L(17) تابع

The relationships between Selected distributions p.74 Textbook

* Imp. limits

1

$$\lim_{\alpha\to\infty} \frac{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{\frac{1}{2}}}{\Gamma(\alpha)} = 1$$

$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt$

Stirling's formula

2

$$\lim_{a\to\infty}\left(1+\frac{x}{a}\right)^{a+b} = \lim_{a\to\infty}\left(1+\frac{x}{a}\right)^{a}\lim_{a\to\infty}\left(1+\frac{x}{a}\right)^{b}$$

$$= e^{x}\cdot 1 = e^{x}$$

Ex: 5.10 p.74

Show that the transformed gamma distribution is a limiting case of the transformed beta dist^n as $\theta\to\infty$, $\alpha\to\infty$, and $\theta/\alpha^{1/\gamma}\to\xi$, a constant.

Ans: The transformed beta pdf is

$$f(x) = \frac{\Gamma(\alpha+\tau)\,\gamma\,x^{\gamma\tau-1}}{\Gamma(\alpha)\Gamma(\tau)\,\theta^{\gamma\tau}\left(1+x^{\gamma}\theta^{-\gamma}\right)^{\alpha+\tau}} \quad (1)$$ given

Let $\theta = \xi\alpha^{1/\gamma}$ (2), at $\theta/\alpha^{1/\gamma} = \xi = \text{Const.}$

$\Rightarrow \theta = \xi\alpha^{\frac{1}{\gamma}}$ (2) given

By using Stirling's formula

$$\lim_{\alpha\to\infty}\frac{e^{-\alpha}\alpha^{\alpha-1/2}(2\pi)^{1/2}}{\Gamma(\alpha)} = 1 \quad (3)$$

$$\Rightarrow \lim_{\alpha+\tau\to\infty}\frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-1/2}(2\pi)^{1/2}}{\Gamma(\alpha+\tau)} = 1 \quad (4)$$

Substitute (2), (3), (4) in (1), we get, where $\theta = \xi\alpha^{1/\gamma}$

$$x^{\gamma}\theta^{-\gamma} = \left(\frac{x}{\theta}\right)^{\gamma}$$

$$(2)\Rightarrow = \left(\frac{x}{\xi\alpha^{1/\gamma}}\right)^{\gamma} = \frac{(x/\xi)^{\gamma}}{\alpha}$$

2

$$f(x) = \frac{e^{-(\alpha+\tau)}(\alpha+\tau)^{\alpha+\tau-\frac{1}{2}}(2\pi)^{\frac{1}{2}}\gamma x^{\gamma\tau-1}}{e^{-\alpha}\alpha^{\alpha-\frac{1}{2}}(2\pi)^{\frac{1}{2}}\Gamma(\tau)\beta^{\gamma\tau}\alpha^{\tau}(1+x^{\gamma}\beta^{-\gamma}\alpha^{-1})^{\alpha+\tau}}$$

$$f(x) = \frac{e^{-\tau}\left(\frac{\alpha+\tau}{\alpha}\right)^{\alpha+\tau-\frac{1}{2}}\gamma x^{\gamma\tau-1}}{\Gamma(\tau)\beta^{\gamma\tau}\left[1+\frac{(x/\beta)^{\gamma}}{\alpha}\right]^{\alpha+\tau}}$$

$$\because \lim_{\alpha\to\infty}\left(\frac{\alpha+\tau}{\alpha}\right)^{\alpha+\tau-\frac{1}{2}} = \lim_{\alpha\to\infty}\left(1+\frac{\tau}{\alpha}\right)^{\alpha+\tau-\frac{1}{2}} = e^{\tau}$$

$$\text{and } \lim_{\alpha\to\infty}\left[1+\frac{(x/\beta)^{\gamma}}{\alpha}\right]^{\alpha+\tau} = e^{(x/\beta)^{\gamma}}$$

$$\therefore \lim_{\alpha\to\infty} f(x) = \frac{e^{-\tau}e^{\tau}\gamma x^{\gamma\tau-1}}{\Gamma(\tau)\beta^{\gamma\tau}e^{(x/\beta)^{\gamma}}}$$

$$\therefore \lim_{\alpha\to\infty} f(x) = \frac{\gamma x^{\gamma\tau-1}e^{-(x/\beta)^{\gamma}}}{\Gamma(\tau)\beta^{\gamma\tau}}$$

which is the pdf of the transformed gamma distn with parameters τ, β and γ.

α τ See p. 496 Textbook