Chapter 3: Random Variables and Probability Distributions

3.1 Concept of a Random Variable:

In a statistical experiment, it is often very important to allocate numerical values to the outcomes.

Example:

- Experiment: testing two components.

D=defective (معيب),

N=non-defective (غير معيب)

- Sample space: S={DD,DN,ND,NN}
- Let X = number of defective components when two components are tested.

Assigned numerical values to the outcomes are:

Sample point (Outcome)	Assigned Numerical Value (x)
DD	2
DN	1
ND	1
NN	0

Notice that, the set of all possible values of the random variable X is {0, 1, 2}.

Definition 3.1:

A random variable X is a function that associates each element in the sample space with a real number (i.e., $X : S \rightarrow \mathbf{R}$.)

Notation:

"X" denotes the random variable .

"x" denotes a value of the random variable X.

Types of Random Variables:

- A random variable X is called a **discrete** random variable if its set of possible values is countable, i.e.,

$x \in \{x_1, x_2, \ldots, x_n\}$ or $x \in \{x_1, x_2, \ldots\}$

- A random variable X is called a **continuous** random variable if it can take values on a continuous scale, i.e.,

.$x \in \{x: a < x < b;\ a, b \in R\}$

3.2 Discrete Probability Distributions

A discrete random variable X assumes each of its values with a certain probability.

Example:

Experiment: tossing a non-balance coin 2 times independently.

- H= head , T=tail
- Sample space: S={HH, HT, TH, TT}
- Suppose P(H)=1/3 and P(T)=2/3

Let X= number of heads

Sample point (Outcome)	Probability	Value of X (x)
HH	P(HH)=P(H) P(H)=1/3x1/3 = 1/9	2
HT	P(HT)=P(H) P(T)=1/3x2/3 = 2/9	1
TH	P(TH)=P(T) P(H)=2/3x1/3 = 2/9	1
TT	P(TT)=P(T) P(T)=2/3x2/3 = 4/9	0

- The possible values of X are: 0, 1, and 2.
- X is a discrete random variable.
- Define the following events:

Event (X=x)	Probability = P(X=x)
(X=0)={TT}	P(X=0) = P(TT)=4/9
(X=1)={HT,TH}	P(X=1) =P(HT)+P(TH)=2/9+2/9=4/9
(X=2)={HH}	P(X=2) = P(HH)= 1/9

The possible values of X with their probabilities are:

x	$P(X = x) = f(x)$
0	$^{4}/_{9}$
1	$^{4}/_{9}$
2	$^{1}/_{9}$
Total	1

The function f(x)=P(X=x) is called the probability function (probability distribution) of the discrete random variable X.

Definition

The function f(x) is a probability function of a discrete random variable X if, for each possible values x, we have:

- $f(x) \geq 0$
- $\sum_{all\ x} f(x) = 1$
- $f(x) = P(X = x)$

Example:

For the previous example, we have:

x	$P(X = x) = f(x)$
0	$^4/_9$
1	$^4/_9$
2	$^1/_9$
Total	1

$P(X<1) = P(X=0) = 4/9$

$P(X\leq 1) = P(X=0) + P(X=1) = 4/9 + 4/9 = 8/9$

$P(X\geq 0.5) = P(X=1) + P(X=2) = 4/9 + 1/9 = 5/9$

$P(X>8) = P(\phi) = 0$

$P(X<10) = P(X=0) + P(X=1) + P(X=2) = P(S) = 1$

Note

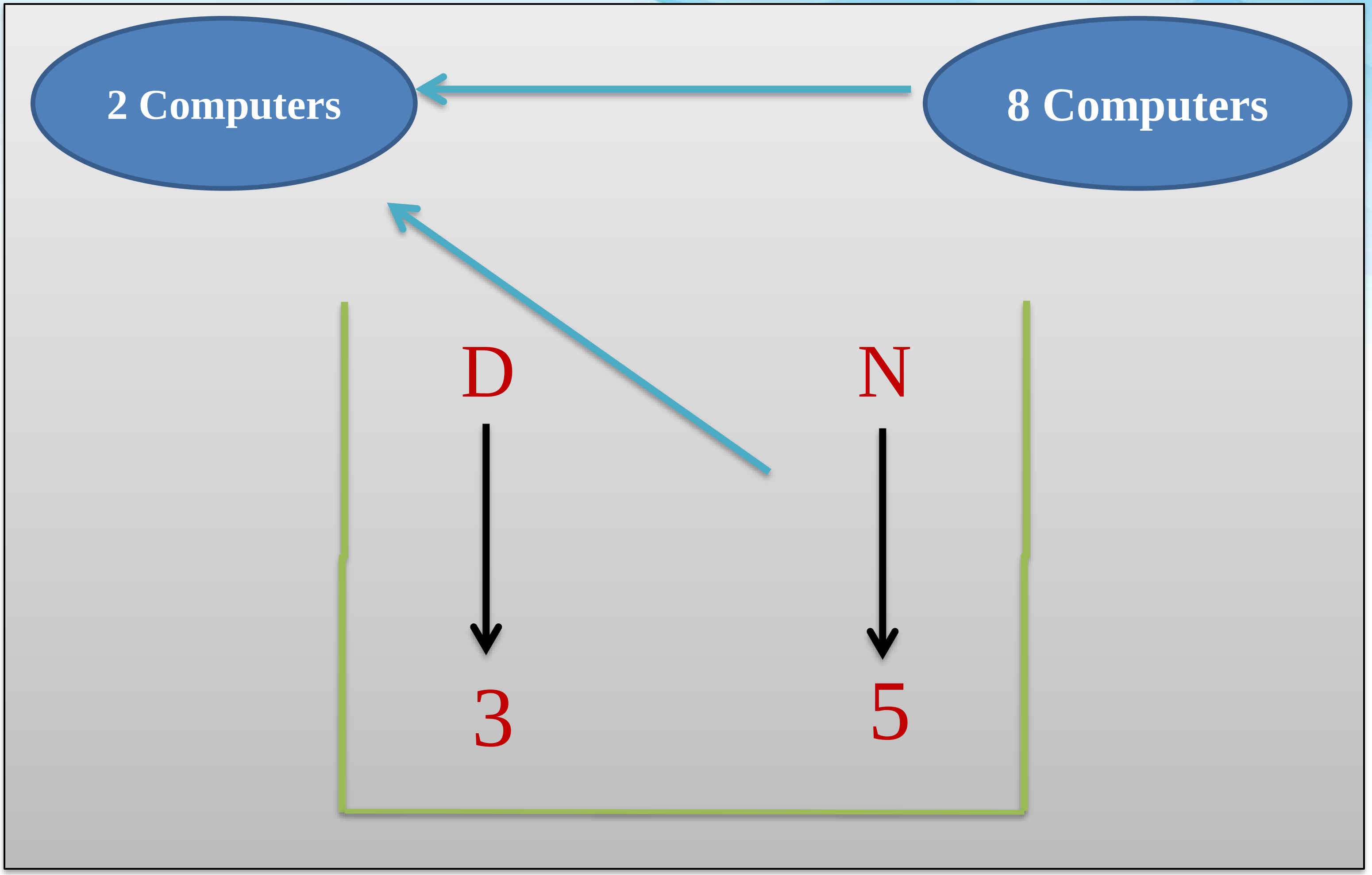
If X is a descrete random variable then

$$P(X < a) \neq P(X \leq a)$$

Example

A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective and 5 are non-defective. If a school makes a random purchase of 2 of these computers, find the probability distribution of the number of defectives.

Answer



Solution:

We need to find the probability distribution of the random variable: X = the number of defective computers purchased.

Experiment: selecting 2 computers at random out of 8

$$n(S) = \binom{8}{2} \text{ equally likely outcomes}$$

The possible values of X are: $x=0, 1, 2$.

