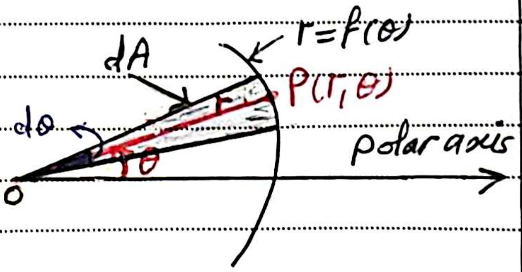
Lecture (31) Areas and lengths in polar Coordinates

The area differential dA for the curve $r=f(\theta)$ is

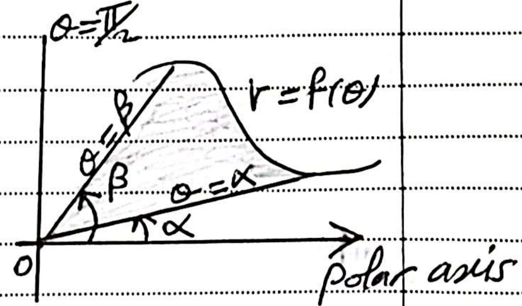
$$dA=\frac{1}{2}r^2\,d\theta=\frac{1}{2}[f(\theta)]^2\,d\theta$$



*** Definition (1)**

Area of the Fan-shaped region between the origin (pole) and the curve $r=f(\theta)$ when $\alpha\le\theta\le\beta$, $r\geq 0$ and $\beta-\alpha\le 2\pi$ is

$$A=\int_{\alpha}^{\beta}\frac{1}{2}r^2\,d\theta=\int_{\alpha}^{\beta}\frac{1}{2}[f(\theta)]^2\,d\theta$$

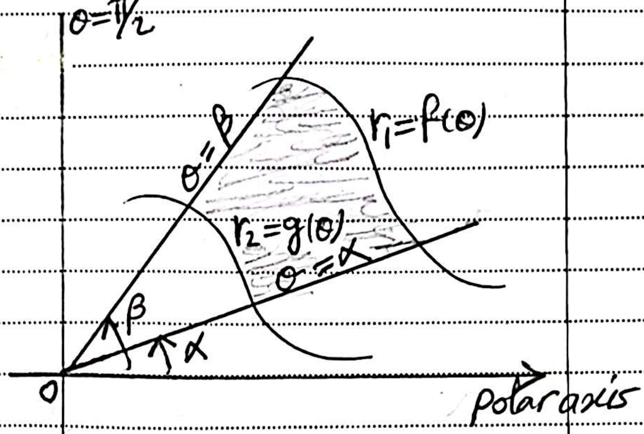


*** Definition (2)**

Area of the region $r_1=f(\theta)$, $r_2=g(\theta)$, $\theta=\alpha$ and $\theta=\beta$ as in the opposite fig. is given by

$$A=\frac{1}{2}\int_{\alpha}^{\beta}(r_1^2-r_2^2)\,d\theta$$

i.e. $$A=\frac{1}{2}\int_{\alpha}^{\beta}\left([f(\theta)]^2-[g(\theta)]^2\right)d\theta$$



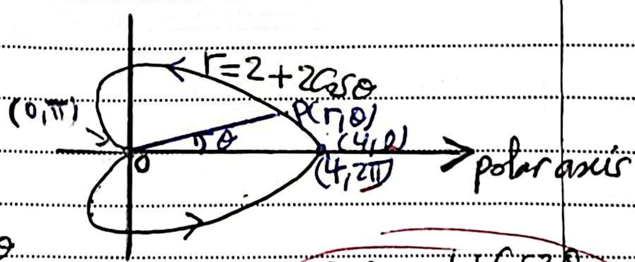
EX (1) Find the area bounded by the graph of the polar Equation $r=2(1+\cos\theta)$

Ans:

the curve $r=2+2\cos\theta$ is the cardioid as in fig.

