Chapter 5:

Some Discrete Probability Distributions

Discrete Uniform Distribution

If the discrete random variable X assumes the values $x_1, x_2, \ldots, x_k$ with equal probabilities, then X has the discrete uniform distribution given by:

$$f(x) = P(X = x) = f(x;k) = \begin{cases} \dfrac{1}{k} & ; x = x_1, x_2, \cdots, x_k \\ 0 & ; \ elsewhere \end{cases}$$

Note:

- $f(x) = f(x; k) = P(X = x)$
- k is called the parameter of the distribution.

Example

Experiment: tossing a balanced die.

- Sample space: $S = \{1,2,3,4,5,6\}$
- Each sample point of S occurs with the same probability 1/6.
- Let X= the number observed when tossing a balanced die.
- The probability distribution of X is:

$$f(x) = P(X = x) = f(x;6) = \begin{cases} \frac{1}{6} \ ; \ x = 1, 2, \cdots, 6 \\ 0 \ ; \ elsewhere \end{cases}$$

Theorem

If the discrete random variable X has a discrete uniform distribution with parameter k, then the mean and the variance of X are:

$$E(X) = \mu = \frac{\sum_{i=1}^{k} x_i}{k}$$

$$Var(X) = \sigma^2 = \frac{\sum_{i=1}^{k} (x_i - \mu)^2}{k}$$

Example

Find E(X) and Var(X) in the previous example

Solution:

$$E(X) = \mu = \frac{\sum_{i=1}^{k} x_i}{k} = \frac{1+2+3+4+5+6}{6} = 3.5$$

$$Var(X) = \sigma^2 = \frac{\sum_{i=1}^{k} (x_i - \mu)^2}{k} = \frac{\sum_{i=1}^{k} (x_i - 3.5)^2}{6}$$

$$= \frac{(1-3.5)^2 + (2-3.5)^2 + \cdots + (6-3.5)^2}{6} = \frac{35}{12}$$

Binomial Distribution

- **Bernoulli trial** is an experiment with only two possible outcomes.
- The two possible outcomes are labeled: success (s) and failure (f)
- The probability of success is $P(s) = p$ and the probability of failure is $P(f) = q = 1 - p$.

Examples:

- Tossing a coin (success=H, failure=T, and $p = P(H)$)
- Inspecting (فحص) an item (success=defective, failure=non-defective, and p =P(defective))

Bernoulli Process

Bernoulli process is an experiment that must satisfy the following properties:

1. The experiment consists of n repeated Bernoulli trials.
2. The probability of success, $P(s) = p$, remains constant from trial to trial.
3. The repeated trials are independent; that is the outcome of one trial has no effect on the outcome of any other trial

Binomial Random Variable

Consider the random variable :

X = The number of successes in the n trials in a Bernoulli process.