Consider the events:

$$(X=0)=\{0D \text{ and } 2N\} \Rightarrow n(X=0)=\binom{3}{0}\times\binom{5}{2}$$

$$(X=1)=\{1D \text{ and } 1N\} \Rightarrow n(X=1)=\binom{3}{1}\times\binom{5}{1}$$

$$(X=2)=\{2D \text{ and } 0N\} \Rightarrow n(X=2)=\binom{3}{2}\times\binom{5}{0}$$

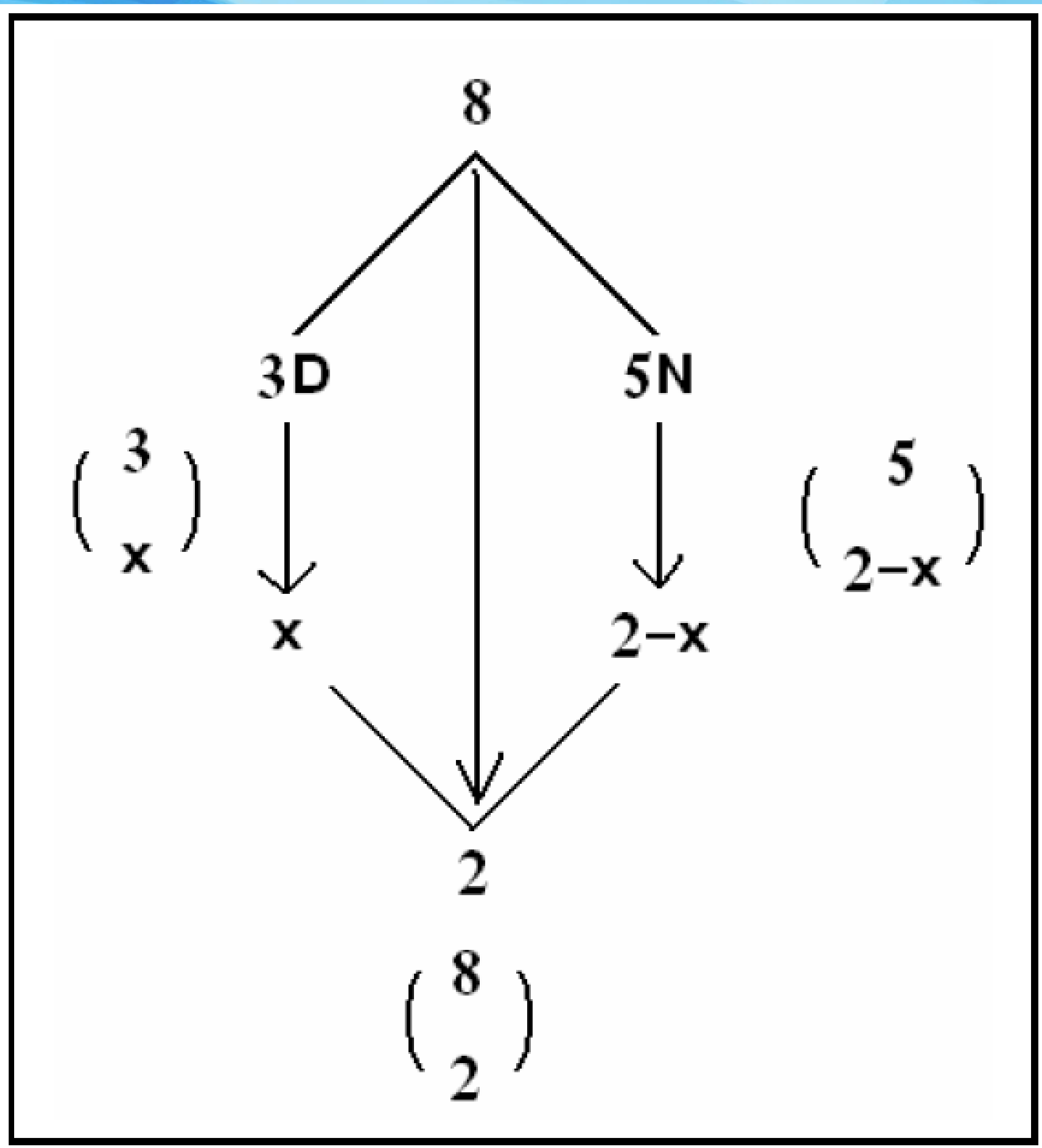
$$f(0)=P(X=0)=\frac{n(X=0)}{n(S)}=\frac{\binom{3}{0}\times\binom{5}{2}}{\binom{8}{2}}=\frac{10}{28}$$

$$f(1)=P(X=1)=\frac{n(X=1)}{n(S)}=\frac{\binom{3}{1}\times\binom{5}{1}}{\binom{8}{2}}=\frac{15}{28}$$

$$f(2)=P(X=2)=\frac{n(X=2)}{n(S)}=\frac{\binom{3}{2}\times\binom{5}{0}}{\binom{8}{2}}=\frac{3}{28}$$

In general, for x=0,1, 2, we have:

$$f(x)=P(X=x)=\frac{n(X=x)}{n(S)}=\frac{\binom{3}{x}\times\binom{5}{2-x}}{\binom{8}{2}}$$


8
3D
5N
$\binom{3}{x}$
x
2−x
$\binom{5}{2-x}$
2
$\binom{8}{2}$

The probability distribution of X can be given in the following table

x	0	1	2	Total
f(x)= P(X=x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$	1.00

The probability distribution of X can be written as a formula as follows:

$$f(x) = P(X = x) = \begin{cases} \dfrac{\binom{3}{x} \times \binom{5}{2-x}}{\binom{8}{2}}; & x = 0, 1, 2 \\ 0; \ otherwise \end{cases}$$

Hypergeometric Distribution

Another Example: see Example 3.8 page 84

Definition 3.5:

The cumulative distribution function (CDF), F(x), of a discrete random variable X with the probability function f(x) is given by:

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \qquad -\infty < x < \infty$$

Example:

Find the CDF of the random variable X with the probability function:

x	0	1	2
f(x)	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

Solution:

