* Q (30) Q.V *

EX 17.16 p. 421 Textbook (EX 17-11 Continued p. 413)

Suppose that $X_1, X_2, \ldots, X_n$ are claims for a policyholder. Show that the Bühlmann estimate of X_{n+1} can be determined as

$$E(X_{n+1} \mid X = x) = \frac{n}{n+\alpha-1}\bar{X} + \frac{\alpha-1}{n+\alpha-1}\left(\frac{\beta}{\alpha-1}\right)$$

Note that $X|\theta$ has an exp distn with mean $1/\theta$. and $\theta \sim$ gamma (α, β) where β is the reciprocal of the usual scale parameter.

Ans:

* $\mu(\theta) = E(X_j \mid \theta=\theta) = \frac{1}{\theta}$ hypothetical mean (1)

* $v(\theta) = \text{Var}(X_j \mid \theta=\theta) = \frac{1}{\theta^2}$ process Variance (2)

* The expected value of hypoth. mean (collective premium) is

$$\mu = E(\theta^{-1}) = \int_0^\infty \frac{1}{\theta} \frac{(\beta\theta)^\alpha e^{-\beta\theta}}{\theta\,\Gamma(\alpha)}\, d\theta$$

$$\therefore \mu = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \theta^{\alpha-2} e^{-\beta\theta}\, d\theta$$

Let $u = \beta\theta \Rightarrow \theta = \frac{u}{\beta}$, $d\theta = \frac{du}{\beta}$

$$\mu = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\frac{u}{\beta}\right)^{\alpha-2} e^{-u} \frac{du}{\beta}$$

$$\mu = \frac{\beta}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-2} e^{-u}\, du = \frac{\beta\,\Gamma(\alpha-1)}{\Gamma(\alpha)}$$

$$\therefore \mu = \frac{\beta(\alpha-2)!}{(\alpha-1)!} = \frac{\beta}{\alpha-1} \quad (3)$$

$X \sim$ gamma (α, θ)

$\Rightarrow f(x) = \frac{(x\theta)^\alpha e^{-x\theta}}{x\,\Gamma(\alpha)}$

For $\theta \sim$ gamma (α, β)

$\pi(\theta) = \frac{(\beta\theta)^\alpha e^{-\beta\theta}}{\theta\,\Gamma(\alpha)}$

$x \to \theta$

$x\theta \to \beta$

$\alpha \to \alpha$

$\beta^\alpha \cdot \frac{1}{\beta^{\alpha-1}} = \beta$

2

* Similarly, the expected value of process Variance is

- $\upsilon = E(\theta^{-2}) = \int_0^\infty \frac{1}{\theta^2} \frac{(\beta\theta)^\alpha e^{-\beta\theta}}{\theta\,\Gamma(\alpha)}\, d\theta$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \theta^{\alpha-3} e^{-\beta\theta}\, d\theta$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(\frac{u}{\beta}\right)^{\alpha-3} e^{-u} \frac{du}{\beta} \quad \text{let } u = \beta\theta$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-3} e^{-u}\, du$$

$$= \frac{\beta^2}{\Gamma(\alpha)} \Gamma(\alpha-2) = \frac{\beta^2 (\alpha-3)!}{(\alpha-1)!}$$

$$\therefore \boxed{\upsilon = \frac{\beta^2}{(\alpha-1)(\alpha-2)}} \quad (4)$$

* Variance of Hypoth. Mean

$$\boxed{a = \mathrm{Var}(\theta^{-1}) = E(\theta^{-2}) - \mu^2}$$

(3), (4) $\Rightarrow$

$$\therefore a = \frac{\beta^2}{(\alpha-1)(\alpha-2)} - \frac{\beta^2}{(\alpha-1)^2}$$

$$= \frac{\beta^2(\alpha-1) - \beta^2(\alpha-2)}{(\alpha-1)^2(\alpha-2)}$$

$$\therefore \boxed{a = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}} \quad (5)$$

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3

④, ⑤ $\Rightarrow K = \frac{v}{a}$

$\therefore K = \frac{\beta^2}{(\alpha-1)(\alpha-2)} \Big/ \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$

$\boxed{K = \alpha - 1}$ ⑥

The Bühlmann credibility factor is

$$Z = \frac{n}{n+k} = \frac{n}{n+\alpha-1} \quad ⑦$$

∴ The Bühlmann credibility premium is given by

$$P_c = Z\bar{X} + (1-Z)\mu$$

③, ⑦ $\Rightarrow$

$$P_c = \frac{n}{n+\alpha-1}\bar{X} + \left(1 - \frac{n}{n+\alpha-1}\right)\frac{\beta}{\alpha-1}$$

$$\therefore P_c = \frac{n}{n+\alpha-1}\bar{X} + \frac{\alpha-1}{n+\alpha-1}\cdot\frac{\beta}{\alpha-1} \quad ⑧$$

which matches the Bayesian estimate in Ex 17.11 p. 413.