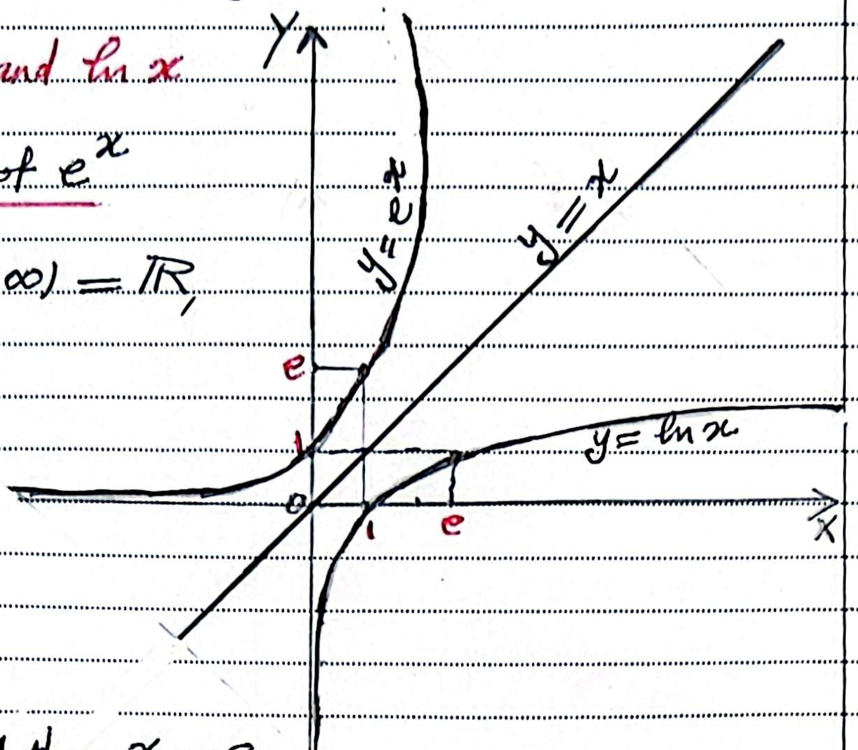
Lecture (9) The Natural Exponential Function

. The inverse function of $\ln x$ is $\ln^{-1} x$ which is also denoted by $\exp x$, and it's called natural exponential function.

i.e. If $y = \ln^{-1} x$ then $x = \ln y$ or $y = e^{x} = \exp x$.

Defn $y = e^{x}$ iff $x = \ln y$ $\forall x \in \mathbb{R}$, $y > 0$.

Graph of e^{x} and $\ln x$



* **Some properties of e^{x}**

- Domain is $(-\infty, \infty) = \mathbb{R}$,
  Range is $(0, \infty)$
- $\lim_{x \to \infty} e^{x} = \infty$,
  $\lim_{x \to -\infty} e^{x} = 0$
- $e^{0} = 1$, $e^{1} = e$

Clearly, if $\ln x = 1$ then $x = e$

where e is an irrational number, $e \approx 2.718\ldots$
It can be defined as follows.
$e = \lim_{n \to \infty} (1 + \frac{1}{n})^{n}$, $n > 0$ or $e = \lim_{h \to 0} (1 + h)^{1/h}$, $h > 0$.

Theorem

(i) $\ln(e^{x}) = x$, $x \in \mathbb{R}$

(ii) $e^{\ln x} = x$, $x > 0$

Rules of exponents for e^{x}

(i) $e^{p} e^{q} = e^{p+q}$

(ii) $\frac{e^{p}}{e^{q}} = e^{p-q}$

(iii) $(e^{p})^{r} = e^{pr}$

where $p, q \in \mathbb{R}$ and r is a rational number.

• Theorem: Derivative of exponential fn

$$\frac{d}{dx} e^x = e^x, \quad \frac{d}{dx} e^u = e^u \frac{du}{dx}, \text{ where}$$

$u = g(x)$, g is differentiable function.

• Theorem: Integral of exponential fn

$$\int e^x dx = e^x + c, \quad \int e^u du = e^u + c$$

* Examples

① Solve for x

(i) $e^{\ln x} = 4$ (ii) $\ln(e^x) = -2$

(iii) $\ln(1/x) = 2$

Ans: (i) $x = 4$ (ii) $x = -2$

(iii) $x = e^{-2}$

② If $y = x^{(e^x)}$, then find y'

Ans: $\ln y = e^x \ln x$,

thus by taking ln for both sides

Diff. both sides w.r.t x,

$\therefore \frac{1}{y}\frac{dy}{dx} = e^x\left(\frac{1}{x}\right) + \ln x \, e^x \quad (\times y)$

$\therefore \frac{dy}{dx} = y e^x \left(\frac{1}{x} + \ln x\right)$

$\therefore y' = x^{(e^x)} e^x \left(\frac{1}{x} + \ln x\right)$.

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③ Use Implicit differentiation to find y' if $xe^y - 10x + 3y = 0$ HW

Ans: $y' = \frac{10 - e^y}{xe^y + 3}$

④ Evaluate $\int_1^2 \frac{e^{3/x}}{x^2} dx$

let $u = \frac{3}{x}$

$x : 1 \to 2 \Rightarrow u : 3 \to \frac{3}{2}$

$du = \frac{-3}{x^2} dx$

$\Rightarrow \frac{1}{x^2} dx = \frac{-1}{3} du$

$\therefore I = \int_1^2 \frac{e^{3/x}}{x^2} dx$

$I = \frac{-1}{3} \int_3^{3/2} e^u du$

$I = \frac{1}{3} \int_{3/2}^3 e^u du$

$I = \frac{1}{3} [e^u]_{3/2}^3$

$\therefore I = \frac{1}{3} (e^3 - e^{3/2})$

≈ 5.2

⑤ Evaluate $\int x e^{-x^2} dx$ HW

Hint $u = -x^2$ ...

Ans: $I = \frac{-1}{2} e^{-x^2} + c$.

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