$$F(x) = P(X \le x) \quad for \;\; -\infty < x < \infty$$

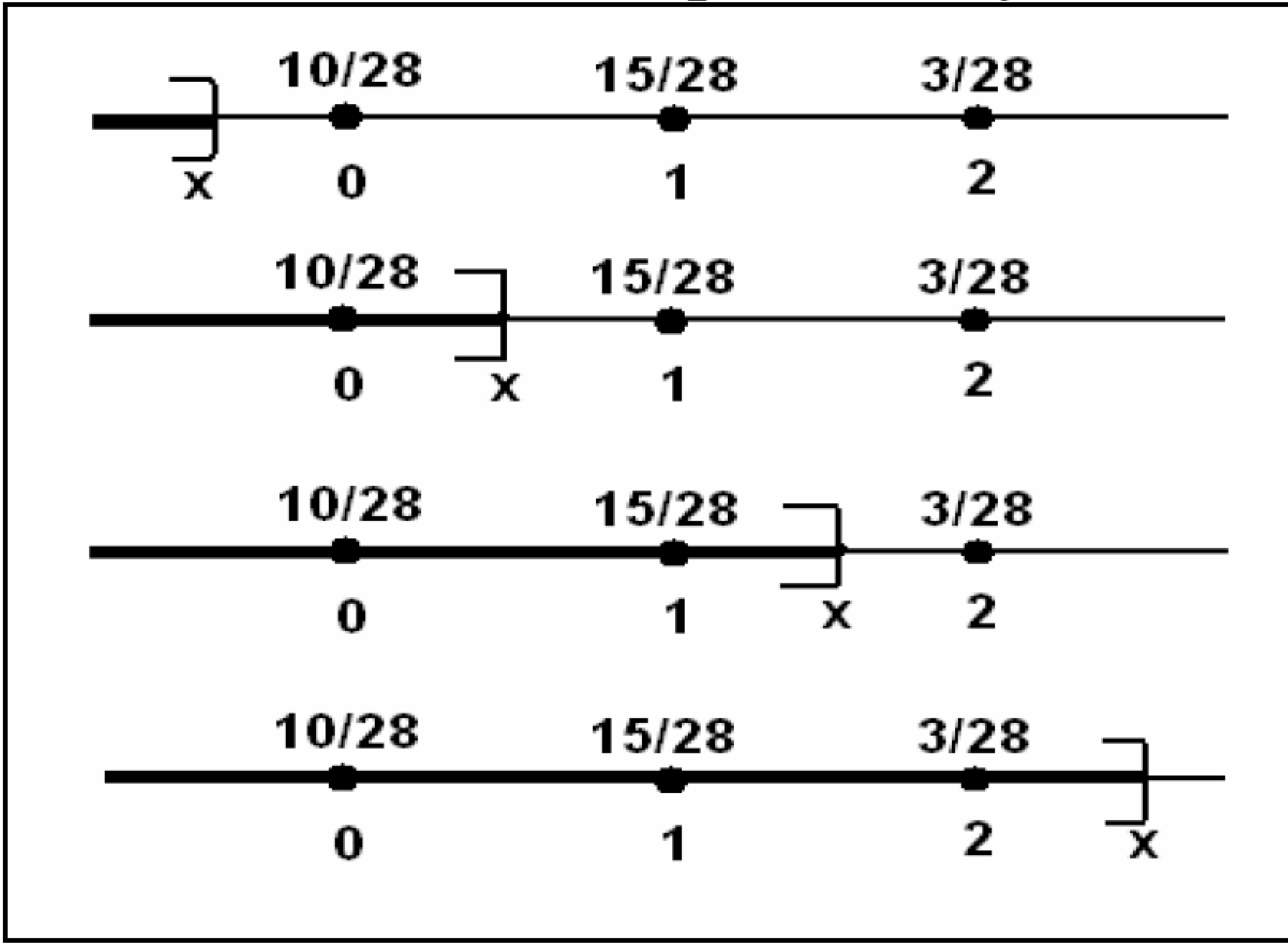
$$for \;\; x < 0: \quad F(x) = 0$$

$$for \;\; 0 \le x < 1: \quad F(x) = P(X = 0) = \frac{10}{28}$$

$$for \;\; 1 \le x < 2: \quad F(x) = P(X = 0) + P(X = 1) = \frac{10}{28} + \frac{15}{28} = \frac{25}{28}$$

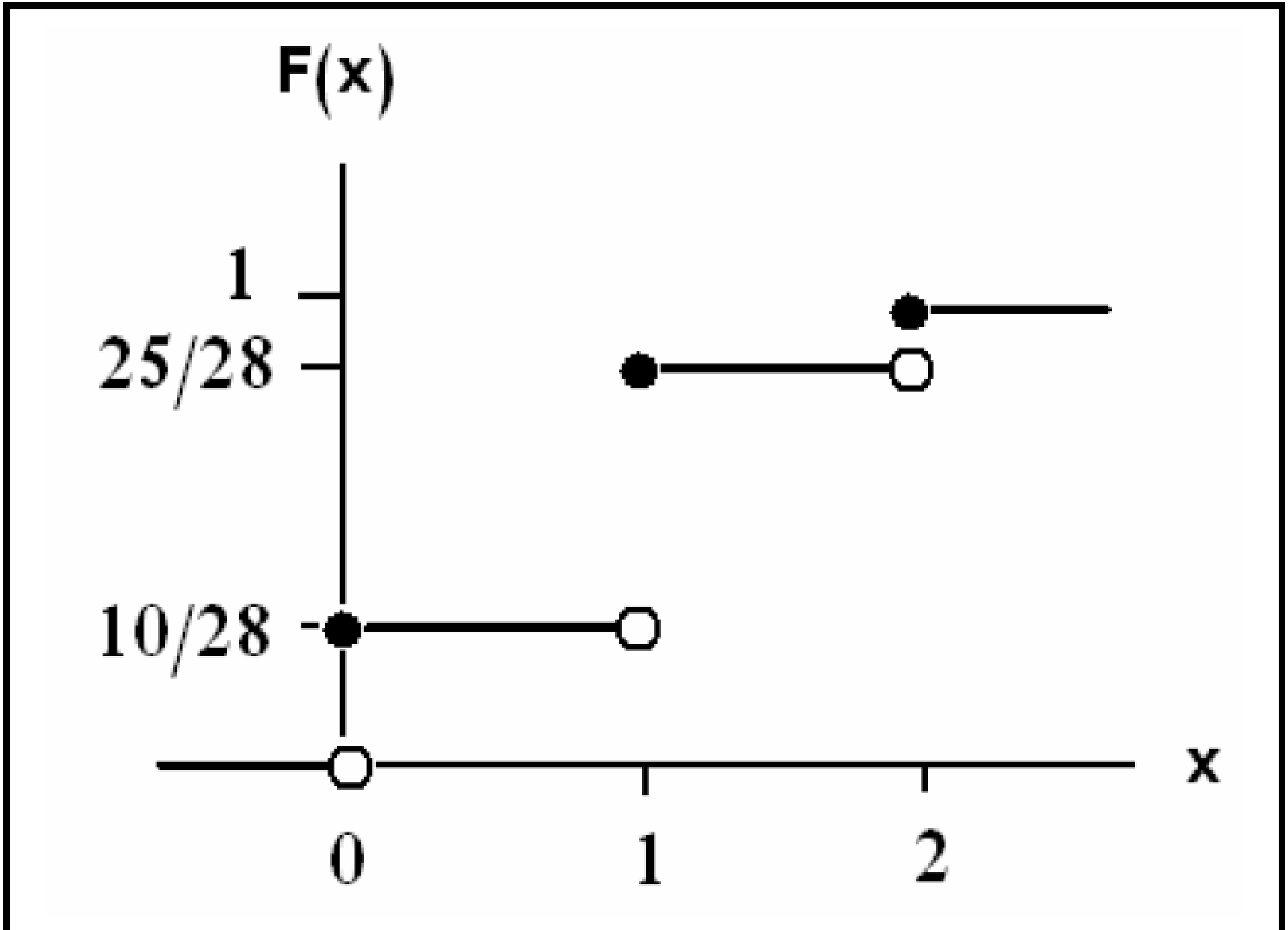
$$for \;\; x \ge 2: \quad F(x) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= \frac{10}{28} + \frac{15}{28} + \frac{3}{28} = 1$$


10/28
15/28
3/28
x
0
1
2
10/28
15/28
3/28
0
x
1
2
10/28
15/28
3/28
0
1
x
2
10/28
15/28
3/28
0
1
2
x

The CDF of the random variable X is:

$$F(x) = P(X \le x) = \begin{cases} 0 \ ; \ x < 0 \\ \frac{10}{28} \ ; \ 0 \le x < 1 \\ \frac{25}{28} \ ; \ 1 \le x < 2 \\ 1 \ ; \ x \ge 2 \end{cases}$$


F(x)
1
25/28
10/28
x
0
1
2

Note:

$$F(-0.5) = P(X \leq -0.5) = 0$$

$$F(1.5) = P(X \leq 1.5) = F(1) = 25/28$$

$$F(3.8) = P(X \leq 3.8) = F(2) = 1$$

Result:

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$$

$$P(a \leq X \leq b) = P(a < X \leq b) + P(X=a) = F(b) - F(a) + f(a)$$

$$P(a < X < b) = P(a < X \leq b) - P(X=b) = F(b) - F(a) - f(b)$$

Result:

Suppose that the probability function of X is:

x	x_1	x_2	x_3	…	x_n
f(x)	$f(x_1)$	$f(x_2)$	$f(x_3)$	…	$f(x_n)$

Where $x_1 < x_2 < \ldots < x_n$. Then:

$$F(x_i) = f(x_1) + f(x_2) + \ldots + f(x_i) \quad ; \quad i=1, 2, \ldots, n$$

$$F(x_i) = F(x_{i-1}) + f(x_i) \quad ; \quad i=2, \ldots, n$$

$$f(x_i) = F(x_i) - F(x_{i-1})$$

Example:

