Lecture ④

1

Summation and Definite Integral

Summation Notation

For a given set of numbers $\{a_1, a_2, \dots, a_k, \dots a_n\}$

the sum is $\sum_{k=1}^{n} a_k = a_1 + a_2 + \dots + a_k + \dots + a_n$

EX Find $\sum_{k=1}^{3} (k+1)^2 k^3$

Ans:

$$\sum_{k=1}^{3} (k+1)^2 k^3 = (1+1)^2(1)^3 + (2+1)^2 2^3 + (3+1)^2 3^3$$

$$= 2^2 + 3^2(2^3) + 4^2(3^3) = 508$$

Theorem $\sum_{k=1}^{n} c = c + c + \dots + c = nc$

EX: $\sum_{i=1}^{50} 10 = 50(10) = 500$

theorem

(1) $\sum_{k=1}^{n} k = 1+2+3+\dots+n = \frac{n(n+1)}{2}$

(2) $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(3) $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

EX: $\sum_{k=1}^{100} k = 1+2+3+\dots+100 = \frac{100(100+1)}{2} = 5050$

$$\sum_{k=1}^{100} k^3 = 1^3 + 2^3 + 3^3 + \dots + 100^3 = \left[\frac{100(101)}{2}\right]^2$$

$$= (5050)^2 = 25,502,500$$

$$\sum_{k=1}^{100} k^2 = \frac{100(101)(201)}{6} = 338,350$$

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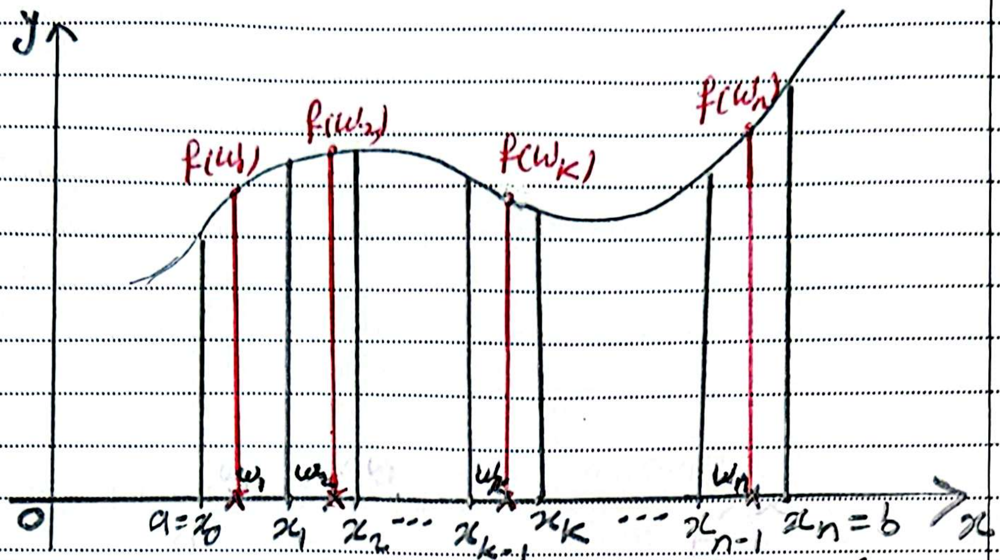
2)

Riemann Sum R_p

Let f be a function defined on a closed interval $[a,b]$ and let $P = \{x_0, x_1, x_2, \dots, x_k, \dots, x_n\}$ be a partition of $[a,b]$. A Riemann sum of f for P, R_p

is defined as $R_p = \sum_{k=1}^{n} f(\omega_k)\,\Delta x_k$,

where $\omega_k \in [x_{k-1}, x_k]$, $k = 1, 2, \dots, n$.



* **Note that** Δx_1 is the Length of 1^{st} subinterval $[a, x_1]$
Δx_2 " " " " 2^{nd} " $[x_1, x_2]$
...
Δx_k " " " " k^{th} " $[x_{k-1}, x_k]$
...
Δx_n " " " " n^{th} " $[x_{n-1}, b]$

} the largest length of these sub-intervals is called the norm of the partition P and is denoted by $\|P\|$.

For a regular partition of a closed interval $[a,b]$

$$\Delta x_k = \Delta x = \frac{b-a}{n}$$

and $x_k = a + k\Delta x$

$$x_k = a + \left(\frac{b-a}{n}\right)k, \quad k = 0, 1, 2, \dots, n$$

Defn: Definite Integral

Let f be a function defined on a closed interval $[a,b]$, the definite Integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{\substack{\|P\|\to 0 \\ \text{or } n\to\infty}} \sum_k f(\omega_k)\,\Delta x_k, \text{ provided the limit exists}$$

i.e. f is integrable on $[a,b]$.

3

- For simplicity, we can find the definite Integral as follows.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x,$$

where $\omega_k = x_k$ and $\Delta x_k = \Delta x$.

Ex ① Evaluate

$\int_0^4 (x+1)\,dx$ by using Riemann sum.

Ans:

$$\because R_n = \sum_{k=1}^{n} f(x_k)\,\Delta x,$$

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}, \text{ and } x_k = a + k\Delta x = 0 + \frac{4k}{n} = \frac{4k}{n}$$

$$\therefore R_n = \sum_{k=1}^{n} \left(\frac{4k}{n}+1\right)\frac{4}{n}$$

$$R_n = \sum_{k=1}^{n} \left(\frac{16k}{n^2} + \frac{4}{n}\right)$$

$$R_n = \frac{16}{n^2}\sum_{k=1}^{n} k + \frac{4}{n}\sum_{k=1}^{n} 1$$

$$R_n = \frac{16}{n^{2}}\left[\frac{n(n+1)}{2}\right] + \frac{4}{\cancel{n}}\cancel{n}$$

$$\therefore R_n = 8\left(\frac{n+1}{n}\right) + 4$$

$$\because \int_0^4 (x+1)\,dx = \lim_{n\to\infty} R_n$$

$$\therefore \int_0^4 (x+1)\,dx = 8 + 4 = 12 \quad \text{where } \lim_{n\to\infty} \frac{n+1}{n} = 1$$

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