Chapter # 6:
The Free Electron Model

Lecture 3: The Free Electron Model: Quantum Description (Sommerfeld Model)

Fermi Gas, Energy Levels, and Density of States

In the previous lecture, we developed the Sommerfeld model and derived the density of states and the Fermi energy for free electrons in metals.

In this lecture, we will use these concepts to study how electrons are distributed among energy levels and how they contribute to measurable physical properties. In particular, we will calculate the electronic heat capacity, electrical conductivity, and thermal conductivity, and compare the results with the predictions of the classical model.

This lecture shows how quantum statistics successfully explains the thermal and electrical behavior of metals.

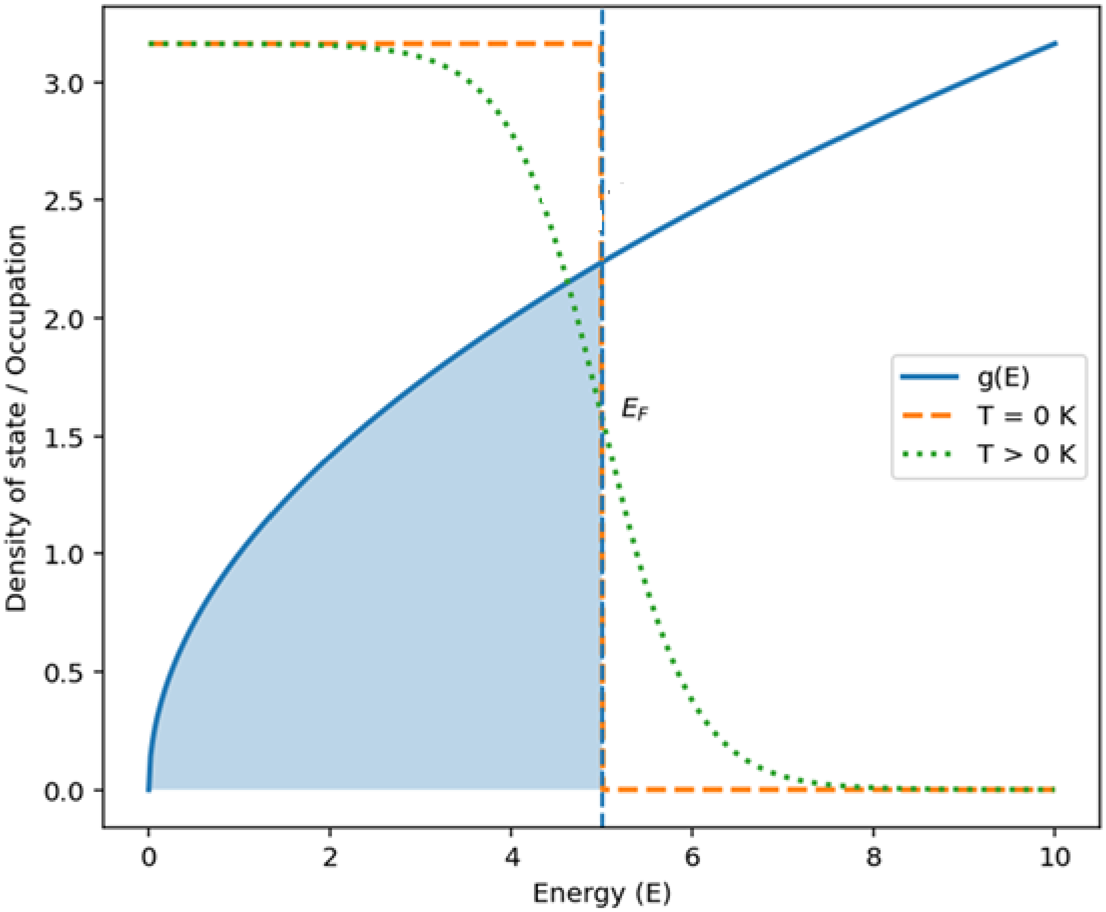
6-8 Fermi–Dirac Distribution

Option:

The distribution is named after Enrico Fermi and Paul Dirac because both scientists independently developed the quantum statistical theory describing particles that obey the Pauli exclusion principle.

Fermi introduced the statistical method for describing the behavior of electrons in a gas of identical particles, while Dirac independently developed the same statistical formulation using quantum mechanics and helped establish its theoretical foundation.

Their combined work led to what is now known as Fermi–Dirac statistics, which applies to particles called fermions, such as electrons, protons, and neutrons.



The distribution of electrons over energy levels is described by the distribution function f(E), which represents the probability that a state of energy E is occupied.