In the previous example,

$$P(0.5 < X \le 1.5) = F(1.5) - F(0.5) = \frac{25}{28} - \frac{10}{28} = \frac{15}{28}$$

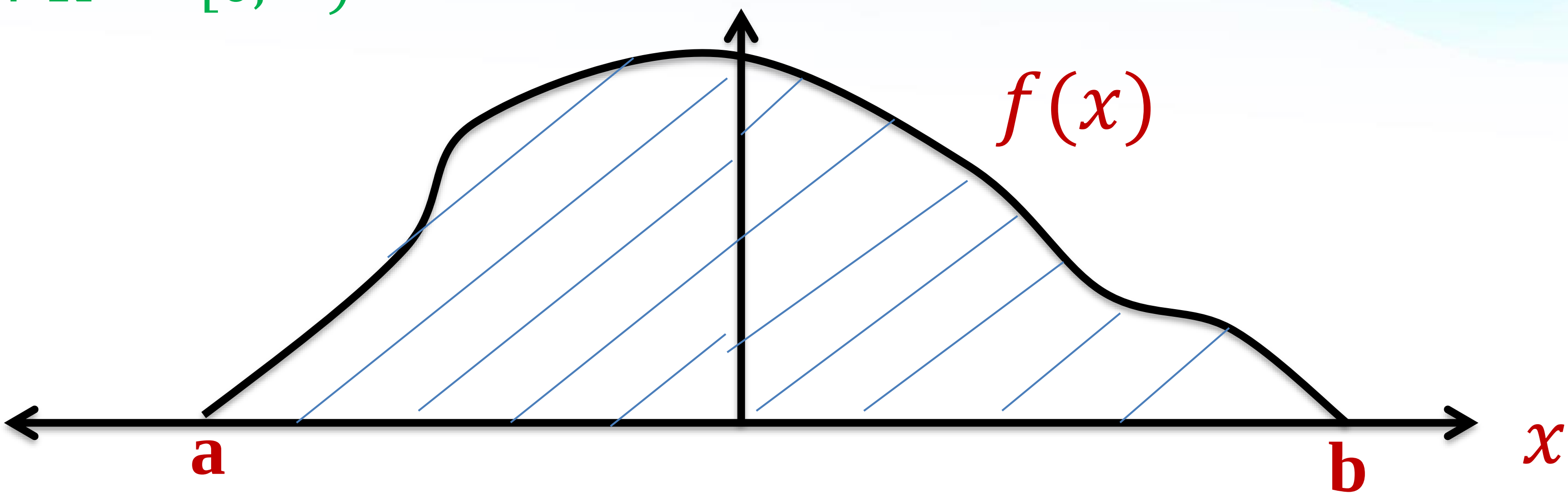
$$P(1 < X \le 2) = F(2) - F(1) = 1 - \frac{25}{28} = \frac{3}{28}$$

Continuous Probability Distributions

For any continuous random variable, X, there exists a non-negative function f(x), called the probability density function (p.d.f) through which we can find probabilities of events expressed in term of X.

For any continuous r. v. X, there exists a function $f(x)$, called the density function of X , for which:
(i) The total area under the curve of $f(x)$=1.

$f:\ \mathbb{R} \longrightarrow [0,\infty)$



$$area = \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

= area under the curve of f(x) and over the interval (a,b)

Definition

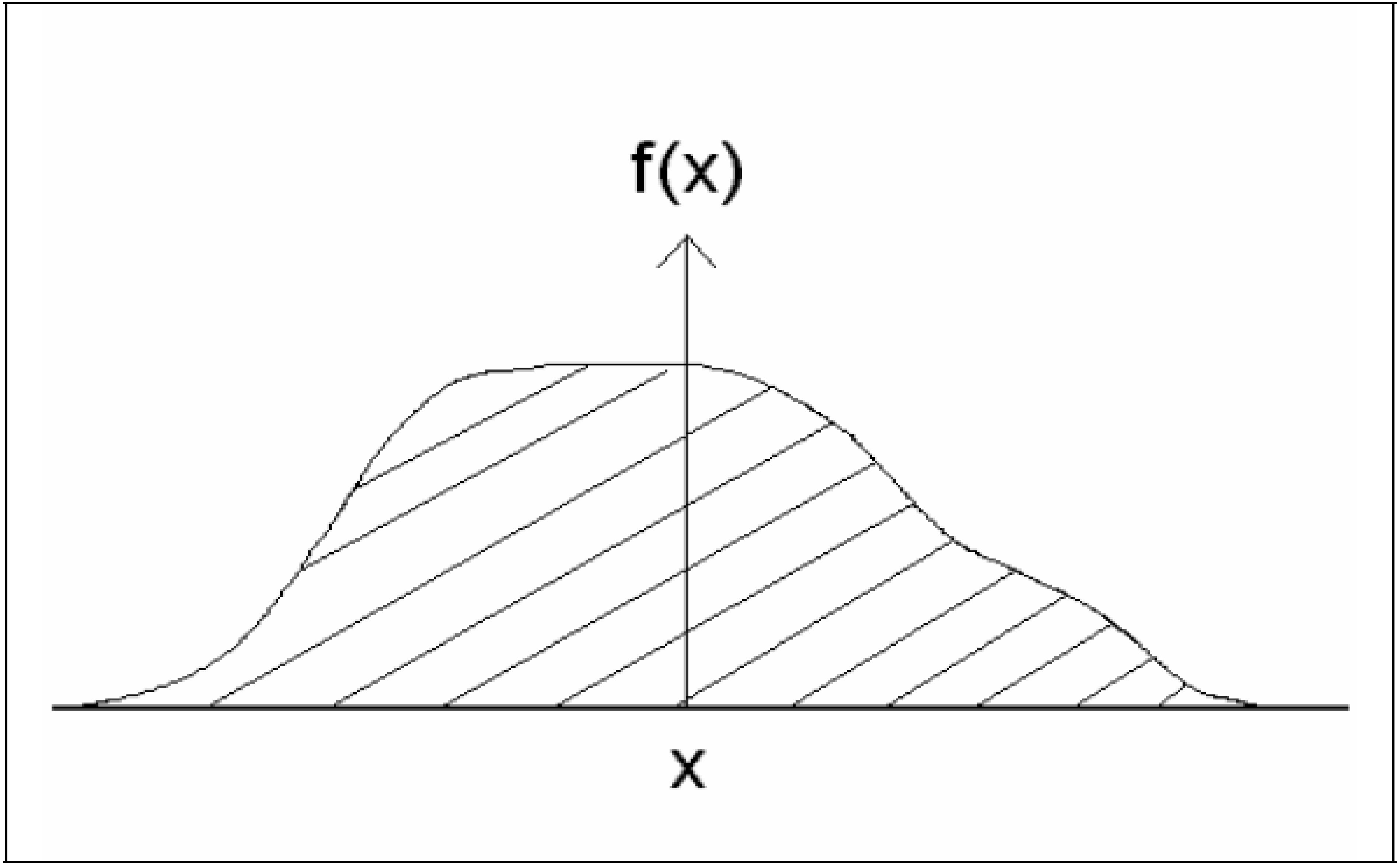
The function f(x) is a probability density function (pdf) for a continuous random variable X, defined on the set of real numbers, if:

1. $f(x) \geq 0 \quad \forall\, x \in R$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$
3. $P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx \quad \forall\, a, b \in R;\ a \leq b$

