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Lecture (28)+

- EX 17.3 p. 403 — Bayesian Methodology (predictive & posterior distns)

The amount of a claim has an exponential distribution with mean $1/\Theta$. Among the class of insureds and potential insureds, the parameter Θ varies according to the gamma distribution with $\alpha = 4$ and scale parameter $\beta = 0.001$. Provide a mathematical description of this model.

Ans: For claims,

$$f_{X|\Theta}(x|\theta) = \theta e^{-\theta x}, \quad x, \theta > 0$$

and for the risk parameter,

$$\pi_\Theta(\theta) = \frac{\theta^4 (1000)^4 e^{-\theta/0.001}}{\theta \, \Gamma(4)}, \quad \theta > 0$$

$$\pi_\Theta(\theta) = \frac{\theta^3 e^{-1{,}000\theta} \, 1{,}000^4}{6}, \quad \theta > 0$$

where $\Gamma(4) = 3!$
$= 6$

$X \sim \exp(\theta)$
$f(x) = \frac{e^{-x/\theta}}{\theta}$ p. 499
$X \sim \text{gamma}(\alpha, \theta)$
$f(x) = \frac{(x/\theta)^\alpha e^{-x/\theta}}{x \, \Gamma(\alpha)}$ p. 497

- EX 17.6 p. 407

If $X|\Theta$ is normally distributed with mean Θ and variance v, where Θ is itself normally distributed with mean μ and variance a, then X is (unconditionally) normally distributed with mean μ and variance $a+v$. (See Ex 5.5 p. 66). Find the mean and variance of X.

Ans: $X|\Theta \sim N(\Theta, v)$ and $\Theta \sim N(\mu, a)$
X ??

$\because E(X) = E[E(X|\Theta)]$ — See Eq 17.3 p. 405

$\therefore E(X) = E[E(X|\Theta)] = E(\Theta) = \mu$

$\because Var(X) = E[Var(X|\Theta)] + Var[E(X|\Theta)]$ — See Eq. 17.6 p. 407

$\therefore Var(X) = E(v) + Var(\Theta)$
$= v + a, \quad v$ is constant.

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EX 17.8 p. 410

(EX 17.3 Continued) Suppose that a person had claims of 100, 950, and 450. Determine the predictive distribution of the fourth claim and the posterior distribution of Θ.

Ans:

② ⇒ The marginal density at the observed values is

In lecture (28)

$$f_X(x) = \int \left[\prod_{j=1}^{n} f_{X_j|\Theta}(x_j|\theta)\right] \pi(\theta)\, d\theta$$

(see EX 17.3)

$$f(100, 950, 450) = \int_0^\infty \theta e^{-100\theta}\, \theta e^{-950\theta}\, \theta e^{-450\theta}\, \frac{1000^4}{6}\theta^3 e^{-1000\theta}\, d\theta \quad (*)$$

$$f(100, 950, 450) = \frac{1000^4}{6}\int_0^\infty \theta^6 e^{-2,500\theta}\, d\theta$$

let $t = 2,500\theta$

$$\therefore f(100, 950, 450) = \frac{1000^4}{6}\int_0^\infty \frac{t^6}{(2500)^6} e^{-t}\, \frac{dt}{2500}$$

$$= \frac{1000^4}{6(2,500)^7}\int_0^\infty t^6 e^{-t}\, dt$$

$$\int_0^\infty t^\alpha e^{-t}\, dt = \Gamma(\alpha+1) = \alpha!, \quad \alpha = 0, 1, \dots$$

$$f(100, 950, 450) = \frac{1,000^4}{6(2500)^7}\Gamma(7)$$

$$\Gamma(7) = 6! = 720$$

$$\therefore f(100, 950, 450) = \frac{1000^4}{6}\,\frac{720}{(2,500)^7} \quad (1)$$

Similarly,

$$f(100, 950, 450, x_4) = \int_0^\infty \theta e^{-100\theta}\, \theta e^{-950\theta}\, \theta e^{-450\theta}\, \theta e^{-x_4\theta}\, \frac{1000^4}{6}\theta^3 e^{-1000\theta}\, d\theta \quad (\;)$$

$$= \frac{1000^4}{6}\int_0^\infty \theta^7 e^{-(2,500+x_4)\theta}\, d\theta$$

$$= \frac{1,000^4}{6}\int_0^\infty \frac{u^7}{(2500+x_4)^8} e^{-u}\, du, \quad u = 2,500 + x_4$$

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$$f(100,950,450,x_4) = \frac{1000^4}{6} \frac{\Gamma(8)}{(2{,}500+x_4)^8}$$

$$f(100,950,450,x_4) = \frac{1{,}000^4}{6} \frac{5040}{(2{,}500+x_4)^8} \quad (2)$$

$\Gamma(8) = 7! = 5040$

Then, the predictive density is

$$f(x_4|100,950,450) = \frac{f(100,950,450,x_4)}{f(100,950,450)} \quad (3)$$

→ Joint density fn
→ marginal ~ ~

Apply (1) and (2) in (3), to get

$$f(x_4|100,950,450) = \frac{\cancel{\frac{1000^4}{6}} \frac{\cancel{5040}^{7}}{(2{,}500+x_4)^8}}{\cancel{\frac{1000^4}{6}} \frac{\cancel{720}}{(2500)^7}}$$

$$\therefore f(x_4|100,950,450) = \frac{7(2{,}500)^7}{(2{,}500+x_4)^8} \quad (4)$$

which is a pareto density with parameter 7 and 2,500. See pareto(α,θ) distn p. 494 #

To get posterior density of θ given X, use (4) in lecture (28)

$$\pi_{\theta|X}(\theta|X) = \frac{f_{X,\theta}(X,\theta)}{f_X(X)}$$

→ integrand of the Integ. (*)
→ determined in (1)

$$\pi(\theta|100,950,450) = \frac{\theta e^{-100\theta}\, \theta e^{-950\theta}\, \theta e^{-450\theta}\, \frac{\cancel{1000^4}}{\cancel{6}} \theta^3 e^{-1000\theta}}{\frac{\cancel{1{,}000^4}}{\cancel{6}} \frac{720}{(2{,}500)^7}}$$

$$\therefore \pi(\theta|100,950,450) = \frac{\theta^6 e^{-2{,}500\theta} (2{,}500)^7}{720} \quad (5)$$

Also, We can determine the predictive density from (6) in lecture (28) as follows.

$$f_{X_{n+1}|X}(x_{n+1}|X) = \int f_{X_{n+1}|\theta}(x_{n+1}|\theta)\, \pi_{\theta|X}(\theta|X)\, d\theta$$

$x_{n+1} = x_4$

$$\therefore f(x_4|100,950,450) = \int_0^\infty \theta e^{-\theta x_4} \frac{\theta^6 e^{-2{,}500\theta} (2500)^7}{720} d\theta$$

$$= \frac{(2500)^7}{720} \int_0^\infty \theta^7 e^{-(2{,}500+x_4)\theta} d\theta$$

$$\therefore f(x_4|100,950,450) = \frac{(2500)^7}{\cancel{720}} \cdot \frac{\cancel{\Gamma(8)}^{7}}{(2500+x_4)^8} = \frac{7(2{,}500)^7}{(2{,}500+x_4)^8} \quad (6)$$

as calculated before in (4) #