The area is

$$A=\int_0^{2\pi}\frac{1}{2}r^2\,d\theta$$



$$A=\frac{1}{2}\int_0^{2\pi}(2+2\cos\theta)^2\,d\theta$$

$$A=\frac{1}{2}\int_0^{2\pi}(4+8\cos\theta+4\cos^2\theta)\,d\theta$$

$$\cos^2\theta=\frac{1+\cos 2\theta}{2}$$

or $2\left(\frac{1}{2}\right)\int_0^{\pi}(4+8\cos\theta+4\cos^2\theta)\,d\theta$

$$A = \frac{1}{2}\int_0^{2\pi}(4 + 8\cos\theta + 2 + 2\cos 2\theta)\,d\theta$$

$$A = \int_0^{2\pi}(3 + 4\cos\theta + \cos 2\theta)\,d\theta$$

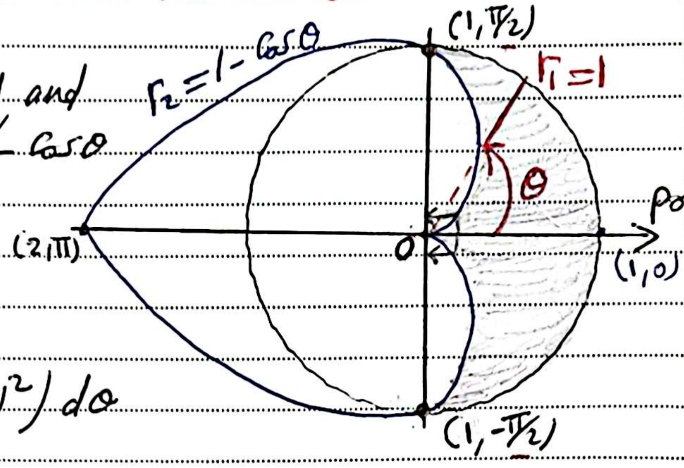
$$\therefore A = \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi$$

EX ② Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

Ans:

The outer curve is $r_1 = 1$ and the inner curve is $r_2 = 1 - \cos\theta$

The required area is



$1 - \cos\theta = 1$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \pm\frac{\pi}{2}$

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(r_1^2 - r_2^2)\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left(1 - (1 - \cos\theta)^2\right)d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[(\cancel{1} - \cancel{1} + 2\cos\theta - \cos^2\theta\right)d\theta$$

$$= \cancel{2}\left(\frac{1}{\cancel{2}}\right)\int_0^{\pi/2}\left(2\cos\theta - \frac{1 + \cos 2\theta}{2}\right)d\theta$$

$$= \int_0^{\pi/2}\left(2\cos\theta - \frac{1}{2} - \frac{\cos 2\theta}{2}\right)d\theta$$

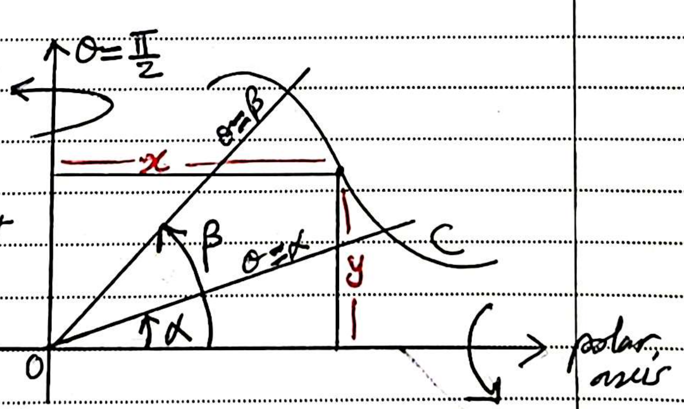
$$\therefore A = \left[2\sin\theta - \frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = (2 - \pi/4) - 0 = 2 - \frac{\pi}{4}.$$

* Surface of revolution in polar coordinates

$$S.A = \int_{\theta=\alpha}^{\theta=\beta} 2\pi y\,ds$$ by revolving about the polar axis.

$$S.A = \int_{\theta=\alpha}^{\theta=\beta} 2\pi x\,ds$$ by revolving about line $\theta = \pi/2$

where $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$, $x = r\cos\theta$ and $y = r\sin\theta$.



Ex ③ Find the area of the surface generated by revolving the graph of $r = 2a\sin\theta$ about the line $\theta = \frac{\pi}{2}$.