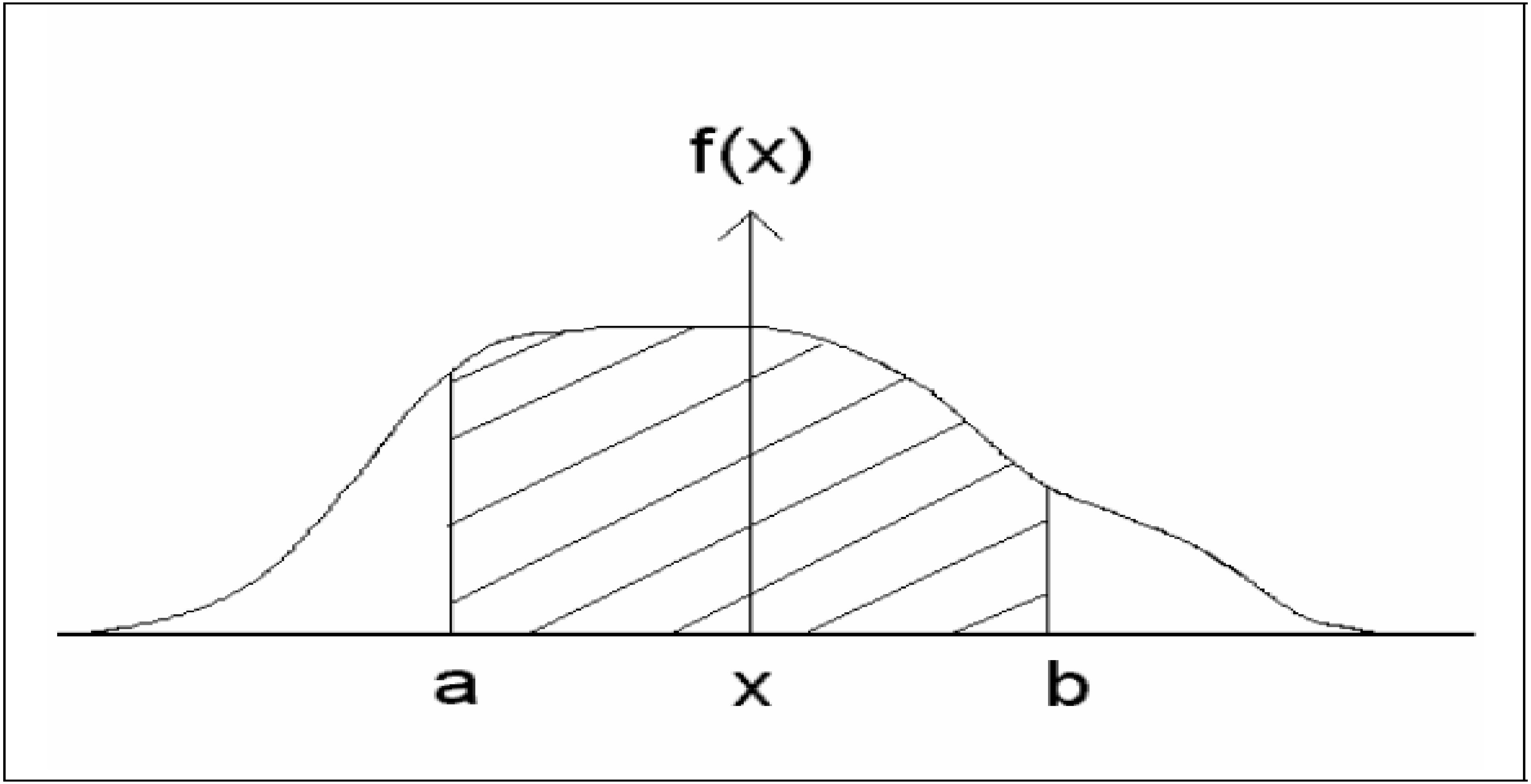
Note:

For a continuous random variable X, we have:

1. $f(x) \neq P(X=x)$ (in general)
2. $P(X=a) = 0$ for any $a \in R$
3. $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$
4. $P(X \in A) = \int_A f(x)\,dx$