- f(E) → 1 level is occupied
- f(E) → 0 level is empty

At absolute zero temperature:

$$f(E)=1 \quad \text{when} \quad E < E_F$$
$$f(E)=0 \quad \text{when} \quad E > E_F \qquad (1)$$

However, when the temperature is greater than zero (T>0), the distribution function is given by the following relation:

$$f(E) = \frac{1}{e^{(E-E_F)/kT}+1} \qquad (2)$$

At $E=E_F$:

$$f(E) = \frac{1}{2}$$

The figure above illustrates the density of states g(E) together with the Fermi–Dirac distribution at T=0 K.

The blue curve represents the density of states g(E), which gives the number of available electronic states at each energy. We observe that g(E) increases with energy, since:

$$g(E) \propto \sqrt{E}$$

This means that the number of available states becomes larger at higher energies.

At absolute zero temperature (T=0 K), represented by the dashed orange curve, all states below the Fermi energy E_F are completely occupied, while all states above E_F are empty. Therefore, the occupation probability changes abruptly at the Fermi level.

When the temperature becomes greater than zero (T>0 K), represented by the dotted green curve, the sharp transition near the Fermi energy becomes smooth. In this case, some electrons near E_F absorb thermal energy and are excited to states above the Fermi level, leaving behind unoccupied states below E_F.

The figure shows that this change occurs only within a narrow energy region of order k_BT around the Fermi energy. Electrons far below E_F remain occupied, while states far above E_F remain nearly empty.

Since $k_BT \ll E_F$ at ordinary temperatures, only a very small fraction of electrons are thermally excited. Therefore, only electrons near the Fermi level contribute significantly to thermal and transport properties such as heat capacity, electrical conductivity, and thermal conductivity. This explains why the electronic heat capacity is much smaller than the classical prediction.

6-9 Heat Capacity of Fermi Gas

It is possible to use the distribution function given in equation (2) to calculate the thermal energy and, consequently, the heat capacity; however, this requires more mathematical effort. Instead, we will attempt to obtain a good approximation with minimal mathematical work.

From the Figure, we observe that the excited electrons have energies on the order of k_BT and are located near the Fermi level. Therefore, we can say that only a fraction of electrons of order k_BT/E_F are affected by an increase in temperature.

Accordingly, the number of excited electrons per mole is approximately $N_A(k_BT/E_F)$. If each electron absorbs an average energy of order kT , then the total thermal energy per mole is:

$$E = N_A(kT/E_F)kT = \frac{N_A(kT)^2}{E_F} \qquad (3)$$

Thus, the heat capacity is given by:

$$C = \frac{\partial E}{\partial T} = \frac{2N_A k^2 T}{E_F} = 2R\frac{kT}{E_F} \qquad (4)$$

In Ordinary Metals, usually E_F=5 eV, T=300 K:
$kT/E_F = 1/200$ agrees with experiment.

Thus, Using $E_F=kT_F$, equation (4) becomes

$$C = 2R\frac{T}{T_F} \qquad (5)$$

Since $kT \ll E_F$ at ordinary temperatures, only a small fraction of electrons are thermally excited.

Thus, we find that the electronic heat capacity is reduced from its classical value by a factor of $\frac{T}{T_F}$

Typically, the Fermi temperature corresponding to E_F=5 eV is about $T_F \approx 60{,}000$ K.

This means the metal would need to be heated to a temperature close to T_F for its heat capacity to reach its classical value. However, this is not possible, since the metal would have already melted and even evaporated before reaching such temperatures.

Therefore, the electronic heat capacity is much smaller than the classical value. The actual expression for the heat capacity is:

$$C = \frac{\pi^2 N_A k^2 T}{2E_F} \qquad (6)$$

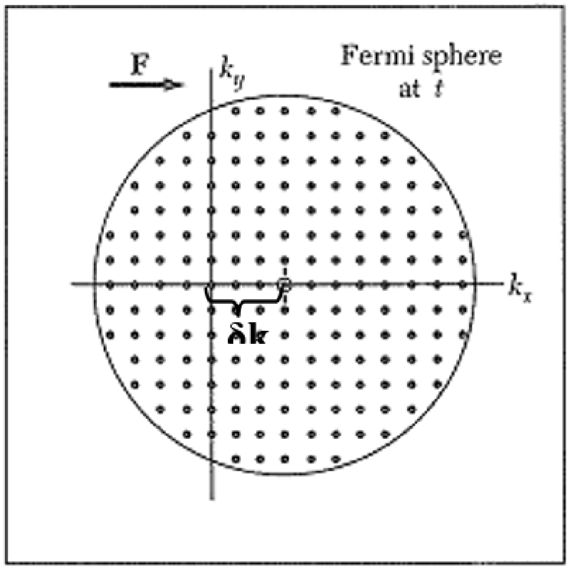
In conclusion, the Sommerfeld model predicts a much smaller electronic heat capacity, in good agreement with experimental observations, unlike the classical theory which greatly overestimates it.