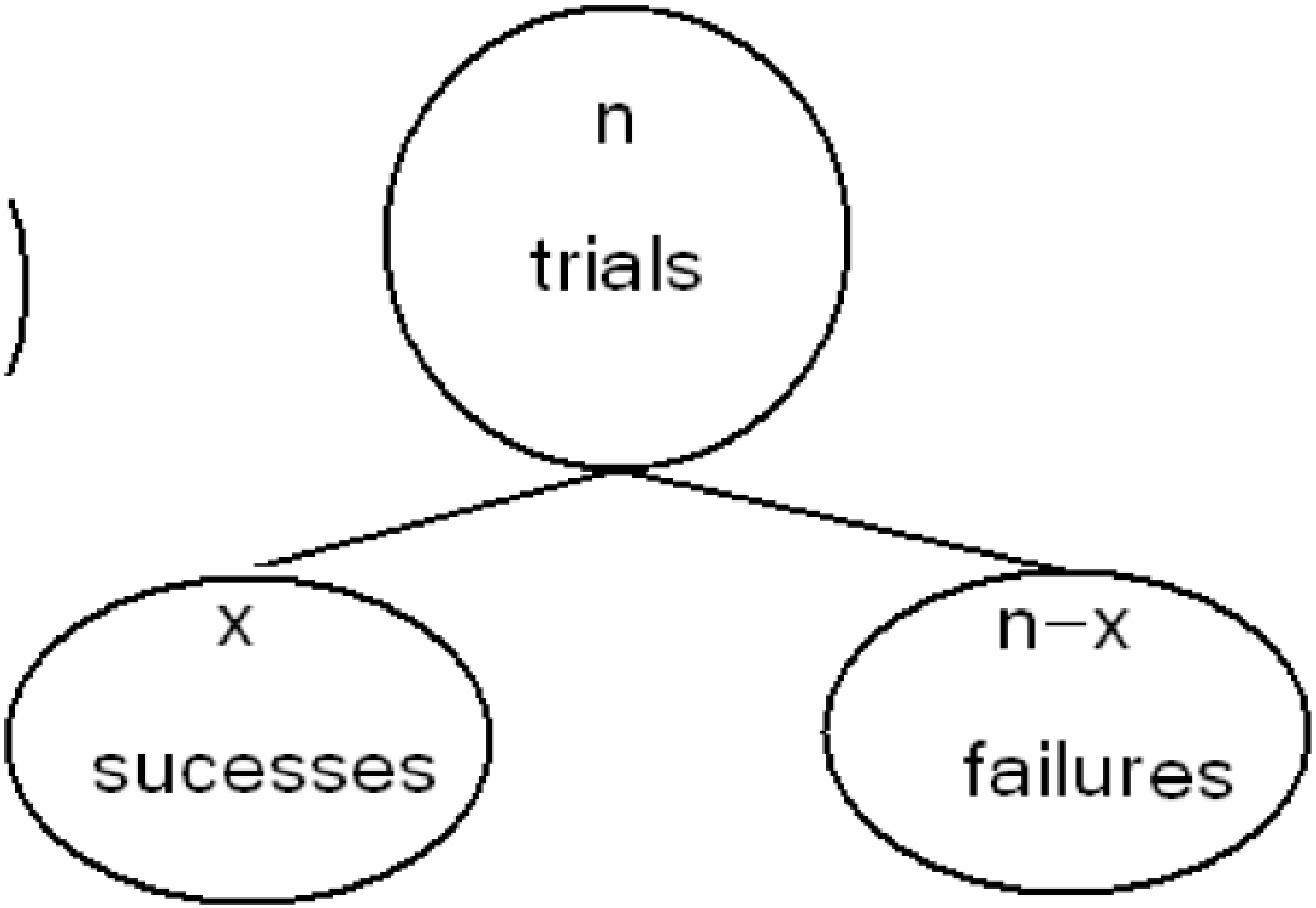
The random variable X has a binomial distribution with parameters n (number of trials) and p (probability of success), and we write:

$$X \sim Binomial(n, p) \text{ or } X \sim b(x; n, p)$$

The probability distribution of X is given by:

$$f(x) = P(X = x) = b(x;n,p) = \begin{cases} \binom{n}{x} p^{x} (1-p)^{n-x} \; ; & x = 0, 1, 2, \ldots, n \\ 0 \; ; & otherwise \end{cases}$$

$\binom{n}{x}$



Probability of successes

$p\,p\,\ldots\,p = p^x$

x times

Probability of failures

$(1-p)\ldots(1-p) = (1-p)^{n-x}$

(n−x) times

We can write the probability distribution of X as a table as follows.

x	f(x)=P(X=x)=b(x;n,p)
0	$\binom{n}{0} p^0 (1-p)^{n-0} = (1-p)^n$
1	$\binom{n}{1} p^1 (1-p)^{n-1}$
2	$\binom{n}{2} p^2 (1-p)^{n-2}$
$\vdots$	$\vdots$
$n-1$	$\binom{n}{n-1} p^{n-1} (1-p)^1$
n	$\binom{n}{n} p^n (1-p)^0 = p^n$
Total	1.00

Example:

Suppose that 25% of the products of a manufacturing process are defective. Three items are selected at random, inspected, and classified as defective (D) or non-defective (N). Find the probability distribution of the number of defective items.

Solution:

- **Experiment:** selecting 3 items at random, inspected, and classified as (D) or (N).
- The sample space is

S={DDD,DDN,DND,DNN,NDD,NDN,NND,NNN}

- Let X = the number of defective items in the sample
- We need to find the probability distribution of X.