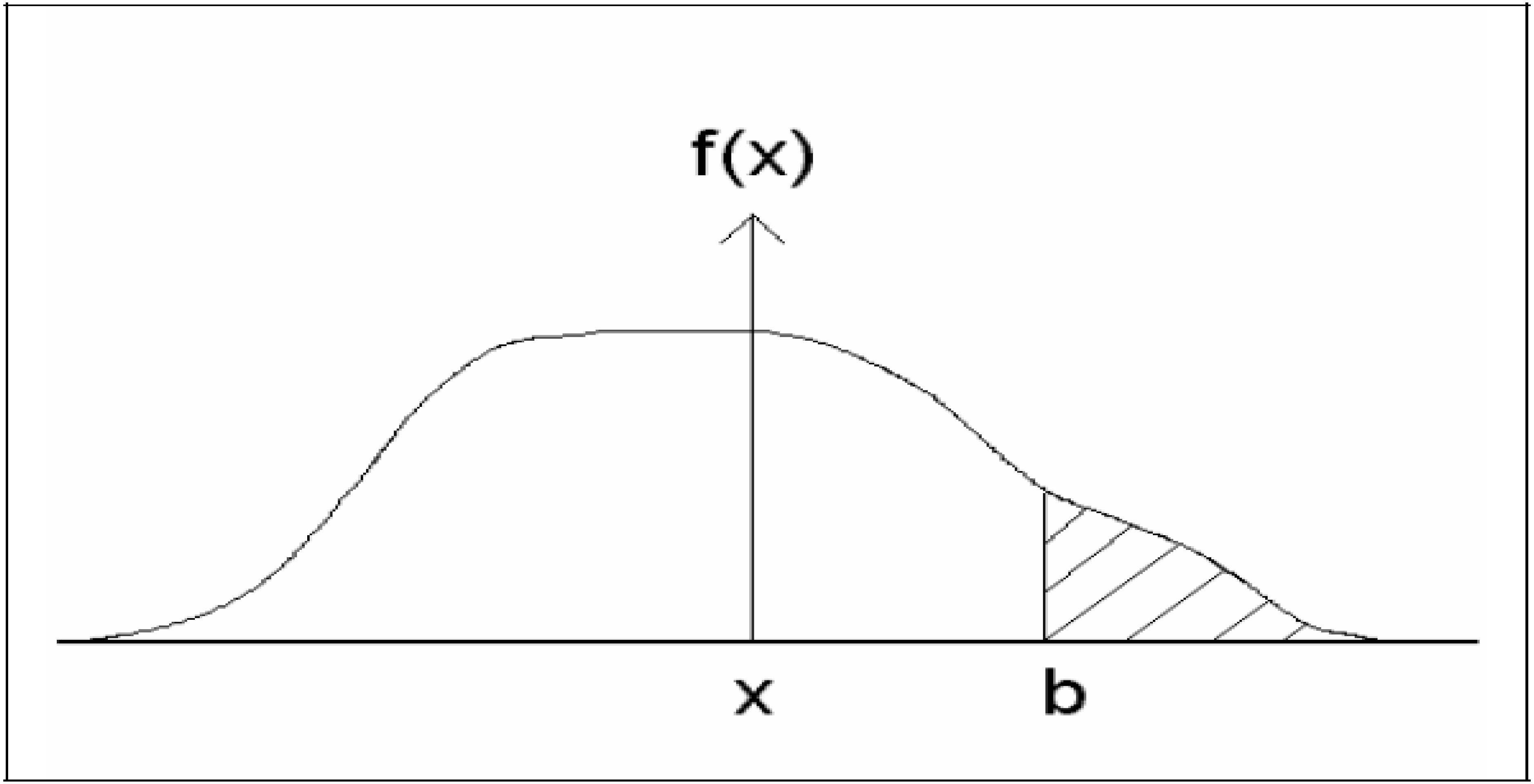


$$Total\ area = \int_{-\infty}^{\infty} f(x)\, dx = 1$$



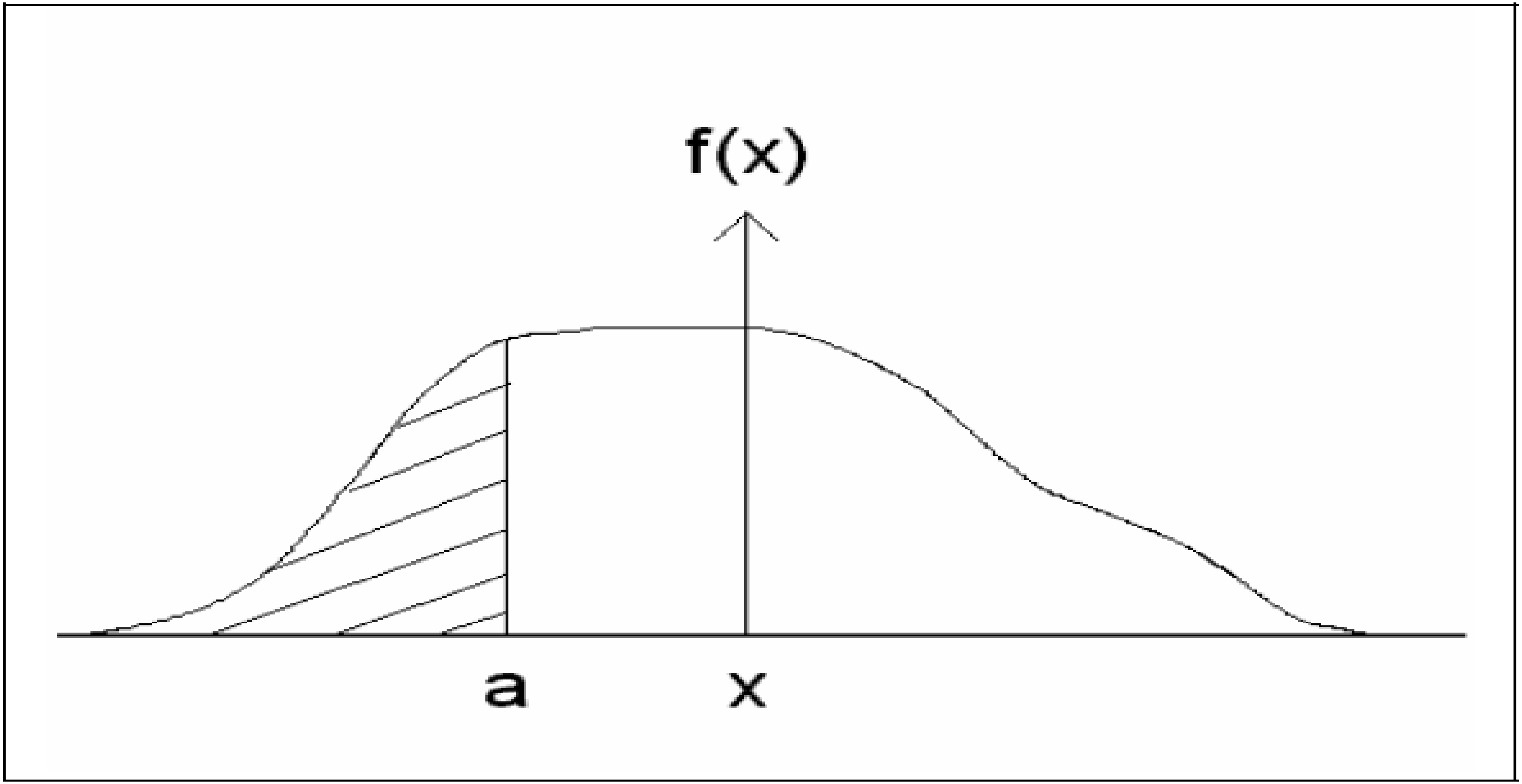
$$area = P(a \le X \le b)$$

$$= \int_a^b f(x)dx$$



$$area = P(X \ge b)$$

$$= \int_b^\infty f(x)\, dx$$



$$area = P(X \le a)$$

$$= \int_{-\infty}^{a} f(x)dx$$

Example

Suppose that the error in the reaction temperature, in °C, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

.

(a) Verify that $f(x)$ is a density function.

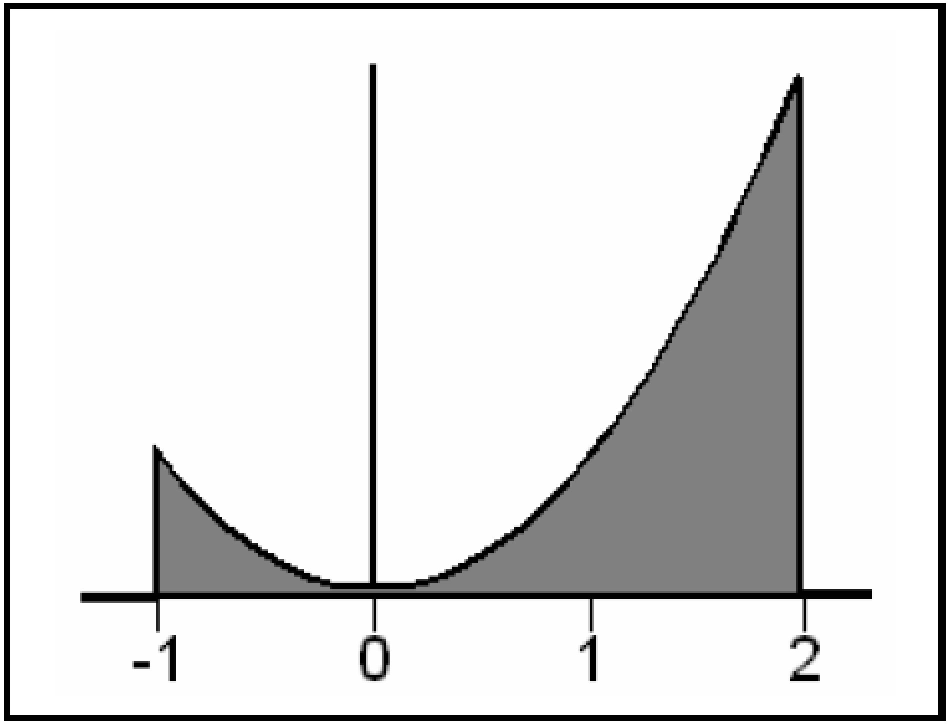
(b) Find $P(0 < X \leq 1)$.

Solution:

X = the error in the reaction temperature in ^{o}C.

X is continuous r. v.

$$f(x)=\begin{cases}\frac{1}{3}x^2 \; ; -1<x<2 \\ 0 \; ; \; elsewhere\end{cases}$$



1. (a) f(x) ≥ 0 because f(x) is a quadratic function.

$$\text{(b)} \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{2} \frac{1}{3}x^2\,dx + \int_{2}^{\infty} 0\,dx$$

$$= \int_{-1}^{2} \frac{1}{3}x^2\,dx = \left[\frac{1}{9}x^3 \,\middle|\, \begin{matrix} x=2 \\ x=-1 \end{matrix}\right]$$

