Credibility Premium
(Bayesian premium- Bühlmann premium)

Example

The amount of a claim X has an exponential distribution with mean $1/\Theta$. Among the class of insureds and potential insureds, the risk parameter Θ varies according to the gamma distribution with $\alpha = 4$ and scale parameter $\beta = 0.001$. Suppose that a person had claims of 100, 950, and 450.

Find each of the following.

(i) The probability density functions for X and risk parameter Θ.

(ii) The predictive distribution of the fourth claim and the posterior distribution of Θ.

(iii) The Bayesian premium.

(iv) The Bühlmann premium.

Hint: For Bühlmann premium estimate in (iv) use, $\mu = \frac{\beta}{\alpha - 1}$, $\upsilon = \frac{\beta^2}{(\alpha-1)(\alpha-2)}$, where β is the reciprocal of the usual scale parameter of gamma distribution.

Answer

(i)

For claims, $f_{X|\Theta}(x|\theta) = \theta e^{-\theta x}, \quad x, \theta > 0$

and for the risk parameter,

$$\pi_\Theta(\theta) = \frac{\theta^4 (1000)^4 e^{-\theta/0.001}}{\theta\,\Gamma(4)}, \qquad \theta > 0$$

$$\therefore \pi_\Theta(\theta) = \frac{\theta^3 e^{-1000\theta} (1000)^4}{6}, \qquad \theta > 0$$

(ii)

The marginal density at the observed values is

$$f_X(x) = \int \left[\prod_{j=1}^{n} f_{X_j|\Theta}(x_j|\theta) \right] \pi(\theta) d\theta$$

$$f(100, 950, 450) = \int_0^\infty \theta e^{-100\theta} \theta e^{-950\theta} \theta e^{-450\theta} \frac{1000^4}{6} \theta^3 e^{-1000\theta} d\theta$$

$$f(100, 950, 450) = \frac{1000^4}{6} \int_0^\infty \theta^6 e^{-2500\theta} d\theta$$

Let $t = 2500\theta$

$$f(100,950,450) = \frac{1000^4}{6}\int_0^\infty \frac{t^6}{(2500)^6}e^{-t}\,\frac{dt}{2500}$$

$$f(100,950,450) = \frac{1000^4}{6(2500)^7}\int_0^\infty t^6 e^{-t}dt$$

$$\therefore f(100,950,450) = \frac{1000^4}{6(2500)^7}\Gamma(7)$$

$$\therefore f(100,950,450) = \frac{1000^4}{6}\frac{720}{(2500)^7}$$

Similarly,

$$f(100,950,450,x_4) = \int_0^\infty \theta e^{-100\theta}\theta e^{-950\theta}\theta e^{-450\theta}\ \theta e^{-x_4\theta}\ \frac{1000^4}{6}\theta^3 e^{-1000\theta}d\theta$$

$$= \frac{1000^4}{6}\int_0^\infty \theta^7 e^{-(2500+x_4)\theta}d\theta$$

$$= \frac{1000^4}{6}\frac{\Gamma(8)}{(2500+x_4)^8}$$

$$= \frac{1000^4}{6}\frac{5040}{(2500+x_4)^8}$$

The predictive density is

$$f(x_4\,|100,950,450) = \frac{f(100,950,450,x_4)}{f(100,950,450)}$$

i.e.

$$f(x_4\,|100,950,450) = \frac{7(2500)^7}{(2500+x_4)^8},$$

which is a Pareto density with parameter 7 and 2500.

To get posterior density of Θ given X, use the formula

$$\pi_{\Theta|\mathrm{X}}(\theta|\mathrm{x}) = \frac{f_{\mathrm{X},\Theta}(\mathrm{x},\theta)}{f_{\mathrm{X}}(\mathrm{x})}$$

$$\therefore \pi(\theta|100,950,450) = \frac{\theta e^{-100\theta}\theta e^{-950\theta}\theta e^{-450\theta}\ \frac{1000^4}{6}\theta^3 e^{-1000\theta}}{\frac{1000^4}{6}\frac{720}{(2500)^7}}$$

$$= \frac{\theta^6 e^{-2500\theta}(2500)^7}{720}$$

(iii) For Bayesian premium,

Since, the amount of a claim has an exponential distribution with mean $1/\Theta$.

$$\therefore \mu_4(\theta) = \theta^{-1} = \frac{1}{\theta}$$

For Bayesian premium estimate, we can use the following Eq.

$$\mathrm{E}(X_{n+1}\,|\mathrm{X}=\mathrm{x}) = \int \mu_{n+1}(\theta)\pi_{\Theta|\mathrm{X}}(\theta\,|\mathrm{x})d\theta$$

$$\therefore \mathrm{E}(X_4 \mid 100,950,450) = \int_0^\infty \frac{1}{\theta} \cdot \frac{\theta^6 e^{-2500\theta}(2500)^7}{720} d\theta$$

$$= \frac{(2500)^7}{720} \int_0^\infty \theta^5 e^{-2500\theta} d\theta$$

$$= \frac{2500}{720} \int_0^\infty u^5 e^{-u} du, \quad u = 2500\theta$$

$$= \frac{2500}{720} \Gamma(6) = \frac{2500(120)}{720}$$

$$\therefore \mathrm{E}(X_4 \mid 100,950,450) = 416.67$$

Another solution

$$\because f(x_4 \mid 100,950,450) = \frac{7(2500)^7}{(2500+x_4)^8},$$

which is a Pareto density function with parameters $(\alpha, \theta) = (7, 2500)$

$$\therefore \mathrm{E}(X_4 \mid 100,950,450) = \frac{\theta}{\alpha - 1} = \frac{2500}{6} = 416,67.$$

(iv) To get the Bühlmann premium,

The hypothetical mean is $\mu(\Theta) = \Theta^{-1}$ where $\Theta \sim gamma(4,1000)$, $\alpha = 4$ and $\beta = 1000$ (the reciprocal of the usual scale parameter).

$\mu = E[\mu(\Theta)] = E(\Theta^{-1}) = \frac{\beta}{\alpha - 1} = \frac{1,000}{3}$ is the expected value of hypothetical mean,

$\upsilon = E[\upsilon(\Theta)] = E(\Theta^{-2}) = \frac{\beta^2}{(\alpha-1)(\alpha-2)} = \frac{500,000}{3}$ is the expected value of process variance and

$a = Var(\Theta^{-1}) = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} = \frac{500,000}{9}$ is the variance of hypothetical mean.

We can also find a as $a = Var(\Theta^{-1}) = \frac{500,000}{3} - \left(\frac{1,000}{3}\right)^2 = \frac{500,000}{9}$

$$\therefore k = \frac{\upsilon}{a} = 3, \quad \mathrm{Z} = \frac{n}{n+k} = \frac{3}{6} = \frac{1}{2}.$$

So, the Bühlmann Premium is

$$P_c = Z\overline{X} + (1 - Z)\mu$$

$$\therefore P_c = \frac{1}{2}(500) + \frac{1}{2}\left(\frac{1000}{3}\right) = 416.67, \quad \text{where } \overline{X} = \frac{100+950+450}{3} = 500,$$

which matches the result of Bayesian premium that introduced in (iii).