(1) First Solution

Outcome	Probability	x
NNN	$\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}=\frac{27}{64}$	0
NND	$\frac{3}{4}\times\frac{3}{4}\times\frac{1}{4}=\frac{9}{64}$	1
NDN	$\frac{3}{4}\times\frac{1}{4}\times\frac{3}{4}=\frac{9}{64}$	1
NDD	$\frac{3}{4}\times\frac{1}{4}\times\frac{1}{4}=\frac{3}{64}$	2
DNN	$\frac{1}{4}\times\frac{3}{4}\times\frac{3}{4}=\frac{9}{64}$	1
DND	$\frac{1}{4}\times\frac{3}{4}\times\frac{1}{4}=\frac{3}{64}$	2
DDN	$\frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}=\frac{3}{64}$	2
DDD	$\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4}=\frac{1}{64}$	3

The probability distribution .of X is

.x	.f(x)=P(X=x)
0	$\frac{27}{64}$
1	$\frac{9}{64}+\frac{9}{64}+\frac{9}{64}=\frac{27}{64}$
2	$\frac{3}{64}+\frac{3}{64}+\frac{3}{64}=\frac{9}{64}$
3	$\frac{1}{64}$

(2) Second Solution

Bernoulli trial is the process of inspecting the item. The results are success=D or failure=N, with probability of success $P(s) = 25/100 = 1/4$ $= 0.25$.

The experiments is a Bernoulli process with:

- number of trials: $n = 3$
- Probability of success: $p = 1/4 = 0.25$
- $X \sim Binomial(n, p) = Binomial(3,1/4)$

The probability distribution of *X* is given by:

$$f(x)=P(X=x)=b(x;3,\frac{1}{4})=\begin{cases}\binom{3}{x}(\frac{1}{4})^{x}(\frac{3}{4})^{3-x}\ ; & x=0,1,2,3\\ 0\ ; & otherwise\end{cases}$$

$$f(0)=P(X=0)=b(0;3,\frac{1}{4})=\binom{3}{0}(\frac{1}{4})^0(\frac{3}{4})^3=\frac{27}{64}$$

$$f(1)=P(X=1)=b(1;3,\frac{1}{4})=\binom{3}{1}(\frac{1}{4})^1(\frac{3}{4})^2=\frac{27}{64}$$

$$f(2)=P(X=2)=b(2;3,\frac{1}{4})=\binom{3}{2}(\frac{1}{4})^2(\frac{3}{4})^1=\frac{9}{64}$$

$$f(3)=P(X=3)=b(3;3,\frac{1}{4})=\binom{3}{3}(\frac{1}{4})^3(\frac{3}{4})^0=\frac{1}{64}$$

The probability distribution of X is

x	f(x)=P(X=x) =b(x;3,1/4)
0	27/64
1	27/64
2	9/64
3	1/64

Theorem

The mean and the variance of the binomial distribution $b(x; n, p)$ are:

$$\mu = n\,p$$

$$\sigma^2 = n\,p\,(1 - p)$$

Example:

In the previous example, find the expected value (mean) and the variance of the number of defective items.

Solution:

We found that $X \sim Binomial(n,p) = Binomial(3,1/4)$

$$n = 3 \; and \; p = 1/4$$

The expected number of defective items is

$$E(X) = \mu = n\,p = (3)\,(1/4) = 3/4 = 0.75$$

The variance of the number of defective items is

$$Var(X) = \sigma^2 = n\,p\,(1 - p) = (3)\,(1/4)\,(3/4)$$

$$= 9/16 = 0.5625$$

Example:

In the previous example, find the following probabilities:

(1) The probability of getting at least two defective items.

(2) The probability of getting at most two defective items.

Solution:

$X \sim \text{Binomial}(3,1/4)$

$$f(x) = P(X=x) = b(x;3,\frac{1}{4}) = \begin{cases} \binom{3}{x} (\frac{1}{4})^x (\frac{3}{4})^{3-x} & for \ \ x = 0, 1, 2, 3 \\ 0 & otherwise \end{cases}$$

.x	.f(x)=P(X=x)=b(x;3,1/4)
0	27/64
1	27/64
2	9/64
3	1/64

(1) The probability of getting at least two defective items:

$$P(X \geq 2) = P(X = 2) + P(X = 3) = f(2) + f(3)$$

$$= \frac{9}{64} + \frac{1}{64} = \frac{10}{64}$$

(2) The probability of getting at most two defective item:

