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Lecture (8) — Integrals and Transcendental Functions

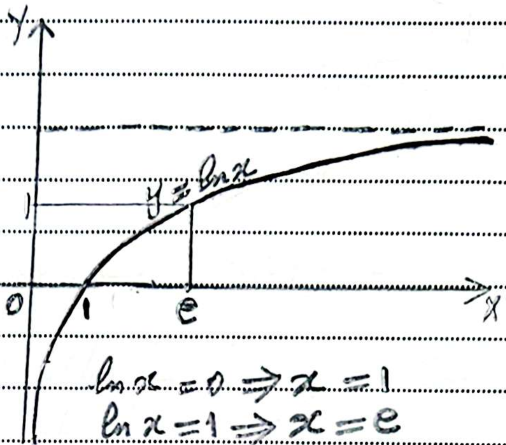
The natural logarithm function

The natural logarithm $\ln x$ is defined as follows.

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad \forall\, x > 0$$

- Note that $\ln 1 = 0$ ...... where $\int_1^1 \frac{1}{t}\,dt = 0$.

* The graph of $y = \ln x$



$\ln x = 0 \Rightarrow x = 1$
$\ln x = 1 \Rightarrow x = e$

- Domain is $(0, \infty)$,
  Range is $\mathbb{R} = (-\infty, \infty)$
- $\lim_{x\to\infty} \ln x = \infty$, $\lim_{x\to 0^+} \ln x = -\infty$,

$x \to 0^+$ تعني عندما x تقترب إلى الصفر من ناحية اليمين

$\ln(1) = 0$ and $\ln e = 1$, $e \approx 2.71828$.
- It's a continuous, increasing and concave downward on $(0, \infty)$.
- As $x \to \infty$, the tangent line is a horizontal asymptote. خط تقاربي أفقي
- Y axis $(x = 0)$ is a vertical asymptote for the graph.
- The derivative of $\ln x$ is given by $\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$, $x > 0$. ... Fundamental th.

- Laws of natural logarithms

If $p > 0$, $q > 0$ and r is a rational number, then

(1) $\ln(pq) = \ln p + \ln q$

(2) $\ln\left(\frac{p}{q}\right) = \ln p - \ln q$

(3) $\ln p^r = r \ln p$

- Theorem

If $u = g(x)$ and g is differentiable function, then

(i) $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$ if $u > 0$

(ii) $\frac{d}{dx}(\ln|u|) = \frac{1}{u}\frac{du}{dx}$ if $u \neq 0$

EX ①

Find $\frac{dy}{dx}$ or $f'(x)$ for each of the following

① $y = \ln(x^2+1)$ ② $f(x) = \ln|2-3x|^5$

③ $f(x) = \ln\sqrt{\frac{4+x^2}{4-x^2}}$ ④ $y = \cos x(\ln x)$ HW

Ans:

① $y = \ln(x^2+1)$

Ans: $\frac{dy}{dx} = \frac{1}{x^2+1}(2x) = \frac{2x}{x^2+1}$

② $f(x) = \ln|2-3x|^5$

$f'(x) = \frac{1}{(2-3x)^{5}} 5(2-3x)^4(-3)$

$\therefore f'(x) = \frac{-15}{2-3x}, \; x \neq \frac{2}{3}$

③ $f(x) = \ln\sqrt{\frac{4+x^2}{4-x^2}}$

$f(x) = \ln\left(\frac{4+x^2}{4-x^2}\right)^{1/2}$

$f(x) = \frac{1}{2}\ln\left(\frac{4+x^2}{4-x^2}\right)$

$f(x) = \frac{1}{2}\left[\ln(4+x^2) - \ln(4-x^2)\right]$

$\therefore f'(x) = \frac{1}{2}\left[\frac{2x}{4+x^2} - \frac{-2x}{4-x^2}\right]$

$f'(x) = \frac{x}{4+x^2} + \frac{x}{4-x^2}$

- $\frac{d}{dx}(\ln x) = \frac{1}{x}, \; x > 0 \Rightarrow \int \frac{1}{x}\,dx = \int \frac{dx}{x}$

$= \ln x + C, \; x > 0$

Theorem

$$\int \frac{1}{u}\,du = \ln|u| + C, \; u \neq 0$$

where $u = g(x)$ and g is differentiable function

EX ② Evaluate ① $\int \frac{3x}{4x^2-3}\,dx$ ② $\int \frac{1}{x\ln x}\,dx$ ③ $\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3+2\sin\theta}\,d\theta$ HW

Ans: $I = \frac{3}{8}\int \frac{8x}{4x^2-3}\,dx$

$I = \frac{3}{8}\ln|4x^2-3| + C$

$I = \int \frac{1/x}{\ln x}\,dx$

$I = \ln|\ln x| + C$

Ans: $2\ln 5$

Hint

$u = 3+2\sin\theta$ ...

EX ③ Evaluate the following Integrals

① $\displaystyle\int_2^4 \frac{1}{9-2x}\,dx$

let $u = 9-2x$

$I \Rightarrow du = -2\,dx \Rightarrow dx = -\frac{1}{2}du$

$x: 2 \to 4 \Rightarrow u: 5 \to 1$

$I =$

$I = -\frac{1}{2}\displaystyle\int_5^1 \frac{1}{u}\,du = \frac{1}{2}\int_1^5 \frac{1}{u}\,du$

$I = \frac{1}{2}\left[\ln u\right]_1^5$

$I = \frac{1}{2}(\ln 5 - \underbrace{\cancel{\ln 1}}_{0}) = \ln\sqrt{5}$

≈ 0.8

② $\displaystyle\int \tan x\,dx$

$I = -\displaystyle\int \frac{-\sin x}{\cos x}\,dx$

$I = -\ln|\cos x| + C$

$(\cos x)^{-1} = \frac{1}{\cos x} = \sec x$

$I = \ln|\sec x| + C$ #

③ $\displaystyle\int \cot x\,dx$

$I = \displaystyle\int \frac{\cos x}{\sin x}\,dx$

$I = \ln|\sin x| + C$ #

④ $\displaystyle\int \sec x\,dx$

$I = \displaystyle\int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx$

$I = \displaystyle\int \left(\frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\right)dx$

$I = \displaystyle\int \left(\frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}\right)dx$

$I = \ln|\sec x + \tan x| + C$ #

⑤ $\displaystyle\int \csc x\,dx$

$I = \displaystyle\int \frac{\csc x(\csc x - \cot x)}{\csc x - \cot x}\,dx$

$I = \displaystyle\int \left(\frac{-\csc x\cot x + \csc^2 x}{\csc x - \cot x}\right)dx$

$I = \ln|\csc x - \cot x| + C$

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⑥ H.W

$\displaystyle\int \frac{\cot^2 x}{\csc x}\,dx$

Hint $1 + \cot^2 x = \csc^2 x$

Ans:

$I = \ln|\csc x - \cot x| + \cos x + C$

Theorem

(1) $\int \tan u\,du = -\ln|\cos u| + C = \ln|\sec u| + C$

(2) $\int \cot u\,du = \ln|\sin u| + C$

(3) $\int \sec u\,du = \ln|\sec u + \tan u| + C$

(4) $\int \csc u\,du = \ln|\csc u - \cot u| + C$