6-10 Electrical Conductivity

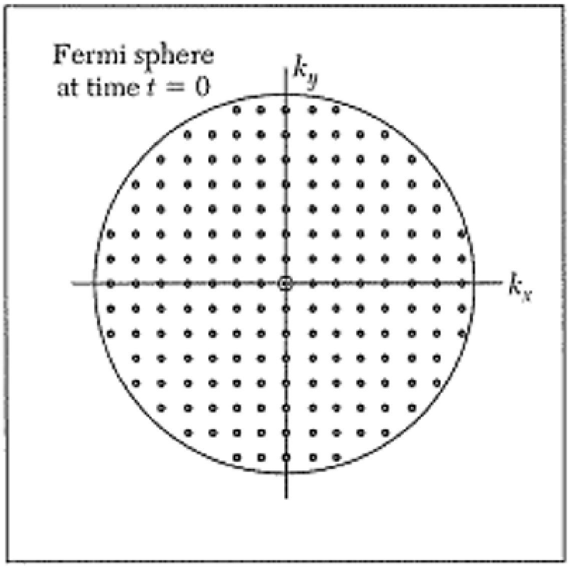
As shown in the figure, each electron carries momentum and contributes to the current. Nevertheless, the total current in equilibrium is zero because electrons moving with velocity v are balanced by electrons moving with velocity −v.

When an external electric field is applied, this symmetry is disturbed, leading to a net electric current.

E≠0



E=0



Therefore, the force exerted by the electric field is given by:

$$\vec{F} = \frac{d\vec{p}}{dt} = \hbar\frac{d\vec{k}}{dt} = -e\vec{E} \qquad (7)$$

In the absence of scattering, the Fermi sphere shifts uniformly in k-space under the influence of the applied electric field. Integrating equation (7), we obtain:

$$\hbar(k(t) - k(t=0)) = -eEt \qquad (8)$$

Hence, the magnitude of the displacement of the Fermi sphere is given by:

$$\delta\vec{k} = \frac{-eEt}{\hbar} \qquad (9)$$

Due to collisions of electrons with impurities inside the crystal, as well as with phonons, the displaced Fermi sphere reaches a steady-state condition under the influence of the applied electric field.

If the relaxation time is τ, then the displacement of the Fermi sphere in the steady state is obtained from equation (9) by setting t=τ, that is:

$$\delta\vec{k} = \frac{-e\tau}{\hbar}\vec{E} \qquad (10)$$

Thus, the drift velocity is given by:

$$\delta \vec{v} = \frac{-e\tau}{m}\vec{E} \qquad (11)$$

If we assume that the metal contains n electrons per unit volume, then the current density is given by the following relation:

$$\vec{J} = -ne\delta\vec{v} = \frac{ne^2\tau}{m}\vec{E} \qquad (12)$$

Thus, the electrical conductivity is given by:

$$\sigma = \frac{ne^2\tau_F}{m} \qquad (13)$$

Here, τ_F denotes the relaxation time of electrons near the Fermi surface, as these are the electrons that respond to the applied electric field. This expression is analogous to equation (7) obtained from the classical Drude model.

In the Sommerfeld model, only electrons near the Fermi surface contribute to conduction; therefore, the relevant relaxation time is denoted by τ_F.

6-11 Thermal Conductivity

For the thermal conductivity, equation (13) -last lecture- will be used again, with the heat capacity C replaced by the expression given in equation (6), as follows:

$$K = \frac{1}{3}Cvl = \frac{1}{3}\left(\frac{\pi^2 N_A k^2 T}{2E_F}\right)v_F l_F \qquad (14)$$

Since: $$E_F = \frac{1}{2}m\mathrm{v}_F^2 \quad \text{and} \quad \tau_F = \frac{l_F}{\mathrm{v}_F}$$

Substituting these relations into equation (14), we obtain:

$$K = \frac{1}{3}\frac{\pi^2 N_A k^2 T}{m}\tau_F \qquad (15)$$

Thermal conductivity depends on:

- temperature
- relaxation time
- electron transport

This leads to what is known as the Wiedemann–Franz law.

Combining the two equations gives:

$$\frac{K}{\sigma T} = constant$$

This is the **Wiedemann–Franz law**.

Good electrical conductors are also good thermal conductors.