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

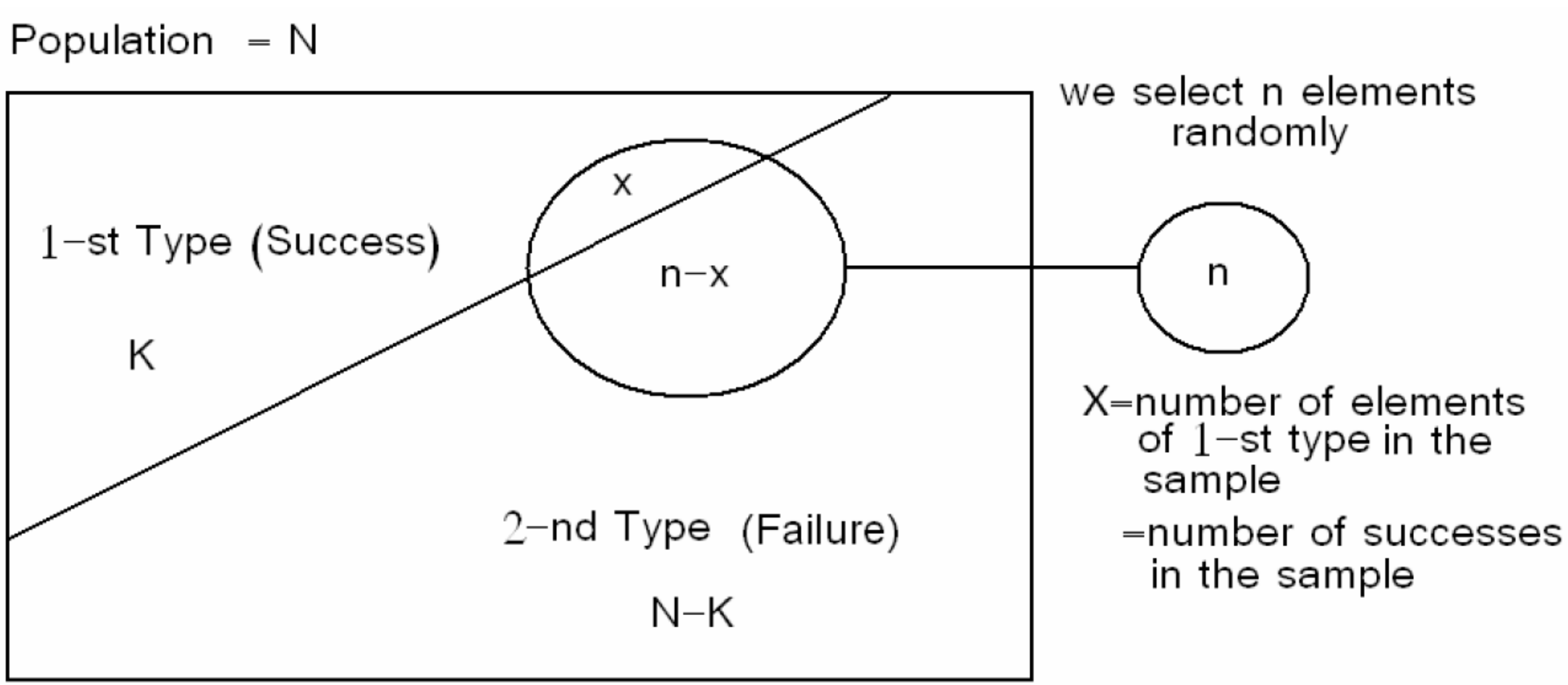
$$= f(0) + f(1) + f(2) = \frac{27}{64} + \frac{27}{64} + \frac{9}{64} = \frac{63}{64}$$

or

$$P(X \leq 2) = 1 - P(X > 2) = 1 - P(X = 3)$$

$$= 1 - f(3) = 1 - \frac{1}{64} = \frac{63}{64}$$

Hypergeometric Distribution



Suppose there is a population with 2 types of elements: 1-st Type = success 2-nd Type = failure

- N= population size • K= number of elements of the 1-st type
- $N-K$ = number of elements of the 2-nd type
- We select a sample of n elements at random from the population
- Let X = number of elements of 1-st type (number of successes) in the sample
- We need to find the probability distribution of X.

There are to two methods of selection:

