Lecture (6) — The definite Integral

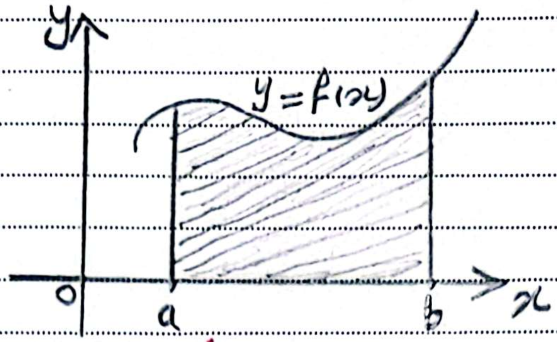
properties & Mean Value theorem and the Fundamental theorem

Theorem

If f is integrable on $[a,b]$ and $f(x) \geq 0 \quad \forall x \in [a,b]$

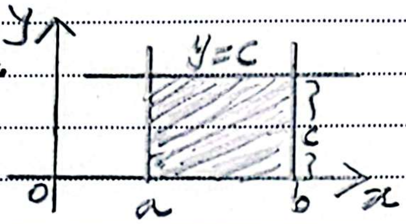
then the area A under the curve f from a to b is

$$A = \int_a^b f(x)\,dx$$



- **Some properties of definite Integrals**

Let f and g be two integrable functions on $[a,b]$, $c \in \mathbb{R}$, the definite integral satisfies the following rules.



① $\int_a^b c\,dx = c(b-a)$

② $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

③ $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

④ If $f(x) \geq 0 \Rightarrow \int_a^b f(x)\,dx \geq 0$

⑤ $\int_a^a f(x)\,dx = 0$ "Zero Width interval"

⑥ If $f(x) \geq g(x) \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

e.g. $\int_{-1}^{2} (x^2+2)\,dx > \int_{-1}^{2} (x-1)\,dx$ where $x^2+2 > x-1 \; \forall x \in [-1,2]$

Check

⑦ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

⑧ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

e.g. $\int_2^5 f(x)\,dx = \int_2^3 f(x)\,dx + \int_3^5 f(x)\,dx$, $\int_2^3 f(x)\,dx = \int_2^5 f(x)\,dx + \int_5^3 f(x)\,dx$ #

Mean Value theorem for definite Integral

Theorem

If f is continuous on $[a,b]$, then there is at least a number c, $c \in (a,b)$ such that

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$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Definition

The average value of f on $[a,b]$ is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

EX ① Find the value of c that satisfies the conclusion of the mean-value theorem of $f(x) = 3x^2+1$ on $[0,2]$, then find the average value of f on $[0,2]$.

Ans: By using the mean value theorem, we have

$$\int_0^2 f(x)\,dx = (2-0)f(c)$$

$$\int_0^2 (3x^2+1)\,dx = 2(3c^2+1)$$

$$\therefore \left[\frac{\cancel{3}x^3}{\cancel{3}} + x\right]_0^2 = 2(3c^2+1)$$

$$\int_0^2 (3x^2+1)\,dx = 10$$

$$10 = 2(3c^2+1) \quad (\div 2)$$

$$\therefore 3c^2+1 = 5 \Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \pm\frac{2}{\sqrt{3}}$$

only $c = \frac{2}{\sqrt{3}} \in (0,2)$, $\quad \frac{2}{\sqrt{3}} \simeq 1.15$

$$av(f) = \frac{1}{2-0}\int_0^2 (3x^2+1)\,dx = \frac{1}{2}(10) = 5.$$

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- Change of Variables in definite Integral.

- Theorem

$$\int_a^b f(g(x))\, g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

EX ② Evaluate

$$\int_2^{10} \frac{3}{\sqrt{5x-1}}\,dx$$

Ans: let $u = 5x-1 \Rightarrow du = 5dx \Rightarrow dx = \frac{du}{5}$

$x=2 \longrightarrow u = 9$

$x=10 \longrightarrow u = 49$

$$I = \frac{3}{5}\int_9^{49} \frac{du}{\sqrt{u}} = \frac{3}{5}\int_9^{49} u^{-1/2}\,du$$

$$I = \frac{3}{5}\left[\frac{u^{1/2}}{1/2}\right]_9^{49} = \frac{6}{5}\left[\sqrt{u}\,\right]_9^{49}$$

$$\therefore\ I = \frac{6}{5}\left(\sqrt{49}-\sqrt{9}\right) = \frac{6}{5}(7-3) = \frac{24}{5}\ \#$$

- Fundamental theorem of calculus

Let f be a continuous function on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

(I) $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$, See Lecture #5

(II) $\displaystyle F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

EX ③ Use the fundamental theorem to find

(a) $\displaystyle\frac{d}{dx}\int_a^x (t^3+1)\,dt$

Ans:

$$\frac{d}{dx}\int_a^x (t^3+1)\,dt = x^3+1$$

(b) $\displaystyle\frac{d}{dx}\int_x^5 3t\sin t\,dt$

Ans:

$$\frac{d}{dx}\int_x^5 3t\sin t\,dt = -\frac{d}{dx}\int_5^x 3t\sin t\,dt$$

$$= -3x\sin x.\ \#$$