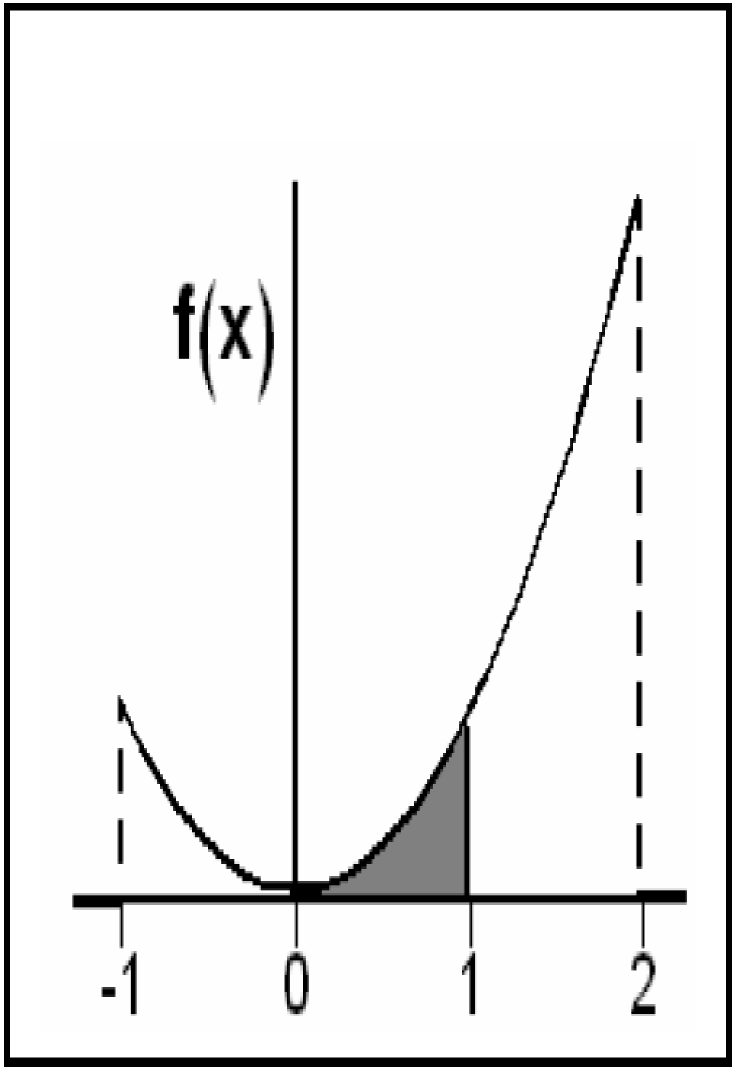
$$= \frac{1}{9}(8-(-1)) = 1$$

2. $P(0<X\leq 1) = \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{3}x^2\,dx$

$$= \left[\frac{1}{9}x^3 \Big|_{x=0}^{x=1}\right]$$

$$= \frac{1}{9}(1-(0))$$

$$= \frac{1}{9}$$



Definition

The cumulative distribution function (CDF), F(x), of a continuous random variable X with probability density function f(x) is given by:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt \; ; \; \text{for } -\infty < x < \infty$$

Result:

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$$

Example:

In the previous example

1. Find the CDF
2. Using the CDF, find $P(0<X\leq 1)$.

$$f(x)=\begin{cases}\frac{1}{3}x^2 \; ; -1<x<2 \\ 0 \; ; \; elsewhere\end{cases}$$

Solution:

(1) Finding F(x):

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt \;;\quad \text{for } -\infty < x < \infty$$

For $x < -1$:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

For $-1 \le x < 2$:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{x} \frac{1}{3} t^2\,dt$$

$$= \int_{-1}^{x} \frac{1}{3} t^2\,dt$$

$$= \left[\frac{1}{9} t^3 \;\middle|\; \begin{matrix} t = x \\ t = -1 \end{matrix} \right] = \frac{1}{9}(x^3 - (-1)) = \frac{1}{9}(x^3 + 1)$$

For x≥2:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{2} \frac{1}{3}t^2\,dt + \int_{2}^{x} 0\,dt = \int_{-1}^{2} \frac{1}{3}t^2\,dt = 1.$$

Therefore, the CDF is:

$$F(x) = P(X \le x) = \begin{cases} 0 & ; x < -1 \\ \frac{1}{9}(x^3 + 1) & ; -1 \le x < 2 \\ 1 & ; x \ge 2 \end{cases}$$

2. Using the CDF,

$$P(0<X\leq 1) = F(1) - F(0) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

Exercises

(1) 3.6 – 3.9 page 92

(2) 3.18 – 3.20 page 93

(3) On a laboratory assignment, if the equipment is working, the density function of the observed outcome, X, is

$$f(x) = k(1 - x), \qquad 0 < x < 1$$

- Determine k that renders $f(x)$ a valid density function.
- Calculate $P(X \leq 1/3)$.
- What is the probability that X will exceed 0.5?

Joint Probability Distributions

In general, if X and Y are two random variables, the probability distribution that defines their simultaneous behavior is called a joint probability distribution.

Note:

If X and Y are 2 discrete random variables, this distribution can be described with a joint probability **mass** function. If X and Y are continuous, this distribution can be described with a joint probability **density** function.

Two Discrete Random Variables:

If X and Y are discrete, with ranges R_X and R_Y , respectively, the joint probability mass function is

$$p(x, y) = P(X = x \text{ and } Y = y),\ x \in R_X\ ,\ y \in R_Y.$$

in the discrete case,

The function $f(x, y)$ is a **joint probability distribution** or **probability mass function** of the discrete random variables X and Y if

1. $f(x, y) \geq 0$ for all (x, y),
2. $\sum_x \sum_y f(x, y) = 1$,
3. $P(X = x, Y = y) = f(x, y)$.

Two Continuous Random Variables:

If X and Y are continuous, the joint probability density function is a function f(x,y) that produces probabilities:

$$P[(X,Y) \in A] = \iint_{A} f(x,y)\,dy\,dx$$

in the continuous case,

The function $f(x, y)$ is a **joint density function** of the continuous random variables X and Y if

1. $f(x, y) \geq 0$, for all (x, y),

2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$,

3. $P[(X, Y) \in A] = \int \int_A f(x, y)\, dx\, dy$, for any region A in the xy plane.

4) $P(a \leq x \leq b, c \leq y \leq d) = \int_a^b \int_c^d f(x, y)\, dy dx$

Example:

Suppose we have the following joint mass function

X \ Y	**-2**	**0**	**5**
1	0.15	*K*	0.20
3	0.20	0.05	0.15

Find the value of *k*?

Answer:

Using

$$\sum_{x}\sum_{y} f(x,y) = 1$$

We get

$$0.15 + 0.20 + k + 0.05 + 0.20 + 0.15 = 1$$

$$0.75 + k = 1$$

$$k = 1 - 0.75 = 0.25$$

Example:

Suppose we have the following joint density function

$$f(x,y)=\begin{cases}\dfrac{6-x-y}{8} & 0\le x\le 2 \quad , \quad 2\le y\le 4\\ 0 & O.W.\end{cases}$$

1) Prove that $f(x,y)$ is a joint probability function?
2) Calculate $P\left(X\le\frac{2}{3}, Y\le\frac{5}{2}\right)$

Answer:

$$1)\ f(x,y) \geq 0$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) = \int_{2}^{4}\int_{0}^{2} \frac{6-x-y}{8} dx\,dy$$

$$= \frac{1}{8}\int_{2}^{4} \left[\int_{0}^{2}(6-x-y)dx\right]dy$$

$$= \frac{1}{8}\int_{2}^{4} \left[6x - \frac{x^2}{2} - yx\right]_{0}^{2} dy$$

$$= \frac{1}{8}\int_{2}^{4} \left[\left(6(2) - \frac{(2)^2}{2} - y(2)\right) - 0\right]dy$$

$$= \frac{1}{8}\int_{2}^{4}(10-2y)dy$$

$$= \frac{1}{8}\left[10y - y^2\right]_{2}^{4} = \left[\left(10(4)-(4)^2\right) - \left(10(2)-(2)^2\right)\right]$$

$$= \frac{1}{8}(40-16) - (20-4) = \frac{1}{8}(8) = 1$$

$$2)\ P\left(x \le \frac{2}{3}, y \le \frac{5}{2}\right) = \int_0^{\frac{2}{3}} \int_2^{\frac{5}{2}} \left(\frac{6 - x - y}{8}\right) dy\,dx$$

$$\vdots$$

$$\vdots$$

$$= \frac{41}{288} = 0.142$$ **Prove that?**

Another example see Ex 3.15 on page 96

The marginal distributions

The **marginal distributions** of X alone and of Y alone are

$$g(x) = \sum_{y} f(x, y) \quad \text{and} \quad h(y) = \sum_{x} f(x, y)$$

for the discrete case, and

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy \quad \text{and} \quad h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

for the continuous case.

Example:

Suppose we have the following joint mass function

X \ Y	-2	0	5
1	0.15	0.25	0.20
3	0.20	0.05	0.15

Find the marginal distributions of X and Y?