1. selection with replacement
2. selection without replacement

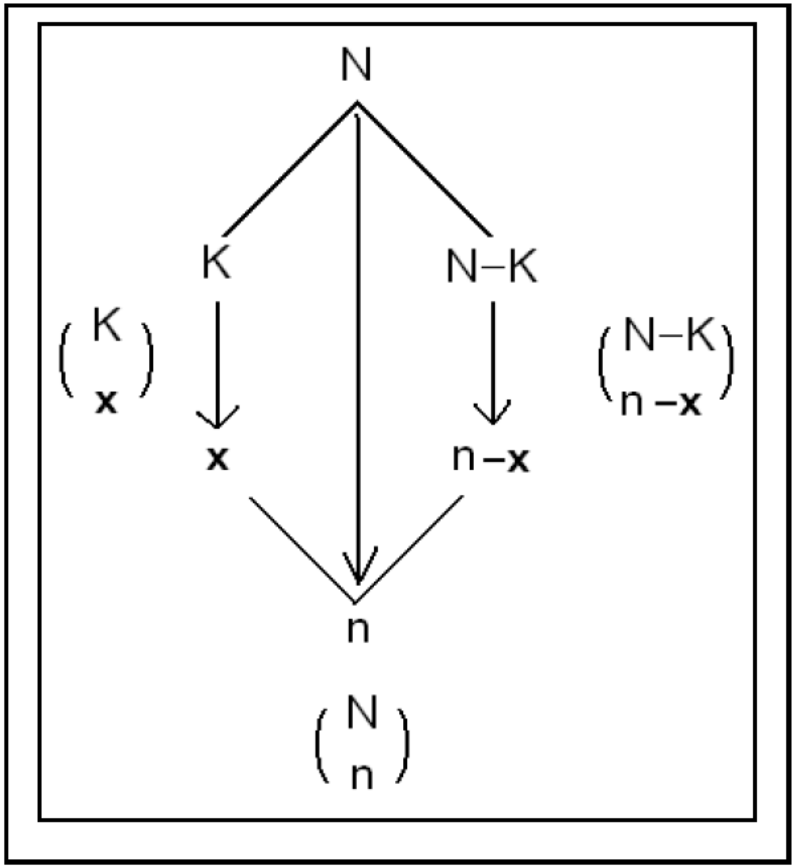
(1) If we select the elements of the sample at random and with replacement, then

$X \sim Binomial(n, p)$; where $p = \frac{k}{N}$

(2) Now, suppose we select the elements of the sample at random and without replacement. When the selection is made without replacement, the random variable X has a hypergeometric distribution with parameters N, n, and K. and we write $X{\sim}h(x; N, n, K)$.

$$f(x) = P(X = x) = h(x; N, n, K)$$

$$= \begin{cases} \dfrac{\dbinom{K}{x} \times \dbinom{N-K}{n-x}}{\dbinom{N}{n}}; & x = 0, 1, 2, \cdots, n \\ 0; \textit{ otherwise} \end{cases}$$



Note that the values of X must satisfy:

$$0 \leq x \leq K \; and \; 0 \leq n - x \leq \; N - K$$

$$\Leftrightarrow$$

$$0 \leq x \leq K \; and \; n - N + K \leq \; x \leq \; n$$

Example

Lots of 40 components each are called acceptable if they contain no more than 3 defectives. The procedure for sampling the lot is to select 5 components at random (without replacement) and to reject the lot if a defective is found. What is the probability that exactly one defective is found in the sample if there are 3 defectives in the entire lot.

Solution:

Let X= number of defectives in the sample

- $N = 40, K = 3, and\ n = 5$
- X has a hypergeometric distribution with

parameters $N = 40, n = 5, and\ K = 3.$

- $X \sim h(x; N, n, K) = h(x; 40,5,3).$
- The probability distribution of X is given by:

$$f(x) = P(X = x) = h(x;40,5,3) = \begin{cases} \dfrac{\binom{3}{x} \times \binom{37}{5-x}}{\binom{40}{5}}; & x = 0, 1, 2, \cdots, 5 \\ 0; & otherwise \end{cases}$$

But the values of X must satisfy:

$$\boldsymbol{0 \leq x \leq K \; and \; n - N + K \leq x \leq n \Leftrightarrow 0 \leq x \leq 3 \; and \; -32 \leq x \leq 5}$$

Therefore, the probability distribution of X is given by:

$$f(x) = P(X = x) = h(x;40,5,3) = \begin{cases} \dfrac{\binom{3}{x} \times \binom{37}{5-x}}{\binom{40}{5}}; & x = 0, 1, 2, 3 \\ 0; \ otherwise \end{cases}$$

Now, the probability that exactly one defective is found in the sample is

$$\text{f(1)=P(X=1)=h(1;40,5,3)=} \frac{\binom{3}{1} \times \binom{37}{5-1}}{\binom{40}{5}} = \frac{\binom{3}{1} \times \binom{37}{4}}{\binom{40}{5}} = 0.3011$$

Theorem

The mean and the variance of the hypergeometric distribution $h(x; N, n, K)$ are:

$$\mu = n\frac{k}{N}$$

$$\sigma^2 = n\frac{k}{N}\left(1 - \frac{k}{N}\right)\frac{N-n}{N-1}$$

Example

In the previous example, find the expected value (mean) and the variance of the number of defectives in the sample.