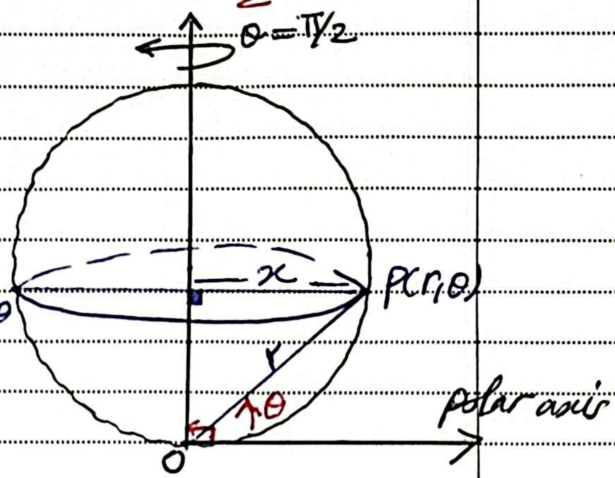
Ans:

$$S.A = \int_{\theta=\alpha}^{\theta=\beta} 2\pi x \, ds$$

$$= 2\pi \int_0^{\pi/2} r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

$$= 2\pi \int_0^{\pi/2} 2a\sin\theta\cos\theta \sqrt{4a^2\sin^2\theta + 4a^2\cos^2\theta} \, . \, d\theta$$

$$= 8\pi a^2 \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$$



$$\therefore S.A = 8\pi a^2 \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = 4\pi a^2$$

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HW

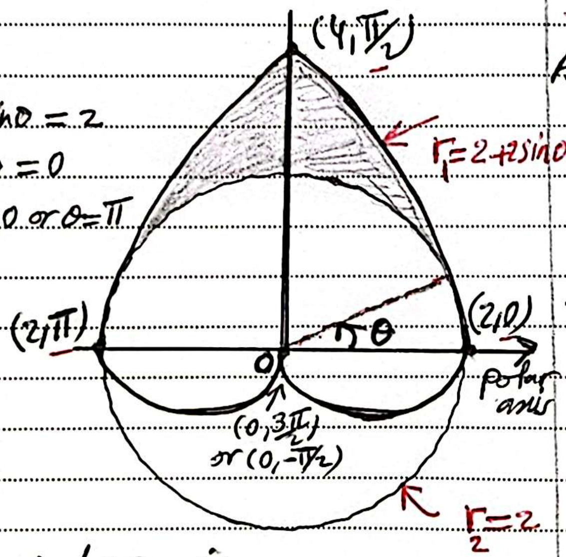
Ex ④ Find the area of the region that lies inside the curve $r = 2 + 2\sin\theta$ (cardioid) and outside the circle $r = 2$.

Ans:

$2 + 2\sin\theta = 2$

$\sin\theta = 0$

$\Rightarrow \theta = 0$ or $\theta = \pi$



The required area is

$$A = \frac{1}{2} \int_0^{\pi} (r_1^2 - r_2^2) \, d\theta$$

or $$A = \cancel{2}\left[\cancel{\frac{1}{2}} \int_0^{\pi/2} (r_1^2 - r_2^2) \, d\theta\right]$$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

Ans: $8 + \pi$

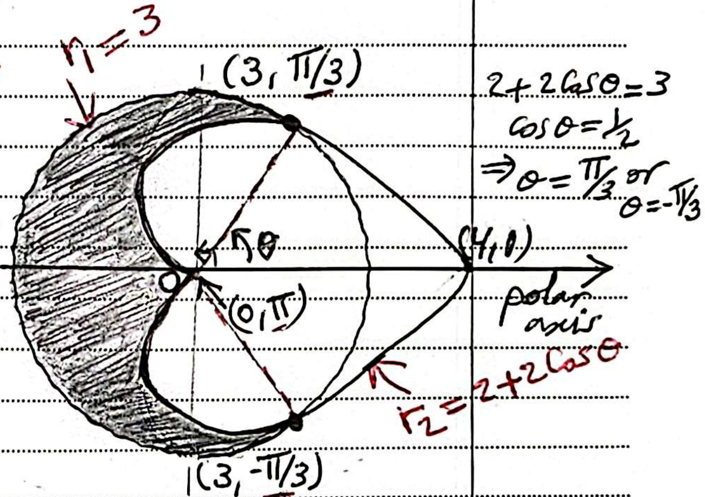
Ex ⑤ Find the area of the region that lies inside the curve $r = 3$ (circle) and outside the curve $r = 2 + 2\cos\theta$ (cardioid).

Ans:

$2 + 2\cos\theta = 3$

$\cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$ or $\theta = -\frac{\pi}{3}$



The required area is

$$A = \frac{1}{2} \int_{-\pi/3}^{\pi/3} (r_1^2 - r_2^2) \, d\theta$$

or $$A = \cancel{2}\left[\cancel{\frac{1}{2}} \int_{\pi}^{\pi/3} (r_1^2 - r_2^2) \, d\theta\right]$$

Ans: $2\pi + \frac{9\sqrt{3}}{2}$