Answer:

Y / X	-2	0	5	Sum
1	0.15	0.25	0.20	0.6
3	0.20	0.05	0.15	0.4
Sum	0.35	0.30	0.35	**1**

So

The marginal distribution of ***X***

x	1	3	Sum
$f(x)$	0.6	0.4	1

The marginal distribution of ***Y***

y	-2	0	5	Sum
$f(y)$	0.35	0.30	0.35	1

Example:

Suppose we have the following joint density function

$$f(x,y)=c(x+y) \quad , \quad 0\le x\le 1,\, 0\le y\le 2$$

Find the value of c ?

Find the marginal distributions of X and Y?

Answer:

$$1)\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_{0}^{2}\int_{0}^{1} c\left(x+y\right)dx\,dy = 1$$

$$\vdots$$

$$\Rightarrow c = \frac{1}{3} \Rightarrow f(x,y) = \frac{1}{3}(x+y)$$

$$2)\ f(x) = \int_{y} f(x,y) = \int_{0}^{2} \frac{1}{3}(x+y)\,dy$$

$$\vdots$$

$$\Rightarrow f(x) = \frac{2}{3}(x+1)$$

$$f(y) = \int_{x} f(x,y) = \int_{0}^{1} \frac{1}{3}(x+y)\,dx$$

$$\vdots$$

$$\Rightarrow f(y) = \frac{1}{3}\left(y+\frac{1}{2}\right)$$

conditional probability distribution

Let X and Y be two random variables, discrete or continuous. The **conditional distribution** of the random variable Y given that $X = x$ is

$$f(y|x) = \frac{f(x,y)}{g(x)}, \text{ provided } g(x) > 0.$$

Similarly, the conditional distribution of X given that $Y = y$ is

$$f(x|y) = \frac{f(x,y)}{h(y)}, \text{ provided } h(y) > 0.$$

Example:

The joint density for the random variables (X, Y), where X is the unit temperature change and Y is the proportion of spectrum shift that a certain atomic particle produces, is

$$f(x,y) = \begin{cases} 10xy^2, & 0 < x < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal densities $g(x)$, $h(y)$, and the conditional density $f(y|x)$.

(b) Find the probability that the spectrum shifts more than half of the total observations, given that the temperature is increased by 0.25 unit.

Solution:

(a) By definition,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_{x}^{1} 10xy^2\, dy$$

$$= \frac{10}{3}xy^3\Big|_{y=x}^{y=1} = \frac{10}{3}x(1-x^3),\ 0 < x < 1,$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_{0}^{y} 10xy^2\, dx = 5x^2y^2\Big|_{x=0}^{x=y} = 5y^4,\ 0 < y < 1.$$

Now

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{10xy^2}{\frac{10}{3}x(1-x^3)} = \frac{3y^2}{1-x^3},\ 0 < x < y < 1.$$

(b) Therefore,

$$P\left(Y > \frac{1}{2} \,\middle|\, X = 0.25\right) = \int_{1/2}^{1} f(y \mid x = 0.25)\, dy = \int_{1/2}^{1} \frac{3y^2}{1 - 0.25^3}\, dy = \frac{8}{9}$$

Another example see Ex 3.20 on page 100

Statistical Independence

Let X and Y be two random variables, discrete or continuous, with joint probability distribution $f(x, y)$ and marginal distributions $g(x)$ and $h(y)$, respectively. The random variables X and Y are said to be **statistically independent** if and only if

$$f(x, y) = g(x)h(y)$$

for all (x, y) within their range.

Example:

Suppose we have the following joint distribution

$$f(x,y)=\begin{cases} 3e^{-x}e^{-3y} & , \quad x \geq 0 \, , \, y \geq 0 \\ 0 & , \quad O.W. \end{cases}$$

Prove that X and Y are independent?

$$f(x,y)=f(x)\cdot f(y)$$

$$1)\ f(x)=\int_{-\infty}^{\infty} f(x,y)dy=\int_{0}^{\infty} 3e^{-x}e^{-3y}dy=3e^{-x}\int_{0}^{\infty} e^{-3y}dy$$

$$=3e^{-x}\left[\frac{e^{-3y}}{-3}\right]_{0}^{\infty}=-e^{-x}\left[0-1\right]=e^{-x}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$2)\ f(y)=\int_{-\infty}^{\infty} f(x,y)dx=\int_{0}^{\infty} 3e^{-x}e^{-3y}dx=3e^{-3y}\int_{0}^{\infty} e^{-x}dx$$

$$=3e^{-3y}\left[-e^{-x}\right]_{0}^{\infty}=-3e^{-3y}\left[0-1\right]=3e^{-3y}\ \ldots\ldots\ldots\ldots\ldots(2)$$

$$From\ (1)\ and\ (2)\Rightarrow$$

$$f(x,y)=f(x)\cdot f(y)$$

Notes:

if X and Y are independent, then

$$1)\ f(x,y) = f(x) \cdot f(y)$$

$$2)\ f\left(x/y\right) = f(x)$$

$$3)\ f\left(y/x\right) = f(y)$$

Example:

Suppose we have the following joint distribution

$$f(x,y) = k(8-x-y),\ 0 \le x \le 4, 1 \le y \le 3$$

Find:

1) The value of k

2) $f(x), f(y)$

3) $f(y/x),\ f(x/y)$

4) $P(x \le 3)$

5) $P(x \le 3/y \le 2)$

Solution:

$$1)\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_{1}^{3}\int_{0}^{4} k(8-x-y)\,dx\,dy = 1$$

$$\Rightarrow k\int_{1}^{3}\left[\int_{0}^{4}(8-x-y)\,dx\right]dy = 1$$

$$\Rightarrow k\int_{1}^{3}\left[8x-\frac{x^{2}}{2}-xy\right]_{0}^{4} = 1$$

$$\Rightarrow k\int_{1}^{3}\left[8(4)-\frac{4^{2}}{2}-4y\right]dy = 1$$

$$\Rightarrow k\int_{1}^{3}\left(-4y+24\right)dy = 1$$

$$\Rightarrow k\left[\frac{-4y^{2}}{2}+24y\right]_{1}^{3} = 1$$

$$\Rightarrow k\left(32\right) = 1 \Rightarrow k = \frac{1}{32} \Rightarrow f(x,y) = \frac{1}{32}(8-x-y)$$

$$2)\ f(x)=\int_{-\infty}^{\infty} f(x,y)\,dy$$

$$\Rightarrow f(x)=\int_{1}^{3}\frac{1}{32}(8-x-y)\,dy$$

$$\vdots$$

$$\Rightarrow f(x)=\frac{1}{32}(12-2x)\quad,\quad 0\le x\le 4$$

$$f(y)=\int_{-\infty}^{\infty} f(x,y)\,dx$$

$$\Rightarrow f(y)=\int_{0}^{4}\frac{1}{32}(8-x-y)\,dx$$

$$\vdots$$

$$\Rightarrow f(y)=\frac{1}{32}(24-4y)\quad,\quad 1\le y\le 3$$

$$3)\ f(y/x)=\frac{f(x,y)}{f(x)}=\frac{\frac{1}{32}(8-x-y)}{\frac{1}{32}(12-2x)}=\frac{(8-x-y)}{(12-2x)}$$

$$f(x/y)=\frac{f(x,y)}{f(y)}=\frac{\frac{1}{32}(8-x-y)}{\frac{1}{32}(24-4y)}=\frac{(8-x-y)}{(24-4y)}$$

$$4)\ P(x\le 3)=\int_0^3 f(x)dx=\frac{27}{32}$$

$$5)\ P\left(x \le 3 / y \le 2\right) = \frac{P(x \le 3, y \le 2)}{p(y \le 2)}$$

$$P(x \le 3, y \le 2) = \int_1^2 \int_0^3 f(x, y)\,dx\,dy = \int_1^2 \int_0^3 \frac{1}{32}\left(8 - x - y\right)dx\,dy = \frac{30}{64}$$

$$p(y \le 2) = \int_1^2 f(y)\,dy = \int_1^2 \frac{1}{32}\left(24 - 4y\right)dy = \frac{18}{32}$$

$$P\left(x \le 3 / y \le 2\right) = \frac{5}{6}$$