Lecture (12)

Ch 5: Continuous Models

Here, we discuss the relations among many Continuous Models that are defined in Appendix A, through some Mathematical Techniques.

(1) Multiplication by a Constant

Theorem (1) Let X be a continuous r.v with pdf $f_X(x)$ and cdf $F_X(x)$. Let $\boxed{Y = \theta X}$ with $\theta > 0$ then

$$F_Y(y) = F_X\left(\frac{y}{\theta}\right) \quad , \quad f_Y(y) = \frac{1}{\theta} f_X\left(\frac{y}{\theta}\right)$$

proof

$$\because F_Y(y) = pr(Y \le y)$$
$$= pr(\theta X \le y)$$
$$\therefore F_Y(y) = pr\left(X \le \frac{y}{\theta}\right) = F_X\left(\frac{y}{\theta}\right)$$

Also,

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$
$$= \frac{d}{dy} F_X\left(\frac{y}{\theta}\right)$$
$$\therefore f_Y(y) = \frac{1}{\theta} f_X\left(\frac{y}{\theta}\right)$$

- **Corollary,** θ is called a scale parameter for the r.v Y.

Ex (1) Let X have pdf $f(x) = e^{-x}$, $x > 0$

Determine the cdf and pdf of $Y = \theta X$

Ans:

$$F_X(x) = 1 - e^{-x}$$

Let $Y = \theta X \Rightarrow F_Y(y) = F_X\left(\frac{y}{\theta}\right)$

$$\therefore F_Y(y) = 1 - e^{-y/\theta} \quad , \text{ and } \quad f_Y(y) = \frac{1}{\theta} f_X\left(\frac{y}{\theta}\right)$$

$$\therefore f_Y(y) = \frac{1}{\theta} e^{-y/\theta}$$

i.e $Y \sim \exp(\theta)$ #

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② Raising to a power

Theorem ② Let X be a continuous r.v with pdf $f_X(x)$ and cdf $F_X(x)$ and with $F_X(0)=0$. Let $Y=X^{1/\tau}$ then

If $\tau>0$, $F_Y(y)=F_X(y^\tau)$, $f_Y(y)=\tau y^{\tau-1} f_X(y^\tau)$

If $\tau<0$, $F_Y(y)=1-F_X(y^\tau)$, $f_Y(y)=-\tau y^{\tau-1} f_X(y^\tau)$

Proof If $\tau>0$, $F_Y(y)=pr(Y\leq y)$

$$=pr(X^{1/\tau}\leq y)$$

$$\therefore F_Y(y)=pr(X\leq y^\tau)=F_X(y^\tau) \quad ①$$

while, If $\tau<0$, $F_Y(y)=pr(X^{1/\tau}\leq y)$

$$=pr(X\geq y^\tau) \quad \tau \text{ is -ve}$$

$$F_Y(y)=1-F_X(y^\tau) \quad ②$$

Also, $f_Y(y)=\frac{d}{dy}F_Y(y)$

$$=\frac{d}{dy}F_X(y^\tau), \quad \tau>0$$

$$=f_X(y^\tau)\cdot\tau y^{\tau-1}$$

Note
$2<3$
$\Rightarrow 2^2<3^2$
$\Rightarrow 2^{-2}>3^{-2}$

$$\therefore f_Y(y)=\tau y^{\tau-1} f_X(y^\tau), \tau>0 \quad ③$$

But for $\tau<0$, $f_Y(y)=\frac{d}{dy}\left[1-F_X(y^\tau)\right]$

$$=-\frac{d}{dy}F_X(y^\tau)$$

$$\therefore f_Y(y)=-\tau y^{\tau-1} f_X(y^\tau), \quad \tau<0 \quad ④$$

Note: For negative τ, We can rewrite ② and ④ as follows:

$$F_Y(y) = 1 - F_X(y^{-\tau}) \text{ and}$$
$$f_Y(y) = \tau y^{-\tau-1} f_X(y^{-\tau})$$

$\tau \to -\tau$, $\tau > 0$ (*)

Defn When raising a distribution to a power, if $\tau > 0$, the resulting distribution is called transformed; if $\tau = -1$, it is called inverse; and if $\tau < 0$ (but is not -1), it is called inverse transformed.

EX ② Suppose that X has an exponential distribution. Determine the cdf of the inverse, transformed and inverse transformed exponential distributions.

Ans: • The inverse exponential distn with no scale parameter (where $\tau = -1$) has cdf

Eq. ② $\Rightarrow$ $F_Y(y) = 1 - F_X(y^{-1})$ From theorem ②

$$= 1 - [1 - e^{-1/y}]$$

$$\therefore F_Y(y) = e^{-1/y}$$

$y \to y/\theta$

$F(x) = 1 - e^{-x}$ exp. distn with no scale parameter

with scale parameter added, it is

$$F(y) = e^{-\theta/y} \quad \text{(inverse exponential distn)}$$

• The transformed exponential distn with no scale parameter has cdf Eq. ① $\Rightarrow$ $F_Y(y) = F_X(y^{\tau})$, $\tau > 0$

$$F(y) = 1 - e^{-y^{\tau}}$$

$$\therefore F(y) = 1 - \exp(-y^{\tau})$$

With scale parameter added, it is

$y \to \frac{y}{\theta}$

$$F(y) = 1 - \exp[-(y/\theta)^{\tau}]$$ which is

Known as the Weibull distribution.

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- The inverse transformed exponential distribution with no scale parameter has cdf

(*) $\Rightarrow$ $F_Y(y) = 1 - F_X(y^{-\tau})$ (for negative τ)

$= \cancel{1} - [\cancel{1} - \exp(-y^{-\tau})]$

$F_Y(y) = \exp(-y^{-\tau})$

$y \to y/\theta$

With the scale parameter added, it is

$y \to \frac{y}{\theta}$

$\Rightarrow \left(\frac{y}{\theta}\right)^{-\tau} = \left(\frac{\theta}{y}\right)^{\tau}$

$F(y) = \exp\left[-(\theta/y)^{\tau}\right]$ which is known as inverse Weibull distribution.

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