Solution:

- X = number of defectives in the sample
- We need to find $E(X)=\mu$ and $Var(X)=\sigma^2$
- We found that $X \sim h(x; 40,5,3)$
- $N = 40, n = 5, and\ K = 3$

The expected number of defective items is

$$E(X)=\mu = n\frac{K}{N} = 5\times\frac{3}{40} = 0.375$$

The variance of the number of defective items is

$$Var(X)=\sigma^2 = n\frac{K}{N}\left(1-\frac{K}{N}\right)\frac{N-n}{N-1} = 5\times\frac{3}{40}\left(1-\frac{3}{40}\right)\frac{40-5}{40-1} = 0.311298$$

Relationship to the binomial distribution

Binomial distribution	**Hypergeometric distribution**
$b(x;n,p)=\binom{n}{x}p^x(1-p)^{n-x};\ x=0,1,\ldots,n$	$h(x;N,n,K)=\dfrac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}};x=0,1,\cdots,n$

If n is small compared to N and K, then the hypergeometric distribution $h(x;N,n,K)$ can be approximated by the binomial distribution $b(x;n,p)$, where $p={}^{K}/_{N}$; i.e., for large N and K and small n, we have:

$$h(x;N,n,K) \approx b\left(x;n,\frac{K}{N}\right)$$

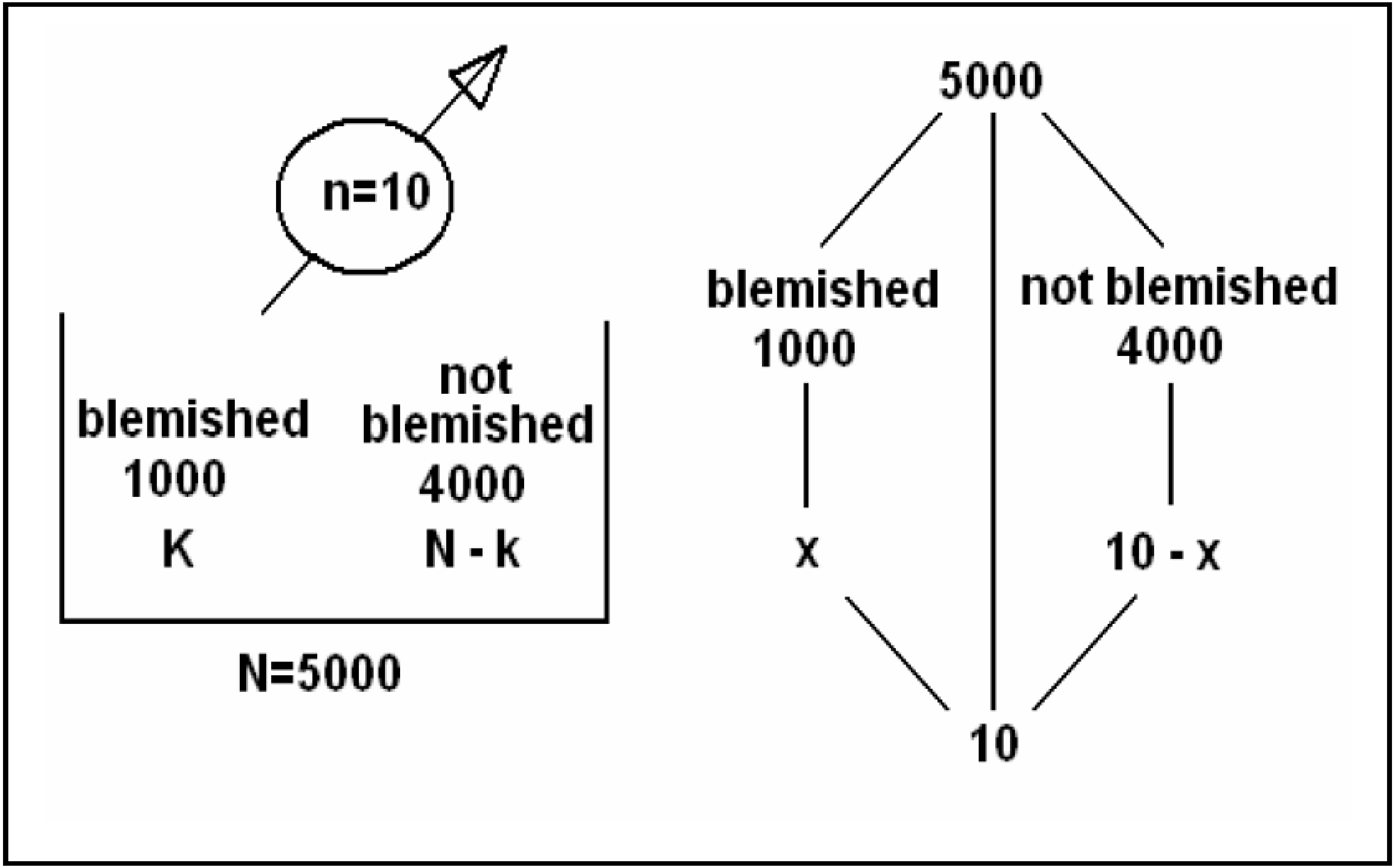
$$\frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \approx \binom{n}{x}\left(\frac{K}{N}\right)^{x}\left(1-\frac{K}{N}\right)^{n-x} \quad ;x = 0,1,\cdots,n$$

