n+\alpha)}\, d\theta$$

$$= \frac{1}{\Gamma(n+\alpha)} \int \frac{1}{\theta^2} [(\sum x_j + \beta)\theta]^{n+\alpha} \exp[-(\sum x_j + \beta)\theta]\, d\theta$$

let $(\sum x_j + \beta)\theta = u$

$$\therefore E(X_{n+1} \mid X = x) = \frac{(\sum x_j + \beta)^{\cancel{2}}}{\Gamma(n+\alpha)} \int_0^{\infty} \frac{u^{n+\alpha}}{u^2} e^{-u} \frac{du}{\cancel{(\sum x_j + \beta)}}$$

$$= \frac{\sum x_j + \beta}{\Gamma(n+\alpha)} \int_0^{\infty} u^{n+\alpha-2} e^{-u}\, du = \frac{(\sum x_j + \beta)\, \Gamma(n+\alpha-1)}{\Gamma(n+\alpha)}$$

$$\therefore E(X_{n+1} \mid X = x) = \frac{(\sum x_j + \beta)(n+\alpha-2)!}{(n+\alpha-1)!} = \frac{\sum x_j + \beta}{n+\alpha-1} = \frac{n}{n+\alpha-1}\bar{x} + \frac{\alpha-1}{n+\alpha-1}\left(\frac{\beta}{\alpha-1}\right)$$

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Example 17.16 p. 421

Determine the Bühlmann estimate for the setting in Ex 17.11

Ans:

For the model discussed in EX 17.11 p. 413

$\mu(\Theta) = \Theta^{-1}$ (1), $v(\Theta) = \Theta^{-2}$ (2)

$\mu(\theta) = E(X_j | \Theta = \theta) = \frac{1}{\theta}$ for exp. dist.

and

$v(\theta) = Var(X_j | \Theta = \theta) = \frac{1}{\theta^2}$ " "

Remember

$X \sim gamma(\alpha, \beta)$, $\beta = \frac{1}{\theta}$

$\Rightarrow f(x) = \frac{(x\beta)^\alpha e^{-\beta x}}{x \Gamma(\alpha)}$

Here, $x \to \theta$, $\Theta \sim gamma(\alpha, \beta)$

$$\mu = E(\Theta^{-1}) = \int_0^\infty \frac{1}{\theta} \frac{(\theta\beta)^\alpha e^{-\beta\theta}}{\theta \Gamma(\alpha)} d\theta$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \theta^{\alpha-2} e^{-\beta\theta} d\theta$$

$$\mu = E(\Theta^{-1}) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\frac{u}{\beta}\right)^{\alpha-2} e^{-u} \frac{du}{\beta}, \quad u = \beta\theta$$

$$\therefore \mu = E(\Theta^{-1}) = \frac{\beta}{\Gamma(\alpha)} \Gamma(\alpha-1) = \frac{\beta(\alpha-2)!}{(\alpha-1)!} = \frac{\beta}{\alpha-1} \quad (3)$$

$$v = E(\Theta^{-2}) = \int_0^\infty \frac{1}{\theta^2} \frac{(\beta\theta)^\alpha e^{-\beta\theta}}{\theta \Gamma(\alpha)} d\theta$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \theta^{\alpha-3} e^{-\beta\theta} d\theta$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\frac{u}{\beta}\right)^{\alpha-3} e^{-u} \frac{du}{\beta}$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-3} e^{-u} du$$

$$v = E(\Theta^{-2}) = \frac{\beta^2}{\Gamma(\alpha)} \Gamma(\alpha-2) = \frac{\beta^2(\alpha-3)!}{(\alpha-1)!}$$

$$\therefore v = E(\Theta^{-2}) = \frac{\beta^2}{(\alpha-1)(\alpha-2)} \quad (4)$$

(3), (4) $\Rightarrow$

$$a = Var(\Theta^{-1}) = \frac{\beta^2}{(\alpha-1)(\alpha-2)} - \frac{\beta^2}{(\alpha-1)^2}$$

$$\therefore a = \frac{\beta^2}{\alpha-1}\left(\frac{1}{\alpha-2} - \frac{1}{\alpha-1}\right) = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} \quad (5)$$

$$K = \frac{v}{a} = \frac{\beta^2}{(\alpha-1)(\alpha-2)} \Big/ \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$$

$$\therefore K = \alpha - 1 \qquad (6)$$

The Bühlmann credibility factor is

$$Z = \frac{n}{n+k} = \frac{n}{n+\alpha-1} \qquad (7)$$

$\therefore$ the Bühlmann credibility premium is given by

(3), (7) $\Rightarrow$

$$P_c = Z\bar{X} + (1-Z)\mu$$

$$P_c = \frac{n}{n+\alpha-1}\bar{X} + \left(1 - \frac{n}{n+\alpha-1}\right)\frac{\beta}{\alpha-1}$$

$$\therefore P_c = \frac{n}{n+\alpha-1}\bar{X} + \frac{\alpha-1}{n+\alpha-1}\,\frac{\beta}{\alpha-1} \qquad (8)$$

Which again matches the Bayesian estimate that introduced in EX. 17.11 p.413.

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