Note:

If n is small compared to N and K, then there will be almost no difference between selection without replacement and selection with replacement

$$\left(\frac{K}{N} \approx \frac{K-1}{N-1} \approx \cdots \approx \frac{K-n+1}{N-n+1}\right).$$

Example



n=10
blemished
1000
K
not
blemished
4000
N - k
N=5000
5000
blemished
1000
not blemished
4000
x
10 - x
10

Solution: (1)

N=5000   K=1000   n=10

X=Number of blemished tires in the Sample

X~h(x;5000,10,1000)

The exact probability is

$$P(X=3) = \binom{1000}{3}\binom{4000}{7} \Big/ \binom{5000}{10}$$

$$= \underline{0.201}477715$$

$$\approx \underline{0.201}$$

Solution: (2)

Since n=10 is small relative to N=5000 and K=4000, we can approximate the hypergeometric probabilities using binomial probabilities as follows:

.n=10 (no. of trials)

.p=K/N=1000/5000=0.2 (probability of success)

$$X \sim h(x;5000,10,1000) \approx b(x;10,0.2)$$

$$P(X=3) \approx \binom{10}{3}(0.2)^3\,(0.8)^7 = \underline{0.201}326592$$

$$\approx \underline{0.201}$$

Geometric Distribution

If repeated independent trials can result in a success with probability p and a failure with probability $q = 1 - p$, then the probability distribution of the random variable X, the number of the trial on which **the first success occurs**, is

$$g(x; p) = pq^{x-1}, \quad x = 1, 2, 3, \ldots.$$

Example

For a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?

***Solution*:** Using the geometric distribution with

$x = 5$ and $p = 0.01$, we have

$$g(5;\ 0.01) = (0.01)(0.99)^4 = 0.0096$$

Theorem

The mean and variance of a random variable following the geometric distribution are

$$\mu = \frac{1}{p} \text{ and } \sigma^2 = \frac{1-p}{p^2}$$

Poisson Distribution:

Poisson experiment is an experiment yielding numerical values of a random variable that count the number of outcomes occurring in a given time interval or a specified region denoted by t.

X = The number of outcomes occurring in a given time interval or a specified region denoted by t.

Examples:

1. X = number of field mice per acre ($t = 1$ acre)

2. X= number of typing errors per page ($t = 1$ page)

3. X=number of telephone calls received every day ($t = 1$ day)

4. X=number of telephone calls received every 5 days ($t = 5$ days)

- Let λ be the average (mean) number of outcomes per unit time or unit region (t = 1).
- The average (mean) number of outcomes (mean of X) in the time interval or region t is:

$$\boldsymbol{\mu = \lambda\, t}$$

The random variable X is called a Poisson random variable with parameter μ $(\mu = \lambda t)$, and we write $X \sim Poisson(\mu)$, if its probability distribution is given by:

$$f(x) = P(X = x) = p(x;\mu) = \begin{cases} \dfrac{e^{-\mu}\mu^{x}}{x!} & ; \quad x = 0, 1, 2, 3, \ldots \\ 0 & ; \quad otherwise \end{cases}$$

Theorem

The mean and the variance of the Poisson distribution $Poisson(x;\mu)$ are:

$$\mu = \lambda . t$$

$$\sigma^2 = \mu = \lambda . t$$

Note:

- λ is the average (mean) of the distribution in the unit time ($t = 1$).
- If X=The number of calls received in a month (unit time $t = 1$ month) and $X \sim Poisson(\lambda)$, then:

(i) Y = number of calls received in a year.

$$Y \sim Poisson\ (\mu);\ \mu = 12\lambda\ (t = 12)$$

(ii) W = number of calls received in a day.

$$W \sim Poisson\ (\mu);\ \mu = \lambda/30\ (t = 1/30)$$

Example:

Suppose that the number of typing errors per page has a Poisson distribution with average 6 typing errors.

(1) What is the probability that in a given page:

(i) The number of typing errors will be 7?

(ii) The number of typing errors will be at least 2?

(2) What is the probability that in 2 pages there will be 10 typing errors?

(3) What is the probability that in a half page there will be no typing errors?

Solution:

(1) X = number of typing errors per page.

$$X \sim Poisson\ (6)\ (t = 1, \lambda = 6, \mu = \lambda t = 6)$$

$$f(x) = P(X = x) = p(x;6) = \frac{e^{-6}6^x}{x!}\ ;\quad x = 0, 1, 2, \ldots$$

(i) $$f(7) = P(X = 7) = p(7;6) = \frac{e^{-6}6^7}{7!} = 0.13768$$

(ii) $P(X \geq 2) = P(X=2) + P(X=3) + \ldots = \sum_{x=2}^{\infty} P(X=x)$

$$P(X \geq 2) = 1 - P(X<2) = 1 - [P(X=0) + P(X=1)]$$

$$= 1 - [f(0) + f(1)] = 1 - \left[\frac{e^{-6}6^{0}}{0!} + \frac{e^{-6}6^{1}}{1!}\right]$$

$$= 1 - [0.00248 + 0.01487]$$

$$= 1 - 0.01735 = 0.982650$$

(2) X = number of typing errors in 2 pages

$X \sim Poisson(12)\ (t = 2, \lambda = 6, \mu = \lambda t = 12)$

$$f(x) = P(X = x) = p(x;12) = \frac{e^{-12}12^{x}}{x!} : \quad x = 0, 1, 2\ldots$$

$$f(10) = P(X = 10) = \frac{e^{-12}12^{10}}{10!} = 0.1048$$

(3) X = number of typing errors in a half page.

$X \sim Poisson\ (3)\ (t = 1/2, \lambda = 6, \mu = \lambda t = 6/2 = 3)$

$$f(x) = P(X = x) = p(x;3) = \frac{e^{-3}\ 3^{x}}{x!} : \quad x = 0\ , 1\ , 2\ldots$$

$$P(X = 0) = \frac{e^{-3}(3)^{0}}{0!} = 0.0497871$$

Poisson approximation for binomial distribution

Let X be a binomial random variable with probability distribution $b(x; n, p)$. If $n \rightarrow \infty, p \rightarrow 0$, and $\mu = np$ remains constant, then the binomial distribution $b(x; n, p)$ can be approximated by Poisson distribution $p(x; \mu)$.

- For large *n* and small *p* we have:

$$b(x; n, p) \approx Poisson(\mu)\ (\mu = np)$$

$$\binom{n}{x} p^x (1-p)^{n-x} \approx \frac{e^{-\mu} \mu^x}{x!}; x = 0, 1, \cdots, n; \quad (\mu = np)$$

Example

X = number of items producing bubbles in a random sample of 8000 items

$n = 8000$ and $p = 1/1000 = 0.001$

$$X \sim b(x; 8000, 0.001)$$

The exact probability is:

$$P(X<7)=P(X\leq 6)=\sum_{x=0}^{6}\binom{8000}{x}(0.001)^{x}(0.999)^{8000-x}=\ldots=\underline{0.313}252$$

The approximated probability using Poisson approximation:

$n = 8000$ (n is large, $i.e., n \to \infty$)

$p = 0.001$ (p is small, $i.e. p \to 0$)

$$\mu = np = 8000(0.001) = 8$$

$$X \approx Poisson\,(8)$$

$$f(x) = P(X = x) = p(x;8) = \frac{e^{-8}\, 8^x}{x!} : \qquad x = 0\,,1\,,2\ldots$$

$$P(X<7)=P(X\leq 6)= \sum_{x=0}^{6} \frac{e^{-8}\, 8^x}{x!} = e^{-8} \sum_{x=0}^{6} \frac{8^x}{x!} = \ldots = \underline{0.313}374$$